Math 553 Homework 8

Due Wed. Oct. 24, 2012

For the next three problems, some of the material in D&F, §§9.1–9.5 will be useful. These sections are mostly review of material from MA 503, and it will be assumed from now on—including exams—that you know what's in them.

1. Let R be a UFD, with fraction field K. Suppose you already have computer algorithms for factoring into primes in R and in the polynomial ring $K[X]$. Describe briefly how you would instruct a computer to factor into primes in $R[X]$.

2. Let k be a field, x, y, and z indeterminates.

(a) Let $f(x)$ and $g(x)$ be relatively prime polynomials in $k[x]$. Show that in the polynomial ring $k(y)[x]$, $f(x) - yg(x)$ is irreducible.

(b) Prove that in $k(y, z)[x]$, the polynomial

$$x^4 - yzx^3 + (y^2z^2 - y)x^2 + (y^2z - y)x + y^2z$$

is irreducible. (Hint. Eisenstein, after rearranging.)

3. Let R be an integral domain with fraction field K, let $R[X]$ be a polynomial ring, and let a and b be nonzero elements in R. Prove:

(a) If R is a UFD and $P \subset R[X]$ is a prime ideal with $P \cap R = (0)$, then P is a principal ideal.

(b) $aR \cap bR = abR$ iff the ring $R[X]/(aX - b)$ is an integral domain.

(c) If $c = aq = bp$ is a nonzero common multiple of a and b then c is an l.c.m. of a and b iff $pX - q$ is a prime element in $R[X]$.

(d) An l.c.m. $[a, b]$ exists iff the kernel of the R-homomorphism $\phi\colon R[X] \to R[\frac{b}{a}] \subset K$ taking X to $\frac{b}{a}$ is a principal ideal.

4. (a) Prove that if $x \neq 0$ and y are elements in a UFD such that x^2 divides y^2, then x divides y.

(b) Let k be a field. In the quotient ring $R = k[X, Y, Z]/(Y^2 - X^2Z)$ let $x = \overline{X}$ and $y = \overline{Y}$ be the natural images of X and Y. Show that x^2 divides y^2 in R, but x does not divide y.

(c) Is R an integral domain? (Why?)