Math 553 Homework 5

Due Fri. Sept. 28, 2012

Work in a fixed commutative monoid M with cancelation.

You may quote, without proof, any result proved in class.

1. Prove that if $[a,b]$ exists then for all c,

(i) $[ac,bc]=[a,b]c$.

(ii) (a,b) exists, and $(ac,bc)=(a,b)c$.

2. Prove that $[a,b]$ exists $\iff$ (ac,bc) exists for all c.

3. Prove: if a is prime and a doesn't divide b then $[a^n,b]=a^nb$ for all n

4. Suppose a is a unit or a product of primes. Prove:

(i) (a,b) and $[a,b]$ exist for all b.

(ii) If (b,c) exists then $(ab,ac)=a(b,c)$.

5. Assume that (x,y) exists for all $x,y\in M$.

(i) Prove that if $(a,b)=1$ then $(a^i,b^j)=1$ for all (i,j).

(ii) Without assuming $(a,b)=1$, deduce from (i) that $(a,b)^n=(a^n,b^n)$ for all n.

(iii) Prove that if $(a,b)=1$ and $ab=c^n$ then $a\sim(a,c)^n$ and $b\sim(b,c)^n$.

6. Assuming all the gcd's and lcm's that appear exist, prove the "distributivity laws":

(i) $\big[a,(b,c)\big]=\big([a,b],[a,c]\big)$, and

(ii) $\big(a,[b,c]\big)=\big[(a,b),(a,c)\big]$.

Hint. In (i), the hard part is to show that $([a,b],[a,c])$ divides $[a,(b,c)]$. For this, divide everything in sight by (a,b,c)—which exists (why?)—to reduce to where $(a,b,c)=1$, in which case $[a,(b,c)]=(ab,ac)$.

Remarks. In a UFM, this problem is easily dealt with by means of prime-power factorizations.

To see all this in an even more abstract setting, look at Thm. 3 on p. 135 (or the note on p. 133) in Garrett Birkhoff's book *Lattice Theory*.