Isomorphisms of semi-direct products.

Proposition. *Let H, K be groups, let $\theta_i \colon K \to \mathrm{Aut}(H)$ $(i = 1, 2)$ be homomorphisms, and let $G_i := H \rtimes_{\theta_i} K$ be the corresponding semi-direct products. Let H_i and K_i be the natural images of H and K respectively in G_i. If*

(C) : there exist isomorphisms $\alpha\colon H \xrightarrow{\sim} H$, $\beta\colon K \xrightarrow{\sim} K$ such that $\forall k \in K$,

$$\theta_2\big(\beta(k)\big) = \alpha \circ \theta_1(k) \circ \alpha^{-1},$$

[that is, the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\theta_1} & \mathrm{Aut}(H) \\ \beta\big\downarrow & & \big\downarrow\text{conjugation by } \alpha \\ K & \xrightarrow[\theta_2]{} & \mathrm{Aut}(H) \end{array}$$]

then there exists an isomorphism $\phi\colon G_1 \to G_2$ such that $\phi(H_1) = H_2$.

Conversely, if H is abelian *and such a ϕ exists, then (C) holds.*

Proof. If (C) holds, define ϕ by

$$\phi(h, k) = \big(\alpha(h), \beta(k)\big) \qquad (h \in H,\ k \in K),$$

and check ...

Now suppose only that ϕ exists. let $\bar\phi\colon G_1/H_1 \xrightarrow{\sim} G_2/H_2$ be the induced isomorphism. Define α and β to be the natural compositions

$$\alpha\colon\ H \xrightarrow{\sim} H_1 \xrightarrow{\phi} H_2 \xrightarrow{\sim} H,$$

$$\beta\colon\ K \xrightarrow{\sim} K_1 \xrightarrow{\sim} G_1/H_1 \xrightarrow{\bar\phi} G_2/H_2 \xrightarrow{\sim} K_2 \xrightarrow[j]{\sim} K.$$

Unraveling the definitions, we find for all $h \in H,\ k \in K$ that:

$$(*) \qquad \begin{aligned} \theta_2\big(\beta(k)\big)(h) &= j\big[(1, \beta(k))(h, 1)(1, \beta(k))^{-1}\big], \\ \alpha \circ \theta_1(k) \circ \alpha^{-1}(h) &= j\phi\big[(1, k)(\alpha^{-1}(h), 1)(1, k^{-1})\big] = j\big[\phi(1, k)(h, 1)\phi(1, k)^{-1}\big]. \end{aligned}$$

Set $x = (1, \beta(k))$ and $y = \phi(1, k)$. These two elements of G_2 have the same image in G_2/H_2, so $x = ya$ for some $a \in H_2$; and *since $H_2 \cong H$ is abelian* we have, with $b = (h, 1) \in H_2$,

$$xbx^{-1} = yaba^{-1}y^{-1} = ybaa^{-1}y^{-1} = yby^{-1},$$

so that both lines in $(*)$ represent the same element, and (C) holds.