Residues, Duality, and the Fundamental Class of a scheme-map

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References

Complete notes for this lecture, with references for many assertions, at

< **http://www.math.purdue.edu/~lipman** >.

So **don't bother taking notes.**

Outline

1. Riemann-Roch and duality on curves.
2. Regular differentials on algebraic varieties.
3. Higher-dimensional residues.
4. Residues, integrals and duality: the Residue Theorem.
5. Closing remarks: fundamental class.

1. Riemann-Roch and duality on curves

V: a projective curve over alg. closed field k.

$h^i(\mathcal{F}) := \dim_k\big(\mathrm{H}^i(V,\mathcal{F})\big) < \infty.$ ($\mathcal{F}$ coherent $\mathcal{O}_V$-module, $i \ge 0$)

$\chi(\mathcal{F}) := h^0(\mathcal{F}) - h^1(\mathcal{F})$ (Euler-Poincaré characteristic).

For *invertible* $\mathcal{F}$, the degree is the integer

$$\deg(\mathcal{F}) := \chi(\mathcal{F}) - \chi(\mathcal{O}_V).$$

This is the Riemann theorem, transformed into a definition.

Properties of degree

Simple manipulations of exact sequences and their cohomology $\Longrightarrow$

Theorem

For invertible $\mathcal{O}_V$-modules $\mathcal{L}_1$ and $\mathcal{L}_2$,

$$\deg(\mathcal{L}_1 \otimes \mathcal{L}_2) = \deg(\mathcal{L}_1) + \deg(\mathcal{L}_2).$$

It follows that if $\mathcal{L} \supset \mathcal{O}_V$ is invertible, and $\mathcal{L}^{-1}$ is the invertible sheaf $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_V}(\mathcal{L}, \mathcal{O}_V) \subset \mathcal{O}_V$, then

$$\deg(\mathcal{L}) = \dim_k(\mathcal{O}_V / \mathcal{L}^{-1}).$$

It results via the correspondence between divisors and invertible sheaves that "degree" has the usual interpretation .

Riemann theorem, classical form.

Riemann problem: *Find the dimension of a complete linear system.*

Translates into finding $h^0(\mathcal{L})$ for invertible $\mathcal{L}$.

With $g := h^1(\mathcal{O}_V)$, the genus of V, rewrite $deg(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O}_V)$ as

$$h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g + h^1(\mathcal{L}).$$

Preceding properties of $\deg(\mathcal{L})$ make it easy to calculate. So need some information about $h^1(\mathcal{L})$.

This is where Roch and duality come in.

Let $\Omega = \Omega_{V/k}$ be the (invertible) sheaf of differential 1-forms on V/k.

Theorem (Roch: Global duality—in modern terms.)

For any invertible $\mathcal{L}$ there is a natural k-linear isomorphism

$$\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{L}, \Omega) \cong \mathrm{H}^0(\mathcal{L}^{-1} \otimes \Omega) \xrightarrow{\ \sim\ } \mathrm{Hom}_k(\mathrm{H}^1(V, \mathcal{L}), k).$$

Thus h^1 becomes a somehow less mysterious h^0.

More elaborately Ω is a *dualizing sheaf*, in the following sense:
A pair (ω, θ) with ω a coherent $\mathcal{O}_V$-module and $\theta\colon \mathrm{H}^1(V, \omega) \to k$ a k-linear map such that for all coherent $\mathcal{O}_V$-modules $\mathcal{F}$, the composition

$$\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}, \omega) \xrightarrow{\text{natural}} \mathrm{Hom}_k\big(\mathrm{H}^1(V, \mathcal{F}), \mathrm{H}^1(V, \omega)\big) \xrightarrow{\text{via } \theta} \mathrm{Hom}_k(\mathrm{H}^1(V, \mathcal{F}), k)$$

is an isomorphism, is called a dualizing pair. Between any two such pairs $\exists$ unique iso. First component of such a pair called a dualizing sheaf.
In fact, there is a canonical k-linear map

$$\textstyle\int_V \colon \mathrm{H}^1(V, \Omega) \xrightarrow{\ \sim\ } k,$$

such that the pair $\big(\Omega, \int_V\big)$ is dualizing—standing out in the isomorphism class of all dualizing pairs.

The residue map

The map $\int_V$ is defined via residues, as follows.

$k(V)$:= the field of rational functions on V.

$\Omega_{k(V)}$ its vector space (one-dimensional) of relative k-differentials.

For any closed point $v \in V$, let H^i_v denote local cohomology supported at the maximal ideal $\mathfrak{m}_v$ of the local ring $\mathcal{O}_{V,v}$.

As the $\mathcal{O}_{V,v}$-module $\Omega_{k(V)}$ is *injective,* the local cohomology sequence associated to the natural exact sequence

$$0 \longrightarrow \Omega_v \longrightarrow \Omega_{k(V)} \longrightarrow \Omega_{k(V)}/\Omega_v \longrightarrow 0$$

gives an isomorphism

$$\Omega_{k(V)}/\Omega_v = \mathrm{H}^0_v(\Omega_{k(V)}/\Omega_v) \xrightarrow{\ \sim\ } \mathrm{H}^1_v(\Omega).$$

Theorem-Definition (Residue map)

There is a unique k-linear map

$$\mathrm{res}_v\colon\ \mathrm{H}^1_v(\Omega) = \Omega_{k(V)}/\Omega_v \to k$$

such that for any local coordinate t at v,

$$\begin{aligned} \mathrm{res}_v(t^{-1}dt + \Omega_v) &= 1 \\ \mathrm{res}_v(t^a dt + \Omega_v) &= 0 \qquad (a < -1). \end{aligned}$$

Classically (when $k = \mathbb{C}$), res_v sends $\nu + \Omega_v$ to $\frac{1}{2\pi i}\oint \nu$.

There are algebraic proofs of the theorem, valid in all characteristics.

Canonical local duality

$\widehat{\ }$ denotes completion at the maximal ideal $\mathfrak{m}_v$ of the local ring $\mathcal{O}_{V,v}$.

Theorem (Canonical local duality)

For all finitely generated $\widehat{\mathcal{O}}_{V,v}$-modules F, the composition

$$\mathrm{Hom}_{\widehat{\mathcal{O}}_{V,v}}(F, \widehat{\Omega}_{V,v}) \xrightarrow{\text{natural}} \mathrm{Hom}_k(\mathrm{H}^1_{\hat{\mathfrak{m}}_v} F,\ \mathrm{H}^1_{\hat{\mathfrak{m}}_v} \widehat{\Omega}_{V,v}) \xrightarrow{\text{via } \mathrm{res}_v} \mathrm{Hom}_k(\mathrm{H}^1_{\hat{\mathfrak{m}}_v} F, k)$$

is an isomorphism.

In other words, $(\widehat{\Omega}_{V,v}, \mathrm{res}_v)$ is a canonical locally dualizing pair.

Globalization; Residue theorem

$\Omega \subset \overline{\Omega} :=$ constant sheaf of meromorphic differentials, sections $\Omega_{k(V)}$.

$\Omega^* := \overline{\Omega}/\Omega$. For open $U \subset V$,

$$\Omega^*(U) := \oplus_{v \in U}\, \Omega_{k(V)}/\Omega_{V,v} \cong \oplus_{v \in U}\, \mathrm{H}^1_v(\Omega).$$

$\overline{\Omega}$ is an *injective* $\mathcal{O}_V$-module, so the cohomology sequence associated to the natural exact sequence $0 \hookrightarrow \Omega \to \overline{\Omega} \to \Omega^* \to 0$ gives the *exact row* in

$$\begin{array}{ccccccc} \Omega_{k(V)} & \longrightarrow & \oplus_{v \in V}\, \mathrm{H}^1_v(\Omega) & \longrightarrow & \mathrm{H}^1(V, \Omega) & \longrightarrow & 0 \\ & & {\scriptstyle \oplus \mathrm{res}_v} \searrow & & \swarrow {\scriptstyle \int_V} & & \\ & & & k & & & \end{array}$$

The key **residue theorem** says that the sum of the residues of a meromorphic differential is zero.

In other words, the map $\oplus \mathrm{res}_v$ annihilates the image of $\Omega_{k(V)}$; that is $\exists$ unique k-linear map $\int_V \colon\ \mathrm{H}^1(V, \Omega) \to k$ making preceding commute.

Remarks

Classically, by Stokes theorem, the sum of the residues of a meromorphic differential ν is the integral of ν around the (empty) boundary of V.

There are, of course, algebraic proofs of the residue theorem. Indeed, we are about to describe, algebraically, a higher-dimensional generalization.

Scholium

For smooth projective curves, differentials and residues give a canonical realization of, and compatibility between, global and local duality.

The present goal is to describe a similar canonical compatibility for arbitrary proper k-varieties.

Some history

V: an n-dimensional variety proper over an algebraically closed field k.

A pair (ω, θ) with ω a coherent $\mathcal{O}_V$-module and $\theta\colon \mathrm{H}^n(V,\omega) \to k$ a k-linear map such that for all coherent $\mathcal{O}_V$-modules $\mathcal{F}$, the composition

$$\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F},\omega) \xrightarrow{\text{natural}} \mathrm{Hom}_k\big(\mathrm{H}^n(V,\mathcal{F}), \mathrm{H}^n(V,\omega)\big) \xrightarrow{\text{via } \theta} \mathrm{Hom}_k(\mathrm{H}^n(V,\mathcal{F}), k)$$

is an isomorphism, is called a dualizing pair. Between any two such pairs $\exists$ unique iso. First component of such a pair called a dualizing sheaf.

Some historical highlights for the above-mentioned generalization.

- Rosenlicht (1952 thesis): V a curve ($n = 1$), dualizing sheaf a certain sheaf of "regular" meromorphic differentials (see below).
- Serre (1955±): V normal, dualizing sheaf $i_* i^* \wedge^n \Omega$, where $i\colon U \hookrightarrow V$ is the inclusion into V of its (open) smooth part U.
- Grothendieck (1957): V embedded in projective N-space, $V \subset \mathbb{P}^N_k$, with (noncanonical) dualizing sheaf $\mathcal{E}xt^{N-n}_{\mathbb{P}^N_k/k}(\mathcal{O}_V, \wedge^n \Omega_{\mathbb{P}^N_k/k})$.

• Grothendieck (1958): V arbitrary (proper/k), with existentially defined dualizing sheaf, the target of a canonical map (the fundamental class) with source $\wedge^n\Omega$, this map being an isomorphism over the smooth part of V—leading to vast generalization.

• Kunz (1975): V projective over k, with dualizing sheaf of regular meromorphic differentials, as explained below. *This dualizing sheaf agrees with those of Rosenlicht and Serre when V is a curve or normal variety, respectively. It contains $\wedge^n\Omega$, with equality at smooth points*

• L— (1984). V arbitrary (proper/k), with dualizing sheaf of regular differentials.

2. Regular differentials

Define "regular differentials."

C: an integral domain finitely generated over k,
$B \subset C$: polynomial k-algebra in n variables over which C is *finite*, and such that the corresponding extension of fraction fields $k(B) \subset k(C)$ is *separable*. (Such B exist, by Noether normalization.)

Setting $\Omega^n := \wedge^n \Omega$, one has the differential trace map

$$\tau\colon \Omega^n_{k(C)/k} = k(C) \otimes_{k(B)} \Omega^n_{k(B)/k} \xrightarrow{\text{trace} \otimes 1} k(B) \otimes_{k(B)} \Omega^n_{k(B)/k} = \Omega^n_{k(B)/k}.$$

The generalized Dedekind complementary module is

$$\omega_{C/B} := \{\, \nu \in \Omega^n_{k(C)/k} \mid \tau(C\nu) \subset \Omega^n_{B/k} \,\}.$$

The C-module $\omega_{C/B}$ does not depend on the choice of B. Kunz proved:

Theorem

Lying between Ω^n and the constant sheaf $\overline{\Omega}^n$ of meromorphic differential n-forms, there is a unique coherent $\mathcal{O}_V$-module ω such that for any affine open subset $U = \operatorname{Spec} C \subset V$ and any $B \subset C$ as above,

$$\Gamma(U, \omega) = \omega_{C/B}.$$

Sections of this ω are called regular differential n-forms.

As before, the stalks Ω^n_v and ω_v are equal at any smooth point $v \in V$.

And again, the sheaf of regular differentials is dualizing. This is not a trivial result, especially for nonprojective varieties.

Moreover, generalizing the above-described case of curves, there is a *canonical* k-linear $\int_V \colon \mathrm{H}^n(V, \omega) \to k$, closely related to residues, such that the pair $(\omega, \int_V)$ is dualizing.

The canonical dualizing pair $(\omega, \int_V)$ is the main actor in this lecture.

Example (Local complete intersection)

Suppose $V \subset X$, an N-dimensional variety, and that $v \in V$ is a smooth point of X. Suppose further that $\mathcal{O}_{V,v} = \mathcal{O}_{X,v}/(f_1, \dots, f_{N-n})$ with $(f_1, \dots, f_{N-n})$ a *regular sequence* in $\mathcal{O}_{X,v}$.
Let $(x_1, \dots, x_N)$ generate the maximal ideal of $\mathcal{O}_{X,v}$, the indexing being such that, with $\bar{x}_i$ the image of x_i in $\mathcal{O}_{V,v}$, the differentials $d\bar{x}_1, \dots, d\bar{x}_n$ generate $\Omega_{k(V)}$.
Then ω_v is freely generated over $\mathcal{O}_{V,v}$ by the meromorphic n-form

$$d\bar{x}_1 d\bar{x}_2 \cdots d\bar{x}_n / [\partial(f_1, \dots, f_{N-n})/\partial(x_{n+1}, \dots, x_N)]^{-},$$

where the denominator is the image in $\mathcal{O}_{V,v}$ of a Jacobian determinant.

3. Higher-dimensional residues

For any closed point $v \in V$, H^n_v is the functor assigning to an $\mathcal{O}_{V,v}$-module its n-th local cohomology module with supports at the maximal ideal $\mathfrak{m}_v$.

For any $\mathcal{O}_V$-module $\mathcal{F}$, set $\mathrm{H}^n_v \mathcal{F} := \mathrm{H}^n_v \mathcal{F}_v$, the local cohomology of the stalk of $\mathcal{F}$ at v.

As above, $\omega :=$ sheaf of regular n-forms on V.
To define the k-linear residue map

$$\mathrm{res}_v \colon \ \mathrm{H}^n_v(\omega) \to k,$$

suppose first that v is a smooth point, so that $\omega_v = \Omega^n_v$.
If $\mathfrak{m}_v$ is generated by $\mathbf{t} := (t_1, t_2, \ldots, t_n)$ then the $\mathcal{O}_{V,v}$-module ω_v is *free of rank one,* with basis

$$dt_1 dt_2 \cdots dt_n := dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n$$

($d \colon \mathcal{O}_{V,v} \to \Omega_{\mathcal{O}_{V,v}/k}$ being the universal derivation).

It is known that $\mathrm{H}^n_v(\Omega^n_v) = \mathrm{H}^n_v(\widehat{\Omega}^n_v)$ is the direct limit of the filtered family

$$(\Omega^n_v/\mathbf{t}^{\mathbf{a}}\Omega^n_v)_{\mathbf{a}} = (\widehat{\Omega}^n_v/\mathbf{t}^{\mathbf{a}}\widehat{\Omega}^n_v)_{\mathbf{a}}$$

where $\mathbf{a} = (a_1, \dots, a_n)$ runs through all n-vectors of positive integers, $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} \dots t_n^{a_n}$, and the transition map from $\mathbf{a}$ to $\mathbf{a}'$ ($a'_i \ge a_i$) is given by multiplication by $\mathbf{t}^{\mathbf{a}'-\mathbf{a}}$. Thus with $\pi_{\mathbf{a}}$ the natural composition

$$\widehat{\Omega}^n_v \twoheadrightarrow \widehat{\Omega}^n_v/\mathbf{t}^{\mathbf{a}}\widehat{\Omega}^n_v \to \mathrm{H}^n_v(\widehat{\Omega}^n_v),$$

every element in $\mathrm{H}^n_v(\widehat{\Omega}^n_v)$ can be represented, non-uniquely, as

$$\begin{bmatrix} \nu \\ t_1^{a_1}, \dots, t_n^{a_n} \end{bmatrix} := \pi_{\mathbf{a}}\nu.$$

Note that $\widehat{\mathcal{O}}_{V,v}$ is a power-series ring $k[[t_1, \dots, t_n]]$, so that any $\nu \in \widehat{\Omega}^n_v$ can be represented as

$$\nu = \sum_{\mathbf{a} \ge (0,\dots,0)} c_{\mathbf{a}}\mathbf{t}^{\mathbf{a}}\, dt_1 dt_2 \cdots dt_n \qquad (c_{\mathbf{a}} \in k).$$

Theorem-Definition

There exists a unique k-linear map

$$\operatorname{res}_v\colon\ \mathrm{H}^n_v(\widehat{\Omega}^n_v) \to k$$

such that for any $(t_1,\dots,t_n)$ *as above and* $\nu = \sum_{\mathbf{a}} c_{\mathbf{a}}\mathbf{t}^{\mathbf{a}}\, dt_1 dt_2\cdots dt_n$,

$$\operatorname{res}_v\begin{bmatrix}\nu\\ t_1^{a_1},\dots,t_n^{a_n}\end{bmatrix} = c_{(a_1-1,\dots,a_n-1)}.$$

Remarks. 1. Classically, res_v is given by some integral.

2. There are a number of algebraic proofs, valid in all characteristics. The difficulty is to show that res_v *does not depend on the choice of* $\mathbf{t}$.

3. When $n = 1$, this res_v is the same as the one in §1

General case: Given $v \in V$, Noether normalization $\Longrightarrow$

$S := \mathcal{O}_{V,v}$ has a local k-subalgebra R that is the local ring of a point $p \in \mathbb{P}^n_k$, and such that S is a localization of a finite separable R-algebra.

It follows easily from the considerations in §2 that

the differential trace map induces a map $\mathrm{H}^n_v(\omega_{V,v}) \to \mathrm{H}^n_p(\Omega^d_{\mathbb{P}^n_k/k})$.

Compose this with res_p (as just defined) to get a k-linear map

$$\mathrm{res}_v \colon \mathrm{H}^n_v(\omega_{V,v}) \to k.$$

This res_v depends only on the k-algebra S, not the choice of V or of a Noether normalization of V. (Must be shown.)

For example, $\omega_{V,v}$ depends only on S, and not on the choice of V; so we can denote this module by ω_v.

Moreover, *there is a definition of* res_v *involving Hochschild homology,* that doesn't need any choices.

4. Residues, integrals and duality: the Residue Theorem

Now here is the **main result**, expressing via residues and integrals a canonical realization of and compatibility between local and global duality.

Theorem (Residue Theorem)

(i) (Canonical local duality). *For all finitely generated $\widehat{\mathcal{O}}_{V,v}$-modules F, the composition*

$$\mathrm{Hom}_{\widehat{\mathcal{O}}_{V,v}}(F, \widehat{\omega}_v) \xrightarrow{\text{natural}} \mathrm{Hom}_k(\mathrm{H}^n_v F,\ \mathrm{H}^n_v \widehat{\omega}_v) \xrightarrow{\text{via res}_v} \mathrm{Hom}_k(\mathrm{H}^n_v F, k)$$

is an isomorphism.

In other words, $(\widehat{\omega}_v, \mathrm{res}_v)$ is a canonical locally dualizing pair.

Residue Theorem (continued)

(ii) (Globalization). *For each proper n-dimensional k-variety V there exists a unique map*

$$\textstyle\int_V \colon \mathrm{H}^n(V, \omega_{V/k}) \to k$$

such that for each $v \in V$, with $\gamma_v \colon \mathrm{H}^n_v(\omega_v) \to \mathrm{H}^n(V, \omega_{V/k})$ the natural map (*derived from the inclusion of the functor of sections supported at v into the functor of all sections*), *the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{H}^n_v(\omega_v) & \xrightarrow{\quad\gamma_v\quad} & \mathrm{H}^n(V, \omega_{V/k}) \\ & \underset{\mathrm{res}_v}{\searrow} \quad \underset{\int_V}{\swarrow} & \\ & k & \end{array}$$

(iii) (Canonical global duality). For each V as in (ii), the pair $(\omega_V, \int_V)$ is dualizing.

5. Closing remarks: fundamental class

Fundamental class. As in the case of curves, what one shows when generalizing to, say, a proper flat map $f\colon X \to Y$ of schemes, of relative dimension n, is first (for example, via Grothendieck duality theory) the existence of some relative dualizing pair (ω, θ), and then (if possible) the existence of a canonical map $\mathbf{c}_f$—the fundamental class of f—from the sheaf Ω of highest order relative differential forms to ω.
This $\mathbf{c}_f$ is the foundation of the role played by differential forms in the abstract Deligne-Verdier approach to Grothendieck duality theory.

In what went before, ω was the sheaf of regular differentials, and $\mathbf{c}_f$ was just the inclusion. In more general situations, regular differentials are not always available. What one seeks then is a map which is compatible in a suitable sense with a certain trace map for differential forms, relative to a factorization of f as smooth $\circ$ finite. This is rather subtle.

If f is *proper* then by Grothendieck duality, $\mathbf{c}_f$ corresponds to a canonical map $\int_f \colon \mathbf{R}^n f_*\Omega \to \mathcal{O}_Y$. But even when f is not proper, one can often construct a fundamental class with the above characteristic properties. There is, for instance, one approach via Hochschild homology.

Locally, the residue theorem will say, very roughly, that $\mathbf{c}_f$ corresponds at each point of x, via a suitable form of local duality, to an intrinsically defined *residue map*, depending only on the local ring in question. Thus the fundamental class is a globalization of locally defined residues. These remarks are necessarily quite vague, given our time constraints. Nor has the theory all been published yet in definitive form. But there do exist some substantial treatments.