About the fundamental class of a flat scheme-map

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Outline

1. Introduction: duality, abstract and concrete.
2. Fundamental class.
3. Fundamental class and traces of differential forms.
4. Pseudofunctoriality of the fundamental class.

1. Introduction

The fundamental class of an essentially-finite-type, separated, flat map of noetherian schemes $f: X \to Y$ with diagonal $\delta: X \to X \times_Y X$ will be explicated below as a $\mathsf{D}(X)(:=$ derived-category)-map

$$\mathcal{H}_f := \mathsf{L}\delta^*\delta_*\mathcal{O}_X \to f^!\mathcal{O}_Y$$

where $\mathcal{H}_f$ is the Hochschild complex of f and $f^!\mathcal{O}_Y$ (with $f^!$ as in Grothendieck duality theory) is the relative dualizing complex.

It is a multifaceted intermediary—via canonical $\mathcal{O}_X$-maps

$$(1) \qquad \Omega^i_f \to H^{-i}\mathsf{L}\delta^*\delta_*\mathcal{O}_X \qquad (i \in \mathbb{Z})$$

from differential i-forms to Hochschild homology—between concrete aspects of differentials (residues, traces...) and abstract duality theory.

We'll go gradually toward stating its definition and some basic properties, beginning with some historical and motivational background.

Duality theory—first 95 years

(1864) **Roch**'s piece of Riemann-Roch (jazzed up):
Let V be a smooth projective curve over $\mathbb{C}$, with sheaf of holomorphic differentials Ω, and F an invertible $\mathcal{O}_V$-module. The finite-dimensional $\mathbb{C}$-vector spaces $\mathrm{H}^1(V, F)$ and $\mathrm{Hom}_V(F, \Omega)$ are dual.

(1931) **Schmidt** (in connection with zeta-functions):
Same with any perfect field in place of $\mathbb{C}$.

(1950s):
Rosenlicht: Same for *any* curve over a perfect field, with Ω replaced by a certain sheaf of meromorphic differentials.

Serre: If V is a normal d-dimensional projective variety over a perfect field k, the reflexive hull of the sheaf $\Omega^d_{V/k}$ of degree-d Kähler differentials represents the functor $\mathrm{Hom}_k(\mathrm{H}^d(V, F), k)$ of coherent $\mathcal{O}_V$-modules F.

Grothendieck: For *any* d-dimensional projective variety V/k (k a field), the functor $\mathrm{Hom}_k(\mathrm{H}^d(V, F), k)$ of coherent $\mathcal{O}_V$-modules F is representable.

Consolidation

Theorem 1

For any d-dimensional variety V proper over a perfect field k, the functor $\mathrm{Hom}_k(\mathrm{H}^d(V,F),k)$ of quasi-coherent $\mathcal{O}_V$-modules F has a canonical representing pair $(\omega_{V/k}, \int_{V/k})$, where the canonical module $\omega := \omega_{V/k}$—the sheaf of "regular differential d-forms"—is an $\mathcal{O}_V$-submodule of the constant sheaf $\Omega^d_{k(V)/k}$, containing the image of the natural map $\Omega^d_{V/k} \to \Omega^d_{k(V)/k}$, with equality over the smooth locus, and $\int_{V/k} \colon \mathrm{H}^d(V,\omega) \to k$ is the unique map t such that for each closed $v \in V$, the composition $\mathrm{H}^d_v(\hat{\omega}_v) \overset{\text{nat'l}}{\twoheadrightarrow} \mathrm{H}^d(V,\omega) \xrightarrow{t} k$ $[(\hat{\ })_v :=$ *completion at* $v]$ is the locally describable, canonical residue map res_v such that $(\hat{\omega}_v, \mathrm{res}_v)$ *represents the functor* $\mathrm{Hom}_k(\mathrm{H}^d_v(G), k)$ *of* $\hat{\mathcal{O}}_{V,v}$*-modules* G.

Thus, differentials and residues underlie a *canonical* realization of, and compatibility between, global and local duality.

This explicit version of duality was worked out for projective varieties by **Kunz** in the mid 1970s, and for arbitrary proper varieties by me in 1984 (*Astérisque 117*). It was generalized in 1993 to equidimensional generically smooth maps of noetherian schemes by **Hübl** and **Sastry** (*Amer. J. Math.* 115). The proofs are lengthy.

Enter derived categories (Grothendieck duality)

Beginning in the late 1950s, Grothendieck formulated a vast generalization of then existing duality theory.

The main result in [RD] (Hartshorne's Springer Lecture Notes (SLN) 20 amended by Conrad in SLN 1750), exposing Grothendieck's ideas—via derived categories, à la Verdier—can be summarized as follows.

First, some conventions:

- Schemes are noetherian; scheme-maps are separated, essentially of finite type.
- $[d]$ denotes "d-fold degree-shift."
- For a scheme S with bounded-below derived category $\mathrm{D}^+(S)$, $\mathrm{D}^+_{\mathrm{c}}(S) \subset \mathrm{D}(S)$ is the full subcategory spanned by the $\mathcal{O}_X$-complexes with coherent homology sheaves; and similarly when S is replaced by a quasi-coherent $\mathcal{O}_S$-algebra.
- For finite $f\colon X \to Y$, with natural ringed-space factorization

$$X \xrightarrow{\bar f} \overline{Y} := (Y, f_*\mathcal{O}_X) \to Y,$$

$\bar f$ is flat and $\bar f^*$ is an equivalence of categories $\mathrm{D}^+_{\mathrm{qc}}(\overline{Y}) \xrightarrow{\approx} \mathrm{D}^+_{\mathrm{qc}}(X)$).

Theorem 2 (cf. SLN20, p. 383, Corollary 3.4)

In the presence of *residual complexes*,
$\exists$ $\mathrm{D}_{\mathrm{c}}^{+}$-valued contravariant pseudofunctor $(-)^!$
over the category of finite-type scheme-maps:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \Longrightarrow \quad f^!\colon \mathrm{D}_{\mathrm{c}}^{+}(Y) \to \mathrm{D}_{\mathrm{c}}^{+}(X), \quad f^!g^! \xrightarrow{\sim} (gf)^!, \ \ldots$$

plus, for each proper f, a trace map $T_f\colon \mathrm{R}f_*f^! \to 1$, all obtained by a (seemingly miraculous) gluing of the two pseudofunctors

(a) $\qquad \Omega_f^d[d] \otimes_X f^*F \qquad (f \text{ smooth, of rel. dim. } d,\ F \in \mathrm{D}_{\mathrm{c}}^{+}(Y)),$

(b) $\qquad \bar{f}^*\mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \qquad (f \text{ finite},\ F \in \mathrm{D}_{\mathrm{c}}^{+}(Y)),$

and their associated trace maps, such that for *proper* f, duality holds: T_f is the counit of an adjunction $\mathrm{R}f_* \dashv f^!$.

We won't dwell on what "residual complexes" are (see Conrad's SLN 1750, §3.2). Commonly occurring (but not all!) noetherian schemes have them.

(Long though it is, the proof in [RD] omits significant details, see top of p. 153 in SLN1750.)

Remarks

• For finite f and $F \in \mathrm{D}_{\mathrm{c}}^{+}(Y)$, $T_f(F)$ is the map ("evaluation at 1")

$$f_* \bar{f}^* \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \cong \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \to \mathrm{R}\mathcal{H}om_Y(\mathcal{O}_Y, F) = F$$

induced by the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

• For projective smooth f, $T_f(\mathcal{O}_Y)\colon \mathrm{R}f_*\Omega_f^d[d] \to \mathcal{O}_Y$ is the map associated with Serre duality [SLN 20, Chap. III].
In particular, if $d = 0$, that is, f is finite and étale, then $T_f(\mathcal{O}_Y)$ is the usual trace $f_*\mathcal{O}_X \to \mathcal{O}_Y$ (which agrees with the above $T_f(\mathcal{O}_Y)$ for finite f via the canonical isomorphism $f_*\mathcal{O}_X \xrightarrow{\;\sim\;} \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)$).

• For arbitrary proper smooth $f\colon X \to Y$, Grothendieck's $T_f(\mathcal{O}_Y)$ is based on a rather sophisticated trace map between certain residual complexes.
(See SLN 1750, §3.4; and for a generalization to Cousin complexes, see **Sastry's** paper in *Contemporary Math.* 375.) Furthermore, for any $F \in \mathrm{D}_{\mathrm{c}}^{+}(Y)$, $T_f(F)$ is the natural composite map

$$\mathrm{R}f_*(\Omega_f^d[d] \otimes_X f^*F) \underset{\text{projn}}{\cong} \mathrm{R}f_*\Omega_f^d[d] \otimes^{\mathsf{L}}_Y F \xrightarrow[T_f(\mathcal{O}_Y)]{\text{via}} \mathcal{O}_Y \otimes^{\mathsf{L}}_Y F = F.$$

Remarks, ct'd

• Grothendieck's earlier result (see above) for any proper $f\colon V \to \operatorname{Spec} k$ (k a field) and coherent $\mathcal{O}_V$-module F results from Theorem 2 via natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_k(\mathrm{H}^d(V,F),k) &\cong \mathrm{Hom}_k(\mathrm{R}f_*F[d],k)\\ &\underset{\text{duality}}{\cong} \mathrm{Hom}_{\mathrm{D}(V)}(F[d],f^!k) \cong \mathrm{Hom}_{\mathcal{O}_V}(F,H^{-d}f^!k).\end{aligned}$$

• For simplicity, we haven't mentioned other basic properties of $(-)^!$, such as its interactions with derived $\otimes$ and $\mathcal{H}om$, with flat base change, etc. Also, there is a property, w.r.t. étale base change of proper maps that guarantees uniqueness of $(-)^!$ and T_f up to unique isomorphism—*but not canonicity.*

• Theorem 2 extends to essentially-finite-type scheme-maps (in particular, to Commutative Local Algebra), see **Nayak**'s paper Advances in Mathematics 222 (2009), 527–546.

Ideal theorem

In the introduction of SLN 20, there is envisioned an "Ideal Theorem," extending Theorem 2 to complexes with *quasi-coherent* homology (replace D^+_c by D^+_qc), even when there are no residual complexes.

This requires, to begin with,

a pseudofunctor $(-)^!$ *on the category of noetherian schemes,*
with $f^!$ right-adjoint to $\mathrm{R}f_*$ for proper f, and $f^! = f^*$ for étale f,
whence, by the duality theorems in SLN20 for smooth and for finite maps, and by the uniqueness of adjoints,
necessarily such that it restricts (up to isomorphism) on the categories of smooth, resp. finite, maps of noetherian schemes to the pseudofunctors (a) and (b) in Theorem 2.

In the appendix to SLN 20, **Deligne**, inspired by work of **Verdier** on duality for locally compact topological spaces, produces such a $(-)^!$.[1]
Deligne's construction is nonconstructive; so we'll call his pseudofunctor *abstract* $(-)^!$, and any pseudofunctor as in Theorem 2 *concrete* $(-)^!$.

[1] For an account of Deligne's arguments, see SLN 1960, §4.1. A more conceptual proof, based on an analog of Brown representability, was given by **Neeman** (1996).

Questions and comments

• Does the Ideal Theorem hold for abstract $(-)^!$?

• Do there exist *canonical* isomorphisms of abstract $(-)^!$ to the concrete pseudofunctors (a) and (b) in Theorem 2? And if so, what are the resulting concrete trace maps for proper f?

Moreover, given the complexity of its proof,

• can one deduce (Hübl and Sastry's generalization of) Theorem 1 from abstract Grothendieck duality theory?

These question fit into an overall project whose goal is

to translate the *abstract* categorical features of Grothendieck duality theory (as initially formulated by Deligne and Verdier, and exposed in detail in SLN 1960[2]) into *canonical, concrete* terms (as in SLN 20 and SLN 1750).

To a large extent, this project is carried out—even for formal schemes—by **Nayak** and **Sastry**, in "Grothendieck Duality and Transitivity II: Traces and Residues via Verdier's isomorphism," arXiv:1903.01783.

Let us elaborate.

[2] see also Neeman's paper, `arXiv:1406.7599`.

Canonical realization of abstract $(-)^!$ for finite maps

The case of a finite map $f\colon X \to Y$ is relatively straightforward.

Recall the equivalence of categories $\bar{f}^*\mathrm{D}^+_{\mathrm{qc}}(\overline{Y}) \xrightarrow{\approx} \mathrm{D}^+_{\mathrm{qc}}(X)$
where $\bar{f}\colon X \to (Y, f_*\mathcal{O}_X)$ is the natural ringed-space map.

Though proving concrete duality as in Theorem 2, (b), for such f is not so hard, one can also *deduce* it from abstract duality by showing that
the sheafified duality isomorphism

$$f_*f^!F = f_*\mathrm{R}\mathcal{H}om_X(\mathcal{O}_X, f^!F) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \qquad (F \in \mathrm{D}_{\mathrm{qc}}(Y))$$

(right-conjugate to the *projection isomorphism* $f_*f^*G \xleftarrow{\sim} G \otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X$)
is actually a $\mathrm{D}(f_*\mathcal{O}_X)$-map, hence is f_* of a unique $\mathrm{D}(X)$-isomorphism
(which, one checks, is pseudofunctorial)

$$\lambda_f(F)\colon f^!F \xrightarrow{\sim} \bar{f}^*\mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F)$$

such that the following $\mathrm{D}(Y)$-diagram commutes:

$$\begin{array}{ccc}
f_*f^!F & \xrightarrow[f_*\lambda_f]{\sim} & \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \\
{\scriptstyle T_f}\big\downarrow & & \big\downarrow{\scriptstyle \text{natural}} \\
F & =\!=\!=\!= & \mathrm{R}\mathcal{H}om_Y(\mathcal{O}_Y, F)
\end{array}$$

Canonical realization of abstract $(-)^!$ for smooth maps

A basic construct is *Verdier's isomorphism*

$$v_f(F)\colon f^!F \xrightarrow{\;\sim\;} \Omega_f^d[d]\otimes_X f^*F \qquad (F\in \mathrm{D_{qc}}(Y))$$

where f is a smooth map of relative dimension d,
Ω_f^d is the invertible $\mathcal{O}_X$-module of relative Kähler d-forms,
and $f^!$ is either abstract or concrete.

This v_f arose in "Base change for twisted inverse images...", *Algebraic Geometry* (Bombay, 1968). Oxford Univ. Press, 1969 (proof of Thm. 3).

A more detailed description appears in §3 of [LN], my paper with Neeman, Alg. Geom. (2018)—where v_f is shown to be a special case of the *fundamental class,* as will be explained below.

Is v_f a *pseudofunctorial* isomorphism? (See section 4 below).

Among other things, Nayak and Sastry investigate v_f's pseudofunctoriality and its interactions with finite maps. They also deal with its relation to *local duality and residues,* whereby (as sketched by Verdier) one explicates, for proper f, the map $\mathrm{R}f_*\Omega_f^d[d]\to\mathcal{O}_Y$ dual to to $v_f(\mathcal{O}_Y)^{-1}$.

Concrete v_f = identity

In *concrete* duality one *identifies* $f^!G$ pseudofunctorially with

$$f^\#(G) := \Omega_f^d[d] \otimes_X f^*(G) \qquad (G \in \mathrm{D}(Y)),$$

so that v_f becomes a functorial automorphism of $f^\#$.

The claim is that then v_f is the identity automorphism.

Proof. (Feel free to skip.) It suffices that $v_f(\mathcal{O}_Y)$ be the identity (see [LN, 3.4.5]).

To check that $v_f(\mathcal{O}_Y)$ is the identity, let $\delta\colon X \to Z := X \times_Y X$ be the diagonal, and let $p_i\colon Z \to X$ $(i = 1, 2)$ be the (smooth) canonical projections.
Let $\omega \in \mathrm{D}(X)$ be the invertible $\mathcal{O}_X$-complex

$$\omega := \Omega_f^d[d].$$

For an $\mathcal{O}_X$-complex E, let $E^\vee$ be the $\mathcal{O}_X$-complex $\mathcal{H}om_X(E, \mathcal{O}_X)$.
One has the isomorphism $\omega^\vee \xrightarrow{\ \sim\ } (\Omega_f^d)^\vee[-d]$ given by multiplication in degree d by $(-1)^{d(d+1)/2}$.

Concrete v_f = identity, ct'd

The definition (see [LN, (3.1.7)]) gives that $v_f(\mathcal{O}_Y)$ is the composite map

$$\mathcal{O}_X \otimes_x \omega \underset{\text{via } \alpha}{\xrightarrow{\sim}} (\omega^\vee \otimes_x \omega) \otimes_x \omega \underset{\text{via } \gamma}{\xrightarrow{\sim}} (\delta^! \mathcal{O}_Z \otimes_x \omega) \otimes_x \omega \underset{\text{via } \vartheta}{\xrightarrow{\sim}} \mathcal{O}_X \otimes_x \omega,$$

where the isomorphism α is the natural one, γ comes from [LN, (3.1.6)], and ϑ from [LN, (3.1.3)]. That this is the identity means that α^{-1} factors as

$$\omega^\vee \otimes_x \omega \underset{\gamma\otimes 1}{\xrightarrow{\sim}} \delta^! \mathcal{O}_Z \otimes_x \omega \underset{\vartheta}{\xrightarrow{\sim}} \mathcal{O}_X,$$

i.e., the natural diagram (with $\otimes := \otimes_x$ and ψ the isomorphism $\psi_{\delta, p_1}(\mathcal{O}_X)$ in SLN 1750, p. 77, (2.7.2))

$$\begin{array}{ccccc}
\omega^\vee \otimes \omega & \xleftarrow{\ \alpha\ } & \mathcal{O}_X & \xrightarrow{\ \psi\ } & \delta^! p_1^\# \mathcal{O}_X \\
{\scriptstyle \gamma\otimes 1}\downarrow & & & & \uparrow \\
\delta^! \mathcal{O}_Z \otimes \omega & \longrightarrow & \delta^! \mathcal{O}_Z \otimes \delta^* p_2^* \omega & \longrightarrow & \delta^! p_2^* \omega
\end{array}$$

commutes. But this commutativity is essentially the definition of ψ. □

Hence for proper f, v_f transforms the abstract trace into Grothendieck's trace.

2. Fundamental class

More generally:

For flat equidimensional $f\colon X \to Y$, of relative dimension d, specify and study a canonical pseudofunctorial fundamental class map

$$c_f(F)\colon \Omega_f^d[d] \otimes_X f^*F \longrightarrow f^!F \qquad (F \in \mathrm{D}^+_{\mathrm{qc}}(Y))$$

which is just $v_f(F)^{-1}$ when f is smooth.

Remarks. 1. For such f, $H^{-e}f^!\mathcal{O}_Y = 0$ for $e > d$, whence a *group isomorphism*

$$\mathrm{Hom}_{\mathrm{D}(X)}(\Omega_f^d[d], f^!\mathcal{O}_Y) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_X}(\Omega_f^d, H^{-d}f^!\mathcal{O}_Y),$$

via which $c_f(\mathcal{O}_Y)$ can be viewed as a canonical map from $\Omega_f^d[d]$ to the canonical (up to isomorphism) module $H^{-d}f^!\mathcal{O}_S$ of f, generically an isomorphism if f is generically smooth.
This suggests, for such f, the existence of a canonical (absolutely) module consisting of meromorphic differential forms ("regular" differentials, see below.)

2. The term "fundamental class" reflects an interpretation of such a $c_f(\mathcal{O}_Y)$ when X is a codimension-e cycle in a smooth Y-scheme Z, as an element of $H^e_X(Z, \Omega^e_{Z/Y})$, see e.g., Grothendieck's Séminaire Bourbaki talk (no. 149), May 1957; or Angéniol's SLN 896 (dealing with Chow schemes).

Fundamental class and $\int$

For *proper* f, such a $c_f(F)$ would be dual to a canonical $\mathsf{D}(Y)$-map

$$\int_f(F)\colon \mathsf{R}f_*(\Omega_f^d[d]\otimes_X f^*F) \underset{\text{projn}}{\xrightarrow{\sim}} \mathsf{R}f_*\Omega_f^d[d]\otimes_Y^{\mathsf{L}} F \longrightarrow F = \mathcal{O}_Y\otimes_Y^{\mathsf{L}} F.$$

Thus it would suffice to specify a suitable family of canonical $\mathsf{D}(Y)$-maps

$$\int_f\colon \mathsf{R}f_*\Omega_f^d[d] \longrightarrow \mathcal{O}_Y$$

or, equivalently, $\mathcal{O}_Y$-maps

$$R^d f_*\Omega_f^d := H^d\mathsf{R}f_*\Omega_f^d \longrightarrow \mathcal{O}_Y.$$

Also, the envisaged $\int_f$ should be tied, *as in Theorem 1*, to concretely describable canonical residue maps

$$H_x^d\Omega_f^d \longrightarrow \mathcal{O}_{Y,y} \qquad (x \text{ closed in } f^{-1}y),$$

closely related to local duality. ("Residue Theorem.")

So, generalizing Theorem 1, this is about setting up a canonical concrete framework for relating (local) algebraic phenomena involving differentials, —e.g., residues and local duality—to (global) abstract duality.

Note. *Non-proper f* are in play too, substantially complicating matters.

As indicated before, at least for smooth maps of formal schemes Nayak and Sastry have done this in arXiv:1903.01783. In fact, the relation between fundamental classes and residues becomes clearer, and more general, over formal schemes, where local and global duality merge into a single theory with fundamental classes and residues conjoined.

The role of higher-dimensional residues in duality theory was hinted at by Grothendieck, in his 1958 Edinburgh talk, where "residual complexes" first appeared; and elaborated in his "prenotes" for [RD]. It was elucidated, without proof, for smooth proper f, in Verdier's above-mentioned paper on Base Change. A detailed exposition for equidimensional f of finite tor-dimension can be found in pp. 86(bottom)–88 in Angéniol and El Zein's Mémoire (see section 3 below). For more on the local theory, see section 5 of my "Lectures on local cohomology and duality," in the book *Local cohomology and its applications,* Dekker, New York.

Fundamental class via Hochschild homology

Recall the Hochschild complex $\mathcal{H}_f := \mathsf{L}\delta^*\delta_*\mathcal{O}_X$ of a flat scheme-map $f\colon X \to Y$ with diagonal $\delta\colon X \to X_1 \times_Y X_2$ ($X_j := X$ for $j = 1, 2$).

One has the composite $\mathsf{D}(X)$-map, with $p_j\colon X_1 \times_Y X_2 \to X$ the projections, from $\mathcal{H}_f$ to the relative dualizing complex $f^!\mathcal{O}_Y$,

$$C_f\colon \mathsf{L}\delta^*\delta_*\mathcal{O}_X \xrightarrow{\sim} \mathsf{L}\delta^*\delta_*\delta^! p_2^!\mathcal{O}_X \longrightarrow \mathsf{L}\delta^* p_2^!\mathcal{O}_X \xrightarrow{\sim} \mathsf{L}\delta^* p_1^* f^!\mathcal{O}_Y \xrightarrow{\sim} f^!\mathcal{O}_Y.$$

The maps here are the natural ones, the third coming from the flat base-change isomorphism. (This is why f needs to be flat.)

Next, I being the kernel of $\mathcal{O}_{X\times_Y X} \twoheadrightarrow \delta_*\mathcal{O}_X$, one has the isomorphism $\partial\colon \Omega^1_f = I/I^2 \xrightarrow{\sim} H^{-1}\mathcal{H}_f$, inverse to the one coming from the natural triangle $\mathsf{L}\delta^* I \longrightarrow \mathsf{L}\delta^*\mathcal{O}_{X\times_Y X} \to \mathsf{L}\delta^*\delta_*\mathcal{O}_X \xrightarrow{+}$

This ∂ extends to a map of alternating graded algebras

$$\oplus_i \Omega^i_f \to \oplus_i H^{-i}\mathcal{H}_f\,,$$

an isomorphism when f is smooth.

Fundamental class via Hochschild homology (ct'd)

If f is equidimensional, rel. dim. d, then $H^{-e}f^!\mathcal{O}_Y = 0$ for all $e > d$, so there is a natural $\mathrm{D}(X)$-map $H^{-d}f^!\mathcal{O}_Y \to f^!\mathcal{O}_Y$.

Thus one has a natural composite $\mathrm{D}(X)$-map, the fundamental class of f:

$$c_f\colon \Omega_f^d[d] \longrightarrow H^{-d}\mathcal{H}_f \xrightarrow{H^{-d}C_f} H^{-d}f^!\mathcal{O}_Y \longrightarrow f^!\mathcal{O}_Y.$$

Examples

1. The maps C_f and c_f "commute" with localization on both X and Y.
2. The map c_f does restrict to $v_f(\mathcal{O}_Y)^{-1}$ when f is smooth.

(See paper by me and Neeman in Algebraic Geometry 5 (2018).)

3. In the affine case, say $f = \mathrm{Spec}(g)$ where $g\colon A \to B$ is a flat ring-map, the map $\delta_*\mathcal{O}_X \to p_2^!\mathcal{O}_X$ underlying the first two arrows in the definition of c_f sheafifies the natural map $B \otimes_{B\otimes_A B} \bar{\mu}$ with $\bar{\mu}$ the natural composite

$$B \xrightarrow{\mu} \mathrm{Hom}_A(B,B) \to \mathrm{RHom}_A(B,B)$$

where μ takes $b \in B$ to "multiplication by b."

(See paper by **Iyengar**, me and **Neeman**, Compositio Math. 151 (2015).)

Examples

4. In 3., if B is *finite* (and flat) over A, then c_f can be identified with the sheafification of the natural composite

$$B \xrightarrow{\mu} \mathrm{Hom}_A(B,B) \to \mathrm{Hom}_A(B,B) \otimes_{B\otimes_A B} B \cong \mathrm{Hom}_A(B,A),$$

which is the B-homomorphism taking $1 \in B$ to the usual trace map.

Using this, and 2., and pseudofunctoriality of c_f (see below), one shows: if f is a locally finitely presented flat map with Cohen-Macaulay fibers, then c_f is an isomorphism *only if* f is smooth.
(First proved algebraically by **Kunz** and **Waldi**.)

5. When $d = 0$ (i.e., $f\colon X \to Y$ is quasi-finite), c_f maps $\mathcal{O}_X$ to $f^!\mathcal{O}_Y$.

Chatzistamatiou and Rülling, in Compositio 151 (2015), used a subtle variation on this theme—one that can be viewed as a globalization of SLN20's "fundamental local isomorphism"—as a basic tool for resolving outstanding questions about *birational invariance of the cohomology of structure sheaves of excellent regular schemes*.

Their result is generalized by **Kovács** in arXiv:1703.02269, section 8.

3. Fundamental class and traces of differential forms

Angéniol and **El Zein** gave a concrete characterization, via standard traces, of (what turns out to be) the fundamental class:

Theorem (Mémoires de la S. M. F., 58 (1978), p. 81)

Given $f\colon X \to S$ equidimensional, rel. dim. d, and of finite tor-dim., suppose either that f is Gorenstein or that S is a $\mathbb{Q}$-scheme. There is a unique $\mathrm{D}(X)$-map $\bar c_f\colon \Omega^d_f[d] \to f^!\mathcal{O}_S$ having the localized trace property: for any commutative

$$\begin{array}{ccc} U & \xrightarrow{\ i\ } & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow[\ g\ ]{} & S \end{array}$$

with i étale, h finite and g smooth of rel. dim. d, the map dual to the natural composite

$$h_*h^*\Omega^d_g[d] \cong h_*\mathcal{O}_U \otimes \Omega^d_g[d] \xrightarrow{\text{trace}\otimes 1} \mathcal{O}_Y \otimes \Omega^d_g[d] = \Omega^d_g[d]$$

is the natural composite

$$h^*\Omega^d_g[d] \to \Omega^d_{gh}[d] \cong i^*\Omega^d_f[d] \xrightarrow{i^*\bar c_f} i^*f^!\mathcal{O}_S \cong (fi)^!\mathcal{O}_S \cong h^!g^!\mathcal{O}_S \xrightarrow{h^!v_f} h^!\Omega^d_g[d].$$

Remarks.

1. The finite tor-dimensionality of *fi* and smoothness of g force $h_*\mathcal{O}_U \in \mathrm{D}(Y)$ to be perfect; so there is a canonical $\mathrm{D}(Y)$-map trace: $h_*\mathcal{O}_U \to \mathcal{O}_Y$.

2. The restriction on the characteristic is due to their using in the *existence* proof a theorem of Bott about Grassmannians. The *uniqueness* proof doesn't need this restriction.

3. For *flat f* we will indicate how the previously described "fundamental class via Hochschild" provides a characteristic-free generalization (not involving Grassmannians or Grothendieck's trace) of A-E's map.

Fundamental class and traces

If f itself factors as $X \xrightarrow{h} Y \xrightarrow{g} S$ (h finite, g smooth) then $\bar{c}_f \colon \Omega_f^d[d] \to h^! g^! \mathcal{O}_S \cong h^! \Omega_g^d[d]$ is dual to an $\mathcal{O}_Y$-map

$$h_* \Omega_f^d[d] \to \Omega_g^d[d]$$

called the trace map for differential forms.

Composing this with the natural map $h_* h^* \Omega_g^d[d] \to h_* \Omega_f^d[d]$ produces the map $h_* h^* \Omega_g^d[d] \to \Omega_g^d[d]$ in the preceding Theorem.

As often happens when abstract duality is interpreted in concrete terms, finding *explicit algebraic formulae* for the trace map is not easy, at least if h is not generically étale. The "Cartier operator" in positive characteristic is a special case. (For details, see §16 in **Kunz's** book *Kähler Differentials.*)

Theorem 1 above, and its generalization by Sastry and Hübl, treat the fundamental class vis-à-vis such traces. In those situations, one can stick to generically étale h.

The general idea can also be gleaned from Prop. 6.3.1 in the book *Calcul différentiel et class caractéristiques* by **Angéniol** and **Lejeune-Jalabert**.

Regular differentials–canonical dualizing sheaf

Suppose S is irreducible, X has no embedded associated points, and $f\colon X \to S$ is flat, equi-d-dimensional and generically smooth. Then with $\omega_f := H^{-d} f^! \mathcal{O}_S$ one has, over a dense open subset $u\colon U \hookrightarrow X$, an isomorphism $u^*c_f\colon u^*\Omega_f^d \xrightarrow{\sim} u^*\omega_f$. One finds that the canonical module ω_f is *torsion-free,* hence isomorphic to its natural image $\tilde{\omega}_f$ in $u_*u^*\Omega_f^d \cong u_*u^*\omega_f$. This $\tilde{\omega}_f$ is a *truly canonical* coherent sheaf of meromorphic d-forms, the sheaf of regular differentials.

For proper f, one has then a *canonical version of non-derived duality*: the sheaf $\tilde{\omega}_f$ represents the functor $\mathrm{Hom}_S(R^d f_* M, \mathcal{O}_S)$ of coherent (or even quasi-coherent) $\mathcal{O}_X$-modules M—because ω_f does so.

There is an extensive *algebraic* study of regular differentials in **Kunz** and **Waldi**'s *Regular differential forms*, *Contemp. Math.* 79. It is motivated by, but has little technical help from, duality theory. In fact, showing that the algebraically defined regular differentials sheafify to $\tilde{\omega}_f$ when X is affine—a concrete local interpretation of the abstractly defined fundamental class—is in essence a major part of (the Hübl–Sastry generalization of) Theorem 1.

Local (algebraic) description of regular differentials

Let R be a universally japanese noetherian ring with artinian total quotient ring. Let $P \to S$ be a finite flat R-algebra map with P finitely generated, smooth and equi-d-dimensional over R, and with the total quotient ring L of S a complete intersection over the total quotient ring K of P.

Then there is an algebraic trace map $t_{L/K} \colon \Omega^d_{L/R} \to \Omega^d_{K/R}$,

that freely generates the L-module $\mathrm{Hom}_K(\Omega^d_{L/R}, \Omega^d_{K/R})$. Moreover,

$(*)$ a d-form $\nu \in \Omega^d_{L/R}$ is regular $\iff$ $t_{L/K}(S\nu) \subset \Omega^d_{P/R}$.

Remarks. 1. $(*)$ says that $\tilde{\omega}_{S/R}$ is S-isomorphic to $\mathrm{Hom}_P(S, \Omega^d_{P/R})$.

2. If S is normal and S_p is smooth over R for any height-1 prime S-ideal p, then with $* := \mathrm{Hom}_S(-, S)$, the S-module $\tilde{\omega}_{S/R}$ of regular differentials is the "double dual" $(\Omega^d_{S/R})^{**}$.

3. When $P = R$, so that $d = 0$, then $\tilde{\omega}_{S/R}$ is just the classical *Dedekind complementary module*.

4. For more details and generality, see §4 of Kunz and Waldi's monograph.

Trace via Hochschild

In §4.5 of *Contemp. Math.* 61 there is defined (in essence), relative to a pair of flat maps

$$\mathrm{Spec}(S) \xrightarrow{h} \mathrm{Spec}(R) \xrightarrow{g} \mathrm{Spec}(A)$$

with h finite and g equidimensional, a concrete $\mathrm{D}(\mathrm{Spec}(R))$-map, the Hochschild trace,

$$\tau_{g,h} \colon h_*\mathcal{H}_{gh} \to \mathcal{H}_g\,.$$

As discussed there, in a number of cases—for instance, if g is smooth, of rel. dim. d—there results a natural commutative diagram

$$\begin{array}{ccc} h_*\Omega^d_{gh}[d] & \xrightarrow{\text{trace}} & \Omega_g[d] \\ \big\downarrow & & \big\downarrow \\ h_*\mathcal{H}_{gh} & \xrightarrow[\tau_{g,h}]{} & \mathcal{H}_g \end{array}$$

Trace via Hochschild (ct'd)

These *local considerations can be globalized,* via simplicial arguments applied to an affine open covering, such as were used by **Swan** in his 1997 paper on Hochschild homology of schemes.

(The difficulty to overcome is that, while the Hochschild homology sheaves are quasicoherent, the sheaf of *bar resolutions* used to define τ is not, so that standard pasting methods aren't sufficient.)

This all suggests the following conjectural assertion stated, for simplicity, for finite-type flat maps, but which can be extended to yield a characteristic-free generalization of the Angéniol-El Zein theorem:

Conjecture: Hochschild trace theorem

Recall the Hochschild fundamental class $C_f\colon \mathcal{H}_f \to f^!\mathcal{O}_Y$ of a flat $f\colon X \to Y$.

It is compatible with étale localization on both X and Y.

Theorem

Let $X \xrightarrow{h} Y \xrightarrow{g} Z$ be flat equidimensional separated finite-type maps of noetherian schemes, with h finite. The following diagram commutes.

$$
\begin{array}{ccc}
h_*\mathcal{H}_{gh} & \xrightarrow{\quad C_{gh}\quad} & h_*(gh)^!\mathcal{O}_Z \\
 & & \| \\
\Big\downarrow{\scriptstyle \tau_{g,h}} & & h_*h^!g^!\mathcal{O}_Z \\
 & & \Big\downarrow{\scriptstyle T_h} \\
\mathcal{H}_g & \xrightarrow[\quad C_g\quad]{} & g^!\mathcal{O}_Z
\end{array}
$$

An *idea for the proof* is to reduce to the case of affine schemes, where concrete descriptions of the maps in play are available, at which point what remains is to prove the commutativity of a concrete (though complicated) diagram in the derived category of a ring.

4. Pseudofunctoriality of the fundamental class

First, ∃ a natural bifunctorial map, compatible with étale localization,

$$\chi_f(E,F)\colon f^!E\otimes^{\mathsf L}_X \mathsf Lf^*F \longrightarrow f^!(E\otimes^{\mathsf L}_Y F) \qquad (E,F\in \mathsf D^+_{\mathrm{qc}}(Y)).$$

For *proper* f, this map is dual to the composite map

$$\mathsf Rf_*(f^!E\otimes^{\mathsf L}_X \mathsf Lf^*F) \xrightarrow[\text{projection}]{\sim} \mathsf Rf_*f^!E\otimes^{\mathsf L}_Y F \xrightarrow[\text{counit}\otimes 1]{} E\otimes^{\mathsf L}_Y F;$$

and it extends to general (essentially-)finite-type maps via (Nayak's generalization of) Nagata's compactification theorem (see Advances in Math. 222 (2009), 527–546.)

This $\chi_f(E,F)$ *is an isomorphism for all* E, F $\iff$ f *has finite tor. dim.*

One verifies that this map is actually pseudofunctorial.

The *pseudofunctoriality* property of the fundamental class is that, w.r.t. composites $X\xrightarrow{f} Y\xrightarrow{g} Z$ where f and g are flat equidimensional maps of rel. dim. d and e respectively (so that gf is flat and equidimensional of rel. dim. $d+e$) the following diagram commutes.

Pseudofunctoriality (ct'd)

$$
\begin{array}{ccc}
\Omega^d_f[d] \otimes_X \mathsf{L}f^*\Omega^e_g[e] & \xrightarrow{\text{via fund'l class}} & f^!\mathcal{O}_Y \otimes_X \mathsf{L}f^* g^!\mathcal{O}_Z \\
 & & \big\downarrow \chi_f(\mathcal{O}_Y, g^!\mathcal{O}_Z) \\
\text{natural}\big\downarrow & & f^! g^! \mathcal{O}_Z \\
 & & \big\downarrow \simeq \\
\Omega^{d+e}_{gf}[d+e] & \xrightarrow[\text{fund'l class}]{} & (gf)^!\mathcal{O}_Z
\end{array}
$$

The commutativity was shown for *proper* f, g in joint work with Sastry, J. Alg. Geom. (1992), 101–130. The proof is nontrivial, making use of the connection between fundamental classes and residues, local duality, etc. At the end of that paper, a sketch is given of how to extend to nonproper maps. But details have yet to appear.

For *smooth* maps of formal schemes, a more advanced approach is worked out by Nayak and Sastry in the above-mentioned arXiv:1903.01783.

Extensions

The theory of $(-)^!$ extends to maps of noetherian formal schemes, see Contemporary Math. 244 (1999);

and to essentially-finite-type separated maps of noetherian schemes (in particular, to Commutative Local Algebra), see Nayak's paper Advances in Mathematics 222 (2009), 527–546;

and to numerous other contexts ...
(as part of Grothendieck's "six operations")

and, *conjecturally*, to derived algebraic geometry, which encodes the homotopical rather than the possibly less basic homological features of the theory. In that context, flatness is more-or-less built in to the notion of base-change, so it should no longer be needed for defining c_f.
For a derived construction of $(-)^!$, see Chapter 6 in **Lurie's** Spectral Algebraic Geometry (2018 version), `www.ias.edu/~lurie/`.
More concretely, if less generally, some steps in this direction were taken by **Shaul** in his study of duality for differential graded rings, Advances in Mathematics 320 (2017), 279–328.