# BÜCHI'S SQUARE PROBLEM

## References

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$$a(a+d)(a+2d)(a+3d) = x^2.$$

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[P] R. G. Pinch, *Squares in quadratic progression.* Math. Comp. **60** (1993), no. 202, 841–845.

[V] P. Vojta, *Diagonal quadratic forms and Hilbert's tenth problem.* Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), 261–274, Contemp. Math., 270, Amer. Math. Soc., Providence, RI, 2000.

# 1. The problem

Find integer sequences

$$0 \le x_1 < x_2 < \cdots < x_n \ (n \ge 3) \tag{1.1}$$

satisfying

$$\boxed{\begin{aligned} x_1^2 - 2x_2^2 + x_3^2 &= 2 \\ x_2^2 - 2x_3^2 + x_4^2 &= 2 \\ &\vdots \\ x_{n-2}^2 - 2x_{n-1}^2 + x_n^2 &= 2 \end{aligned}} \tag{1.2}$$

"Trivial" case: $x_1 < x_2 < \cdots < x_n$ successive integers ($x_{i+1} = x_i + 1$). Otherwise, call $(x_1, x_2, \ldots, x_n)$ a *Büchi n-tuple.*

**Example 1.3.** $(6, 23, 32, 39)$ is a Büchi 4-tuple.

Equivalent guises occurring in the literature:

**Example 1.4.** (1.1) *is a Büchi n-tuple* $\iff$ *the sequence* $x_1^2 < x_2^2 < \cdots < x_n^2$ *differs termwise by a nonzero constant from a sequence of squares of successive integers*

$$a < a+1 < a+2 < \ldots \qquad (a > 3).$$

E.g., $246^2 - 6^2 = 247^2 - 23^2 = 248^2 - 32^2 = 249^2 - 39^2 = 60480.$

**Variant**: (1.1) *is a Büchi n-tuple* $\iff$

$$\exists\, g(X) := X^2 + bX + c \ (b, c \in \mathbb{Z},\ b \ge 3,\ b^2 \ne 4c)$$

*such that* $g(i) = x_i^2$ $(i = 1, 2, \ldots, n)$.

E.g., $X^2 + 490X - 455$ takes values $6^2$, $23^2$, $32^2$, $39^2$ at $X = 1, 2, 3, 4$.

Why this, from among the endless recreational puzzles about integers? Büchi found an application of the nonexistence of Büchi $n$-tuples, for some $n$, to a strengthening of a theorem of Matijasevic (based on work of Davis and Robinson) asserting nonexistence of a single algorithm for deciding integer solvability of all polynomial equation with integer coefficients. (Hilbert's tenth problem.) Büchi showed, for any such equation, that the existence of an integer solution is equivalent to the existence of an integer solution of a system of equations each of which is either "diagonal quadratic," i.e., of the form

$$\sum_i a_i x_i^2 = b$$

or of the form

$$x_i = \pm x_j \pm 1.$$

Now if for some $n$ there exist no Büchi $n$-tuples, then $x_1 = \pm x_2 \pm 1$ is equivalent to $\exists x_3, \ldots, x_n \in \mathbb{Z}$ such that (1.2) holds. Thus the solvability of any polynomial equation (or system of equations, since $P_1 = P_2 = \cdots = P_m = 0$ is equivalent to $\Sigma P_i^2 = 0$) would be equivalent to that of some diagonal quadratic system.

One would then have the impossibility of finding an algorithm to decide diagonal quadratic systems.

Büchi's argument is explained in a few lines, e.g., in §1 of [V].

**Example 1.5.** If there existed an algorithm for deciding diagonal quadratic systems, one could decide whether or not there exists a "perfect Euler brick," i.e., a rectangular parallelopiped for which any line segment joining two vertices has integer length. (So far, no one knows).

## 2. What's known?

- The formulas

$$\boxed{x_1 = \frac{q^2-1}{2p} - p, \quad x_2 = \frac{q^2-1}{2p} + q + p, \quad x_3 = \frac{q^2-1}{2p} + 2q + p}$$

give a bijection between Büchi 3-tuples and pairs of positive integers $(p, q)$ such that $q^2 \equiv 1 \pmod{2p}$ and $q > \sqrt{2}p$.

For instance, $(p, q) = (4, 9)$ corresponds to $(x_1, x_2, x_3) = (6, 23, 32)$, and $(p, q) = (1, 7)$ corresponds to $(x_1, x_2, x_3) = (23, 32, 39)$.

*Proof.* That the formulas define a Büchi 3-tuple is simple to check. Conversely, looking mod 4, one sees that in any Büchi $n$-tuple, successive members have opposite parities; and one checks that the succesive differences $x_{i+1} - x_i$ form a decreasing sequence. So for any Büchi 3-tuple $x_1 < x_2 < x_3$ we can set $x_2 - x_1 = 2p + q$, $x_3 - x_2 = q$, and substitute into (1.2)... □

- There are infinitely many Büchi 4-tuples. For example:

$S_0 = (1, 2, 3, 4)$, $S_1 = (6, 23, 32, 39)$, and $S_{i+1} = 10S_i - S_{i-1}$ $(i > 0)$.

Buell (1987) related finding Büchi 4-tuples to questions about when certain indefinite binary quadratic forms represent 1. In essence, he transformed solving (1.2) (nontrivially) for $n = 4$ into solving

$$(*) \qquad 1 = j^2y^2 - 2(l^2 + jl - j^2)yz + j^2z^2$$

for positive integers $j < l, y, z$.

As a quadratic form in $y$ and $z$, the right side has discriminant

$$D := 4(l^2 + jl - j^2)^2 - 4j^4 = 4(l - j)l(l + j)(l + 2j) > 0.$$

This integer is not a square (see [E]).

For fixed $j < l$, if $(y, z)$ satisfies $(*)$ then $y/z$ is a continued fraction convergent to $j^2 - jl - l^2 \pm \sqrt{D}$; and if there are any such pairs $(y, z)$ then there are infinitely many. So the Büchi 4-tuples come in infinite families (not necessarily disjoint).

There are lots of "good" $(j, l)$ for which solutions exist. To check, one need only look at a finite set of convergents, of cardinality the "period" of $\sqrt{D}$. This is easy to do with Mathematica or Maple. Buell examines thousands of random pairs, and finds that about 40% are good. But I don't know any nice characterization of which pairs are good.

For example, taking $j = 1$ yields a doubly indexed family of Büchi 4-tuples $S_n(l)$ $(n > 0,\ l > 1)$, specified by

$$S_0 = (1, 2, 3, 4),$$

$$S_1(l) = \left(2l^3 - 5l,\ 2l^3 + 2l^2 - l + 1,\ 2l^3 + 4l^2 + l - 2,\ 2(l + 1)^3 - 5(l + 1)\right),$$

$$S_{i+1}(l) = 2(l^2 + l - 1)S_i - S_{i-1} \ (i > 0).$$

- It is *not known* whether there exist Büchi 5-tuples.

## 3. More on 3- and 4-tuples

Naive strategy: use 3-tuples to learn about 4-tuples then 5-tuples.

- # of 3-tuples with given $x_1$ is 0 or $\infty$. (Known for centuries.)
- # of such 4-tuples $< \infty$: equations (1.2) $\implies$

$$w^2 := (x_3 x_4)^2 = (2x_2^2 + 2 - x_1^2)(3x_2^2 + 6 - x_1^2);$$

and for fixed $x_1$, an old theorem of Siegel guarantees that only finitely many integer pairs $(w, x_2)$ satisfy this. (Google "elliptic curve".)

- In 1993, Pinch computed all 36 Büchi 4-tuples with $x_1 < 1000$.

Note: Any bound on $x_2$ forces bounds on $x_3 = 2x_2^2 + 2 - x_1^2$ and $x_4 = 3x_2^2 + 6 - x_1^2$. But:

**(?)** Could $x_2$ be huge w.r.t. $x_1$

Pinch bounded $x_2$ in terms of $x_1$ via Baker's theory of linear combinations of logarithms.

Pinch's computation showed that if $x_1 < 1000$, then $x_2 \leq 26605$. (The corresponding 4-tuple is (916, 26605, 37614, 46063)). Knowing this, one can nowadays reproduce his table, essentially by brute-force checking, in a fraction of a second—but that's cheating, since one doesn't have any such bound *a priori*. Brute force means just list all $x_1$ and $x_2$ within certain bounds (for instance using the above formula for Büchi triples), and then check via (1.2) whether $x_3$ and $x_4$ exist. The same brute-force method produces, in a couple of hours, all the 754 4-tuples with $x_1 < x_2 < 10^7$. (And none of these extends to a 5-tuple: neither $2 + 2x_1^2 - x_2^2$ nor $2 + 2x_4^2 - x_3^2$ is an integer square.)

Among these 754 4-tuples, 60 have $x_1 < \sqrt{10^7}$.

**(??)** Could there be any more 4-tuples with $x_1 < \sqrt{10^7}$ ?

The ratios $x_2/x_1^2$ for these sixty range from 0.831444...
for $(1413, 1660036, 2347645, 2875266)$ down to 0.000426...
for (2607, 2894, 3155, 3396).
For $x_1 > 1637$, the ratio is $< 0.1$.

This suggests (at least):

**Conjecture:** $x_2 < x_1^2$ for all Büchi 4-tuples.

For bounding $x_2$ in terms of $x_1$, this would be much simpler than the approach via Baker.
In any case, thinking about it could conceivably teach one something about 4-tuples that might be useful in thinking about 5-tuples.
(Actually, it is not hard to show that for a 5-tuple, $x_2 < .75x_1^2$; but one would hope to learn other things.)

Computations (see below) show:

*The conjecture holds if* $x_2 < 10^{1000}$.

So a Büchi 4-tuple with $x_1 < \sqrt{10^7} = 3162.28\ldots$ and $x_2 > 10^7$ would have an $x_2$ with more than 1000 decimal digits. This seems unlikely to occur, but I can't yet exclude it. Perhaps Pinch's methods could be applied, but I would stubbornly prefer a proof of the conjecture.

Here is a quick sketch of the calculation.

If $(w, x, y, z)$ is a Büchi 4-tuple then $w^2 = 2x^2 + 2 - y^2$; so if this 4-tuple were a counterexample to the conjecture then

$$0 < 2x^2 + 2 - y^2 < x.$$

Call a Büchi 3-tuple

$$(x, y, z) = \left(\frac{q^2-1}{2p} - p,\ \frac{q^2-1}{2p} + q + p,\ \frac{q^2-1}{2p} + 2q + p\right)$$

*exotic* if $0 < 2x^2 + 2 - y^2 < x$.

We could then prove the conjecture by showing that for any exotic triple $(x, y, z)$, $2x^2 + 2 - y^2$ is *not* a perfect square.

An exotic $(x, y, z)$ is determined by $x$, because $y = \llcorner \sqrt{2}x \lrcorner$ and $z = \llcorner \sqrt{3}x \lrcorner$. (Easy proof.)

The calculation gives that that there are about 2000 exotic triples with $x < 10^{1000}$—so they are very sparse—and for each of these, $2x^2 + 2 - y^2$ is not a perfect square.

This calculation takes a few minutes with Mathematica on a Mac. What makes it feasible is, very roughly, that one can express $(p, q)$ for an exotic triple via continued fraction convergents to $(\sqrt{2}-1)(\sqrt{3}-1)$ or $(\sqrt{2} - 1)(\sqrt{3} - 1)/2$, and the numerators and denominators of these convergents grow very quickly. I went up to the two-thousandth convergent, where the numerator and denominator have more than 1000 digits.

Unfortunately, the calculation did not suggest any patterns on which some general proof could be based.

Let me mention in passing one really exotic triple, where

$x =$

26352185630150586644443921684341997297798285735502455029231965414
24150799193864238326819400897385118209853983553454991283146348016
122730027059861106783826343262184743120915074572081708445291792412

and

$2x^2 + 2 - y^2 =$

354027662012117690947418275043702269114057822800970066354286280224
715623765511042431270791922404483840907750149254855485888403317369
645680964373668042375162209795945459741916418433124023707961

so that

$$x/(2x^2 + 2 - y^2) \doteq 7443.54,$$

whereas if $\sqrt{2x^2 + 2 - y^2} \in \mathbb{Z}$, the conjecture would say that the ratio is $< 1$ !

Here $2x^2 + 2 - y^2 = (\sqrt{2}x - y)(\sqrt{2}x + y) + 2$ is relatively small because $\sqrt{2}x - y \doteq 0.0000475$.

In fact the conjecture is equivalent to:

$3 < x < y \in \mathbb{Z}$, $2x^2 + 2 - y^2$ and $2y^2 + 2 - x^2$ both squares $\Longrightarrow$

$$\sqrt{2}x - y > 1/\sqrt{8}.$$

# 4. Arithmetic Geometry

Let $B_m$ be the surface in $\mathbb{P}^m$ whose homogeneous equations are gotten by writing $2x_0^2$ in place of each 2 in (1.2). It is a nonsingular complete intersection, of general type when $m \geq 6$. It contains the $2^m$ lines

$$[u, \pm t, \pm(t+u), \cdots, \pm(t+(m-1)u)]$$

parametrized by $(t, u) \in \mathbb{P}^1$.

There is a **conjecture of Lang** related to Falting's theorem that there are only finitely many rational points on curves of general type (i.e., of genus $> 1$). Lang's conjecture says that a surface of general type should have only finitely many rational points outside the union of its curves of genus 0 or 1.

Vojta showed that for $m \geq 8$, the only curves on $B_m$ of genus 0 or 1 are the above lines. However, any Büchi $m$-tuple would give a rational point outside those lines.

Hence Lang's conjecture implies that for such $m$ there are only finitely many—say $M$—Büchi $m$-tuples, and hence there could be no Büchi $(m+M)$-tuple.

This would at least give the logic application.

The surface $B_5$ is what's called a *Kummer surface.* Projecting from any of the 32 lines we get a degree 4 surface in $\mathbb{P}^3$ containing 25 lines:

$$\det\begin{vmatrix} 2xyz & wyz & -wxz & -2wxy \\ w & x & y & z \\ 2 & 1 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

the so-called *Weddle surface* (treated at length in a 1916 book by Jessop). The pencil of planes through any of the 25 lines cuts the surface in a family of degree-3 plane curves, i.e., the surface is fibered by a pencil of elliptic curves. The rational points on these curves form an abelian group. By a theorem of Mazur, the torsion subgroup has order at most 16. But with help from the known lines one can explicitly write down at least 17 rational points on the general member of the pencil, so its group has infinite order. Hence $B_5$ has infinitely many rational points, each of which gives a rational solution of (1.2).

I calculated quite a few of these rational Büchi 5-tuples, hoping of course to come up with one where the common denominator was 1, but the smallest denominators found were for

$$(11, 50, 71, 88, 103)/9, \quad (13656, 26317, 34622, 41289, 47020)/11,$$
$$(4937, 9265, 12137, 14449, 16439)/12, \ (445, 1179, 1607, 1943, 2229)/14.$$

(whence, incidentally, Büchi 5-tuples exist in $\mathbb{Z}/n$ for all $n$).

These calculations suggest the conjecture that *there are at most finitely many rational Büchi 5-tuples with a given denominator* $d$.

Taking $d = 1$ this would imply that for some $m$, no (integral) Büchi $m$-tuple exists.