Math 553 Homework 3

Due Wed. Sept. 12, 2012

1. Let p be a prime number. For any finite group H, $\mathcal{P}_H$ denotes the set of all Sylow p-subgroups of H. (If p doesn't divide the order $|H|$, define the Sylow p-subgroup to be the subgroup consisting of the identity element alone). Let N be a normal subgroup of the finite group G. Prove that

$$\mathcal{P}_N = \{\, P \cap N \mid P \in \mathcal{P}_G \,\}$$

and that

$$\mathcal{P}_{G/N} = \{\, PN/N \mid P \in \mathcal{P}_G \,\}.$$

2. (a) A *Hall subgroup* of a finite group G is a subgroup whose order is relatively prime to its index. Prove that if H is a *normal* Hall subgroup of G then H contains every subgroup of G whose order is relatively prime to the index $[G:H]$.

(b) Let H and K be groups, and S the set of group homomorphisms from K to $\mathrm{Aut}(H)$. Show that for $\alpha \in \mathrm{Aut}(H)$, $\beta \in \mathrm{Aut}(K)$ and $\theta \in S$, the map $(\alpha,\beta)\cdot\theta$ from K to $\mathrm{Aut}(H)$ given by

$$\big((\alpha,\beta)\cdot\theta\big)(k) := \alpha \circ \theta(\beta^{-1}k) \circ \alpha^{-1}$$

is a group homomorphism; and that there results an action of the group $\mathrm{Aut}(H)\times\mathrm{Aut}(K)$ on the set S.

(c) Let H be a finite abelian group and let K be a finite group of order relatively prime to that of H. Prove that the isomorphism classes of semi-direct products $H \rtimes_\theta K$ correspond one-one with the orbits of the action in (b). (You may assume the results appearing in the notes "Isomorphisms of semi-direct products" (on the course webpage).)

3. (a) Show that the center of the group of transformations

$$x \mapsto ax + b \qquad (a, b \in \mathbb{Z}/5\mathbb{Z},\ a \neq 0)$$

is trivial (i.e., consists of the identity alone).

(b) Let $\zeta = e^{\frac{2\pi i}{5}}$, a fifth root of unity. Is the group (of order 20) generated by the complex matrices

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad (i^2 = -1).$$

isomorphic to the group in (a)?

(c) Show that the Sylow 5-subgroup of a group of order 20 is normal.

(d) Show that there are exactly five distinct groups of order 20.

(OVER)

4. Groups of order 30 are classified in the first example on p. 182 in D&F. Show that the four different groups of order 30 have 1, 3, 5, and 15 elements of order 2, respectively.

5. Groups of order 12 are classified in the second example on p. 182 in D&F. For each of the following groups, determine which of the groups in that example it's isomorphic to.

(a) The multiplicative group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad (a, b, c \in \mathbb{Z}_3,\ ac \neq 0).$$

(b) The multiplicative group generated by the complex matrices

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad (i^2 = -1,\ \omega^3 = 1,\ \omega \neq 1).$$

(c) The transformations of the form $x \mapsto ax + b$ $(a \neq 0)$ of the field $\mathbb{F}_4$ into itself.

($\mathbb{F}_4$ is a field with four elements, whose existence and uniqueness is discussed in the example in D&F, p. 549. We'll come to that later; for now you just have to know what a field is, and you may assume that there is one with four elements, and that in it, 1+1=0— or, if you're ambitious, construct $\mathbb{F}_4$ by defining a suitable multiplication on the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

(d) The dihedral group D_{12}.

(e) A non-abelian semidirect product of a group of order 4 by a group of order 3.

6. Let p be an odd prime. Consider the group, of order p^3, consisting of all matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with a, b, and c in the field $\mathbb{Z}/p\mathbb{Z}$. (The group operation is the usual multiplication of matrices.) To which of the two non-abelian groups discussed in the Example in D&F, p. 183 is this group isomorphic? (Why?)