Supplement on imaginary quadratic UFDs

(a) Let $\omega \in \mathbb{C}$ be such that $q := \omega - \omega^2$ is a non-prime integer ≥ 2. Show that any $\mathbb{Z}$-prime divisor of q is irreducible in $\mathbb{Z}[\omega]$ but not prime in $\mathbb{Z}[\omega]$ (so that $\mathbb{Z}[\omega]$ is not a UFD).

(b) Show that if the integer q in (a) is prime, then $x + \omega$ is irreducible in $\mathbb{Z}[\omega]$ for all integers x such that $0 < x < q - 1$. Deduce that if $q + 2$ is not prime then $\mathbb{Z}[\omega]$ is not a UFD.

(c) Show that if $q = 29$ in (b) then $\mathbb{Z}[\omega]$ is not a UFD.

Remark: In just a few minutes, Mathematica tells you that as q runs through the first ten million primes, only $q = 17$ and $q = 41$ have the property that $\text{Norm}(x+\omega) = x^2 + x + q$ is prime for $0 \leq x \leq 10$. (See below.)

It follows that the only possible values of $q \leq 179,424,673$ (the ten millionth prime) for which $\mathbb{Z}[\omega]$ is a UFD are $1, 2, 3, 5, 11, 17, 41$; and it is not too difficult to show that these values of q do indeed give rise to UFD's. In fact Gauss did that around 1800, did extensive calculations (without Mathematica), and then conjectured that there are no other values of q at all giving UFDs. This was worked on by professional mathematicians for 150 years before an "amateur" Kurt Heegner, proved it. (It was not recognized for 20 years that his published proof was correct.) See <`http://mathworld.wolfram.com/HeegnerNumber.html`>

Here is another strange fact which, surprisingly, is tied in with the fact that $\mathbb{Z}[\omega] = \mathbb{Z}[(1+\sqrt{-163})/2]$ is a UFD (this is the case $q = 41$):

The number $e^{\pi\sqrt{163}}$ differs from an integer by less than 10^{-12}, see below. (So a calculator would tell you that it is an integer!)

The connection can be explained via the theory of elliptic and modular curves—the same theory about the arithmetic of curves of degree 3 which underlies Heegner's proof, and even Wiles' proof of Fermat's last theorem.

OVER

Some Mathematica Output

```
Do[If[ PrimeQ[Prime[n]+2]
         &&PrimeQ[Prime[n]+6]
         &&PrimeQ[Prime[n]+12]
         &&PrimeQ[Prime[n]+20]
         &&PrimeQ[Prime[n]+30]
         &&PrimeQ[Prime[n]+42]
         &&PrimeQ[Prime[n]+56]
         &&PrimeQ[Prime[n]+72]
         &&PrimeQ[Prime[n]+90]
         &&PrimeQ[Prime[n]+110],
      Print[Prime[n]]],
{n, 10000000}]
//Timing
17
41
{358.63 Second}
```

(* $x^2 + x + 844427$ is prime for $x = 0, 1, 2, ...9$, but not for $x = 10$ *)

```
PrimeQ[844427 + {2,6,12,20,30,42,56,72,90,110}]
{True,True,True,True,True,True,True,True,True,False}
```

```
Prime[10000000]
179424673
```

```
N[E ^ (Pi Sqrt[163]), 35]
262537412640768743.99999999999925007
```