Math 553 Homework 9

Due Wed. Oct. 31, 2012

1. Let $K \subset L$ be a field extension of finite degree, and let $f \in K[X]$ be an irreducible polynomial whose degree is relatively prime to $[L : K]$. Show that f is irreducible in $L[X]$.
Hint: Consider the degree $[F : K]$ where F is obtained from L by adjoining a root of f.

2. Read D&F pp. 587-8, starting with Prop. 18 (to be proved in class).

(a) Factor $X^{16} - X$ into irreducible polynomials over $\mathbb{F}_4$ and over $\mathbb{F}_8$.
Represent the elements of $\mathbb{F}_4$ as 0, 1, α, and $\alpha^2 = \alpha + 1$.

(b) Optional. In MAPLE type `Factor (X^80 - 1) mod 3;` ⟨ return ⟩.
How many irreducible polynomials of degree 4 are there in $\mathbb{Z}/3[X]$?
What about degree 3? (After answering, type `? mipolys`).

(c) Optional. Type `? Factor` . Display the set of all irreducible polynomials of degree 2 over a field of cardinality 9.

3. (a) Show that the polynomials $f(X) = X^3 + X^2 + 1$ and $g(X) = X^3 + X + 1$ are irreducible over the field $\mathbb{F}_4$ with four elements. (Cf. 1.)

(b) The fields $\mathbb{F}_4[X]/\big(f(X)\big)$ and $\mathbb{F}_4[Y]/\big(g(Y)\big)$ must be isomorphic, since both have cardinality 64. Describe *explicitly* an isomorphism between them.

4. Determine the minimal polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over each of the following:
(a) $\mathbb{Q}$, (b) $\mathbb{Q}(\sqrt{5})$, (c) $\mathbb{Q}(\sqrt{10})$, (d) $\mathbb{Q}(\sqrt{15})$.

5. Let F be a field, and let α be an element which generates a field extension of F of degree 5. Prove that α^2 generates the same extension.

6. Decide whether or not $i := \sqrt{-1}$ is in the field
(a) $\mathbb{Q}(\sqrt{-2})$, (b) $\mathbb{Q}(\sqrt[4]{-2})$, (c) $\mathbb{Q}(\sqrt[4]{-4})$, (d) $\mathbb{Q}(\alpha)$ where $\alpha^3 + \alpha = -1$.

7. Let α, β be complex numbers of degree 3 over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha, \beta)$. Determine the possibilities for $[K : \mathbb{Q}]$.

8. Determine whether or not the regular 9-gon is constructible by ruler and compass.

9. (a) Let F be a finite field of characteristic p, and let $\varphi\colon F \to F$ be the map defined by $\varphi(x) = x^p$. Show that φ is an automorphism of F.

(b) Show that every automorphism of F is a power of φ.
Hint: Let E be the prime subfield of F. There is a β such that $F = E(\beta)$. (Why?). Show that every root of the minimal polynomial of β over E has the form $\varphi^n(\beta)$.