NOTES ON
DERIVED FUNCTORS
AND
GROTHENDIECK DUALITY

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In *Foundations of Grothendieck Duality for Diagrams of Schemes,*
Lecture Notes in Math., no. 1960, Springer-Verlag, New York, 2009, 1–259.

Abstract: This is a polished version of notes begun in the late 1980s, largely available from my home page since then, meant to be accessible to mid-level graduate students. The first three chapters treat the basics of derived categories and functors, and of the rich formalism, over ringed spaces, of the derived functors, for unbounded complexes, of the sheaf functors $\otimes$, $\mathcal{H}om$, f_* and f^* (where f is a ringed-space map). Included are some enhancements, for concentrated (= quasi-compact and quasi-separated) schemes, of classical results such as the projection and Künneth isomorphisms. The fourth chapter presents the abstract foundations of Grothendieck Duality—existence and tor-independent base change for the right adjoint of the derived functor $\mathbf{R}f_*$ when f is a quasi-proper map of concentrated schemes, the twisted inverse image pseudofunctor for separated finite-type maps of noetherian schemes, some refinements for maps of finite tor-dimension, and a brief discussion of dualizing complexes.

Contents

Introduction

(0.1) The first three chapters of these notes[1] treat the basics of derived categories and functors, and of the formalism of four of Grothendieck's "six operations" ([**Ay**], [**Mb**]), over, say, the category of ringed spaces (topological spaces equipped with a sheaf of rings)—namely the derived functors, for complexes which need not be bounded, of the sheaf functors $\otimes$, $\mathcal{H}om$, and of the direct and inverse image functors f_* and f^* relative to a map f. The axioms of this formalism are summarized in §3.6 under the rubric *adjoint monoidal* Δ*-pseudofunctors,* with values in closed categories (§3.5).

Chapter 4 develops the abstract theory of the *twisted inverse image* functor $f^!$ associated to a finite-type separated map of schemes $f\colon X \to Y$. (Suppose for now that Y is noetherian and separated, though for much of what we do, weaker hypotheses will suffice.) This functor maps the derived category of cohomologically bounded-below $\mathcal{O}_Y$-complexes with quasi-coherent homology to the analogous category over X. Its characterizing properties are:

- *Duality.* If f is proper then $f^!$ is right-adjoint to the derived direct image functor $\mathbf{R}f_*$.
- *Localization.* If f is an open immersion (or even étale), then $f^!$ is the usual inverse image functor f^*.
- *Pseudofunctoriality* (or 2-*functoriality*). To each composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ we can assign a natural functorial isomorphism $(gf)^! \xrightarrow{\sim} f^!g^!$, in such a way that a kind of associativity holds with respect to any composition of three maps, see §(3.6.5).

Additional basic properties of $f^!$ are its compatibility with *flat base change* (Theorems (4.4.3), (4.8.3)), and the existence of canonical functorial maps, for $\mathcal{O}_Y$-complexes E and F having quasi-coherent homology:

$$\mathbf{R}\mathcal{H}om(\mathbf{L}f^*E, f^!F) \to f^!\mathbf{R}\mathcal{H}om(E, F)$$

$$\mathbf{L}f^*E \underline{\otimes} f^!F \to f^!(E \underline{\otimes} F)$$

(where $\underline{\otimes}$ denotes the left-derived tensor product), of which the first is an isomorphism when E is cohomologically bounded above, with coherent homology, and F is cohomologically bounded below, (Exercise (4.9.3)(b)), and the second is an isomorphism whenever f has finite tor-dimension (Theorem (4.9.4)) or E is a bounded flat complex (Exercise (4.9.6)(a)).

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[1] that are a polished version of notes written largely in the late 1980s, available in part since then from < `www.math.purdue.edu/~lipman` >. I am grateful to Bradley Lucier for his patient instruction in some of the finer points of TeX, and for setting up the appearance macros in those days when canned style files were not common—and when compilation was several thousand times slower than nowadays.

The existence and uniqueness, up to isomorphism, of the twisted inverse image pseudofunctor is given by Theorem (4.8.1), and compatibility with flat base change by Theorem (4.8.3). These are culminating results in the notes. Various approximations to these theorems have been known for decades, see, e.g., [**H**, p. 383, 3.4]. At present, however, the proofs of the theorems, as stated here, seem to need, among other things, a compactification theorem of Nagata, that any finite-type separable map of noetherian schemes factors as an open immersion followed by a proper map, a fact whose proof was barely accessible before the appearance of [**Lt**] and [**C′**] (see also [**Vj**]); and even with that compactification theorem, I am not aware of any complete, detailed exposition of the proofs in print prior to the recent one by Nayak [**Nk**].[2] There must be a more illuminating treatment of this awesome pseudofunctor in the Plato-Erdös Book!

(0.2) The theory of $f^!$ was conceived by Grothendieck [**Gr′**, pp. 112-115], as a generalization of Serre's duality theorems for smooth projective varieties over fields. Grothendieck also applied his ideas in the context of étale cohomology. The fundamental technique of derived categories was developed by Verdier, who used it in establishing a duality theorem for locally compact spaces that generalizes classical duality theorems for topological manifolds. Deligne further developed the methods of Grothendieck and Verdier (cf. [**De′**] and its references).

Hartshorne gave an account of the theory in [**H**]. The method there is to treat separately several distinctive special situations, such as smooth maps, finite maps, and regular immersions (local complete intersections), where $f^!$ has a nice explicit description; and then to do the general case by pasting together special ones (locally, a general f can be factored as smooth$\,\circ\,$finite). The fact that this approach works is indicative of considerable depth in the underlying structure, in that the special cases, that don't *a priori* have to be related at all, can in fact be melded; and in that the reduction from general to special involves several choices (for example, in the just-mentioned factorization) of which the final results turn out to be independent. Proving the existence of $f^!$ and its basic properties in this manner involves many compatibilities among those properties in their various epiphanies, a notable example being the "Residue Isomorphism" [**H**, p. 185]. The proof in [**H**] also makes essential use of a pseudofunctorial theory of dualizing complexes,[3] so that it does not apply, e.g., to arbitrary separated noetherian schemes.

[2] In fact Nayak's methods, which are less dependent on compactifications, apply to other contexts as well, for example flat finitely-presentable separated maps of not-necessarily-noetherian schemes, or separated maps of noetherian formal schemes, see [**Nk**, §7]. See also the summary of Nayak's work in [**S′**, §§3.1–3.3].

[3] This enlightening theory—touched on in §4.10 below—is generalized to Cousin complexes over formal schemes in [**LNS**]. A novel approach, via "rigidity," is given in [**YZ**], at least for schemes of finite type over a fixed regular one.

On first acquaintance, [**De**′] appears to offer a neat way to cut through the complexity—a direct abstract proof of the existence of $f^!$, with indications about how to derive the concrete special situations (which, after all, motivate and enliven the abstract formalism). Such an impression is bolstered by Verdier's paper [**V**′]. Verdier gives a reasonably short proof of the flat base change theorem, sketches some corollaries (for example, the finite tor-dimension case is treated in half a page [*ibid.,* p. 396], as is the smooth case [*ibid.,* pp. 397–398]), and states in conclusion that "all the results of [**H**], except the theory of dualizing and residual complexes, are easy consequences of the existence theorem." In short, Verdier's concise summary of the main features, together with some background from [**H**] and a little patience, should suffice for most users of the duality machine.

Personally speaking, it was in this spirit—not unlike that in which many scientists use mathematics—that I worked on algebraic and geometric applications in the late 1970s and early 1980s. But eventually I wanted to gain a better understanding of the foundations, and began digging beneath the surface. The present notes are part of the result. They show, I believe, that there is more to the abstract theory than first meets the eye.

(0.3) There are a number of treatments of Grothendieck duality for the Zariski topology (not to mention other contexts, see e.g., [**Gl**′], [**De**], [**LO**]), for example, Neeman's approach via Brown representability [**N**], Hashimoto's treatment of duality for *diagrams* of schemes (in particular, schemes with group actions) [**Hsh**], duality for *formal* schemes [**AJL**′], as well as various substantial enhancements of material in Hartshorne's classic [**H**], such as [**C**], [**S**], [**LNS**] and [**YZ**]. Still, some basic results in these notes, such as Theorem (3.10.3) and Theorem (4.4.1) are difficult, if not impossible, to find elsewhere, at least in the present generality and detail. And, as indicated below, there are in these notes some significant differences in emphasis.

It should be clarified that the word "Notes" in the title indicates that the present exposition is neither entirely self-contained nor completely polished. The goal is, basically, to guide the willing reader along one path to an understanding of all that needs to be done to prove the fundamental Theorems (4.8.1) and (4.8.3), and of how to go about doing it. The intent is to provide enough in the way of foundations, yoga, and references so that the reader can, more or less mechanically, fill in as much of what is missing as motivation and patience allow.

So what is meant by "foundations and yoga"?

There are innumerable interconnections among the various properties of the twisted inverse image, often expressible via commutativity of some diagram of natural maps. In this way one can encode, within a formal functorial language, relationships involving higher direct images of quasi-coherent sheaves, or, more generally, of complexes with quasi-coherent homology, relationships whose treatment might otherwise, on the whole, prove discouragingly complicated.

As a strategy for coping with duality theory, disengaging the underlying category-theoretic skeleton from the algebra and geometry which it supports has the usual advantages of simplification, clarification, and generality. Nevertheless, the resulting fertile formalism of adjoint monoidal pseudofunctors soon sprouts a thicket of rather complicated diagrams whose commutativity is an essential part of the development—as may be seen, for example, in the later parts of Chapters 3 and 4. Verifying such commutativities, fun to begin with, soon becomes a tedious, time-consuming, chore. Such chores must, eventually, be attended to.[4]

Thus, these notes emphasize purely formal considerations, and attention to detail. On the whole, statements are made, whenever possible, in precise category-theoretic terms, canonical isomorphisms are not usually treated as equalities, and commutativity of diagrams of natural maps—a matter of paramount importance—is not taken for granted unless explicitly proved or straightforward to verify. The desire is to lay down transparently secure foundations for the main results. A perusal of §2.6, which treats the basic relation "adjoint associativity" between the derived functors $\underline{\underline{\otimes}}$ and $\mathbf{R}\mathcal{H}om$, and of §3.10, which treats various avatars of the tor-independence condition on squares of quasi-compact maps of quasi-separated schemes, will illustrate the point. (In both cases, total understanding requires a good deal of preceding material.)

> *Computer-aided proofs are often more convincing than many standard proofs based on diagrams which are claimed to commute, arrows which are supposed to be the same, and arguments which are left to the reader.*
>
> —J.-P. Serre [**R**, pp. 212–213].

In practice, the techniques used to decompose diagrams successively into simpler ones until one reaches those whose commutativity is axiomatic do not seem to be too varied or difficult, suggesting that sooner or later a computer might be trained to become a skilled assistant in this exhausting task. (For the general idea, see e.g., [**Sm**].) Or, there might be found a theorem in the vein of "coherence in categories" which would help even more.[5] Though I have been saying this publicly for a long time, I have not yet made a serious enough effort to pursue the matter, but do hope that someone else will find it worthwhile to try.

(0.4) Finally, the present exposition is incomplete in that it does not include that part of the "Ideal Theorem" of [**H**, pp. 6–7] involving concrete realizations of the twisted inverse image, particularly through differential forms. Such interpretations are clearly important for applications. Moreover, connections between different such realizations—isomorphisms forced

[4] Cf. [**H**, pp. 117–119], which takes note of the problem, but entices readers to relax their guard so as to make feasible a hike over the seemingly solid crust of a glacier.

[5] Warning: see Exercise (3.4.4.1) below.

by the uniqueness properties of the twisted inverse image—give rise to some fascinating maps, such as residues, with subtle properties reflecting pseudofunctoriality and base change (see [**H**, pp. 197–199], [**L′**]).

Indeed, the theory as a whole has two complementary aspects. Without the enlivening concrete interpretations, the abstract functorial approach can be rather austere—though when it comes to treating complex relationships, it can be quite advantageous. While the theory can be based on either aspect (see e.g., [**H**] and [**C**] for the concrete foundations), bridging the concrete and abstract aspects is not a trivial matter. For a simple example (recommended as an exercise), over the category of open-and-closed immersions f, it is easily seen that the functor $f^!$ is naturally isomorphic to the inverse image functor f^*; but making this isomorphism pseudofunctorial, and proving that the flat base-change isomorphism is the "obvious one," though not difficult, requires some effort.

More generally, consider smooth maps, say with d-dimensional fibers. For such $f\colon X \to Y$, and a complex $A^\bullet$ of $\mathcal{O}_Y$-modules, there is a natural isomorphism

$$f^*A^\bullet \otimes_{\mathcal{O}_X} \Omega^d_{X/Y}[d] \xrightarrow{\;\sim\;} f^!A^\bullet$$

where $\Omega^d_{X/Y}[d]$ is the complex vanishing in all degrees except $-d$, at which it is the sheaf of relative d-forms (Kähler differentials).[6] For proper such f, where $f^!$ is right-adjoint to $\mathbf{R}f_*$, there is, correspondingly, a natural map $\int(A^\bullet)\colon \mathbf{R}f_*f^!A^\bullet \to A^\bullet$. In particular, when $Y = \mathrm{Spec}(k)$, k a field, these data give *Serre Duality,* i.e., the existence of natural isomorphisms

$$\mathrm{Hom}_k(H^i(X,F),\,k) \xrightarrow{\;\sim\;} \mathrm{Ext}^{d-i}_X(F,\,\Omega^d_{X/Y})$$

for quasi-coherent $\mathcal{O}_X$-modules F.

Pseudofunctoriality of $^!$ corresponds here to the standard isomorphism

$$\Omega^d_{X/Y} \otimes_{\mathcal{O}_X} f^*\Omega^e_{Y/Z} \xrightarrow{\;\sim\;} \Omega^{d+e}_{X/Z}$$

attached to a pair of smooth maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ of respective relative dimensions d, e. For a map $h\colon Y' \to Y$, and $p_X\colon X' := X \times_Y Y' \to X$ the projection, the abstractly defined base change isomorphism ((4.4.3) below) corresponds to the natural isomorphism

$$\Omega^d_{X'/Y'} \xrightarrow{\;\sim\;} p_X^*\Omega^d_{X/Y}.$$

The proofs of these down-to-earth statements are not easy, and will not appear in these notes.

[6] A striking definition of this isomorphism was given by Verdier [**V′**, p. 397, Thm. 3]. See also [**S′**, §5.1] for a generalization to formal schemes.

Thus, there is a *canonical* dualizing pair $(f^!, \int\colon \mathbf{R}f_*f^! \to \mathbf{1})$ when f is smooth; and there are explicit descriptions of its basic properties in terms of differential forms. But it is not at all clear that there is a canonical such pair for all f, let alone one which restricts to the preceding one on smooth maps. At the (homology) level of dualizing sheaves the case of varieties over a fixed perfect field is dealt with in [**Lp**, §10], and this treatment is generalized in [**HS**, §4] to generically smooth equidimensional maps of noetherian schemes without embedded components.

All these facts should fit into a general theory of the *fundamental class* of an arbitrary separated finite-type flat map $f\colon X \to Y$ with d-dimensional fibers, a canonical derived-category map $\Omega^d_{X/Y}[d] \to f^!\mathcal{O}_Y$ which globalizes the local residue map, and expresses the basic relation between differentials and duality. It is hoped that a "Residue Theorem" dealing with these questions in full generality will appear not too many years after these notes do.

Chapter 1

Derived and Triangulated Categories

In this chapter we review foundational material from [**H**, Chap. 1][7] (see also [**De**, §1]) insofar as seems necessary for understanding what follows. The main points are summarized in (1.9.1).

Why derived categories? We postulate an interest in various homology objects and their functorial behavior. Homology is defined by means of complexes in appropriate abelian categories; and we can often best understand relations among homology objects as shadows of relations among their defining complexes. Derived categories provide a supple framework for doing so.

To construct the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, we begin with the category $\mathbf{C} = \mathbf{C}(\mathcal{A})$ of complexes in $\mathcal{A}$. Being interested basically in homology, we do not want to distinguish between homotopic maps of complexes; and we want to consider a morphism of complexes which induces homology isomorphisms (i.e., a *quasi-isomorphism*) to be an "equivalence" of complexes. So force these two considerations on $\mathbf{C}$: first *factor out* the homotopy-equivalence relation to get the category $\mathbf{K}(\mathcal{A})$ whose objects are those of $\mathbf{C}$ but whose morphisms are homotopy classes of maps of complexes; and then *localize* by formally adjoining an inverse morphism for each quasi-isomorphism. The resulting category is $\mathbf{D}(\mathcal{A})$, see §1.2 below. The category $\mathbf{D}(\mathcal{A})$ is no longer abelian; but it carries a supplementary structure given by *triangles,* which take the place of, and are functorially better-behaved than, exact sequence of complexes, see 1.4, 1.5.[8]

Restricting attention to complexes which are bounded (above, below, or both), or whose homology is bounded, or whose homology groups lie in some plump subcategory of $\mathcal{A}$, we obtain corresponding derived categories, all of which are in fact isomorphic to full triangulated subcategories of $\mathbf{D}(\mathcal{A})$, see 1.6, 1.7, and 1.9.

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[7] an expansion of some of [**V**], for which [**Do**] offers some motivation. See the historical notes in [**N′**, pp. 70–71]. See also [**I′**]. Some details omitted in [**H**] can be found in more recent exposés such as [**Gl**], [**Iv**, Chapter XI], [**KS**, Chapter I], [**W**, Chapter 10], [**N′**, Chapters 1 and 2], and [**Sm**].

[8] All these constructs are *Verdier quotients* with respect to the triangulated subcategory of $\mathbf{K}(\mathcal{A})$ whose objects are the exact complexes, see [**N′**, p. 74, 2.1.8].

In 1.8 we describe some equivalences among derived categories. For example, any choice of injective resolutions, one for each homologically bounded-below complex, gives a triangle-preserving equivalence from the derived category of such complexes to its full subcategory whose objects are bounded-below injective complexes (and whose morphisms can be identified with homotopy-equivalence classes of maps of complexes). Similarly, any choice of flat resolutions gives a triangle-preserving equivalence from the derived category of homologically bounded-above complexes to its full subcategory whose objects are bounded-above flat complexes. (For flat complexes, however, quasi-isomorphisms need not have homotopy inverses). Such equivalences are useful, for example, in treating derived functors, also for unbounded complexes, see Chapter 2.

The truncation functors of 1.10 and the "way-out" lemmas of 1.11 supply repeatedly useful techniques for working with derived categories and functors. These two sections may well be skipped until needed.

1.1. The homotopy category $\mathbf{K}$

Let $\mathcal{A}$ be an abelian category [**M**, p. 194]. $\mathbf{K} = \mathbf{K}(\mathcal{A})$ denotes the additive category [**M**, p. 192] whose objects are complexes of objects in $\mathcal{A}$:

$$C^\bullet \qquad \cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \cdots \qquad (n \in \mathbb{Z},\ d^n \circ d^{n-1} = 0)$$

and whose morphisms are homotopy-equivalence classes of maps of complexes [**H**, p. 25]. (The maps d^n are called the *differentials* in $C^\bullet$.)

We always assume that $\mathcal{A}$ comes equipped with a *specific choice* of the zero-object, of a kernel and cokernel for each map, and of a direct sum for any two objects. Nevertheless we will often abuse notation by allowing the symbol 0 to stand for *any* initial object in $\mathcal{A}$; thus for $A \in \mathcal{A}$, $A = 0$ means only that A is isomorphic to the zero-object.

For a complex $C^\bullet$ as above, since $d^n \circ d^{n-1} = 0$ therefore d^{n-1} induces a natural map

$$C^{n-1} \to (\text{kernel of } d^n)\,,$$

the cokernel of which is defined to be the homology $H^n(C^\bullet)$. A map of complexes $u\colon A^\bullet \to B^\bullet$ obviously induces maps

$$H^n(u)\colon H^n(A^\bullet) \to H^n(B^\bullet) \qquad (n \in \mathbb{Z}),$$

and these maps depend only on the homotopy class of u. Thus we have a family of functors

$$H^n\colon \mathbf{K} \to \mathcal{A} \qquad (n \in \mathbb{Z}).$$

We say that u (or its homotopy class $\bar{u}$, which is a morphism in $\mathbf{K}$) is a *quasi-isomorphism* if for every $n \in \mathbb{Z}$, the map $H^n(u) = H^n(\bar{u})$ is an isomorphism.

1.2. The derived category D

The *derived category* $\mathbf{D} = \mathbf{D}(\mathcal{A})$ is the category whose objects are the same as those of $\mathbf{K}$, but in which each morphism $A^\bullet \to B^\bullet$ is the equivalence class f/s of a pair (s,f)

$$A^\bullet \xleftarrow{\ s\ } C^\bullet \xrightarrow{\ f\ } B^\bullet$$

of morphisms in $\mathbf{K}$, with s a quasi-isomorphism, where two such pairs (s,f), (s',f') are equivalent if there is a third such pair (s'',f'') and a commutative diagram in $\mathbf{K}$:

$$\begin{tikzcd}
 & C^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
A^\bullet & C''^\bullet \arrow[l, "s''"'] \arrow[r, "f''"] \arrow[u] \arrow[d] & B^\bullet \\
 & C'^\bullet \arrow[ul, "s'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

see [**H**, p. 30]. The composition of two morphisms $f/s\colon A^\bullet \to B^\bullet$, $f'/s'\colon B^\bullet \to B'^\bullet$, is $f'g/st$, where (t,g) is a pair (which always exists) such that $ft = s'g$, see [**H**, pp. 30–31, 35–36]:

$$\begin{tikzcd}
 & & C_1^\bullet \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "g"] & & \\
A^\bullet & C^\bullet \arrow[l, "s"'] \arrow[dr, "f"'] & & C'^\bullet \arrow[dl, "s'"] \arrow[r, "f'"] & B'^\bullet \\
 & & B^\bullet & &
\end{tikzcd}$$

In particular, with (s,f) as above and $1_{C^\bullet}$ the homotopy class of the identity map of $C^\bullet$, we have

$$f/s = (f/1_{C^\bullet}) \circ (1_{C^\bullet}/s) = (f/1_{C^\bullet}) \circ (s/1_{C^\bullet})^{-1}.$$

There is a natural functor $Q\colon \mathbf{K} \to \mathbf{D}$ with $Q(A^\bullet) = A^\bullet$ for each complex $A^\bullet$ in $\mathbf{K}$ and $Q(f) = f/1_{A^\bullet}$ for each map $f\colon A^\bullet \to B^\bullet$ in $\mathbf{K}$. If f is a quasi-isomorphism then $Q(f) = f/1_{A^\bullet}$ is an isomorphism (with inverse $1_{A^\bullet}/f$); and in this respect, Q is universal: *any functor* $Q'\colon \mathbf{K} \to \mathbf{E}$ *taking quasi-isomorphisms to isomorphisms factors uniquely via* Q, i.e., there is a unique functor $\widetilde{Q}'\colon \mathbf{D} \to \mathbf{E}$ such that $Q' = \widetilde{Q}' \circ Q$ (so that $\widetilde{Q}'(A^\bullet) = Q'(A^\bullet)$ and $\widetilde{Q}'(f/s) = Q'(f) \circ Q'(s)^{-1}$).

This characterizes the pair $(\mathbf{D}, Q)$ up to canonical isomorphism.[9]

Moreover [**H**, p. 33, Prop. 3.4]: *any morphism* $Q'_1 \to Q'_2$ *of such functors extends uniquely to a morphism* $\widetilde{Q}'_1 \to \widetilde{Q}'_2$. In other words, composition with Q gives, for any category $\mathbf{E}$, an isomorphism of the functor category $\mathbf{Hom}(\mathbf{D}, \mathbf{E})$ onto the full subcategory of $\mathbf{Hom}(\mathbf{K}, \mathbf{E})$ whose objects are the functors $\mathbf{K} \to \mathbf{E}$ which transform quasi-isomorphisms in $\mathbf{K}$ into isomorphisms in $\mathbf{E}$.

One checks that the category $\mathbf{D}$ supports a *unique additive structure* such that the canonical functor $Q \colon \mathbf{K} \to \mathbf{D}$ is additive; and accordingly we will always regard $\mathbf{D}$ as an additive category. If the category $\mathbf{E}$ and the above functor $Q' \colon \mathbf{K} \to \mathbf{E}$ are both additive, then so is $\widetilde{Q}'$.

Remarks. **(1.2.1).** The homology functors $H^n \colon \mathbf{K} \to \mathcal{A}$ defined in (1.1) transform quasi-isomorphisms into isomorphisms, and hence may be regarded as functors on $\mathbf{D}$.

(1.2.2). *A morphism* $f/s \colon A^\bullet \to B^\bullet$ *in* $\mathbf{D}$ *is an isomorphism if and only if*

$$H^n(f/s) = H^n(f) \circ H^n(s)^{-1} \colon\ H^n(A^\bullet) \to H^n(B^\bullet)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Indeed, if $H^n(f/s)$ is an isomorphism for all n, then so is $H^n(f)$, i.e., f is a quasi-isomorphism; and then s/f is the inverse of f/s.

(1.2.3). *There is an isomorphism of* $\mathcal{A}$ *onto a full subcategory of* $\mathbf{D}$, *taking any object* $X \in \mathcal{A}$ *to the complex* $X^\bullet$ *which is* X *in degree zero and* 0 *elsewhere, and taking a map* $f \colon X \to Y$ *in* $\mathcal{A}$ *to* $f^\bullet/1_{X^\bullet}$, *where* $f^\bullet \colon X^\bullet \to Y^\bullet$ *is the homotopy class whose sole member is the map of complexes which is* f *in degree zero.*

Bijectivity of the indicated map $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X^\bullet, Y^\bullet)$ is a straightforward consequence of the existence of a natural functorial isomorphism $Z \xrightarrow{\sim} H^0(Z^\bullet)$ $(Z \in \mathcal{A})$.

1.3. Mapping cones

An important construction is that of the *mapping cone* $C_u^\bullet$ of a map of complexes $u \colon A^\bullet \to B^\bullet$ in $\mathcal{A}$. (For this construction we need only assume that the category $\mathcal{A}$ is additive.) $C_u^\bullet$ is the complex whose degree n component is

$$C_u^n = B^n \oplus A^{n+1}$$

[9] The set Σ of quasi-isomorphisms in $\mathbf{K}$ admits a calculus of left and of right fractions, and $\mathbf{D}$ is, up to canonical isomorphism, the *category of fractions* $\mathbf{K}[\Sigma^{-1}]$, see e.g., [**Sc**, Chapter 19.] The set-theoretic questions arising from the possibility that Σ is "too large," i.e., a class rather than a set, are dealt with in *loc. cit.* Moreover, there is often a construction of a universal pair $(\mathbf{D}, Q)$ which gets around such questions (but may need the axiom of choice), cf. (2.3.2.2) and (2.3.5) below.

and whose differentials $d^n\colon C_u^n \to C_u^{n+1}$ satisfy

$$d^n|_{B^n} = d_B^n, \qquad d^n|_{A^{n+1}} = u|_{A^{n+1}} - d_A^{n+1} \qquad (n \in \mathbb{Z})$$

where the vertical bars denote "restricted to," and d_B, d_A are the differentials in $B^\bullet$, $A^\bullet$ respectively.

$$\begin{tikzcd}
C_u^{n+1} \arrow[r, phantom, "="] & B^{n+1} \arrow[r, phantom, "\oplus"] & A^{n+2} \\
C_u^n \arrow[u, "d"] \arrow[r, phantom, "="] & B^n \arrow[u, "d_B"] \arrow[r, phantom, "\oplus"] & A^{n+1} \arrow[u, "-d_A"'] \arrow[ul, "u"']
\end{tikzcd}$$

For any complex $A^\bullet$, and $m \in \mathbb{Z}$, $A^\bullet[m]$ will denote the complex having degree n component

$$(A^\bullet[m])^n = A^{n+m} \qquad (n \in \mathbb{Z})$$

and in which the differentials $A^n[m] \to A^{n+1}[m]$ are $(-1)^m$ times the corresponding differentials $A^{n+m} \to A^{n+m+1}$ in $A^\bullet$. There is a natural "translation" functor T from the category of $\mathcal{A}$-complexes into itself satisfying $TA^\bullet = A^\bullet[1]$ for all complexes $A^\bullet$.

To any map u as above, we can then associate the sequence of maps of complexes

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C_u^\bullet \xrightarrow{w} A^\bullet[1] \tag{1.3.1}$$

where v (resp. w) is the natural inclusion (resp. projection) map. The sequence (1.3.1) could also be represented in the form

$$\begin{tikzcd}
& C_u^\bullet \arrow[dl, "{[1]}"'] & \\
A^\bullet \arrow[rr, "u"'] & & B^\bullet \arrow[ul]
\end{tikzcd} \tag{1.3.2}$$

and so we call such a sequence a *standard triangle.*

A commutative diagram of maps of complexes

$$\begin{tikzcd}
A^\bullet \arrow[r, "u"] \arrow[d] & B^\bullet \arrow[d] \\
A'^\bullet \arrow[r, "u'"'] & B'^\bullet
\end{tikzcd}$$

gives rise naturally to a commutative diagram of associated g triangles (each arrow representing a map of complexes):

$$\begin{tikzcd}
A^\bullet \arrow[r, "u"] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C_u^\bullet \arrow[r] \arrow[d] & A^\bullet[1] \arrow[d] \\
A'^\bullet \arrow[r, "u'"'] & B'^\bullet \arrow[r] & C_{u'}^\bullet \arrow[r] & A'^\bullet[1]
\end{tikzcd}$$

Most of the basic properties of standard triangles involve homotopy, and so are best stated in $\mathbf{K}(\mathcal{A})$. For example, the mapping cone $C_1^\bullet$ of the identity map $A^\bullet \to A^\bullet$ is homotopically equivalent to zero, a homotopy between the identity map of $C_1^\bullet$ and the zero map being as indicated:

$$\begin{tikzcd}
C_1^{n+1} \arrow[d, "h^{n+1}"'] & = & A^{n+1} \arrow[drr, "1"] & \oplus & A^{n+2} \\
C_1^{n} & = & A^{n} & \oplus & A^{n+1}
\end{tikzcd}$$

(i.e., for each $n \in \mathbb{Z}$, h^{n+1} restricts to the identity on A^{n+1} and to 0 on A^{n+2}; and $d^{n-1}h^n + h^{n+1}d^n$ is the identity of C_1^n). Other properties can be found e.g., in [**Bo**, pp. 102–105], [**Iv**, pp. 22–33]. For subsequent developments we need to axiomatize them, as follows.

1.4. Triangulated categories (Δ-categories)

A *triangulation* on an arbitrary additive category $\mathbf{K}$ consists of an additive automorphism T (the *translation functor*) of $\mathbf{K}$, and a collection $\mathcal{T}$ of diagrams of the form

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA\,. \tag{1.4.1}$$

A *triangle* (*with base* u *and summit* C) is a diagram (1.4.1) in $\mathcal{T}$. (See (1.3.2) for a more picturesque—but typographically less convenient—representation of a triangle.) The following conditions are required to hold:

(Δ1)$'$ *Every diagram of the following form is a triangle:*

$$A \xrightarrow{\text{identity}} A \longrightarrow 0 \longrightarrow TA\,.$$

(Δ1)$''$ *Given a commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & TA \arrow[d, "T\alpha"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA'
\end{tikzcd}$$

if α, β, γ are all isomorphisms and the top row is a triangle then the bottom row is a triangle.

(Δ2) *For any triangle* (1.4.1) *consider the corresponding infinite diagram* $(1.4.1)^\infty$:

$$\cdots \longrightarrow T^{-1}C \xrightarrow{-T^{-1}w} A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB \longrightarrow \cdots$$

in which every arrow is obtained from the third preceding one by applying $-T$. *Then any three successive maps in* $(1.4.1)^\infty$ *form a triangle.*

($\Delta3)'$ *Any morphism* $A \xrightarrow{u} B$ *in* $\mathbf{K}$ *is the base of a triangle* (1.4.1).

($\Delta3)''$ *For any diagram*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, dashed, "(\exists\gamma)"'] & TA \arrow[d, "T\alpha"] \\
A' \arrow[r, "u'"'] & B' \arrow[r] & C' \arrow[r] & TA'
\end{tikzcd}$$

whose rows are triangles, and with maps α, β *given such that* $\beta u = u'\alpha$, *there exists a morphism* $\gamma\colon C \to C'$ *making the entire diagram commute, i.e., making it a morphism of triangles.*[10]

As a consequence of these conditions we have [**H**, p. 23, Prop. 1.1 c]:

($\Delta3)^*$ *If in* ($\Delta3)''$ *both* α *and* β *are isomorphisms, then so is* γ.

Thus, and by ($\Delta3)'$:

Every morphism $A \xrightarrow{u} B$ *is the base of a triangle, uniquely determined up to isomorphism by* u.

---

[10] ($\Delta3)''$ is implied by a stronger "octahedral" axiom, which states that for a composition $A \xrightarrow{u} B \xrightarrow{\beta} B'$ and triangles Δ_u, $\Delta_{\beta u}$, Δ_β with respective bases u, βu, β, there exist morphisms of triangles $\Delta_u \to \Delta_{\beta u} \to \Delta_\beta$ extending the diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & B \arrow[d, "\beta"] \\
A \arrow[r, "\beta u"'] \arrow[d, "u"'] & B' \arrow[d, equal] \\
B \arrow[r, "\beta"'] & B'
\end{tikzcd}$$

and such that the induced maps on summits $C_u \to C_{\beta u} \to C_\beta$ are themselves the sides of a triangle, whose third side is the composed map $C_\beta \to TB \to TC_u$. This axiom is incompletely stated in [**H**, p. 21], see [**V**, p. 3] or [**Iv**, pp. 453–455]. We omit it here because it plays no role in these notes (nor, as far as I can tell, in [**H**]). Thus the adjective "pre-triangulated" may be substituted for "triangulated" throughout, see [$\mathbf{N}'$, p. 51, Definition 1.3.13 and p. 60, Remark 1.4.7].

Definition (1.4.2). A *triangulated category* (Δ-*category* for short) is an additive category together with a triangulation.

EXERCISE **(1.4.2.1).** (Cf. [**N**$'$, pp. 42–45].) For any triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

in a Δ-category $\mathbf{K}$, and any object M, the induced sequence of abelian groups

$$\mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$$

is *exact* [**H**, p. 23, 1.1 b)]. Using this and (Δ2) (or otherwise), show that u *is an isomorphism iff* $C \cong 0$. More generally, *the following conditions are equivalent:*

(a) u is a monomorphism.
(b) v is an epimorphism.
(c) $w = 0$.
(d) There exist maps $A \xleftarrow[t]{} B \xleftarrow[s]{} C$ such that

$$1_A = tu, \quad 1_B = sv + ut, \quad 1_C = vs$$

(so that $B \cong A \oplus C$).

Consequently, in view of (Δ3)$'$, any monomorphism in $\mathbf{K}$ has a left inverse and any epimorphism has a right inverse. And incidentally, the existence of finite direct sums in $\mathbf{K}$ follows from the other axioms of Δ-categories.

EXAMPLE **(1.4.3): $\mathbf{K}(\mathcal{A})$.** For any abelian (or just additive) category $\mathcal{A}$, the homotopy category $\mathbf{K} := \mathbf{K}(\mathcal{A})$ of (1.1) has a triangulation, with translation T such that

$$TA^\bullet = A^\bullet[1] \qquad (A^\bullet \in \mathbf{K})$$

(i.e., T is induced by the translation functor on complexes, see (1.3), a functor which respects homotopy), and with triangles all those diagrams (1.4.1) which are isomorphic (in the obvious sense, see (Δ3)*) to the image in $\mathbf{K}$ of some standard triangle, see (1.3) again. The properties (Δ1)$'$, (Δ1)$''$, and (Δ3)$'$ follow at once from the discussion in (1.3). To prove (Δ3)$''$ we may assume that $C = C_u^\bullet$, $C' = C_{u'}^\bullet$, and the rows of the diagram are standard triangles. By assumption, βu is homotopic to $u'\alpha$, i.e., there is a family of maps $h^n \colon A^n \to B'^{n-1}$ $(n \in \mathbb{Z})$ such that

$$\beta^n u^n - u'^n \alpha^n = d_{B'}^{n-1} h^n + h^{n+1} d_A^n .$$

Define γ by the family of maps

$$\gamma^n \colon C^n = B^n \oplus A^{n+1} \longrightarrow B'^n \oplus A'^{n+1} = C'^n \qquad (n \in \mathbb{Z})$$

such that for $b \in B^n$ and $a \in A^{n+1}$,

$$\gamma^n(b, a) = \big(\beta^n(b) + h^{n+1}(a),\, \alpha^{n+1}(a)\big) ,$$

and then check that γ is as desired.

For establishing the remaining property ($\Delta 2$), we recall some facts about cylinders of maps of complexes (see e.g., [**B**, §2.6]—modulo sign changes leading to isomorphic complexes).

Let $u\colon A^\bullet \to B^\bullet$ be a map of complexes, and let $w\colon C_u^\bullet \to A^\bullet[1]$ be the natural map, see §1.3. We define the *cylinder of* u, $\widetilde{C}_u^\bullet$, to be the complex

$$\widetilde{C}_u^\bullet := C_w^\bullet[-1]\,.$$

($\widetilde{C}_u^\bullet$ is also the cone of the map $(-1,u)\colon A \to A \oplus B$.) One checks that there is a map of complexes $\varphi\colon \widetilde{C}_u^\bullet \to B^\bullet$ given in degree n by the map

$$\varphi^n\colon \widetilde{C}_u^n = A^n \oplus B^n \oplus A^{n+1} \to B^n$$

such that

$$\varphi^n(a,b,a') = u(a) + b\,.$$

The map φ is a *homotopy equivalence,* with homotopy inverse ψ given in degree n by

$$\psi^n(b) = (0,b,0)\,.$$

[If $d^n\colon \widetilde{C}_u^n \to \widetilde{C}_u^{n+1}$ is the differential and $h^{n+1}\colon \widetilde{C}_u^{n+1} \to \widetilde{C}_u^n$ is given by $h^{n+1}(a,b,a') = (0,0,-a)$, then $1_{\widetilde{C}_u^n} - \psi^n\varphi^n = d^{n-1}h^n + h^{n+1}d^n \dots$]

There results a diagram of maps of complexes

(1.4.3.1)
$$\begin{tikzcd}
A^\bullet \arrow[r, "\tilde u"] \arrow[d, equal] & \widetilde{C}_u^\bullet \arrow[r, "\tilde v"] \arrow[d, "\varphi"'] & C_u^\bullet \arrow[r, "w"] \arrow[d, equal] & A^\bullet[1] \arrow[d, equal] \\
A^\bullet \arrow[r, "u"'] & B^\bullet \arrow[r, "v = \tilde v\psi"'] & C_u^\bullet \arrow[r, "w"'] & A^\bullet[1]
\end{tikzcd}$$

in which $\tilde u$ and $\tilde v$ are the natural maps, the bottom row is a standard triangle, the two outer squares commute, and the middle square is *homotopy-commutative,* i.e., $\tilde v - v\varphi = \tilde v(1-\psi\varphi)$ is homotopic to 0.

Now, (1.4.3.1) implies that the diagram

$$C_u^\bullet[-1] \xrightarrow{-w[-1]} A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C_u^\bullet$$

is isomorphic in $\mathbf{K}$ to the diagram

$$C_u^\bullet[-1] \xrightarrow{-w[-1]} A^\bullet \xrightarrow{\tilde u} \widetilde{C}_u^\bullet \xrightarrow{\tilde v} C_u^\bullet$$

which is a standard triangle, since $\widetilde{C}_u^\bullet = C_w^\bullet[-1] = C^\bullet_{-w[-1]}$.

Hence if

$$A^\bullet \xrightarrow{u'} B^\bullet \xrightarrow{v'} C^\bullet \xrightarrow{w'} A^\bullet[1]$$

is any triangle in $\mathbf{K}$, then

$$C^\bullet[-1] \xrightarrow{-w'[-1]} A^\bullet \xrightarrow{u'} B^\bullet \xrightarrow{v'} C^\bullet$$

is a triangle, and—by the same reasoning—so is

$$B^\bullet[-1] \xrightarrow{-v'[-1]} C^\bullet[-1] \xrightarrow{-w'[-1]} A^\bullet \xrightarrow{u'} B^\bullet\,,$$

and consequently so is

$$B^\bullet \xrightarrow{v'} C^\bullet \xrightarrow{-w'} A^\bullet[1] \xrightarrow{u'[1]} B^\bullet[1]$$

(because if $A^\bullet \cong C^\bullet_{-v'[-1]} = C^\bullet_{v'}[-1]$, then $A^\bullet[1] \cong C^\bullet_{v'}$), as is the isomorphic diagram

$$B^\bullet \xrightarrow{v'} C^\bullet \xrightarrow{w'} A^\bullet[1] \xrightarrow{-u'[1]} B^\bullet[1]\,.$$

Property $(\Delta 2)$ for $\mathbf{K}$ results.[11]

We will always consider $\mathbf{K}$ *to be a* Δ*-category, with this triangulation.*

There is a close relation between triangles in $\mathbf{K}$ and certain exact sequences. For any exact sequence of complexes in an abelian category $\mathcal{A}$

$$0 \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0\,, \tag{1.4.3.2}$$

if u_0 is the isomorphism from $A^\bullet$ onto the kernel of v induced by u, then we have a natural exact sequence of complexes

$$0 \longrightarrow C^\bullet_{u_0} \xrightarrow{\text{inclusion}} C^\bullet_u \xrightarrow{\chi} C^\bullet \longrightarrow 0 \tag{1.4.3.3}$$

where $\chi^n\colon C^n_u \to C^n$ $(n \in \mathbb{Z})$ is the composition

$$\chi^n : C^n_u = B^n \oplus A^{n+1} \xrightarrow{\text{natural}} B^n \xrightarrow{\quad v \quad} C^n$$

(see (1.3)). It is easily checked—either directly, or because $C^\bullet_{u_0}$ is isomorphic to the cone of the identity map of $A^\bullet$—that $H^n(C^\bullet_{u_0}) = 0$ for all n; and then from the long exact cohomology sequence associated to (1.4.3.3) we conclude that χ *is a quasi-isomorphism.*

[11] For other treatments of $(\Delta 2)$ and $(\Delta 3)''$ see [**Bo**, pp. 102–104] or [**Iv**, p. 27, 4.16; and p. 30, 4.19]. And for the octahedral axiom, use triangle (4.22) in [**Iv**, p. 32], whose vertices are the cones of two composable maps and of their composition.

If the exact sequence (1.4.3.2) is *semi-split,* i.e., for every $n \in \mathbb{Z}$, the restriction $v^n\colon B^n \to C^n$ of v to B^n has a left inverse, say φ^n, then with

$$\Phi^n = \varphi^n \oplus (\varphi^{n+1} d_C^n - d_B^n \varphi^n)\colon C^n \to B^n \oplus A^{n+1}$$

(where A^{n+1} is identified with $\ker(v^{n+1})$ via u), the map of complexes $\Phi := (\Phi^n)_{n\in\mathbb{Z}}$ is a *homotopy inverse* for χ: $\chi\circ\Phi$ is the identity map of $C^\bullet$, and also the map $(1_{C_u^\bullet} - \Phi\circ\chi)\colon C_u^\bullet \to \ker(\chi) = C_{u_0}^\bullet \cong 0$ vanishes in $\mathbf{K}$. [More explicitly, if $h^{n+1}\colon C_u^{n+1} \to C_u^n$ is given by

$$h^{n+1}(b,a) := b - \phi^{n+1} v^{n+1} b \in A^{n+1} \subset C_u^n \qquad (b \in B^{n+1},\ a \in A^{n+2})$$

and d is the differential in $C_u^\bullet$, then $1_{C_u^n} - \Phi^n \circ \chi^n = (d^{n-1}h^n + h^{n+1}d^n)$.] Thus χ induces a *natural isomorphism in* $\mathbf{K}$

$$C_u^\bullet \xrightarrow{\ \sim\ } C^\bullet\,,$$

and hence by $(\Delta 1)''$ we have a triangle

(1.4.3.4) $$A^\bullet \xrightarrow{\bar u} B^\bullet \xrightarrow{\bar v} C^\bullet \xrightarrow{\bar w} A^\bullet[1]$$

where $\bar u$, $\bar v$ are the homotopy classes of u, v respectively, and $\bar w$ is the homotopy class of the composed map

(1.4.3.5) $$(\varphi^{n+1} d_C^n - d_B^n \varphi^n)_{n\in\mathbb{Z}}\colon C^\bullet \xrightarrow{\Phi} C_u^\bullet \xrightarrow{\text{natural}} A^\bullet[1]\,,$$

a class independent of the choice of splitting maps φ^n, because χ does not depend on that choice, so that neither does its inverse Φ, up to homotopy. This $\bar w$ is called the *homotopy invariant* of (1.4.3.2) (assumed semi-split).[12]

Moreover, *any triangle in* $\mathbf{K}$ *is isomorphic to one so obtained.*

This is shown by the image in $\mathbf{K}$ of (1.4.3.1) (in which the bottom row is any standard triangle, and the homotopy equivalence φ becomes an isomorphism) as soon as one checks that the top row is in fact of the form specified by (1.4.3.4) and (1.4.3.5).

[12] The category $\mathcal{A}$ need only be additive for us to define the homotopy invariant of a semi-split sequence of complexes $A^\bullet \underset{\psi}{\overset{u}{\rightleftarrows}} B^\bullet \underset{\varphi}{\overset{v}{\rightleftarrows}} C^\bullet$ (i.e., $B^n \cong A^n \oplus C^n$ for all n, and $u^n, \psi^n, v^n, \varphi^n$ are the usual maps associated with a direct sum): it's the homotopy class of the map

$$\psi(\varphi d_C - d_B\varphi)\colon C^\bullet \to A^\bullet[1],$$

a class depending, as above, only on u and v. [More directly, note that if φ' is another family of splitting maps then

$$\psi(\varphi d_C - d_B\varphi) - \psi(\varphi' d_C - d_B\varphi') = d_{A[1]}\psi(\varphi-\varphi') + \psi(\varphi-\varphi')d_C\,.]$$

By way of illustration here is an often used fact, whose proof involves triangles. (See also [**H**, pp. 35–36].)

Lemma (1.4.3.6). *Any diagram* $A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet$ *in* $\mathbf{K}(\mathcal{A})$, *with* s *a quasi-isomorphism, can be embedded in a commutative diagram*

(1.4.3.7)
$$\begin{tikzcd} C^\bullet \arrow[r, "f"] \arrow[d, "s"'] & B^\bullet \arrow[d, "s'"] \\ A^\bullet \arrow[r, "f'"'] & D^\bullet \end{tikzcd}$$

with s' *a quasi-isomorphism.*

Proof. By $(\Delta 3)'$ there exists a triangle

(1.4.3.8)
$$C^\bullet \xrightarrow{(s,-f)} A^\bullet \oplus B^\bullet \longrightarrow D^\bullet \longrightarrow C^\bullet[1]\,.$$

If f' is the natural composition $A^\bullet \to A^\bullet \oplus B^\bullet \to D^\bullet$, and s' is the composition $B^\bullet \to A^\bullet \oplus B^\bullet \to D^\bullet$, then commutativity of (1.4.3.7) results from the easily-verifiable fact that the composition of the first two maps in a standard triangle is homotopic to 0.[13] And if s is a quasi-isomorphism, then from (1.4.3.8) we get exact homology sequences

$$0 \to H^n(C^\bullet) \to H^n(A^\bullet) \oplus H^n(B^\bullet) \to H^n(D^\bullet) \to 0 \qquad (n \in \mathbb{Z})$$

(see (1.4.5) below) which quickly yield that s' is a quasi-isomorphism too.

Example **(1.4.4):** $\mathbf{D}(\mathcal{A})$. The above triangulation on $\mathbf{K}$ leads naturally to one on the derived category $\mathbf{D}$ of 1.2. The translation functor $\widetilde{T}$ is determined by the relation $QT = \widetilde{T}Q$, where $Q\colon \mathbf{K} \to \mathbf{D}$ is the canonical functor, and T is the translation functor in $\mathbf{K}$ (see (1.4.3)): note that QT transforms quasi-isomorphisms into isomorphisms, and use the universal property of Q given in 1.2. In particular $\widetilde{T}(A^\bullet) = A^\bullet[1]$ for every complex $A^\bullet \in \mathbf{D}$. ($\widetilde{T}$ is additive, by the remarks just before (1.2.1).) The triangles are those diagrams which are isomorphic—in the obvious sense, see $(\Delta 3)^*$—to those coming from $\mathbf{K}$ via Q, i.e., diagrams isomorphic to natural images of standard triangles.

Conditions $(\Delta 1)'$, $(\Delta 1)''$, and $(\Delta 2)$ are easily checked.

[13] In fact in any Δ-category, any two successive maps in a triangle compose to 0 [**H**, p. 23, Prop. 1.1 a)].

Next, given $f/s\colon A^\bullet \to B^\bullet$ in $\mathbf{D}$, represented by $A^\bullet \xleftarrow{s} X^\bullet \xrightarrow{f} B^\bullet$ in $\mathbf{K}$ (see 1.2), we have, by $(\Delta 3)'$ for $\mathbf{K}$, a triangle $X^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} X^\bullet[1]$ in $\mathbf{K}$, whose image is the top row of a commutative diagram in $\mathbf{D}$, as follows:

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{Q(f)} & B^\bullet & \xrightarrow{Q(g)} & C^\bullet & \xrightarrow{Q(h)} & X^\bullet[1] \\ {\scriptstyle Q(s)}\downarrow{\scriptstyle\simeq} & & \| & & \| & & {\scriptstyle\simeq}\downarrow{\scriptstyle \widetilde{T}Q(s)} \\ A^\bullet & \xrightarrow[f/s]{} & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & X^\bullet[1] \end{array} \tag{1.4.4.1}$$

Condition $(\Delta 3)'$ for $\mathbf{D}$ results. As for $(\Delta 3)''$, we can assume, via isomorphisms, that the rows of the diagram in question come from $\mathbf{K}$, via Q. Then we check via definitions in 1.2 that the commutative diagram

$$\begin{array}{ccc} A^\bullet & \xrightarrow{u} & B^\bullet \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ A'^\bullet & \xrightarrow[u']{} & B'^\bullet \end{array}$$

in $\mathbf{D}$ can be expanded to a commutative diagram of the form

$$\begin{array}{ccc} A^\bullet & \xrightarrow{u} & B^\bullet \\ {\scriptstyle\alpha_1}\uparrow{\scriptstyle\simeq} & & {\scriptstyle\beta_1}\uparrow{\scriptstyle\simeq} \\ X^\bullet & \longrightarrow & Y^\bullet \\ {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\beta_2}\downarrow \\ A'^\bullet & \xrightarrow[u']{} & B'^\bullet \end{array}$$

(i.e., $\alpha = \alpha_2\alpha_1^{-1}$, $\beta = \beta_2\beta_1^{-1}$), where all the arrows represent maps coming from $\mathbf{K}$, i.e., maps of the form $Q(f)$. By $(\Delta 3)'$ and $(\Delta 3)''$ for $\mathbf{K}$, this diagram embeds into a larger commutative one whose middle row also comes from $\mathbf{K}$:

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A^\bullet[1] \\ {\scriptstyle\alpha_1}\uparrow{\scriptstyle\simeq} & & {\scriptstyle\beta_1}\uparrow{\scriptstyle\simeq} & & {\scriptstyle\gamma_1}\uparrow & & {\scriptstyle\simeq}\uparrow{\scriptstyle\widetilde{T}\alpha_1} \\ X^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\ {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\beta_2}\downarrow & & {\scriptstyle\gamma_2}\downarrow & & \downarrow{\scriptstyle\widetilde{T}\alpha_2} \\ A'^\bullet & \xrightarrow[u']{} & B'^\bullet & \longrightarrow & C'^\bullet & \longrightarrow & A'^\bullet[1] \end{array}$$

Using (1.2.2) and the exact homology sequences associated to the top two rows (see (1.4.5) below), we find that γ_1 is an isomorphism. Then $\gamma := \gamma_2\gamma_1^{-1}$ fulfills $(\Delta 3)''$.

So we have indeed defined a triangulation; and from $(\Delta 1)''$, $(\Delta 3)^*$, and (1.4.4.1) we conclude that this is the *unique* triangulation on $\mathbf{D}$ with translation $\widetilde{T}$ and such that Q transforms triangles into triangles.

We will always consider $\mathbf{D}$ *to be a* Δ*-category, with this triangulation.*

Now for any exact sequence of complexes in $\mathcal{A}$

$$0 \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0 \tag{1.4.4.2}$$

the quasi-isomorphism χ of (1.4.3.3) becomes an isomorphism $\tilde{\chi}$ in $\mathbf{D}$, so that in $\mathbf{D}$ there is a natural composed map

$$\tilde{w}\colon C^\bullet \xrightarrow{\tilde{\chi}^{-1}} C_u^\bullet \longrightarrow A^\bullet[1]\,;$$

and then with $\tilde{u}$ and $\tilde{v}$ corresponding to u and v respectively, the diagram

$$A^\bullet \xrightarrow{\tilde{u}} B^\bullet \xrightarrow{\tilde{v}} C^\bullet \xrightarrow{\tilde{w}} A^\bullet[1] \tag{1.4.4.2$^\sim$}$$

is a triangle in $\mathbf{D}$. If the sequence (1.4.4.2) is semi-split, then $(1.4.4.2)^\sim$ is the image in $\mathbf{D}$ of the triangle (1.4.3.4) in $\mathbf{K}$. Since every triangle in $\mathbf{K}$ is isomorphic to one coming from a semi-split exact sequence (see end of example (1.4.3)), therefore *every triangle in* $\mathbf{D}$ *is isomorphic to one of the form* $(1.4.4.2)^\sim$ *arising from an exact sequence of complexes in* $\mathcal{A}$ *(in fact, from a semi-split such sequence).*

(1.4.5). To any triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$ in $\mathbf{K}$ or in $\mathbf{D}$, we can apply the homology functors H^n (see (1.2.1)) to obtain an associated *exact homology sequence*

$(1.4.5)^{\mathrm{H}}$

$$\begin{aligned}\cdots \longrightarrow H^{i-1}(C^\bullet) \xrightarrow{H^{i-1}(w)} H^i(A^\bullet) \xrightarrow{H^i(u)} H^i(B^\bullet)&\\ \xrightarrow{H^i(v)} H^i(C^\bullet) \xrightarrow{H^i(w)} H^{i+1}(A^\bullet) \longrightarrow \cdots&\end{aligned}$$

Exactness is verified by reduction to the case of standard triangles.

For an exact sequence (1.4.4.2), the usual connecting homomorphism

$$H^i(C^\bullet) \to H^{i+1}(A^\bullet) \qquad (i \in \mathbb{Z})$$

is easily seen to be $-H^i(\tilde{w})$ (see $(1.4.4.2)^\sim$). Thus $(1.4.5)^{\mathrm{H}}$ (for $(1.4.4.2)^\sim$) is, except for signs, the usual homology sequence associated to (1.4.4.2).

It should now be clear why it is that we can replace exact sequences of complexes in $\mathcal{A}$ by triangles in $\mathbf{D}$. And the following notion of "Δ-functor" will eventually make it quite advantageous to do so.

1.5. Triangle-preserving functors (Δ-functors)

Let $\mathbf{K_1}$, $\mathbf{K_2}$ be Δ-categories (1.4.2) with translation functors T_1, T_2 respectively. A (covariant) Δ-*functor* is defined to be a pair (F,θ) consisting of an additive functor $F\colon \mathbf{K_1} \to \mathbf{K_2}$ together with an isomorphism of functors

$$\theta\colon FT_1 \xrightarrow{\sim} T_2F$$

such that for every triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T_1A$$

in $\mathbf{K_1}$, the corresponding diagram

$$FA \xrightarrow{\ Fu\ } FB \xrightarrow{\ Fu\ } FC \xrightarrow{\ \theta\circ Fw\ } T_2FA$$

is a triangle in $\mathbf{K_2}$.

These are the exact functors of [$\mathbf{V}$, p. 4], and also the ∂-functors of [$\mathbf{H}$, p. 22]; it should be kept in mind however that θ is not always the identity transformation (see Examples (1.5.3), (1.5.4) below—but see also Exercise (1.5.5)). In practice, for given F if there is *some* θ such that (F,θ) is a Δ-functor then there will usually be a natural one, and after specifying such a θ we will simply say (abusing language) that F is a Δ-functor.

Let $\mathbf{K_3}$ be a third Δ-category, with translation T_3. If each of $(F,\theta)\colon \mathbf{K_1} \to \mathbf{K_2}$ and $(H,\chi)\colon \mathbf{K_2} \to \mathbf{K_3}$ is a Δ-functor, then so is

$$(H\circ F,\ \chi\circ\theta)\colon \mathbf{K_1} \to \mathbf{K_3}$$

where $\chi\circ\theta$ is defined to be the composition

$$HFT_1 \xrightarrow{\text{via }\theta} HT_2F \xrightarrow{\text{via }\chi} T_3HF\,.$$

A *morphism* $\eta\colon (F,\theta) \to (G,\psi)$ *of* Δ-*functors* (from $\mathbf{K_1}$ to $\mathbf{K_2}$) is a morphism of functors $\eta\colon F \to G$ such that for all objects X in $\mathbf{K_1}$, the following diagram commutes:

$$\begin{tikzcd}
FT_1(X) \arrow[r, "\theta(X)"] \arrow[d, "\eta(T_1(X))"'] & T_2F(X) \arrow[d, "T_2(\eta(X))"] \\
GT_1(X) \arrow[r, "\psi(X)"'] & T_2G(X)
\end{tikzcd}$$

The set of all such η can be made, in an obvious way, into an abelian group. If $\mu\colon (G,\psi) \to (G',\psi')$ is also a morphism of Δ-functors, then so is the composition $\mu\eta\colon (F,\theta) \to (G',\psi')$. And if $(H,\chi)\colon \mathbf{K_2} \to \mathbf{K_3}$ [respectively $(H',\chi')\colon \mathbf{K_3} \to \mathbf{K_1}$] is, as above, another Δ-functor then η naturally induces a morphism of composed Δ-functors

$$(H\circ F,\ \chi\circ\theta) \to (H\circ G,\ \chi\circ\psi)$$
$$[\text{respectively }\ (F\circ H',\ \theta\circ\chi') \to (G\circ H',\ \psi\circ\chi')\,]\,.$$

We find then that:

Proposition. *The Δ-functors from $\mathbf{K_1}$ to $\mathbf{K_2}$, and their morphisms, form an additive category $\mathbf{Hom}_\Delta(\mathbf{K_1}, \mathbf{K_2})$; and the composition operation*

$$\mathbf{Hom}_\Delta(\mathbf{K_1}, \mathbf{K_2}) \times \mathbf{Hom}_\Delta(\mathbf{K_2}, \mathbf{K_3}) \longrightarrow \mathbf{Hom}_\Delta(\mathbf{K_1}, \mathbf{K_3})$$

is a biadditive functor.

A morphism η as above has an inverse in $\mathbf{Hom}_\Delta(\mathbf{K_1}, \mathbf{K_2})$ if and only if $\eta(X)$ is an isomorphism in $\mathbf{K_2}$ for all $X \in \mathbf{K_1}$. We call such an η a Δ-*functorial isomorphism.*

Similarly, a *contravariant Δ-functor* is a pair (F, θ) with $F\colon \mathbf{K_1} \to \mathbf{K_2}$ a contravariant additive functor and

$$\theta\colon T_2^{-1}F \xrightarrow{\sim} FT_1$$

an isomorphism of functors such that for every triangle in $\mathbf{K_1}$ as above, the corresponding diagram

$$FA \xleftarrow{Fu} FB \xleftarrow{Fv} FC \xleftarrow{-Fw\circ\theta} T_2^{-1}FA$$

is a triangle in $\mathbf{K_2}$. Composition and morphisms etc. of contravariant Δ-functors are introduced in the obvious way.

Exercise. A contravariant Δ-functor is the same thing as a covariant Δ-functor on the opposite (dual) category $\mathbf{K_1^{op}}$ [**M**, p. 33], suitably triangulated. (For example, $\mathbf{D}(\mathcal{A})^{\mathsf{op}}$ is Δ-isomorphic to $\mathbf{D}(\mathcal{A}^{\mathsf{op}})$, see (1.4.4).)

Examples. **(1.5.1)** (see [**H**, p. 33, Prop. 3.4]). By (1.4.4), the natural functor $Q\colon \mathbf{K} \to \mathbf{D}$ of §1.2, together with $\theta =$ identity, is a Δ-functor. Moreover, as in 1.2: *composition with Q gives, for any Δ-category $\mathbf{E}$, an isomorphism of the category of Δ-functors $\mathbf{Hom}_\Delta(\mathbf{D}, \mathbf{E})$ onto the full subcategory of $\mathbf{Hom}_\Delta(\mathbf{K}, \mathbf{E})$ whose objects are the Δ-functors (F, θ) such that F transforms quasi-isomorphisms in $\mathbf{K}$ to isomorphisms in $\mathbf{E}$.*[14]

(1.5.2). Let $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ be an additive functor of abelian categories, and set $\mathbf{K_1} = \mathbf{K}(\mathcal{A}_1)$, $\mathbf{K_2} = \mathbf{K}(\mathcal{A}_2)$. Then F extends in an obvious way to an additive functor $\bar{F}\colon \mathbf{K_1} \to \mathbf{K_2}$ which commutes with translation, and which (together with $\theta =$ identity) is easily seen to be a Δ-functor, essentially because $\bar{F}$ takes cones to cones, i.e., for any map u of complexes in $\mathcal{A}_1$ we have

(1.5.2.1) $$\bar{F}(C_u^\bullet) = C_{\bar{F}(u)}^\bullet\,.$$

[14] *Equivalently* $(*)$: $F(C^\bullet) \cong 0$ *for every exact complex* $C^\bullet \in \mathbf{K}$. ("$C^\bullet$ exact" means $H^i(C^\bullet) = 0$ for all i, i.e., the zero map $C^\bullet \to 0$ is a quasi-isomorphism). Exactness of the homology sequence $(1.4.5)^{\mathrm{H}}$ of a standard triangle shows that a map u in $\mathbf{K}$ is a quasi-isomorphism iff the cone $C_u^\bullet$ is exact. Also, the base of a triangle is an isomorphism iff the summit is 0, see (1.4.2.1). So since $F(C_u^\bullet)$ is the summit of a triangle with base $F(u)$, $(*)$ implies that if u is a quasi-isomorphism then $F(u)$ is an isomorphism.

(1.5.3) (expanding [**H**, p. 64, line 7] and illustrating [**De**, p. 265, Prop. 1.1.7]). For complexes $A^\bullet$, $B^\bullet$ in the abelian category $\mathcal{A}$, the complex of abelian groups $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ is given in degree n by

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{gr}}(A^\bullet[-n], B^\bullet) = \prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j, B^{j+n})$$

(" $\mathrm{Hom}_{\mathrm{gr}}$ " denotes "homomorphisms of graded groups") and the differential $d^n\colon \mathrm{Hom}^n \to \mathrm{Hom}^{n+1}$ takes $f \in \mathrm{Hom}_{\mathrm{gr}}(A^\bullet[-n], B^\bullet)$ to

$$d^n(f) := (d_B \circ f)[-1] + f \circ d_{A[-n-1]} \in \mathrm{Hom}_{\mathrm{gr}}(A^\bullet[-n-1], B^\bullet).$$

In other words, if $f = (f^j)_{j\in\mathbb{Z}}$ with $f^j \in \mathrm{Hom}(A^j, B^{j+n})$ then

$$d^n(f) = \big(d_B^{n+j} \circ f^j + (-1)^{n+1} f^{j+1} \circ d_A^j\big)_{j\in\mathbb{Z}}.^{15}$$

For fixed $C^\bullet$, the additive functor of complexes

$$F_1(A^\bullet) = \mathrm{Hom}^\bullet(C^\bullet, A^\bullet)$$

preserves homotopy, and so gives an additive functor (still denoted by F_1) from $\mathbf{K} = \mathbf{K}(\mathcal{A})$ into $\mathbf{K}(\mathfrak{Ab})$ (where $\mathfrak{Ab}$ is the category of abelian groups). One checks that $F_1 T = T_* F_1$, (T = translation in $\mathbf{K}$, T_* = translation in $\mathbf{K}(\mathfrak{Ab})$) and that F_1 takes cones to cones (cf. (1.5.2.1)); and hence F_1 (together with θ_1 = identity) is a Δ-functor.

Similarly, for fixed $D^\bullet$,

$$F_2(A^\bullet) = \mathrm{Hom}^\bullet(A^\bullet, D^\bullet)$$

gives a contravariant additive functor from $\mathbf{K}$ into $\mathbf{K}(\mathfrak{Ab})$. But now we run into sign complications: the complexes $T_*^{-1}F_2(A^\bullet)$ and $F_2T(A^\bullet)$, while coinciding as graded objects, are *not* equal, the differential in one being the negative of the differential in the other. We define a functorial isomorphism

$$\theta_2(A^\bullet)\colon T_*^{-1}F_2(A^\bullet) \xrightarrow{\ \sim\ } F_2T(A^\bullet)$$

to be multiplication in each degree n by $(-1)^n$, and claim that *the pair* (F_2, θ_2) *is a contravariant* Δ-functor.

Indeed, if $u\colon A^\bullet \to B^\bullet$ is a morphism of complexes in $\mathcal{A}$, then we check (by writing everything out explicitly) that, with $F = F_2$, $\theta = \theta_2$, the map of graded objects

$$C_{Fu}^\bullet = FA^\bullet \oplus T_*FB^\bullet \xrightarrow{T_*(\theta(A^\bullet))\oplus(-1)} T_*FTA^\bullet \oplus T_*FB^\bullet = T_*FC_u^\bullet$$

is an isomorphism of *complexes,* whence, $v\colon B^\bullet \to C_u^\bullet$ and $w\colon C_u^\bullet \to TA^\bullet$ being the canonical maps, the diagram

$$FB^\bullet \xrightarrow{\ Fu\ } FA^\bullet \xrightarrow{(T_*Fw)\circ T_*(\theta(A^\bullet))} T_*FC_u^\bullet \xrightarrow{-T_*Fv} T_*FB^\bullet$$

is a triangle in $\mathbf{K}(\mathfrak{Ab})$, i.e.,

$$T_*^{-1}FA^\bullet \xrightarrow{(-Fw)\circ\theta(A^\bullet)} F(C_u^\bullet) \xrightarrow{\ Fv\ } FB^\bullet \xrightarrow{\ Fu\ } FA^\bullet$$

is a triangle (see ($\Delta 2$) in §1.4); and the claim follows.

[15] This standard d^n differs from the one in [**H**, p. 64] by a factor of $(-1)^{n+1}$.

(**1.5.4**) (see again [**De**, p. 265, Prop. 1.1.7]). Let U be a topological space, $\mathcal{O}$ a sheaf of rings—say, for simplicity, commutative—and $\mathcal{A}$ the abelian category of sheaves of $\mathcal{O}$-modules. For complexes $A^\bullet$, $B^\bullet$ in $\mathcal{A}$, the complex $A^\bullet \otimes B^\bullet$ is given in degree n by

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{p\in\mathbb{Z}}(A^p \otimes B^{n-p}) \qquad (\otimes = \otimes_{\mathcal{O}})$$

and the differential

$$d^n\colon (A^\bullet \otimes B^\bullet)^n \to (A^\bullet \otimes B^\bullet)^{n+1}$$

is the unique map whose restriction to $A^p \otimes B^{n-p}$ is

$$d^n|(A^p \otimes B^{n-p}) = d_A^p \otimes 1 + (-1)^p \otimes d_B^{n-p} \qquad (p \in \mathbb{Z}).$$

With the usual translation functor T, we have for each $i, j \in \mathbb{Z}$ a unique isomorphism of complexes

$$\theta_{ij} : T^iA^\bullet \otimes T^jB^\bullet \xrightarrow{\ \sim\ } T^{i+j}(A^\bullet \otimes B^\bullet)$$

satisfying, for every $p, q \in \mathbb{Z}$,

$$\theta_{ij}|(A^{p+i} \otimes B^{q+j}) = \text{multiplication by } (-1)^{pj}\,.$$

[Note that $A^{p+i} \otimes B^{q+j}$ is contained in both $(T^iA^\bullet \otimes T^jB^\bullet)^{p+q}$ and $(T^{i+j}(A \otimes B))^{p+q}$.]

For fixed $A^\bullet$, we find then that the functor of complexes taking $B^\bullet$ to $B^\bullet \otimes A^\bullet$ preserves homotopy and takes cones to cones, giving an additive functor from $\mathbf{K}(\mathcal{A})$ into itself, which, together with θ_{10} = identity, is a Δ-functor.

Similarly, for fixed $A^\bullet$ the functor taking $B^\bullet$ to $A^\bullet \otimes B^\bullet$ induces a functor of $\mathbf{K}(\mathcal{A})$ into itself which, together with $\theta_{01} \neq$ identity, is a Δ-functor. And for fixed $A^\bullet$, the family of isomorphisms

(1.5.4.1) $$\theta(B^\bullet)\colon A^\bullet \otimes B^\bullet \xrightarrow{\ \sim\ } B^\bullet \otimes A^\bullet$$

defined locally by

$$\theta(B^\bullet)(a \otimes b) = (-1)^{pq}(b \otimes a) \qquad (a \in A^p,\ b \in B^q)$$

constitutes an isomorphism of Δ-functors.

Exercise **(1.5.5).** Let $\mathbf{K_1}$, $\mathbf{K_2}$ be Δ-categories with respective translation functors T_1, T_2; and let $(F,\theta)\colon \mathbf{K_1} \to \mathbf{K_2}$ be a Δ-functor. An object A in $\mathbf{K_1}$ is *periodic* if there is an integer $m > 0$ such that $T_1^m(A) = A$. Suppose that 0 is the only periodic object in $\mathbf{K_1}$. (For example, $\mathbf{K_1}$ could be any one of the Δ-categories $\mathbf{K^*}$ of §1.6 below.) Then we can choose a function ν: (objects of $\mathbf{K_1}$) $\to \mathbb{Z}$ such that $\nu(0) = 0$ and $\nu(T_1A) = \nu(A) - 1$ for all $A \neq 0$; and using θ, we can define isomorphisms

$$\eta_A\colon F(A) \xrightarrow{\ \sim\ } T_2^{-\nu(A)}F(T_1^{\nu(A)}A) =: f(A) \qquad (A \in \mathbf{K_1}).$$

Note that $f(T_1A) = T_2f(A)$. Verify that there is a unique way of extending f to a functor such that the η_A form an isomorphism of Δ-functors $(F,\theta) \xrightarrow{\ \sim\ } (f, \text{identity})$.

1.6. Δ-subcategories

A *full additive subcategory* $\mathbf{K}'$ of a Δ-category $\mathbf{K}$ carries at most one triangulation for which the translation is the restriction of that on $\mathbf{K}$, and such that the inclusion functor $\iota : \mathbf{K}' \hookrightarrow \mathbf{K}$ (together with the identity transformation from ιT to $T\iota$) is a Δ-functor. For the existence of such a triangulation it is necessary and sufficient that $\mathbf{K}'$ be stable under the translation automorphism and its inverse, and that the summit of any triangle in $\mathbf{K}$ with base in $\mathbf{K}'$ be isomorphic to an object in $\mathbf{K}'$; the triangles in $\mathbf{K}'$ are then precisely the triangles of $\mathbf{K}$ whose vertices are all in $\mathbf{K}'$. (Details left to the reader.) Such a $\mathbf{K}'$ is called a Δ-*subcategory* of $\mathbf{K}$.

For example, if $\mathbf{K} = \mathbf{K}(\mathcal{A})$ is as in (1.4.3), then a full additive subcategory $\mathbf{K}'$ is a Δ-subcategory if and only if:

(i) for every complex $A^\bullet \in \mathbf{K}$ we have $A^\bullet \in \mathbf{K}' \Leftrightarrow A^\bullet[1] \in \mathbf{K}'$, and

(ii) the mapping cone of any $\mathcal{A}$-morphism of complexes $u\colon A^\bullet \to B^\bullet$ with $A^\bullet$ and $B^\bullet$ in $\mathbf{K}'$ is homotopically equivalent to a complex in $\mathbf{K}'$.

Example **(1.6.1).** We consider various full additive subcategories $\mathbf{K}^+$, $\mathbf{K}^-$, $\mathbf{K}^{\mathsf{b}}$, $\overline{\mathbf{K}}^+$, $\overline{\mathbf{K}}^-$, $\overline{\mathbf{K}}^{\mathsf{b}}$, of $\mathbf{K} = \mathbf{K}(\mathcal{A})$.

The objects of $\mathbf{K}^+$ are complexes $A^\bullet$ which are *bounded below,* i.e., there is an integer n_0 (depending on $A^\bullet$) such that $A^n = 0$ for $n < n_0$. The objects of $\overline{\mathbf{K}}^+$ are complexes $B^\bullet$ whose *homology* is bounded below, i.e., $H^m(B^\bullet) = 0$ for all $m < m_0(B^\bullet)$. The objects of $\mathbf{K}^-$ and $\overline{\mathbf{K}}^-$ (respectively $\mathbf{K}^{\mathsf{b}}$ and $\overline{\mathbf{K}}^{\mathsf{b}}$) are specified similarly, with "*bounded above*" (resp. "*bounded above and below*") in place of "bounded below." We have, obviously,

$$\mathbf{K}^{\mathsf{b}} = \mathbf{K}^+ \cap \mathbf{K}^-, \qquad \overline{\mathbf{K}}^{\mathsf{b}} = \overline{\mathbf{K}}^+ \cap \overline{\mathbf{K}}^-;$$

and if * stands for any one of $^+$, $^-$, or $^{\mathsf{b}}$, then

$$\mathbf{K}^* \subset \overline{\mathbf{K}}^*.$$

Using the natural exact sequence (see (1.3))

$$0 \to B^\bullet \to C_u^\bullet \to A^\bullet[1] \to 0 \tag{1.6.2}$$

associated with a morphism $u\colon A^\bullet \to B^\bullet$ of complexes in $\mathcal{A}$, we find that if both $A^\bullet$ and $B^\bullet$ satisfy one of the above boundedness conditions then so does the cone $C_u^\bullet$, whence $\mathbf{K}^*$ *and* $\overline{\mathbf{K}}^*$ *are* Δ-*subcategories of* $\mathbf{K}$.

Remark **(1.6.3).** In (1.4.3.6) and its proof, we can replace $\mathbf{K}(\mathcal{A})$ by any Δ-subcategory.

1.7. Localizing subcategories of K; Δ-equivalent categories

In the description of the derived category $\mathbf{D}$ given in §1.2, we can replace $\mathbf{K}$ by any Δ-subcategory $\mathbf{L}$, and obtain a derived category $\mathbf{D_L}$ together with a functor $Q_L\colon \mathbf{L} \to \mathbf{D_L}$ which is universal among all functors transforming quasi-isomorphisms into isomorphisms. (Here, as in 1.2, for checking details one needs [**H**, p. 35, Prop. 4.2].) Then, just as in (1.4.4), $\mathbf{D_L}$ has a unique triangulation for which the translation functor is the obvious one and for which Q_L is a Δ-functor; and (1.5.1) remains valid with Q_L in place of Q.

If $\mathbf{L}' \subset \mathbf{L}''$ are Δ-subcategories of $\mathbf{K}$ and $j\colon \mathbf{L}' \to \mathbf{L}''$ is the inclusion, then there exists a natural commutative diagram of Δ-functors

$$\begin{array}{ccc}
\mathbf{L}' & \xrightarrow{\ j\ } & \mathbf{L}'' \\
{\scriptstyle Q' := Q_{L'}}\big\downarrow & & \big\downarrow{\scriptstyle Q_{L''} =: Q''} \\
\mathbf{D}' := \mathbf{D}_{\mathbf{L}'} & \xrightarrow[\tilde{\jmath}]{} & \mathbf{D}_{\mathbf{L}''} =: \mathbf{D}''
\end{array}$$

Note that on objects of $\mathbf{D}'$ (= objects of $\mathbf{L}'$), $\tilde{\jmath}$ is just the inclusion map to objects of $\mathbf{D}''$.

Recalling that passage to derived categories is a kind of localization in categories (§1.2, footnote), we say that $\mathbf{L}'$ localizes to a Δ-subcategory of $\mathbf{D}''$, or more briefly, that $\mathbf{L}'$ is a *localizing subcategory of* $\mathbf{L}''$, if the functor $\tilde{\jmath}$ is fully faithful, i.e., the natural map is an isomorphism

$$\mathrm{Hom}_{\mathbf{D}'}(A^\bullet, B^\bullet) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}''}(\tilde{\jmath}A^\bullet, \tilde{\jmath}B^\bullet)$$

for all $A^\bullet$ and $B^\bullet$ in $\mathbf{D}'$.

When this condition holds, $\tilde{\jmath}$ is an additive isomorphism of $\mathbf{D}'$ onto the full subcategory $\tilde{\jmath}(\mathbf{D}')$ of $\mathbf{D}''$, so $\tilde{\jmath}$ carries the triangulation on $\mathbf{D}'$ over to a triangulation on $\tilde{\jmath}(\mathbf{D}')$; and then since $\tilde{\jmath}$ is a Δ-functor, the inclusion functor $\tilde{\jmath}(\mathbf{D}') \hookrightarrow \mathbf{D}''$, together with $\theta =$ identity, is a Δ-functor, i.e., $\tilde{\jmath}(\mathbf{D}')$ is a Δ-subcategory of $\mathbf{D}''$. Thus *if* $\mathbf{L}'$ *is localizing in* $\mathbf{L}''$, *then we can identify* $\mathbf{D}'$ *with the* Δ*-subcategory of* $\mathbf{D}''$ *whose objects are the complexes in* $\mathbf{L}'$, *and* Q' *with the restriction of* Q'' *to* $\mathbf{L}'$.

(1.7.1). From definitions in §1.2, we deduce easily the following simple sufficient condition for $\mathbf{L}'$ to be localizing in $\mathbf{L}''$:

For every quasi-isomorphism $X^\bullet \to B^\bullet$ *in* $\mathbf{L}''$ *with* $B^\bullet$ *in* $\mathbf{L}'$, *there exists a quasi-isomorphism* $A^\bullet \to X^\bullet$ *with* $A^\bullet$ *in* $\mathbf{L}'$.

(1.7.1)$^{\mathsf{op}}$. A "dual" argument (see [**H**, p. 32, proof of 3.2]) yields:

The same condition with arrows reversed is also sufficient.

For example, *if the objects in* $\mathbf{L}'$ *are precisely those complexes in* $\mathbf{K}$ *which satisfy some condition on their homology* (for instance, if $\mathbf{L}'$ is any one of the categories $\overline{\mathbf{K}}^*$ of (1.6.1)), *then* $\mathbf{L}'$ *is localizing in* $\mathbf{L}''$.

This follows at once from (1.7.1) (take $A^\bullet = X^\bullet$).

The following results will provide a useful interpretation of various kinds of *resolutions* (injective, flat, flasque, etc.) as defining an *equivalence of Δ-categories.*

(1.7.2). *If for every $X^\bullet \in \mathbf{L}''$ there exists a quasi-isomorphism $A^\bullet \to X^\bullet$ with $A^\bullet \in \mathbf{L}'$ then $\tilde{\jmath}$ is an equivalence of categories, i.e., there exists a functor $\rho\colon \mathbf{D}'' \to \mathbf{D}'$ together with functorial isomorphisms*

$$\mathbf{1}_{\mathbf{D}''} \xrightarrow{\sim} \tilde{\jmath}\rho\,, \qquad \mathbf{1}_{\mathbf{D}'} \xrightarrow{\sim} \rho\tilde{\jmath} \tag{1.7.2.1}$$

(see [**M**, p. 91]). *Moreover, for the usual translation T there is then a unique functorial isomorphism*

$$\theta\colon \rho T \xrightarrow{\sim} T\rho$$

such that the pair (ρ,θ) is a Δ-functor and the isomorphisms (1.7.2.1) *are isomorphisms of Δ-functors* (§1.5).

We say then that $\tilde{\jmath}$ and ρ—or more precisely $(\tilde{\jmath}, \text{identity})$ and (ρ,θ)—are Δ-*equivalences of categories, quasi-inverse to each other.*

(1.7.2)$^{\mathsf{op}}$. *Same as* (1.7.2), *with $A^\bullet \to X^\bullet$ replaced by $X^\bullet \to A^\bullet$.*

To prove $(1.7.2)^{\mathsf{op}}$, for example, suppose that we have a family of quasi-isomorphisms ("right $\mathbf{L}'$-resolutions")

$$\varphi_{X^\bullet}\colon X^\bullet \to A^\bullet_{X^\bullet} \in \mathbf{L}' \qquad (X^\bullet \in \mathbf{L}'')\,.$$

Then by $(1.7.1)^{\mathsf{op}}$, $\mathbf{L}'$ is localizing in $\mathbf{L}''$. So finding an additive functor ρ with isomorphisms (1.7.2.1) is equivalent to finding for each object $X^\bullet$ of $\mathbf{D}''$ an isomorphism to an object in $\mathbf{D}' \subset \mathbf{D}''$ (see [**M**, p. 92, (iii)$\Rightarrow$(ii)]). But $Q''(\varphi_{X^\bullet})$ is such an isomorphism. Thus we have $\rho\colon \mathbf{D}'' \to \mathbf{D}'$ with

$$\rho(X^\bullet) = A^\bullet_{X^\bullet} \qquad (X^\bullet \in \mathbf{D}'')\,.$$

Next, define $\theta(X^\bullet)$ to be the unique map making the following diagram (with all arrows representing *isomorphisms* in $\mathbf{D}''$) commute:

$$\begin{tikzcd} & TX^\bullet \arrow[dl, "Q''(\varphi_{TX^\bullet})"'] \arrow[dr, "TQ''(\varphi_{X^\bullet})"] & \\ \rho TX^\bullet = A^\bullet_{TX^\bullet} \arrow[rr, "\theta(X^\bullet)"'] & & TA^\bullet_{X^\bullet} = T\rho X^\bullet \end{tikzcd} \tag{1.7.2.2}$$

Then, one checks, the family $\theta(X^\bullet)$ constitutes an isomorphism of functors $\theta\colon \rho T \xrightarrow{\sim} T\rho$.

Furthermore, if

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} TX^\bullet$$

is a triangle in $\mathbf{D}''$, then $(\Delta 1)''$ (see §1.4) applied to the commutative diagram in $\mathbf{D}''$

$$\begin{tikzcd}
X^\bullet \arrow[r, "u"] \arrow[d, "Q''\varphi_{X^\bullet}"'] & Y^\bullet \arrow[r, "v"] \arrow[d, "Q''\varphi_{Y^\bullet}"'] & Z^\bullet \arrow[r, "w"] \arrow[d, "Q''\varphi_{Z^\bullet}"] & TX^\bullet \arrow[d, "TQ''\varphi_{X^\bullet}"] \\
A^\bullet_{X^\bullet} \arrow[r, "\rho(u)"'] & A^\bullet_{Y^\bullet} \arrow[r, "\rho(v)"'] & A^\bullet_{Z^\bullet} \arrow[r, "\theta(X^\bullet)\circ\rho(w)"'] & TA^\bullet_{X^\bullet}
\end{tikzcd}$$

guarantees that the bottom row is a triangle; and so (ρ,θ) is a Δ-functor.

Finally, the fact that the isomorphisms in (1.7.2.1) (induced by the family $\varphi_{X^\bullet}$) are isomorphisms of Δ-functors is nothing but the commutativity of (1.7.2.2). Thus the family $\theta := \{\theta(X^\bullet)\}$ is the unique functorial isomorphism having the properties stated in $(1.7.2)^{\mathsf{op}}$.

Remark **(1.7.2.3).** It is sometimes possible to choose the functor ρ so that $\rho T = T\rho$ and $\theta =$ identity, i.e., to find a family of quasi-isomorphisms $\varphi_{X^\bullet}\colon X^\bullet \to A^\bullet_{X^\bullet}$ commuting with translation (see (1.8.1.1), (1.8.2), and (1.8.3) below).

1.8. Examples

(1.8.1). *If* $\mathbf{L}' \subset \mathbf{K}$ *is any one of the* Δ*-subcategories* $\overline{\mathbf{K}}^*$ *of* (1.6.1) *and if* $\mathbf{L}''$ *is any* Δ*-subcategory of* $\mathbf{K}$ *containing* $\mathbf{L}'$*, then* $\mathbf{L}'$ *is localizing in* $\mathbf{L}''$*. The same holds for* $\mathbf{L}' = \mathbf{K}^{+}$ *or* $\mathbf{L}' = \mathbf{K}^{-}$*; and also for* $\mathbf{L}' = \mathbf{K}^{\mathsf{b}}$ *if* $\mathbf{L}''$ *is localizing in* $\mathbf{K}$*.*

For $\mathbf{L}' = \overline{\mathbf{K}}^*$ the assertion follows at once from (1.7.1). For the rest (and for other purposes) we need the *truncation operators* τ^+, τ^-, defined as follows:

For any $B^\bullet \in \mathbf{K}$, set

$$i = i(B^\bullet) := \inf\{\, m \mid H^m(B^\bullet) \neq 0\,\}$$

and let $\tau^+(B^\bullet)$ be the complex

$$\cdots \to 0 \to 0 \to \operatorname{coker}(B^{i-1} \to B^i) \to B^{i+1} \to B^{i+2} \to \cdots \quad .$$

(When $i = \infty$, i.e., when $B^\bullet$ is exact, this means $\tau^+(B^\bullet) = 0^\bullet$; and when $i = -\infty$, $\tau^+(B^\bullet) = B^\bullet$.) There is an obvious *quasi-isomorphism*

$$B^\bullet \to \tau^+(B^\bullet)\,. \qquad (1.8.1)^+$$

Dually, for any $C^\bullet \in \mathbf{K}$ set

$$s = s(C^\bullet) := \sup\{\, n \mid H^n(C^\bullet) \neq 0 \,\}$$

and let $\tau^-(C^\bullet)$ be the complex

$$\cdots \to C^{s-2} \to C^{s-1} \to \ker(C^s \to C^{s+1}) \to 0 \to 0 \to \cdots \quad .$$

There is an obvious *quasi-isomorphism*

$$(1.8.1)^- \qquad \tau^-(C^\bullet) \to C^\bullet .$$

Now if $C^\bullet \to B^\bullet$ is a quasi-isomorphism in $\mathbf{L}''$ with $B^\bullet \in \mathbf{K}^-$ then $C^\bullet \in \overline{\mathbf{K}}^-$, and we have the quasi-isomorphism $(1.8.1)^-$ with $\tau^-(C^\bullet) \in \mathbf{K}^-$. So (1.7.1) with $\mathbf{L}' = \mathbf{K}^- \subset \mathbf{L}''$ shows that $\mathbf{K}^-$ is localizing in $\mathbf{L}''$.

Dually, via $(1.8.1)^+$, $(1.7.1)^{\mathsf{op}}$ implies that $\mathbf{K}^+$ is localizing in any Δ-subcategory $\mathbf{L}''$ of $\mathbf{K}$ containing $\mathbf{K}^+$.

And again via $(1.8.1)^-$, (1.7.1) shows that $\mathbf{K}^{\mathsf{b}}$ is localizing in $\mathbf{K}^+$; and since as above $\mathbf{K}^+$ is localizing in $\mathbf{K}$, the natural functors $\mathbf{D}^{\mathsf{b}} \to \mathbf{D}^+ \to \mathbf{D}$ between the corresponding derived categories are both fully faithful, whence so is their composition, i.e., $\mathbf{K}^{\mathsf{b}}$ *is localizing in* $\mathbf{K}$. It follows at once that $\mathbf{K}^{\mathsf{b}}$ is localizing in any $\mathbf{L}'' \supset \mathbf{K}^{\mathsf{b}}$ such that $\mathbf{L}''$ is localizing in $\mathbf{K}$.

Consequently, as in (1.7): *the derived category* $\mathbf{D}^*$ (*resp.* $\overline{\mathbf{D}}^*$) *of* $\mathbf{K}^*$ (*resp.* $\overline{\mathbf{K}}^*$) *can be identified in a natural way with a* Δ*-subcategory of* $\mathbf{D}$.

Then the inclusion $\mathbf{D}^+ \hookrightarrow \overline{\mathbf{D}}^+$ is a Δ-equivalence of categories. Indeed, as in the proof of $(1.7.2)^{\mathsf{op}}$, with $\mathbf{L}' = \mathbf{K}^+$, $\mathbf{L}'' = \overline{\mathbf{K}}^+$, and $\varphi_{B^\bullet} = (1.8.1)^+$, we can see that τ^+—which commutes with translation—extends to a Δ-functor

$$(1.8.1.1) \qquad (\tau^+, 1) \colon \overline{\mathbf{D}}^+ \to \mathbf{D}^+$$

which is quasi-inverse to the inclusion.

Similarly the inclusions $\mathbf{D}^- \hookrightarrow \overline{\mathbf{D}}^-$, $\mathbf{D}^{\mathsf{b}} \hookrightarrow \overline{\mathbf{D}}^{\mathsf{b}}$ are Δ-equivalences, with respective quasi-inverses τ^- and $\tau^{\mathsf{b}} = \tau^- \circ \tau^+ = \tau^+ \circ \tau^-$. More precisely, τ^{b} is the composition

$$\overline{\mathbf{D}}^{\mathsf{b}} \xrightarrow{\tau^+} \overline{\mathbf{D}}^{\mathsf{b}} \cap \mathbf{D}^+ \xrightarrow{\tau^-} \mathbf{D}^- \cap \mathbf{D}^+ = \mathbf{D}^{\mathsf{b}} .$$

(1.8.2) Let $\mathbf{I}$ be a full additive subcategory of $\mathcal{A}$ such that every object of $\mathcal{A}$ admits a monomorphism into an object in $\mathbf{I}$. Then there exists a family of quasi-isomorphisms

$$\varphi_{B^\bullet}\colon B^\bullet \to I^\bullet_{B^\bullet} \qquad \big(B^\bullet \in \overline{\mathbf{K}}^+ = \overline{\mathbf{K}}^+(\mathcal{A})\big)$$

where each $I^\bullet = I^\bullet_{B^\bullet}$ is a bounded-below $\mathbf{I}$-complex (i.e., $I^n \in \mathbf{I}$ for all n, and $I^n = (0)$ for $n \ll 0$); and such that moreover with the usual translation functor T we have

$$I^\bullet_{TB^\bullet} = TI^\bullet_{B^\bullet}\,, \qquad \varphi_{TB^\bullet} = T(\varphi_{B^\bullet})\,. \tag{1.8.2.1}$$

To see this, first construct quasi-isomorphisms $\varphi_{B^\bullet}$ as in [**H**, p. 42, 4.6, 1)] for those $B^\bullet$ such that $H^0(B^\bullet) \neq 0$ and $B^m = 0$ for $m < 0$. Then (1.8.2.1) forces the definition of $\varphi_{B^\bullet}$ for any $B^\bullet$ such that there exists $i \in \mathbb{Z}$ with $H^i(B^\bullet) \neq 0$ and $B^m = 0$ for all $m < i$ (i.e., $0^\bullet \neq B^\bullet = \tau^+ B^\bullet$, see (1.8.1)). Set $I_{0^\bullet} = 0^\bullet$, and finally for any $B^\bullet \in \overline{\mathbf{K}}^+$ set

$$\varphi_{B^\bullet} = (\varphi_{\tau^+ B^\bullet}) \circ (1.8.1)^+\,.$$

Now let $\mathbf{K}^+_{\mathbf{I}}$ be the full subcategory of $\mathbf{K}^+$ whose objects are the bounded-below $\mathbf{I}$-complexes. Since the additive subcategory $\mathbf{I} \subset \mathcal{A}$ is closed under finite direct sums, one sees that $\mathbf{K}^+_{\mathbf{I}}$ is a Δ-subcategory of $\mathbf{K}^+$. According to $(1.7.2)^{\mathsf{op}}$, the derived category $\mathbf{D}^+_{\mathbf{I}}$ of $\mathbf{K}^+_{\mathbf{I}}$ can be identified with a Δ-subcategory of $\overline{\mathbf{D}}^+$, and the above family $\varphi_{B^\bullet}$ gives rise to an $\mathbf{I}$-*resolution functor*

$$\rho\colon \overline{\mathbf{D}}^+ \to \mathbf{D}^+_{\mathbf{I}} \tag{1.8.2.2}$$

which is, together with $\theta =$ identity, a Δ-*equivalence of categories,* quasi-inverse to the inclusion $\mathbf{D}^+_{\mathbf{I}} \hookrightarrow \overline{\mathbf{D}}^+$.

For example, if $\mathbf{I}$ is the full subcategory of $\mathcal{A}$ whose objects are all the injectives in $\mathcal{A}$, then by [**H**, p. 41, Lemma 4.5] every quasi-isomorphism in $\mathbf{K}^+_{\mathbf{I}}$ is an isomorphism, so that $\mathbf{K}^+_{\mathbf{I}}$ can be identified with its derived category $\mathbf{D}^+_{\mathbf{I}}$. Thus, if $\mathcal{A}$ has enough injectives (i.e., every object of $\mathcal{A}$ admits a monomorphism into an injective object), then the natural composition

$$\mathbf{D}^+_{\mathbf{I}} = \mathbf{K}^+_{\mathbf{I}} \hookrightarrow \overline{\mathbf{K}}^+ \to \overline{\mathbf{D}}^+$$

is a Δ-equivalence, having as quasi-inverse an injective resolution functor (1.8.2.2) (cf. [**H**, p. 46, Prop. 4.7]).

(1.8.3). Let $\mathbf{P}$ be a full additive subcategory of $\mathcal{A}$ such that for every object $B \in \mathcal{A}$ there exists an epimorphism $P_B \to B$ with $P_B \in \mathbf{P}$. An argument dual to that in (1.8.2) yields that *there exists a family of quasi-isomorphisms*

$$\psi_{B^\bullet}\colon P^\bullet_{B^\bullet} \to B^\bullet \qquad \big(B^\bullet \in \overline{\mathbf{K}}^-(\mathcal{A})\big)$$

commuting with translation, and such that each $P^\bullet_{B^\bullet}$ *is a bounded-above* $\mathbf{P}$*-complex.*

According to (1.7.2), we have then a $\mathbf{P}$-*resolution functor* which is a Δ-*equivalence* into $\overline{\mathbf{D}}^-(\mathcal{A})$ from its Δ-subcategory whose objects are bounded-above $\mathbf{P}$-complexes.

For example, if U is a topological space, $\mathcal{O}$ is a sheaf of rings on U, and $\mathcal{A}$ is the abelian category of (sheaves of) left $\mathcal{O}$-modules, then we can take $\mathbf{P}$ to be the full subcategory of $\mathcal{A}$ whose objects are all the *flat* $\mathcal{O}$-modules [**H**, p. 86, Prop. 1.2].

1.9. Complexes with homology in a plump subcategory

(1.9.1). Here, in brief, are some essential basic facts.

Let $\mathcal{A}^{\#}$ be a *plump* subcategory of the abelian category $\mathcal{A}$, i.e., a full subcategory containing 0 and such that for every exact sequence in $\mathcal{A}$

$$X_1 \to X_2 \to X \to X_3 \to X_4\,,$$

if X_1, X_2, X_3, and X_4 all lie in $\mathcal{A}^{\#}$ then so does X. Then the kernel and cokernel (in $\mathcal{A}$) of any map in $\mathcal{A}^{\#}$ must lie in $\mathcal{A}^{\#}$ (whence $\mathcal{A}^{\#}$ is abelian), and any object of $\mathcal{A}$ isomorphic to an object in $\mathcal{A}^{\#}$ must itself be in $\mathcal{A}^{\#}$.

Considering only complexes in $\mathcal{A}$ whose homology objects all lie in $\mathcal{A}^{\#}$, we obtain full subcategories $\mathbf{K}_{\#}$ of $\mathbf{K}$, $\mathbf{K}^*_{\#}$ of $\mathbf{K}^*$, and $\overline{\mathbf{K}}^*_{\#}$ of $\overline{\mathbf{K}}^*$ (see (1.6.1)). Via the exact homology sequence $(1.4.5)^{\mathrm{H}}$ of a standard triangle (1.3.1), we find that these subcategories are all Δ-subcategories (see (i) and (ii) in §1.6), and indeed, by (1.7.1), *localizing subcategories.* From (1.8.1) it follows then that $\mathbf{K}_{\#}$, $\mathbf{K}^*_{\#}$, and $\overline{\mathbf{K}}^*_{\#}$ are localizing subcategories of $\mathbf{K}$, from which we derive Δ-subcategories $\mathbf{D}_{\#}$, $\mathbf{D}^*_{\#}$, and $\overline{\mathbf{D}}^*_{\#}$ of $\mathbf{D}$, with universal properties analogous to (1.5.1). As in (1.8.1) the inclusion $\mathbf{D}^*_{\#} \hookrightarrow \overline{\mathbf{D}}^*_{\#}$ is a Δ-equivalence of categories, with quasi-inverse τ^*.

(1.9.2). The following isomorphism test will be useful.

Lemma. *If* $\mathcal{A}^{\#}$ *is a plump subcategory of* $\mathcal{A}$, *and* $u\colon A^\bullet_1 \to A^\bullet_2$ *is a map in* $\overline{\mathbf{D}}^+_{\#}$ *such that for all* $B^\bullet \in \mathbf{D}^{\mathrm{b}}_{\#}$ *the induced map*

$$\mathrm{Hom}_{\mathbf{D}}(B^\bullet, A^\bullet_1) \to \mathrm{Hom}_{\mathbf{D}}(B^\bullet, A^\bullet_2)$$

is an isomorphism, then u *is an isomorphism.*

Proof. Let $C^\bullet \in \overline{\mathbf{D}}^+_{\#}$ be the summit of a triangle with base u, so that by (1.4.2.1), u is an isomorphism iff $C^\bullet \cong 0$, i.e., iff $\tau^+(C^\bullet) = 0^\bullet$, see (1.8.1), (1.2.2).

For each $m \in \mathbb{Z}$ and each object $M \in \mathcal{A}^{\#}$ we have, by (1.4.2.1) and $(\Delta 2)$ in §1.4, an exact sequence (with $\mathrm{Hom} = \mathrm{Hom}_{\mathbf{D}}$):

$$\begin{aligned}\mathrm{Hom}(M[-m], A_1^\bullet) \xrightarrow[\text{via } u]{\ \sim\ } \mathrm{Hom}(M[-m], A_2^\bullet) &\longrightarrow \mathrm{Hom}(M[-m], C^\bullet)\\ \longrightarrow \mathrm{Hom}(M[-m], A_1^\bullet[1]) &\xrightarrow[\text{via } -u[1]]{\ \sim\ } \mathrm{Hom}(M[-m], A_2^\bullet[1]).\end{aligned}$$

The two labeled maps are, by hypothesis, isomorphisms, and hence

$$\mathrm{Hom}(M[-m], C^\bullet) = 0\,.$$

Were $\tau^+(C^\bullet) \neq 0^\bullet$, then with $m := i(C^\bullet)$ (see (1.8.1) and

$$M := H^m(C^\bullet) = \ker\bigl(\tau^+(C^\bullet)^m \to \tau^+(C^\bullet)^{m+1}\bigr) \neq 0\,,$$

the inclusion $M \hookrightarrow \tau^+(C^\bullet)^m$ would lead to a map $j\colon M[-m] \to \tau^+(C^\bullet)$ with $H^m(j)$ the (non-zero) identity map of M, so we'd have

$$\mathrm{Hom}\bigl(M[-m], C^\bullet\bigr) \xrightarrow[(1.8.1)^+]{\ \sim\ } \mathrm{Hom}\bigl(M[-m], \tau^+(C^\bullet)\bigr) \neq 0\,,$$

contradiction. Thus $\tau^+(C^\bullet) = 0^\bullet$. Q.E.D.

1.10. Truncation functors

Let $\mathcal{A}$ be an abelian category, and let $\mathbf{D} = \mathbf{D}(\mathcal{A})$ be the derived category. For any complex $A^\bullet$ in $\mathcal{A}$, and $n \in \mathbb{Z}$, we let $\tau_{\le n}A^\bullet$ be the truncated complex

$$\cdots \longrightarrow A^{n-2} \longrightarrow A^{n-1} \longrightarrow \ker(A^n \to A^{n+1}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\quad,$$

and dually we let $\tau_{\ge n}A$ be the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{coker}(A^{n-1} \to A^n) \longrightarrow A^{n+1} \longrightarrow A^{n+2} \longrightarrow \cdots\quad.$$

Note that

$$\begin{aligned}H^m(\tau_{\le n}A^\bullet) &= H^m(A^\bullet) &&\text{if } m \le n\,,\\ &= 0 &&\text{if } m > n\,,\end{aligned}$$

and that

$$\begin{aligned}H^m(\tau_{\ge n}A^\bullet) &= H^m(A^\bullet) &&\text{if } m \ge n\,,\\ &= 0 &&\text{if } m < n\,.\end{aligned}$$

One checks that $\tau_{\geq n}$ (respectively $\tau_{\leq n}$) extends naturally to an additive functor of complexes which preserves homotopy and takes quasi-isomorphisms to quasi-isomorphisms, and hence induces an additive functor $\mathbf{D} \to \mathbf{D}$, see §1.2. In fact if $\mathbf{D_{\leq n}}$ (resp. $\mathbf{D_{\geq n}}$) is the full subcategory of $\mathbf{D}$ whose objects are the complexes $A^\bullet$ such that $H^m(A^\bullet) = 0$ for $m > n$ (resp. $m < n$) then we have additive functors

$$\tau_{\leq n} \colon \mathbf{D} \longrightarrow \mathbf{D_{\leq n}} \subset \mathbf{D}$$
$$\tau_{\geq n} \colon \mathbf{D} \longrightarrow \mathbf{D_{\geq n}} \subset \mathbf{D}$$

together with obvious functorial maps

$$i_A^n \colon \tau_{\leq n} A^\bullet \longrightarrow A^\bullet$$
$$j_A^n \colon A^\bullet \longrightarrow \tau_{\geq n} A^\bullet \ .$$

Proposition (1.10.1). *The preceding maps i_A^n, j_A^n induce functorial isomorphisms*

$$(1.10.1.1) \qquad \mathrm{Hom}_{\mathbf{D_{\leq n}}}(B^\bullet, \tau_{\leq n} A^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(B^\bullet, A^\bullet) \qquad (B^\bullet \in \mathbf{D_{\leq n}}),$$
$$(1.10.1.2) \qquad \mathrm{Hom}_{\mathbf{D_{\geq n}}}(\tau_{\geq n} A^\bullet, C^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(A^\bullet, C^\bullet) \qquad (C^\bullet \in \mathbf{D_{\geq n}}).$$

Proof. Bijectivity of (1.10.1.1) means that any map $\varphi \colon B^\bullet \to A^\bullet$ (in $\mathbf{D}$) with $B^\bullet \in \mathbf{D_{\leq n}}$ factors uniquely via $i_A := i_A^n$.

Given φ, we have a commutative diagram

$$\begin{tikzcd} \tau_{\leq n} B^\bullet \arrow[r, "\tau_{\leq n}\varphi"] \arrow[d, "i_B"'] & \tau_{\leq n} A^\bullet \arrow[d, "i_A"] \\ B^\bullet \arrow[r, "\varphi"'] & A^\bullet \end{tikzcd}$$

and since $B^\bullet \in \mathbf{D_{\leq n}}$, therefore i_B is an isomorphism in $\mathbf{D}$, see (1.2.2), so we can write $\varphi = i_A \circ (\tau_{\leq n}\varphi \circ i_B^{-1})$, and thus (1.10.1.1) is *surjective.*

To prove that (1.10.1.1) is also *injective,* we assume that $i_A \circ \tau_{\leq n}\varphi = 0$ and deduce that $\tau_{\leq n}\varphi = 0$. As in §1.2, the assumption means that there is a commutative diagram in $\mathbf{K}(\mathcal{A})$

$$\begin{tikzcd} & C^\bullet \arrow[dl, "s"'] \arrow[r, "f"] & \tau_{\leq n} A^\bullet \arrow[d, "i_A"] \\ \tau_{\leq n} B^\bullet & C''^\bullet \arrow[l, "s''"] \arrow[u] \arrow[r, "0"'] & A^\bullet \end{tikzcd}$$

where s and s'' are quasi-isomorphisms, and $f/s = \tau_{\leq n}\varphi$.

Applying the (idempotent) functor $\tau_{\le n}$, we get a commutative diagram

$$\begin{tikzcd}
& \tau_{\le n}C^\bullet \arrow[r, "\tau_{\le n}f"] \arrow[dl, "\tau_{\le n}s"'] & \tau_{\le n}A^\bullet \\
\tau_{\le n}B^\bullet & \tau_{\le n}C''^\bullet \arrow[l, "\tau_{\le n}s''"] \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

Since $\tau_{\le n}s$ and $\tau_{\le n}s''$ are quasi-isomorphisms, we have

$$\tau_{\le n}\varphi = \tau_{\le n}f/\tau_{\le n}s = 0/\tau_{\le n}s'' = 0\,,$$

as desired.

A similar argument proves the bijectivity of (1.10.1.2).

Remarks **(1.10.2).** Let $n \in \mathbb{Z}$, $A^\bullet \in \mathbf{D}(\mathcal{A})$.

(i) There exist natural isomorphisms

$$\tau_{\le n}\tau_{\ge n}A^\bullet \cong H^n(A^\bullet)[-n] \cong \tau_{\ge n}\tau_{\le n}A^\bullet\,.$$

(ii) The cokernel of $i_A^{n-1}\colon \tau_{\le n-1}A^\bullet \to A^\bullet$ maps quasi-isomorphically to $\tau_{\ge n}A^\bullet$; and hence there are natural triangles in $\mathbf{D}(\mathcal{A})$ (see (1.4.4.2)$^\sim$):

$$\tau_{\le n-1}A^\bullet \xrightarrow{i_A^{n-1}} A^\bullet \xrightarrow{j_A^n} \tau_{\ge n}A^\bullet \to (\tau_{\le n-1}A^\bullet)[1]\,, \tag{1.10.2.1}$$

$$\tau_{\le n-1}A^\bullet \to \tau_{\le n}A^\bullet \to H^n(A^\bullet)[-n] \to (\tau_{\le n-1}A^\bullet)[1]\,. \tag{1.10.2.2}$$

Details are left to the reader.

1.11. Bounded functors; way-out lemma

Many of the main results in subsequent chapters will be to the effect that some natural map or other is a functorial *isomorphism.* So we'll need isomorphism criteria. In (1.11.3) we review some commonly used ones ("Lemma on way-out functors," [**H**, p. 68, Prop. 7.1]).

Throughout this section, $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, $\mathcal{A}^\#$ is a plump subcategory of $\mathcal{A}$, and $\overline{\mathbf{D}}{}^*_\#(\mathcal{A}) \subset \mathbf{D}(\mathcal{A})$ is as in (1.9.1). We identify $\mathcal{A}^\#$ with a full subcategory of $\overline{\mathbf{D}}{}^*_\#(\mathcal{A})$, see (1.2.3).

For a subcategory $\mathbf{E}$ of $\mathbf{D}(\mathcal{A})$, $\mathbf{E}_{\le \mathbf{n}}$ (resp. $\mathbf{E}_{\ge \mathbf{n}}$) will denote the full subcategory of $\mathbf{E}$ whose objects are those complexes $A^\bullet$ such that $H^m(A^\bullet) = 0$ for $m > n$ (resp. $m < n$).

Definition (1.11.1). Let $\mathbf{E}$ be a subcategory of $\mathbf{D}(\mathcal{A})$, and let F (resp. F'): $\mathbf{E} \to \mathbf{D}(\mathcal{B})$ be a covariant (resp. contravariant) additive functor. The *upper dimension* $\dim^{+}$ and *lower dimension* $\dim^{-}$ of these functors are:

$$\begin{aligned}
\dim^{+}F &:= \inf\{\, d \mid F(\mathbf{E}_{\leq \mathbf{n}}) \subset \mathbf{D}_{\leq \mathbf{n}+\mathbf{d}}(\mathcal{B}) \ \text{ for all } n \in \mathbb{Z}\,\},\\
\dim^{+}F' &:= \inf\{\, d \mid F'(\mathbf{E}_{\geq -\mathbf{n}}) \subset \mathbf{D}_{\leq \mathbf{n}+\mathbf{d}}(\mathcal{B}) \ \text{ for all } n \in \mathbb{Z}\,\},\\
\dim^{-}F &:= \inf\{\, d \mid F(\mathbf{E}_{\geq \mathbf{n}}) \subset \mathbf{D}_{\geq \mathbf{n}-\mathbf{d}}(\mathcal{B}) \ \text{ for all } n \in \mathbb{Z}\,\},\\
\dim^{-}F' &:= \inf\{\, d \mid F'(\mathbf{E}_{\leq -\mathbf{n}}) \subset \mathbf{D}_{\geq \mathbf{n}-\mathbf{d}}(\mathcal{B}) \ \text{ for all } n \in \mathbb{Z}\,\}.
\end{aligned}$$

The functor F is *bounded above*[16] (resp. *bounded below*)[17] if $\dim^{+}F < \infty$ (resp. $\dim^{-}F < \infty$); and similarly for F'. F (resp. F') is *bounded* if it is both bounded-above and bounded-below.

REMARKS **(1.11.2).** (i) Let T_1 and T_2 be the translation functors in $\mathbf{D}(\mathcal{A})$ and $\mathbf{D}(\mathcal{B})$ respectively. Suppose that $T_1\mathbf{E} = \mathbf{E}$ and that there is a functorial isomorphism $FT_1 \xrightarrow{\sim} T_2F$ (resp. $T_2^{-1}F' \xrightarrow{\sim} F'T_1$). (For example, $\mathbf{E}$ could be a Δ-subcategory of $\mathbf{D}(\mathcal{A})$ and F' a Δ-functor.) Then, for instance, $F'(\mathbf{E}_{\geq -\mathbf{n}}) \subset \mathbf{D}_{\leq \mathbf{n}+\mathbf{d}}(\mathcal{B})$ holds for all $n \in \mathbb{Z}$ as soon as it holds for one single n.

(ii) If $\mathbf{E}$ is a Δ-subcategory of $\mathbf{D}(\mathcal{A})$ such that for all $n \in \mathbb{Z}$, $\tau_{\leq n}\mathbf{E} \subset \mathbf{E}$ and $\tau_{\geq n}\mathbf{E} \subset \mathbf{E}$ (e.g., $\mathbf{E} = \overline{\mathbf{D}}^{*}_{\#}(\mathcal{A})$), and if F (resp. F') is a Δ-functor, then:

$$\dim^{+}F \leq d \iff H^iF(A^\bullet) \xrightarrow[j_A^n]{\sim} H^iF(\tau_{\geq n}A^\bullet)$$
$$\text{for all } A^\bullet \in \mathbf{E},\ n \in \mathbb{Z},\ \text{and } i \geq n+d.$$

(The display signifies that the map $H^i(j_A^n)$ (see §1.10) is an isomorphism; and as in (i), we can restrict attention to a single n.) The implication $\Rightarrow$ follows from the exact homology sequence $(1.4.5)^{\mathrm{H}}$ of the triangle gotten by applying F to (1.10.2.1); while $\Leftarrow$ is obtained by taking $A^\bullet$ to be an arbitrary complex in $\mathbf{E}_{\leq \mathbf{n-1}}$. An equivalent condition is that *if* $\alpha\colon A_1^\bullet \to A_2^\bullet$ *is a map in* $\mathbf{E}$ *such that* $H^i(\alpha)$ *is an isomorphism for all* $i \geq n$, (that is, if α induces an isomorphism $\tau_{\geq n}A_1^\bullet \xrightarrow{\sim} \tau_{\geq n}A_2^\bullet$), *then* $H^i(F\alpha)$ *is an isomorphism for all* $i \geq n+d$.

Similarly:

$$\dim^{+}F' \leq d \iff H^iF'(A^\bullet) \xrightarrow[i_A^{-n}]{\sim} H^iF'(\tau_{\leq -n}A^\bullet) \qquad (i \geq n+d),$$

$$\dim^{-}F \leq d \iff H^iF(\tau_{\leq n}A^\bullet) \xrightarrow[i_A^{n}]{\sim} H^iF(A^\bullet) \qquad (i \leq n-d),$$

$$\dim^{-}F' \leq d \iff H^iF'(\tau_{\geq -n}A^\bullet) \xrightarrow[j_A^{-n}]{\sim} H^iF'(A^\bullet) \qquad (i \leq n-d).$$

[16] *way-out left* in the terminology of [**H**, p. 68]

[17] *way-out right*

(iii) If $\mathbf{E} = \mathcal{A}^{\#}$ (so that $\mathbf{E}_{\geq \mathbf{0}} = \mathbf{E} = \mathbf{E}_{\leq \mathbf{0}}$), then $\dim^+ F \leq d \Leftrightarrow H^j F(A) = 0$ for all $j > d$ and all $A \in \mathcal{A}^{\#}$. Similarly, $\dim^- F \leq d \Leftrightarrow H^j F(A) = 0$ for $j < -d$ and $A \in \mathcal{A}^{\#}$. These assertions remain true when F is replaced by F'.

(iv) If $\mathbf{E} = \overline{\mathbf{D}}^{+}_{\#}(\mathcal{A})$ and F is a Δ-functor, then $\dim^+ F = \dim^+ F_0$ where F_0 is the restriction $F|_{\mathcal{A}^{\#}}$. A similar statement holds for $\dim^- F'$; and analogous statements hold for $\dim^- F$ or $\dim^+ F'$ when $\mathbf{E} = \overline{\mathbf{D}}^{-}_{\#}(\mathcal{A})$.

Here is a typical proof: we deal with $\dim^- F'$ when $\mathbf{E} = \overline{\mathbf{D}}^{+}_{\#}(\mathcal{A})$.

Obviously $\dim^- F' \geq \dim^- F'_0$. To prove the opposite inequality, suppose that $\dim^- F'_0 \leq d < \infty$, fix an $n \in \mathbb{Z}$, and let us show for any $A^\bullet \in \mathbf{E}_{\leq -\mathbf{n}}$ that $H^j F'(A^\bullet) = 0$ whenever $j < n - d$.

We proceed by induction on the number $\nu = \nu(A^\bullet)$ of non-vanishing homology objects of $A^\bullet$, the case $\nu = 0$ being trivial. If $\nu = 1$, say $H^{-m}(A^\bullet) =: H \neq 0$ $(m \geq n)$, then $A^\bullet \cong \tau^- \tau^+ A^\bullet \cong H[m]$ (see (1.8.1)), and since F' is a contravariant Δ-functor, $F'(A^\bullet) \cong F'(H)[-m]$; so by definition of $\dim^- F'_0$,

$$H^j F'(A^\bullet) \cong H^{j-m} F'(H) = 0 \quad \text{if} \quad j - m < -d,$$

whence the conclusion. When $\nu > 1$, choose any integer s such that there exist integers $p < s \leq q$ with $H^p(A^\bullet) \neq 0$, $H^q(A^\bullet) \neq 0$ (so that $\nu(\tau_{\leq s-1} A^\bullet) < \nu(A^\bullet)$ and $\nu(\tau_{\geq s} A^\bullet) < \nu(A^\bullet)$). Then apply F' to (1.10.2.1) to get a triangle

$$F'(\tau_{\leq s-1} A^\bullet) \longleftarrow F'(A^\bullet) \longleftarrow F'(\tau_{\geq s} A^\bullet) \longleftarrow F'(\tau_{\leq s-1} A^\bullet)[-1]$$

whose associated homology sequence $(1.4.5)^{\mathrm{H}}$ yields the inductive step.

Lemma (1.11.3). *Let (F, θ) and (G, ψ) be covariant Δ-functors from $\overline{\mathbf{D}}^{*}_{\#}(\mathcal{A})$ to $\mathbf{D}(\mathcal{B})$, and assume one of the following sets of conditions:*

(i) $* = \mathsf{b}$.

(ii) $* = +$ *and both F and G are bounded below.*

(iii) $* = -$ *and both F and G are bounded above.*

(iv) $* =$ blank *and F and G are bounded above and below.*

Then for a morphism $\eta \colon F \to G$ of Δ-functors to be an isomorphism it suffices that $\eta(X)$ be an isomorphism for all objects $X \in \mathcal{A}^{\#}$.

A similar assertion holds for contravariant functors if we interchange "bounded above" and "bounded below."

Complement (1.11.3.1). *Let $\mathbf{I}$ (resp. $\mathbf{P}$) be a set of objects in $\mathcal{A}^{\#}$ such that every object in $\mathcal{A}^{\#}$ admits a monomorphism into one in $\mathbf{I}$ (resp. is the target of an epimorphism out of one in $\mathbf{P}$). If $* = +$ and F and G are bounded below (resp. $* = -$ and F and G are bounded above) and if $\eta(X)$ is an isomorphism for all objects $X \in \mathbf{I}$ (resp. $X \in \mathbf{P}$), then η is an isomorphism.*

A similar assertion holds for contravariant functors if we interchange "bounded above" and "bounded below."

Proof. We deal first with the covariant case.

(i) Using the definition of "morphism of Δ-functors" (§1.5) we see by induction on $|n|$ that $\eta(X[-n])$ is an isomorphism for all $X \in \mathcal{A}^{\#}$ and $n \in \mathbb{Z}$. In showing that $\eta(A^\bullet)$ is an isomorphism for all $A^\bullet \in \overline{\mathbf{D}}^{\mathrm{b}}_{\#}(\mathcal{A})$, we may replace $A^\bullet$ by the isomorphic complex $\tau^-(A^\bullet) = \tau_{\le n}A^\bullet$ with $n := s(A^\bullet)$, see (1.8.1). From (1.10.2.2), and (Δ2) of §1.4, we obtain a map of triangles, induced by η:

$$\begin{CD}
F(H^n(A^\bullet)[-n-1]) @>>> F(\tau_{\le n-1}A^\bullet) @>>> F(\tau_{\le n}A^\bullet) @>>> F(H^n(A^\bullet)[-n])\\
@VVV @VVV @VVV @VVV\\
G(H^n(A^\bullet)[-n-1]) @>>> G(\tau_{\le n-1}A^\bullet) @>>> G(\tau_{\le n}A^\bullet) @>>> G(H^n(A^\bullet)[-n])
\end{CD}$$

and then we can conclude by $(\Delta 3)^*$ of §1.4 and induction on the number of non-vanishing homology objects of $A^\bullet$ (a number which is less for $\tau_{\le n-1}A^\bullet$ than for $A^\bullet$ whenever n is finite).

(ii) By (1.2.2), it suffices to show that $\eta(A^\bullet)$ induces an isomorphism from $H^iF(A^\bullet)$ to $H^iG(A^\bullet)$ for all $A^\bullet \in \overline{\mathbf{D}}^{+}_{\#}(\mathcal{A})$ and all $i \in \mathbb{Z}$. For this, remark (1.11.2)(ii) lets us replace $A^\bullet$ by $\tau_{\le i+d}A^\bullet \in \overline{\mathbf{D}}^{\mathrm{b}}_{\#}(\mathcal{A})$ for any $d \ge \max(\dim^- F, \dim^- G)$, and then (i) applies.

(iii) Similar to (ii).

(iv) As in the proof of (i), (1.10.2.1) with $n = 0$ gives rise to a map of triangles, induced by η:

$$\begin{CD}
F(\tau_{\ge 0}A^\bullet)[-1]) @>>> F(\tau_{\le -1}A^\bullet) @>>> F(A^\bullet) @>>> F(\tau_{\ge 0}A^\bullet)\\
@VV{\simeq}V @VV{\simeq}V @V{?}VV @V{\simeq}VV\\
G((\tau_{\ge 0}A^\bullet)[-1]) @>>> G(\tau_{\le -1}A^\bullet) @>>> G(A^\bullet) @>>> G(\tau_{\ge 0}A^\bullet)
\end{CD}$$

in which the maps other than ? are isomorphisms by (ii) and (iii), whence, by $(\Delta 3)^*$ of §1.4, so is ?.

For (1.11.3.1), it now suffices to show that $\eta(X)$ is an isomorphism for all objects $X \in \mathcal{A}^{\#}$. By a standard resolution argument (see [**H**, p. 43]), X is isomorphic in $\mathbf{D}_{\#}(\mathcal{A})$ to a bounded-below complex $I^\bullet$ of objects of $\mathbf{I}$ (resp. bounded-above complex $P^\bullet$ of objects of $\mathbf{P}$), and so it suffices to show that $\eta(I^\bullet)$ (resp. $\eta(P^\bullet)$) is an isomorphism for any such $I^\bullet$ (resp. $P^\bullet$). This is done as above, except that in the inductive step in (i), say for bounded $I^\bullet$, one uses instead of (1.10.2.2) the triangle associated as in (1.4.3) to the natural semi-split exact sequence

$$0 \longrightarrow I^n[-n] \longrightarrow \tau'_{\le n}I^\bullet \longrightarrow \tau'_{\le n-1}I^\bullet \longrightarrow 0$$

where for any $A^\bullet$ and $m \in \mathbb{Z}$, $\tau'_{\le m}A^\bullet$ is the complex

$$\cdots \longrightarrow A^{m-2} \longrightarrow A^{m-1} \longrightarrow A^m \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\,;$$

and in (ii), for example, one replaces $I^\bullet$ by the bounded complex $\tau'_{\le i+d+1}I^\bullet$.

Similar arguments settle the contravariant case. (Or, use the exercise just before (1.5.1).) Q.E.D.

Chapter 2

Derived Functors

Derived functors are Δ-functors out of derived categories, giving rise, upon application of homology, to functors such as Ext, Tor, and their sheaf-theoretic variants—in particular sheaf cohomology. Derived functors are characterized in §2.1 below by a universal property, and conditions for their existence are given in 2.2, leading up to the construction of right-derived functors via injective resolutions in 2.3 and, dually, of some left-derived functors via flat resolutions in 2.5. We use ideas of Spaltenstein [**Sp**] to deal throughout with unbounded complexes. The basic examples $\mathsf{R}\mathcal{H}om^{\bullet}$ and $\underline{\underline{\otimes}}$ are described in 2.4 and 2.5 respectively. Illustrating all that has gone before, their relation "adjoint associativity" is given in 2.6, which includes an abbreviated discussion of what is, in all conscience, involved in constructing natural transformations of multivariate derived functors: a host of underlying category-theoretic trivialities, usually ignored, but of whose existence one should at least be aware. The last section 2.7 develops further refinements.

2.1. Definition of derived functors

Fix an abelian category $\mathcal{A}$, let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$, let $\mathbf{D_J}$ be the corresponding derived category, and let

$$Q = Q_J\colon \mathbf{J} \to \mathbf{D_J}$$

be the canonical Δ-functor (see (1.7)). For any Δ-functors F and G from $\mathbf{J}$ to another Δ-category $\mathbf{E}$, or from $\mathbf{D_J}$ to $\mathbf{E}$, $\mathrm{Hom}(F,\,G)$ will denote the abelian group of Δ-functor morphisms from F to G.

Definition (2.1.1). A Δ-functor $F\colon \mathbf{J} \to \mathbf{E}$ is *right-derivable* if there exists a Δ-functor

$$\mathsf{R}F\colon \mathbf{D_J} \to \mathbf{E}$$

and a morphism of Δ-functors

$$\zeta\colon F \to \mathsf{R}F \circ Q$$

such that for every Δ-functor $G\colon \mathbf{D_J} \to \mathbf{E}$ the composed map

$$\mathrm{Hom}(\mathsf{R}F,\,G) \xrightarrow{\text{natural}} \mathrm{Hom}(\mathsf{R}F \circ Q,\,G \circ Q) \xrightarrow{\text{via }\zeta} \mathrm{Hom}(F,\,G \circ Q)$$

is an isomorphism (i.e., by (1.5.1), the map "via ζ" is an isomorphism).

The Δ-functor F is *left-derivable* if there exists a Δ-functor

$$\mathbf{L}F\colon \mathbf{D_J} \to \mathbf{E}$$

and a morphism of Δ-functors

$$\xi\colon \mathbf{L}F \circ Q \to F$$

such that for every Δ-functor $G\colon \mathbf{D_J} \to \mathbf{E}$ the composed map

$$\mathrm{Hom}(G, \mathbf{L}F) \xrightarrow{\text{natural}} \mathrm{Hom}(G \circ Q, \mathbf{L}F \circ Q) \xrightarrow{\text{via } \xi} \mathrm{Hom}(G \circ Q, F)$$

is an isomorphism (i.e., by (1.5.1), the map "via ξ" is an isomorphism).

Such a pair $(\mathbf{R}F, \zeta)$ $\big($respectively: $(\mathbf{L}F, \xi)\big)$ is called a *right-derived* (respectively: *left-derived*) *functor of* F.

As in (1.5.1), composition with Q gives an embedding of Δ-functor categories

(2.1.1.1) $$\mathbf{Hom}_\Delta(\mathbf{D_J}, \mathbf{E}) \hookrightarrow \mathbf{Hom}_\Delta(\mathbf{J}, \mathbf{E}),$$

with image the full subcategory whose objects are the Δ-functors which transform quasi-isomorphisms into isomorphisms. Consequently we can regard a right-(left-)derived functor of F as an *initial* (*terminal*) *object* [**M**, p. 20] in the category of Δ-functor morphisms $F \to G'$ $(G' \to F)$ where G' ranges over all Δ-functors from $\mathbf{J}$ to $\mathbf{E}$ which transform quasi-isomorphisms into isomorphisms. As such, the pair $(\mathbf{R}F, \zeta)$ (or $(\mathbf{L}f, \xi)$)—if it exists—is unique up to canonical isomorphism.

Complement (2.1.2). Let $\mathcal{A}'$ be another abelian category. Any additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ extends to a Δ-functor $\bar{F}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}')$ (see (1.5.2)). $Q'\colon \mathbf{K}(\mathcal{A}') \to \mathbf{D}(\mathcal{A}')$ being the canonical map, we will refer to derived functors of $Q'\bar{F}$, or of the restriction of $Q'\bar{F}$ to some specified Δ-subcategory $\mathbf{J}$ of $\mathbf{K}(\mathcal{A})$, as being "derived functors of F" and denote them by $\mathbf{R}F$ or $\mathbf{L}F$.

Example **(2.1.3).** If $F\colon \mathbf{J} \to \mathbf{E}$ transforms quasi-isomorphisms into isomorphisms then $F = \widetilde{F} \circ Q$ for a unique $\widetilde{F}\colon \mathbf{D_J} \to \mathbf{E}$; and $(\widetilde{F}, \text{identity})$ is both a right-derived and a left-derived functor of F.

Remark **(2.1.4).** Let $\mathcal{A}'$ be an abelian category, and in (2.1.1) suppose that $\mathbf{E}$ is a Δ-subcategory of $\mathbf{K}(\mathcal{A}')$ or of $\mathbf{D}(\mathcal{A}')$. If $\mathbf{R}F$ exists we can set

$$\mathbf{R}^i F(A) := H^i(\mathbf{R}F(A)) \qquad (A \in \mathbf{J},\ i \in \mathbb{Z}).$$

Since $\mathbf{R}F$ is a Δ-functor, any triangle $A \to B \to C \to A[1]$ in $\mathbf{J}$ is transformed by $\mathbf{R}F$ into a triangle in $\mathbf{E}$, and hence we have an exact homology sequence (see $(1.4.5)^{\mathrm{H}}$):

$(2.1.4)^{\mathrm{H}}$

$$\cdots \to \mathbf{R}^{i-1}F(C) \to \mathbf{R}^i F(A) \to \mathbf{R}^i F(B) \to \mathbf{R}^i F(C) \to \mathbf{R}^{i+1}F(A) \to \cdots$$

This applies in particular to the triangle $(1.4.4.2)^{\sim}$ associated to an exact sequence of $\mathcal{A}$-complexes

$$0 \to A \to B \to C \to 0 \qquad (A, B, C \in \mathbf{J}).$$

A similar remark can be made for $\mathbf{L}F$.

2.2. Existence of derived functors

Derivability of a given functor is often proved by reduction, via suitable Δ-equivalences of categories, to the trivial example (2.1.3), as we now explain—and summarize in (2.2.6).

We consider, as in (1.7), a diagram

$$\begin{tikzcd} \mathbf{J}' \arrow[r, "j"] \arrow[d, "Q'"'] & \mathbf{J}'' \arrow[d, "Q''"] \\ \mathbf{D}' \arrow[r, "\tilde{\jmath}"'] & \mathbf{D}'' \end{tikzcd}$$

where $\mathbf{J}' \subset \mathbf{J}''$ are Δ-subcategories of $\mathbf{K}(\mathcal{A})$, $\mathbf{D}'$ and $\mathbf{D}''$ are the corresponding derived categories, Q' and Q'' are the canonical Δ-functors, j is the inclusion, and $\tilde{\jmath}$ is the unique Δ-functor making the diagram commute; and we assume that the conditions of (1.7.2) or of $(1.7.2)^{\mathsf{op}}$ obtain. In other words we have a family of quasi-isomorphisms

(2.2.1) $$\psi_X \colon A_X \to X, \qquad X \in \mathbf{J}'',\ A_X \in \mathbf{J}', \quad \text{(see (1.7.2))},$$

or a family of quasi-isomorphisms

$(2.2.1)^{\mathsf{op}}$ $$\varphi_X \colon X \to A_X, \qquad X \in \mathbf{J}'',\ A_X \in \mathbf{J}', \quad \text{(see } (1.7.2)^{\mathsf{op}}).$$

In either situation, $\tilde{\jmath}$ identifies $\mathbf{D}'$ with a Δ-subcategory of $\mathbf{D}''$; there is a Δ-functor $(\rho, \theta) \colon \mathbf{D}'' \to \mathbf{D}'$ with

$$\rho(X) = A_X \qquad (X \in \mathbf{J}'');$$

and there are isomorphisms of Δ-functors

(2.2.2) $$\mathbf{1}_{\mathbf{D}''} \xrightarrow{\sim} \tilde{\jmath}\rho, \qquad \mathbf{1}_{\mathbf{D}'} \xrightarrow{\sim} \rho\tilde{\jmath}$$

induced by ψ or by φ.

Proposition (2.2.3). *With preceding notation, let* $\mathbf{E}$ *be a* Δ*-category, let* $F\colon \mathbf{J}'' \to \mathbf{E}$ *be a* Δ*-functor, and suppose that the restricted functor*

$$F' := F \circ j\colon \mathbf{J}' \to \mathbf{E}$$

has a right-derived functor

$$\mathbf{R}F'\colon \mathbf{D}' \to \mathbf{E}, \qquad \zeta'\colon F' \to \mathbf{R}F' \circ Q'.$$

If there exists a family $\varphi_X\colon X \to A_X$ *as in* $(2.2.1)^{\mathsf{op}}$, *whence a functor* ρ *as above, then* F *has the right-derived functor* $(\mathbf{R}F, \zeta)$ *where*

$$\mathbf{R}F = \mathbf{R}F' \circ \rho\colon \mathbf{D}'' \to \mathbf{E}$$

so that

$$\mathbf{R}F(X) = \mathbf{R}F'(A_X) \qquad (X \in \mathbf{J}''),$$

and where for each $X \in \mathbf{J}''$, $\zeta(X)$ *is the composition*

$$F(X) \xrightarrow{F(\varphi_X)} F(A_X) = F'(A_X) \xrightarrow{\zeta'(A_X)} \mathbf{R}F'(A_X) = \mathbf{R}F(X)\,.$$

A similar statement holds for left-derived functors when there exists a family ψ_X *as in* (2.2.1).

Proof. We check first that ζ is actually a morphism of Δ-functors. Consider a map $u\colon X \to Y$ in $\mathbf{J}''$. Since $Q''(\varphi_X)$ is an isomorphism, there is a unique map $\tilde u\colon A_X \to A_Y$ in $\mathbf{D}''$ (and hence in the full subcategory $\mathbf{D}'$) making the following $\mathbf{D}''$-diagram commute:

$$\begin{tikzcd}
X \arrow[r, "Q''(\varphi_X)"] \arrow[d, "Q''(u)"'] & A_X \arrow[d, "\tilde u"] \\
Y \arrow[r, "Q''(\varphi_Y)"'] & A_Y
\end{tikzcd}$$

By the definition of the functor ρ (see proof of (1.7.2)), that ζ is a morphism of functors means that the following diagram $\mathcal{D}(u)$ commutes for all u:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\varphi_X)"] \arrow[d, "F(u)"'] & F(A_X) \arrow[r, "\zeta'(A_X)"] \arrow[d, dashed, "?"'] & \mathbf{R}F'(A_X) \arrow[d, "\mathbf{R}F'(\tilde u)"] \\
F(Y) \arrow[r, "F(\varphi_Y)"'] & F(A_Y) \arrow[r, "\zeta'(A_Y)"'] & \mathbf{R}F'(A_Y)
\end{tikzcd}$$

If there were a $\mathbf{J}'$-map $u'\colon A_X \to A_Y$ such that $u'\varphi_X = \varphi_Y u$, whence $Q''(u')Q''(\varphi_X) = Q''(\varphi_Y)Q''(u)$ and $\tilde u = Q''(u') = Q'(u')$, then the broken arrow in $\mathcal{D}(u)$ could be replaced by the map $F(u')$, making both resulting subdiagrams of $\mathcal{D}(u)$, and hence $\mathcal{D}(u)$ itself, commute. We don't know that such a u' exists; but, I claim, *there exists a quasi-isomorphism* $v\colon Y \to Z$ *such that* (*with self-explanatory notation*) *both* v' *and* $(vu)'$ *exist.* This being so, both diagrams $\mathcal{D}(v)$ and $\mathcal{D}(vu)$ commute; and since $\tilde v$ is an isomorphism (because v is a quasi-isomorphism), therefore $\mathbf{R}F'(\tilde v)$ is an isomorphism, and it follows easily that $\mathcal{D}(u)$ also commutes, as desired.

To verify the claim, use (1.6.3) to construct in $\mathbf{J}''$ a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi_X"] \arrow[d, "u"'] \arrow[dr] & A_X \arrow[dr, "w"] & & \\
Y \arrow[r, "\varphi_Y"'] & A_Y \arrow[r, "\varphi"'] & Z \arrow[r, "\varphi_Z"'] & A_Z
\end{tikzcd}$$

with φ a quasi-isomorphism, and set

$$\begin{aligned}
v &:= \varphi \circ \varphi_Y \\
v' &:= \varphi_Z \circ \varphi \\
(vu)' &:= \varphi_Z \circ w.
\end{aligned}$$

Then $v'\varphi_Y = \varphi_Z v$ and $(vu)'\varphi_X = \varphi_Z(vu)$, as desired.

Thus ζ is a morphism of functors; and it is straightforward to check, via commutativity of (1.7.2.2), that ζ is in fact a morphism of Δ-functors.

Now we need to show (see (2.1.1)) that for every Δ-functor $G\colon \mathbf{D}'' \to \mathbf{E}$ the composed map

$$\operatorname{Hom}(\mathbf{R}F,\, G) \xrightarrow{(1.5.1)} \operatorname{Hom}(\mathbf{R}F \circ Q'',\, G \circ Q'') \xrightarrow{\text{via } \zeta} \operatorname{Hom}(F,\, G \circ Q'')$$

is *bijective.* For this it suffices to check that the following natural composition is an inverse map:

$$\begin{aligned}
\operatorname{Hom}(F,\, G \circ Q'') &\longrightarrow \operatorname{Hom}(F \circ j,\, G \circ Q'' \circ j) \\
&=\!=\!= \operatorname{Hom}(F',\, G \circ \tilde{\jmath} \circ Q') \\
&\xrightarrow{(2.1.1)} \operatorname{Hom}(\mathbf{R}F',\, G \circ \tilde{\jmath}) \\
&\longrightarrow \operatorname{Hom}(\mathbf{R}F' \circ \rho,\, G \circ \tilde{\jmath} \circ \rho) \\
&\xrightarrow{(2.2.2)} \operatorname{Hom}(\mathbf{R}F' \circ \rho,\, G) \\
&=\!=\!= \operatorname{Hom}(\mathbf{R}F,\, G)\,.
\end{aligned}$$

This checking is left to the reader, as is the proof for left-derived functors. Q.E.D.

Example **(2.2.4)** [**H**, p. 53, Thm. 5.1]. Let $j\colon \mathbf{J}' \hookrightarrow \mathbf{J}''$, $F\colon \mathbf{J}'' \to \mathbf{E}$, and $\varphi_X\colon X \to A_X$ be as above, and suppose that the restricted functor $F' := F \circ j$ transforms quasi-isomorphisms into isomorphisms (or, equivalently, $F(C) \cong 0$ for every exact complex $C \in \mathbf{J}'$, see (1.5.1)). Then by (2.1.3), F' has a right-derived functor $(\mathbf{R}F',\, \mathbf{1})$ where $F' = \mathbf{R}F' \circ Q'$ and $\mathbf{1}$ is the identity morphism of F'.

So by (2.2.3), F has a right-derived functor $(\mathbf{R}F, \zeta)$ with

$$\mathbf{R}F(X) = F(A_X)$$

and

$$\zeta(X) = F(\varphi_X)\colon F(X) \to F(A_X) = \mathbf{R}F(X)$$

for all $X \in \mathbf{J}''$. Note that if $X \in \mathbf{J}'$ then φ_X is a quasi-isomorphism in $\mathbf{J}'$, whence $\zeta(X)$ is an isomorphism.

The action of $\mathbf{R}F$ on maps can be described thus: if $u\colon X \to Y$ is a map in $\mathbf{J}''$ then with v' and $(vu)'$ as in the preceding proof,

$$\mathbf{R}F(u/1) = F(v')^{-1} \circ F((vu)')\,;$$

and for any map f/s in $\mathbf{D}''$ (see §1.2), we have

$$\mathbf{R}F(f/s) = \mathbf{R}F(f/1) \circ \mathbf{R}F(s/1)^{-1}.$$

As for the Δ-structure on $\mathbf{R}F$, one has for each X the isomorphism

$$\theta(X)\colon \mathbf{R}F(X[1]) = F(A_{X[1]}) \xrightarrow[F(\eta_X)]{\sim} F(A_X[1]) \xrightarrow[\theta_F]{\sim} F(A_X)[1] = \mathbf{R}F(X)[1]$$

where

$$\eta_X := Q''(\varphi_X[1]) \circ Q''(\varphi_{X[1]})^{-1}\colon A_{X[1]} \xrightarrow{\sim} A_X[1]\,,$$

and where the isomorphism θ_F comes from the Δ-functoriality of F.

(2.2.5). Let $\mathcal{A}$ be an abelian category, let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$, and let F be a Δ-functor from $\mathbf{J}$ to a Δ-category $\mathbf{E}$. We will say that a complex $X \in \mathbf{J}$ is *right-F-acyclic* if for each quasi-isomorphism $u\colon X \to Y$ in $\mathbf{J}$ there exists a quasi-isomorphism $v\colon Y \to Z$ in $\mathbf{J}$ such that the map $F(vu)\colon F(X) \to F(Z)$ is an isomorphism. Left-F-acyclicity is defined similarly, with arrows reversed.

For example, if $\mathbf{J} := \mathbf{J}''$ in (2.2.4), then every complex $X \in \mathbf{J}'$ is right-F-acyclic—just take $Z := A_Y$ and $v := \varphi_Y$. Conversely:

Lemma (2.2.5.1). *The right-F-acyclic complexes in $\mathbf{J}$ are the objects of a localizing subcategory* (§1.7). *Moreover, the restriction of F to this subcategory transforms quasi-isomorphisms into isomorphisms; in other words, if the complex X is both exact and right-F-acyclic, then $F(X) \cong 0$* (*see* (1.5.1)).

Proof. Since F commutes with translation—up to isomorphism—it is clear that X is right-F-acyclic iff so is $X[1]$.

Next, suppose we have a triangle $X \to X_1 \to X_2 \to X[1]$ in which X_1 and X_2 are right-F-acyclic. We will show that then X is right-F-acyclic. Any quasi-isomorphism $u\colon X \to Y$ can be embedded into a map of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & X_1 \arrow[r] \arrow[d, "u_1"'] & X_2 \arrow[r] \arrow[d, "u_2"'] & X[1] \arrow[d, "{u[1]}"] \\ Y \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & Y[1] \end{tikzcd}$$

where u_1 is a quasi-isomorphism whose existence is given by (1.6.3), and where u_2 is then given by $(\Delta 3)'$ and $(\Delta 3)''$ in §1.4. Such a u_2 is also a quasi-isomorphism, as one sees by applying the five-lemma to the natural map between the homology sequences of the two triangles (see $(1.4.5)^{\mathrm{H}}$). Similarly, from the definition of right-F-acyclic we deduce a triangle-map

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, "v_1"'] & Y_2 \arrow[r] \arrow[d, "v_2"'] & Y[1] \arrow[r] \arrow[d, "{v[1]}"'] & Y_1[1] \arrow[d, "{v_1[1]}"] \\ Z_1 \arrow[r] & Z_2 \arrow[r] & Z[1] \arrow[r] & Z_1[1] \end{tikzcd}$$

where v_1, v_2, and v are quasi-isomorphisms such that $F(v_1u_1)$ and $F(v_2u_2)$ are isomorphisms. (Here $(\Delta 2)$ in §1.4 should be kept in mind.) We can then apply the Δ-functor F to the map of triangles

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "v_1u_1"'] & X_2 \arrow[r] \arrow[d, "v_2u_2"'] & X[1] \arrow[r] \arrow[d, "{(vu)[1]}"'] & X_1[1] \arrow[d, "{(v_1u_1)[1]}"] \\ Z_1 \arrow[r] & Z_2 \arrow[r] & Z[1] \arrow[r] & Z_1[1] \end{tikzcd}$$

and deduce from $(\Delta 3)^*$ that $F((vu)[1])$, and hence $F(vu)$, is also an isomorphism. Thus X is indeed right-F-acyclic.

In particular, the direct sum of two right-F-acyclic complexes is right-F-acyclic, because the direct sum is the summit of a triangle whose base is the zero-map from one to the other, see (1.4.2.1). Also, $0 \in \mathbf{J}$ is clearly right-F-acyclic. We see then that the right-F-acyclic complexes are the objects of a Δ-subcategory of $\mathbf{J}$.

For this subcategory to be localizing it suffices, by $(1.7.1)^{\mathsf{op}}$, that if $X \to Y \to Z$ is as in the definition of right-F-acyclic, then Z is right-F-acyclic; and this follows from:

Lemma (2.2.5.2). *If X is right-F-acyclic and if there exists a quasi-isomorphism $\alpha\colon X \to Z$ such that $F(\alpha)\colon F(X) \to F(Z)$ is an epimorphism, then Z is right-F-acyclic.*

Proof. Given a quasi-isomorphism $Z \to Y'$, there exists a quasi-isomorphism $Y' \to Z'$ such that $F(X) \to F(Z) \to F(Z')$ is an isomorphism (since X is right-F-acyclic); and since $F(X) \to F(Z)$ is an epimorphism, therefore $F(Z) \to F(Z')$ is an isomorphism. Q.E.D.

To justify the last assertion in (2.2.5.1), take $Y := 0$ in the definition of right-F-acyclicity. Q.E.D.

We leave it to the reader to establish a corresponding statement for left-F-acyclic complexes.

In summary:

Proposition (2.2.6). *Let $\mathcal{A}$ be an abelian category, let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$, and let F be a Δ-functor from $\mathbf{J}$ to a Δ-category $\mathbf{E}$. Suppose $\mathbf{J}$ contains a family of quasi-isomorphisms $\varphi_X \colon X \to A_X$ $(X \in \mathbf{J})$ such that A_X is right-F-acyclic for all X, see* (2.2.5). *Then F has a right-derived functor $(\mathsf{R}F, \zeta)$ such that for all $X \in \mathbf{J}$,*

$$\mathsf{R}F(X) = F(A_X) \quad \text{and} \quad \zeta(X) = F(\varphi_X)\colon F(X) \to F(A_X) = \mathsf{R}F(X)\,.$$

Moreover, X is right-F-acyclic $\Leftrightarrow \zeta(X)$ is an isomorphism.

Proof. Everything is contained in (2.2.4) and (2.2.5), except for the fact that if $\zeta(X)$ is an isomorphism then X is right-F-acyclic, which is proved by taking, in (2.2.5), $Z := A_Y$, $v := \varphi_Y$, and noting that then $F(vu)$ is the composite isomorphism

$$F(X) \xrightarrow[\zeta(X)]{\ \sim\ } \mathsf{R}F(X) \xrightarrow{\ \sim\ } \mathsf{R}F(Y) = F(Z).$$

Q.E.D.

Corollary (2.2.6.1). *With assumptions as in* (2.2.6), *if $G\colon \mathbf{E} \to \mathbf{E}'$ is any Δ-functor then $(G \circ \mathsf{R}F,\, G(\zeta))$ is a right-derived functor of GF.*

Proof. Clearly, right-F-acyclic complexes are right-(GF)-acyclic. It follows then from (2.2.4) and (2.2.5) that the assertion need only be proved for the restriction of F to the subcategory of right-F-acyclic complexes, in which case it follows from (2.1.3). Q.E.D.

Corollary (2.2.7). *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, let $\mathbf{J} \subset \mathbf{K}(\mathcal{A})$ and $\mathbf{J}' \subset \mathbf{K}(\mathcal{A}')$ be Δ-subcategories with canonical functors $Q\colon \mathbf{J} \to \mathbf{D}_\mathbf{J}$, $Q'\colon \mathbf{J}' \to \mathbf{D}_{\mathbf{J}'}$ to their respective derived categories, and let $F\colon \mathbf{J} \to \mathbf{J}'$ and $G\colon \mathbf{J}' \to \mathbf{E}$ be Δ-functors. Assume that G has a right-derived functor $\mathsf{R}G$ and that every complex $X \in \mathbf{J}$ admits a quasi-isomorphism into a right-$(Q'F)$-acyclic complex A_X such that $F(A_X)$ is right-G-acyclic. Then $Q'F$ and GF have right-derived functors, denoted $\mathsf{R}F$ and $\mathsf{R}(GF)$, and there is a unique Δ-functorial isomorphism*

$$\alpha\colon \mathsf{R}(GF) \xrightarrow{\ \sim\ } \mathsf{R}G\,\mathsf{R}F$$

such that the following natural diagram commutes for all $X \in \mathbf{J}$:

(2.2.7.1)
$$\begin{array}{ccc} GF(X) & \longrightarrow & \mathsf{R}(GF)(QX) \\ \big\downarrow & & \simeq\big\downarrow \alpha(QX) \\ \mathsf{R}GQ'F(X) & \longrightarrow & \mathsf{R}G\mathsf{R}F(QX) \end{array}$$

PROOF. Derivability of $Q'F$ results from (2.2.6). Derivability of GF results similarly once we show, as follows, that A_X is right-(GF)-acyclic: just note for any quasi-isomorphism $A_X \to Y$ in $\mathbf{J}$ that, by (2.2.5.1), the resulting composed map $F(A_X) \to F(Y) \to F(A_Y)$ is a quasi-isomorphism and so $GF(A_X) \xrightarrow{\sim} GF(A_Y)$. The existence of a unique Δ-functorial α making (2.2.7.1) commute follows from the definition of right-derived functor. Since A_X is right-(GF)-acyclic and right-$(Q'F)$-acyclic, and $F(A_X)$ is right-G-acyclic, (2.2.6) implies that $\alpha(QX)$ is isomorphic to the identity map of $GF(A_X)$. Thus α is an isomorphism. Q.E.D.

We leave the corresponding statements for left-F-acyclic complexes and left-derived functors to the reader.

Incidentally, (2.2.6) generalizes in a simple way to triangulation-compatible multiplicative systems in any Δ-category (see [**H**, p. 31]). It is of course of little interest unless we can construct a family (φ_X). That matter is addressed in the following sections.

EXERCISES **(2.2.8).** (a) Verify that F transforms quasi-isomorphisms into isomorphisms iff every complex $X \in \mathbf{J}$ is right-F-acyclic.

(b) Verify that if $X \in \mathbf{J}$ is exact then X is right-F-acyclic iff $F(X) \cong 0$.

(c) Let F be a Δ-functor from $\mathbf{J}$ to a Δ-category $\mathbf{E}$. Let $\mathbf{J}'$ be the full subcategory of $\mathbf{J}$ whose objects are all the complexes in $\mathbf{J}$ admitting a quasi-isomorphism to a right-F-acyclic complex. Then $\mathbf{J}'$ is a Δ-subcategory of $\mathbf{J}$.

(d) X is right-F-acyclic iff every map $C \to X$ in $\mathbf{J}$ with C exact factors as $C \to C' \to X$ with C' exact and $F(C') \cong 0$.

(e) X is said to be "unfolded for F" if for every $Z \in \mathbf{E}$ the natural map

$$\mathrm{Hom}_{\mathbf{E}}(Z, F(X)) \to \varinjlim_{X \to Y} \mathrm{Hom}_{\mathbf{E}}(Z, F(Y))$$

is an isomorphism, where the $\varinjlim$ is taken over the category of all quasi-isomorphisms $X \to Y$ in $\mathbf{J}$ [**De**, p. 274, (iv)]. Check that any right-F-acyclic X is unfolded for F; and that the converse holds under the hypotheses of (2.2.6).

(f) Show: X is unfolded for F iff every map $C \to X$ in $\mathbf{J}$ with C exact factors as $C \to C' \to X$ with C' is exact and $F(C) \to F(C')$ the zero map. (For this, the octahedral axiom in $\mathbf{E}$ may be needed, see §1.4.)

2.3. Right-derived functors via injective resolutions

The basic example of a family (φ_X) as in (2.2.6) arises when $\mathcal{A}$ has enough injectives, i.e., every object of $\mathcal{A}$ admits a monomorphism into an injective object. Then every complex $X \in \overline{\mathbf{K}}^{+}(\mathcal{A})$ admits a quasi-isomorphism $\varphi_X \colon X \to I_X$ into a bounded-below complex of injectives (see (1.8.2)); and by (2.3.4) and (2.3.2.1) below, this I_X is right-F-acyclic for *every* Δ-functor $F \colon \overline{\mathbf{K}}^{+}(\mathcal{A}) \to \mathbf{E}$, whence F is right-derivable.

Later on, however, it will become important for us to be able to deal with *unbounded* complexes; and for this purpose the following more general injectivity notion is, via (2.3.5), essential.

Definition (2.3.1). Let $\mathcal{A}$ be an abelian category, and let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$. A complex $I \in \mathbf{J}$ is said to be *q-injective in* $\mathbf{J}$ (*or* $\mathbf{J}$-*q-injective*) if for every diagram $Y \xleftarrow{s} X \xrightarrow{f} I$ in $\mathbf{J}$ with s a quasi-isomorphism, there exists $g\colon Y \to I$ such that $gs = f$.[18]

Lemma (2.3.2). *$I \in \mathbf{J}$ is $\mathbf{J}$-q-injective iff every quasi-isomorphism $I \to Y$ in $\mathbf{J}$ has a left inverse.*

Proof. In (2.3.1) take $X := I$ and $f :=$ identity to see that if I is q-injective then the quasi-isomorphism s has a left inverse. Conversely, by (1.6.3) any diagram $Y \xleftarrow{s} X \xrightarrow{f} I$ is part of a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & I \arrow[d, "s'"] \\ Y \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in which s' is a quasi-isomorphism; and then if t is a left inverse for s' and $g := tf'$, we have $gs = f$. Q.E.D.

Corollary (2.3.2.1). *$I \in \mathbf{J}$ is $\mathbf{J}$-q-injective iff I is right-F-acyclic for every Δ-functor $F\colon \mathbf{J} \to \mathbf{E}$.*

Proof. If any quasi-isomorphism $I \to Y$ has a left inverse, then setting $X := I$ in (2.2.5) we see at once that I is right-F-acyclic. Conversely, if I is right-F-acyclic for the identity functor $\mathbf{J} \to \mathbf{J}$, then every quasi-isomorphism $I \to Y$ has a left inverse. Q.E.D.

Taking $F :=$ identity in (2.2.5.1), we deduce:

Corollary (2.3.2.2). *The $\mathbf{J}$-q-injective complexes are the objects of a localizing subcategory $\mathbf{I}$. Every quasi-isomorphism in $\mathbf{I}$ is an isomorphism, so the pair* $(\mathbf{I}, \text{identity})$ *has the characteristic universal property of the derived category $\mathbf{D_I}$* (§1.2)*, and therefore $\mathbf{I} \cong \mathbf{D_I}$ can be identified with a Δ-subcategory of $\mathbf{D_J}$.*

Corollary (2.3.2.3). *Suppose that there exists a family of* q-injective resolutions $\varphi_X\colon X \to I_X$ $(X \in \mathbf{J})$, *i.e., for each X, φ_X is a quasi-isomorphism and I_X is $\mathbf{J}$-q-injective. Then any Δ-functor $F\colon \mathbf{J} \to \mathbf{E}$ has a right-derived functor* $(\mathbf{R}F, \zeta)$[19] *with*

$$\mathbf{R}F(X) = F(I_X) \quad \textit{and} \quad \zeta(X) = F(\varphi_X)\colon F(X) \to F(I_X) = \mathbf{R}F(X)\,,$$

[18] Here "q" stands for the class of quasi-isomorphisms. The equivalent term "K-injective" in [**Sp**, p. 127] seems to me less suggestive.

[19] So the embedding functor (2.1.1.1) has a *left adjoint,* taking F to $\mathbf{R}F$.

and such that for any morphism $f/s\colon X_1 \xleftarrow{s} X \xrightarrow{f} X_2$ *in* $\mathbf{D_J}$,

$$\mathbf{R}F(f/s) = F(f')\circ F(s')^{-1}$$

where f' *is the unique map in* $\mathbf{I}$ *making the following square in* $\mathbf{J}$ *commute*

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "f"'] & I_X \arrow[d, "f'"] \\ X_2 \arrow[r, "\varphi_{X_2}"'] & I_{X_2} \end{tikzcd}$$

and similarly for s'.

Proof. Since φ_X becomes an isomorphism in $\mathbf{D_J}$, the map f' exists uniquely in $\mathbf{D_J}$, hence in $\mathbf{I}$ (2.3.2.2). For the rest see (2.2.4), with $\mathbf{J}' := \mathbf{I}$, $\mathbf{J}'' := \mathbf{J}$, and $v :=$ identity. Q.E.D.

Example **(2.3.3).** An object I in $\mathcal{A}$ is injective iff when considered as a complex vanishing in all nonzero degrees it is q-injective in $\mathbf{K}(\mathcal{A})$ (or in $\mathbf{K}^{\mathsf{b}}(\mathcal{A})$).

Sufficiency: for any $\mathcal{A}$-diagram $Y^0 \xleftarrow{s^0} X \xrightarrow{f} I$ with s^0 a monomorphism, take Y to be the complex which looks like the natural map $Y^0 \to \operatorname{coker}(s^0)$ in degrees 0 and 1, and vanishes elsewhere, and take $s\colon X \to Y$ to be the obvious quasi-isomorphism; then deduce from (2.3.1) that if I is q-injective there exists $g^0\colon Y^0 \to I$ such that $g^0s^0 = f$—so that I is $\mathcal{A}$-injective.

For necessity, use (2.3.2): to find a left inverse in $\mathbf{K}(\mathcal{A})$ for a quasi-isomorphism $\beta\colon I \to Y$ we may replace Y by the complex $\tau_{\geq 0}Y$, to which Y maps quasi-isomorphically (§1.10), i.e., we may assume that Y vanishes in all negative degrees; then β induces a monomorphism (in $\mathcal{A}$) $\beta^0\colon I \to Y^0$, which has a left inverse if I is $\mathcal{A}$-injective, and that gives rise, obviously, to a left inverse for β. (One could also use (iv) in (2.3.8) below.)

Example **(2.3.4).** Any bounded-below complex I of $\mathcal{A}$-injectives is q-injective in $\mathbf{K}(\mathcal{A})$. Indeed, by [**H**, p. 41, Lemma 4.5], I satisfies the condition in (2.3.2). (One could also use (2.3.8)(iv).) Thus (2.3.2.3) applies to $\mathbf{J} := \overline{\mathbf{K}}^{+}(\mathcal{A})$ whenever $\mathcal{A}$ has enough injectives (see beginning of this §2.3). In that case, further, every $\overline{\mathbf{K}}^{+}(\mathcal{A})$-q-injective complex admits a quasi-isomorphism, hence, by (2.3.2.2), an *isomorphism,* to a bounded-below complex of $\mathcal{A}$-injectives.

Example **(2.3.5).** Let U be a topological space, $\mathcal{O}$ a sheaf of rings on U, and $\mathcal{A}$ the abelian category of left $\mathcal{O}$-modules. Then a theorem of Spaltenstein [**Sp**, p. 138, Theorem 4.5] asserts that *every complex in* $\mathbf{K}(\mathcal{A})$ *admits a q-injective resolution.* Hence by (2.3.2.3), every Δ-functor out of $\mathbf{K}(\mathcal{A})$ is right-derivable.

More generally, a q-injective resolution exists for every complex in any Grothendieck category, i.e., an abelian category with exact direct limits and having a generator [**AJS**, p. 243, Theorem 5.4]. For example, injective Cartan-Eilenberg resolutions [**EGA**, III, Chap. 0, (11.4.2)] always exist in Grothendieck categories; and their totalizations—which generally require countable direct products—give q-injective resolutions when such products of epimorphisms are epimorphisms (a condition which holds in the category of modules over a fixed ring, but fails, for instance, in most categories of sheaves on topological spaces).

Example **(2.3.6).** Let $\mathcal{A}_1$, $\mathcal{A}_2$ be abelian categories, $\mathcal{A}_1$ having enough injectives. As in (1.5.2) any additive functor $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ extends to a Δ-functor $\bar F\colon \mathbf{K}^+(\mathcal{A}_1) \to \mathbf{K}^+(\mathcal{A}_2)$ which has, by (2.3.4), a right-derived functor

$$\mathbf{R}^+\bar F\colon \mathbf{D}^+(\mathcal{A}_1) \to \mathbf{K}^+(\mathcal{A}_2)$$

satisfying, for a given family $\varphi_X\colon X \to I_X$ of injective resolutions,

$$\mathbf{R}^+\bar F(X) = \bar F(I_X)\,.$$

We can extend the domain of $\mathbf{R}^+\bar F$ to $\overline{\mathbf{D}}^+(\mathcal{A}_1)$ by composing with the equivalence τ^+ defined in (1.8.1). Moreover, if every $\mathcal{A}_1$-complex has a q-injective resolution, then there is a further extension to a derived functor $\mathbf{R}\bar F\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{K}(\mathcal{A}_2)$—whose composition with the canonical map $\mathbf{K}(\mathcal{A}_2) \to \mathbf{D}(\mathcal{A}_2)$ is $\mathbf{R}F$, see (2.1.2).

With H^i the usual homology functor, let $R^iF\colon \mathcal{A}_1 \to \mathcal{A}_2$ $(i \in \mathbb{Z})$ be the composition

$$\mathcal{A}_1 \xrightarrow{(1.2.2)} \mathbf{D}^+(\mathcal{A}_1) \xrightarrow{\mathbf{R}^+F} \mathbf{K}^+(\mathcal{A}_2) \xrightarrow{H^i} \mathcal{A}_2$$

(cf. (2.1.4)). Then $R^iF = 0$ for $i < 0$, and there is a natural map of functors $F \to R^0F$ which is an isomorphism if and only if F is left-exact.

Example **(2.3.7).** Let $f\colon U_1 \to U_2$ be a continuous map of topological spaces. Let $\mathcal{A}_i$ be the category of sheaves of abelian groups on U_i $(i = 1, 2)$. Then $\mathcal{A}_i$ is abelian, and has enough injectives. The direct image functor $f_*\colon \mathcal{A}_1 \to \mathcal{A}_2$ is left-exact, and has, as in (2.3.6), a derived functor

$$\mathbf{R}^+\overline{f_*}\colon \overline{\mathbf{D}}^+(\mathcal{A}_1) \to \mathbf{K}^+(\mathcal{A}_2)\,.$$

By (2.3.5), the composition $\mathbf{K}(\mathcal{A}_1) \xrightarrow{\overline{f_*}} \mathbf{K}(\mathcal{A}_2) \xrightarrow{Q} \mathbf{D}(\mathcal{A}_2)$ has a derived functor $\mathbf{R}f_*$, whose restriction to $\overline{\mathbf{D}}^+(\mathcal{A}_1)$ is isomorphic to $Q \circ \mathbf{R}^+\overline{f_*}$.

In particular, when U_2 is a single point then $\mathcal{A}_2 = \mathfrak{Ab}$, the category of abelian groups, and f_* is the global section functor $\Gamma = \Gamma(U_1, -)$. In this case one usually sets, for $i \in \mathbb{Z}$, see (2.1.4),

$$\mathbf{R}f_* = \mathbf{R}\Gamma, \qquad \mathbf{R}^if_* = \mathbf{R}^i\Gamma = \mathbf{H}^i, \qquad R^if_*(-) = H^i(U_1, -)\,.$$

Here are some other characterizations of q-injectivity, see [**Sp**, p. 129, Prop. 1.5], [**BN**, Def. 2.6 etc.].

Proposition (2.3.8). *Let $\mathcal{A}$ be an abelian category, and let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$. The following conditions on a complex $I \in \mathbf{J}$ are equivalent:*

(i) *I is q-injective in $\mathbf{J}$.*

(i)′ *For every diagram $Y \xleftarrow{s} X \xrightarrow{f} I$ in $\mathbf{J}$ with s a quasi-isomorphism there is a* unique *$g\colon Y \to I$ such that $gs = f$.*

(ii) *Every quasi-isomorphism $I \to Y$ in $\mathbf{J}$ has a left inverse.*

(ii)′ *Every quasi-isomorphism $I \to Y$ in $\mathbf{J}$ is a monomorphism.*

(iii) *I is right-F-acyclic for every Δ-functor $F\colon \mathbf{J} \to \mathbf{E}$.*

(iii)′ *I is right-F-acyclic for F the identity functor $\mathbf{J} \to \mathbf{J}$.*

(iv) *For every exact complex $X \in \mathbf{J}$, we have $\mathrm{Hom}_{\mathbf{J}}(X, I) = 0$.*

(iv)′ *The Δ-functor $\mathrm{Hom}^\bullet(-, I)\colon \mathbf{J} \to \mathbf{K}(\mathfrak{Ab})$ of (1.5.3) takes quasi-isomorphisms into quasi-isomorphisms.*

(v) *For every complex $X \in \mathbf{J}$, the natural map $\mathrm{Hom}_{\mathbf{J}}(X, I) \to \mathrm{Hom}_{\mathbf{D_J}}(X, I)$ is bijective.*

Proof. The equivalence of (i), (ii), (iii) and (iii)′ has already been shown (see (2.3.2) and the proof of (2.3.2.1)). For (ii) $\Leftrightarrow$ (ii)′ see (1.4.2.1). Taking $Y := 0$ in (2.3.1), we see that (i) $\Rightarrow$ (iv). The equivalence of (iv) and (iv)′ results from the footnote in (1.5.1) and the easily-checked relation

$$(2.3.8.1) \qquad H^n\big(\mathrm{Hom}^\bullet(X, I)\big) \cong \mathrm{Hom}_{\mathbf{J}}(X[-n], I) \qquad (n \in \mathbb{Z},\ X \in \mathbf{J}).$$

The implications (v) $\Rightarrow$ (i)′ $\Rightarrow$ (i) are simple to verify.

We show next that (iv) $\Rightarrow$ (ii). Let X be the summit of a triangle T in $\mathbf{J}$ whose base is a quasi-isomorphism $I \to Y$. By [**H**, p. 23, 1.1 b)], the resulting sequence

$$\mathrm{Hom}(X, I) \to \mathrm{Hom}(Y, I) \to \mathrm{Hom}(I, I) \to \mathrm{Hom}(X[-1], I)$$

is *exact.* Moreover, the exact homology sequence $(1.4.5)^{\mathrm{H}}$ of T shows that X is exact. So if (iv) holds, then $\mathrm{Hom}(Y, I) \to \mathrm{Hom}(I, I)$ is bijective, and (ii) follows.

Finally, we show that (ii) $\Rightarrow$ (v). For any map $f/s\colon X \to I$ in $\mathbf{D_J}$, (1.6.3) yields a commutative diagram in $\mathbf{J}$, with s' a quasi-isomorphism:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & I \arrow[d, "s'"] \\ X \arrow[r, "f'"'] & B \end{tikzcd}$$

If $ts' =$ identity, then $f/s = (s'/1)^{-1}(f'/1) = (tf')/1$, and so the map $\mathrm{Hom}_{\mathbf{J}}(X, I) \to \mathrm{Hom}_{\mathbf{D_J}}(X, I)$ is surjective. For the injectivity, given $f\colon X \to I$ in $\mathbf{J}$, note that $f/1 = 0 \implies$ there exists a quasi-isomorphism $t\colon X' \to X$ such that $ft = 0$ (see §1.2) $\implies$ there exists a quasi-isomorphism $s\colon I \to Y$ such that $sf = 0$ [**H**, p. 37]; and if s has a left inverse, then $sf = 0 \implies f = 0$. Q.E.D.

Exercise **(2.3.9).** Show: If $\mathcal{A}$ is a Grothendieck category then $\mathbf{D}(\mathcal{A})$ is equivalent to the homotopy category of q-injective complexes. Hence if $\mathcal{A}$ has inverse limits then so does $\mathbf{D}(\mathcal{A})$.

2.4. Derived homomorphism functors

Let $\mathcal{A}$ be an abelian category, and let $\mathbf{L}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$ in which there exists a family of quasi-isomorphisms $\varphi_X\colon X \to I_X$ $(X \in \mathbf{L})$ such that $I_X \in \mathbf{L}$ is q-injective in $\mathbf{K}(\mathcal{A})$ for every X. Then for any quasi-isomorphism $s\colon X \to Y$ with Y in $\mathbf{K}(\mathcal{A})$ there exists, by (2.3.1), a map $g\colon Y \to I_X$, necessarily a quasi-isomorphism, such that $gs = \varphi_X$; and hence by $(1.7.1)^{\mathsf{op}}$, $\mathbf{L}$ is a localizing subcategory of $\mathbf{K}(\mathcal{A})$, i.e., the derived category $\mathbf{D_L}$ identifies naturally with a Δ-subcategory of $\mathbf{D}(\mathcal{A})$.

For example, if $\mathcal{A}$ has enough injectives we could take $\mathbf{L} := \overline{\mathbf{K}}^{+}(\mathcal{A})$, see (2.3.4). Or, if U is a topological space with a sheaf of rings $\mathcal{O}$ and $\mathcal{A}$ is the category of left $\mathcal{O}$-modules, we could take $\mathbf{L} := \mathbf{K}(\mathcal{A})$, see (2.3.5).

By (2.3.2.3), every Δ-functor $F\colon \mathbf{L} \to \mathbf{E}$ is right-derivable. So for any fixed object $A \in \mathbf{K}(\mathcal{A})$, the Δ-functor $F_A\colon \mathbf{L} \to \mathbf{K}(\mathfrak{Ab})$ given by

$$F_A(B) = \mathrm{Hom}^\bullet(A,\, B) \qquad (B \in \mathbf{L})$$

(see (1.5.3)) has a right-derived functor

$$\mathbf{R}F_A\colon \mathbf{D_L} \to \mathbf{K}(\mathfrak{Ab})$$

with

$$\mathbf{R}F_A(B) = \mathrm{Hom}^\bullet(A,\, I_B).$$

For fixed B and variable A, $\mathrm{Hom}^\bullet(A,\, I_B)$ is a contravariant Δ-functor from $\mathbf{K}(\mathcal{A})$ to $\mathbf{K}(\mathfrak{Ab})$ (see 1.5.3), which takes quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ to quasi-isomorphisms in $\mathbf{K}(\mathfrak{Ab})$ ((2.3.8)(iv)$'$) and hence—after composition with the natural functor $Q'\colon \mathbf{K}(\mathfrak{Ab}) \to \mathbf{D}(\mathfrak{Ab})$—to isomorphisms in $\mathbf{D}(\mathfrak{Ab})$. So by (1.5.1)—and the exercise preceding it—there results a Δ-functor $\mathbf{D}(\mathcal{A})^{\mathsf{op}} \to \mathbf{D}(\mathfrak{Ab})$. Thus we obtain a functor of two variables

$$\mathbf{R}\mathrm{Hom}^\bullet(A,\, B)\colon \mathbf{D}(\mathcal{A})^{\mathsf{op}} \times \mathbf{D_L} \to \mathbf{D}(\mathfrak{Ab})$$

which, together with appropriate θ (see (1.5.3)), is a Δ-functor in each variable separately:

$$\mathbf{R}\mathrm{Hom}^\bullet(A,\, B) = Q'\mathrm{Hom}^\bullet(A,\, I_B) \tag{2.4.1}$$

for all objects $A \in \mathbf{D}(\mathcal{A})^{\mathsf{op}}$, $B \in \mathbf{D_L}$; and we leave it to the reader to make explicit the effect of $\mathbf{R}\mathrm{Hom}^\bullet$ on morphisms in $\mathbf{D}(\mathcal{A})^{\mathsf{op}}$ and $\mathbf{D_L}$ respectively.

From (2.3.8)(v) and (2.3.8.1) (with $\mathbf{J} := \mathbf{K}(\mathcal{A})$), we deduce canonical isomorphisms (*Yoneda theorem*):

$$(2.4.2)\qquad H^n(\mathbf{R}\mathrm{Hom}^\bullet(X,\,B)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,\,B[n]) \qquad (n \in \mathbb{Z}).$$

This leads, in particular, to an elementary interpretation of the exact sequence $(2.1.4)^{\mathrm{H}}$ when $F := F_X$, see [**H**, p. 23, Prop. 1.1, b)].

(2.4.3). The variables A, B are treated quite differently in the above definition of $\mathbf{R}\mathrm{Hom}^\bullet$. But there is a more symmetric characterization of this derived functor, analogous to the one in (2.1.1). This is given in (2.4.4), after the necessary preparation.

Let $\mathbf{K_1}$, $\mathbf{K_2}$, $\mathbf{E}$ be Δ-categories, with respective translation functors T_1, T_2, T. A Δ-functor from $\mathbf{K_1} \times \mathbf{K_2}$ to $\mathbf{E}$ is defined to be a triple (F, θ_1, θ_2) with

$$F\colon \mathbf{K_1} \times \mathbf{K_2} \to \mathbf{E}$$

a functor and

$$\theta_1\colon F\circ(T_1 \times 1) \xrightarrow{\sim} T\circ F, \qquad\qquad \theta_2\colon F\circ(1\times T_2) \xrightarrow{\sim} T\circ F$$

isomorphisms of functors, such that for each $B \in \mathbf{K_2}$ the functor

$$F_B(A) := F(A,B)$$

together with θ_1 is a Δ-functor from $\mathbf{K_1}$ to $\mathbf{E}$, and for each $A \in \mathbf{K_1}$ the functor

$$F_A(B) := F(A,B)$$

together with θ_2 is a Δ-functor from $\mathbf{K_2}$ to $\mathbf{E}$; and such that furthermore the composed functorial isomorphisms

$$F(T_1\times T_2) = F(T_1\times 1)(1\times T_2) \xrightarrow{\text{via } \theta_1} TF(1\times T_2) \xrightarrow{\text{via } \theta_2} TTF$$
$$F(T_1\times T_2) = F(1\times T_2)(T_1\times 1) \xrightarrow{\text{via } \theta_2} TF(T_1\times 1) \xrightarrow{\text{via } \theta_1} TTF$$

are *negatives* of each other. Similarly, we can define Δ-functors of three or more variables—with a condition indicated by the equation

$$(\text{via } \theta_i)\circ(\text{via } \theta_j) = -(\text{via } \theta_j)\circ(\text{via } \theta_i) \qquad (i \neq j).$$

Morphisms of Δ-functors are defined in the obvious way, see (1.5).

For example, let $\mathbf{L} \subset \mathbf{K} := \mathbf{K}(\mathcal{A})$ be as above, with respective derived categories $\mathbf{D_L} \subset \mathbf{D}$, and consider the functor

$$\mathrm{Hom}^\bullet \colon \mathbf{K}^{\mathsf{op}} \times \mathbf{L} \to \mathbf{K}(\mathfrak{Ab}).$$

As in the exercise preceding (1.5.1), we can consider the opposite category $\mathbf{K}^{\mathsf{op}}$ to be triangulated, with translation inverse to that in $\mathbf{K}$, in such a way that the canonical contravariant functor $\mathbf{K} \to \mathbf{K}^{\mathsf{op}}$ and its inverse, together with $\theta =$ identity, are both Δ-functors. This being so, one checks then that $\mathrm{Hom}^\bullet$ is a Δ-functor (see (1.5.3)).

Similarly

$$\mathbf{R}\mathrm{Hom}^\bullet \colon \mathbf{D}^{\mathsf{op}} \times \mathbf{D_L} \to \mathbf{D}(\mathfrak{Ab})$$

is a Δ-functor. Furthermore, the q-injective resolution maps $\varphi_B \colon B \to I_B$ induce a natural morphism of Δ-functors

$$\eta \colon Q'\mathrm{Hom}^\bullet(A,\, B) \to Q'\mathrm{Hom}^\bullet(A,\, I_B) \overset{(2.4.1)}{=} \mathbf{R}\mathrm{Hom}^\bullet(QA,\, QB)$$

where $Q \colon \mathbf{K} \to \mathbf{D}$ is the canonical functor. This η is, in the following sense, *universal* (hence unique up to isomorphism):

Lemma (2.4.4). *Let*

$$G \colon \mathbf{D}^{\mathsf{op}} \times \mathbf{D_L} \to \mathbf{D}(\mathfrak{Ab})$$

be a Δ-functor, and let

$$\mu \colon Q'\mathrm{Hom}^\bullet(A,\, B) \to G(QA, QB) \qquad (A \in \mathbf{K}^{\mathsf{op}},\ B \in \mathbf{L})$$

be a morphism of Δ-functors. Then there exists a unique morphism of Δ-functors

$$\overline{\mu} \colon \mathbf{R}\mathrm{Hom}^\bullet \to G$$

such that $\mu = \overline{\mu}\eta$.

Proof. $\overline{\mu}$ is the composition

$$\mathbf{R}\mathrm{Hom}^\bullet(QA,\, QB) = Q'\mathrm{Hom}^\bullet(A,\, I_B) \xrightarrow{\mu} G(QA, QI_B) \xrightarrow{\ \sim\ } G(QA, QB)\,.$$

The rest is left to the reader. (See also (2.6.5) below.)

(2.4.5). Next we discuss the *sheafified version* of the above. Let U be a topological space, $\mathcal{O}$ a sheaf of commutative rings, and $\mathcal{A}$ the abelian category of (sheaves of) $\mathcal{O}$-modules. The "sheaf-hom" functor

$$\mathcal{H}\mathsf{om}\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{A}$$

extends naturally to a Δ-functor

$$\mathcal{H}\mathsf{om}^\bullet\colon \mathbf{K}(\mathcal{A})^{\mathsf{op}} \times \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$$

(essentially because everything in (1.5.3) is compatible with restriction to open subsets—details left to the reader).

Taking note of the following Lemma, we can proceed as above to derive a Δ-functor

$$\mathsf{R}\mathcal{H}\mathsf{om}^\bullet\colon \mathbf{D}(\mathcal{A})^{\mathsf{op}} \times \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})\,.$$

Lemma (2.4.5.1). *If I is a q-injective complex in $\mathbf{K}(\mathcal{A})$ then the functor $\mathcal{H}\mathsf{om}^\bullet(-, I)$ takes quasi-isomorphisms to quasi-isomorphisms.*

Proof. For $A \in \mathbf{K}(\mathcal{A})$ and $i \in \mathbb{Z}$, the homology $H^i(\mathcal{H}\mathsf{om}^\bullet(A, I))$ is the sheaf associated to the presheaf

$$V \mapsto H^i\big(\Gamma(V, \mathcal{H}\mathsf{om}^\bullet(A,\, I)\big) = H^i\big(\mathrm{Hom}^\bullet(A|V,\, I|V)\big) \qquad (V \text{ open in } U).$$

We can then apply (2.3.8)(iv)′ to the category $\mathcal{A}_V$ of $(\mathcal{O}|V)$-modules, as soon as we know:

Lemma (2.4.5.2). *Let V be an open subset of U, with inclusion map $i\colon V \hookrightarrow U$. Then for any q-injective complex $I \in \mathbf{K}(\mathcal{A})$, the restriction $i^*I = I|_V$ is q-injective in $\mathbf{K}(\mathcal{A}_V)$.*

Proof. The *extension by zero* of an $\mathcal{O}_V$-module M is the sheaf $i_!M$ associated to the presheaf on U which assigns $M(W)$ to any open $W \subset V$ and 0 to any open $W \not\subset V$. The restriction $i^*i_!M$ can be identified with M; and the stalk of $i_!M$ at any point $w \notin V$ is 0. So $i_!$ is an exact functor.

Now from any diagram $Y \xleftarrow{s} X \xrightarrow{f} i^*I$ of maps of $\mathcal{A}_V$-complexes with s a quasi-isomorphism, we get the diagram

$$i_!Y \xleftarrow{i_!s} i_!X \xrightarrow{i_!f} i_!i^*I \overset{\alpha}{\hookrightarrow} I$$

where $i_!s$ is a quasi-isomorphism (since $i_!$ is exact) and α is the natural map. By (2.3.1), there exists a map $g\colon i_!X \to I$ such that $g \circ i_!s = \alpha \circ i_!f$ in $\mathbf{K}(\mathcal{A})$; and then we have, in $\mathbf{K}(\mathcal{A}_V)$,

$$i^*g \circ s = i^*g \circ i^*i_!s = i^*\alpha \circ i^*i_!f = 1 \circ f = f\,.$$

Thus i^*I is indeed q-injective. Q.E.D.

(2.4.5.3). Similarly, any functor having an exact left adjoint preserves q-injectivity.

2.5. Derived tensor product

Let U be a topological space, $\mathcal{O}$ a sheaf of commutative rings, and $\mathcal{A}$ the abelian category of (sheaves of) $\mathcal{O}$-modules. Recall from (1.5.4) the definition of the tensor product (over $\mathcal{O}$) of two complexes in $\mathbf{K}(\mathcal{A})$, and its Δ-functorial properties. The standard theory of the derived tensor product, via resolutions by complexes of flat modules, applies to complexes in $\overline{\mathbf{D}}^{-}(\mathcal{A})$, see e.g., [**H**, p. 93]. Following Spaltenstein [**Sp**] we can use direct limits to extend the theory to *arbitrary* complexes in $\mathbf{D}(\mathcal{A})$. Before defining, in (2.5.7), the derived tensor product, we need to develop an appropriate acyclicity notion, "q-flatness."

Definition (2.5.1). A complex $P \in \mathbf{K}(\mathcal{A})$ is *q-flat* if for every quasi-isomorphism $Q_1 \to Q_2$ in $\mathbf{K}(\mathcal{A})$, the resulting map $P \otimes Q_1 \to P \otimes Q_2$ is also a quasi-isomorphism; or equivalently (see footnote under (1.5.1)), if for every exact complex $Q \in \mathbf{K}(\mathcal{A})$, the complex $P \otimes Q$ is also exact.

Example **(2.5.2).** $P \in \mathbf{K}(\mathcal{A})$ is q-flat iff for each point $x \in U$, the stalk P_x is q-flat in $\mathbf{K}(\mathcal{A}_x)$, where $\mathcal{A}_x$ is the category of modules over the ring $\mathcal{O}_x$. (In verifying this statement, note that an exact $\mathcal{O}_x$-complex Q_x is the stalk at x of the exact $\mathcal{O}$-complex Q which associates Q_x to those open subsets of U which contain x, and 0 to those which don't.)

For instance, a complex P which vanishes in all degrees but one (say n) is q-flat if and only if tensoring with the degree n component P^n is an exact functor in the category of $\mathcal{O}$-modules, i.e., P^n is a flat $\mathcal{O}$-module, i.e., for each $x \in U$, P^n_x is a flat $\mathcal{O}_x$-module.

Example **(2.5.3).** Tensoring with a fixed complex Q is a Δ-functor, and so the exact homology sequence $(1.4.5)^{\mathrm{H}}$ of a triangle yields that the q-flat complexes are the objects of a Δ-subcategory of $\mathbf{K}(\mathcal{A})$.

A bounded complex

$$P: \quad \cdots \to 0 \to 0 \to P^m \to \cdots \to P^n \to 0 \to 0 \to \dots$$

fits into a triangle $P' \to P \to P'' \to P'[1]$ where P' is P^n in degree n and 0 elsewhere, and where P'' is the cokernel of the obvious map $P' \to P$. So starting with (2.5.2) we see by induction on $n - m$ that any bounded complex of flat $\mathcal{O}$-modules is q-flat.

Example **(2.5.4).** Since (filtered) direct limits commute with both tensor product and homology, therefore any such limit of q-flat complexes is again q-flat.

A bounded-above complex

$$P: \quad \cdots \to P^m \to \cdots \to P^n \to 0 \to 0 \to \cdots$$

is the limit of the direct system $P_0 \to P_1 \to \cdots \to P_i \to \cdots$ where P_i is obtained from P by replacing all the components P^j with $j < n - i$ by 0, and the maps are the obvious ones. Hence, any bounded-above complex of flat $\mathcal{O}$-modules is q-flat.

A *q-flat resolution* of an $\mathcal{A}$-complex C is a quasi-isomorphism $P \to C$ with P q-flat. The totality of such resolutions (with variable P and C) is the class of objects of a category, whose morphisms are the obvious ones.

Proposition (2.5.5). *Every $\mathcal{A}$-complex C is the target of a quasi-isomorphism ψ_C from a q-flat complex P_C, which can be constructed to depend functorially on C, and so that $P_{C[1]} = P_C[1]$ and $\psi_{C[1]} = \psi_C[1]$.*

Proof. Every $\mathcal{O}$-module is a quotient of a flat one; in fact there exists a functor P_0 from $\mathcal{A}$ to its full subcategory of flat $\mathcal{O}$-modules, together with a functorial epimorphism $P_0(\mathcal{F}) \twoheadrightarrow \mathcal{F}$ $(\mathcal{F} \in \mathcal{A})$. Indeed, for any open $V \subset U$ let $\mathcal{O}_V$ be the extension of $\mathcal{O}|V$ by zero, (i.e., the sheaf associated to the presheaf taking an open W to $\mathcal{O}(W)$ if $W \subset V$ and to 0 otherwise), so that $\mathcal{O}_V$ is flat, its stalk at $x \in U$ being $\mathcal{O}_x$ if $x \in V$ and 0 otherwise. There is a canonical isomorphism

$$\psi\colon \mathcal{F}(V) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}_V, \mathcal{F}) \qquad (\mathcal{F} \in \mathcal{A})$$

such that $\psi(\lambda)$ takes $1 \in \mathcal{O}_V(V)$ to λ. With $\mathcal{O}_\lambda := \mathcal{O}_V$ for each $\lambda \in \mathcal{F}(V)$, the maps $\psi(\lambda)$ define an epimorphism, with flat source,

$$P_0(\mathcal{F}) := \Big(\bigoplus_{V \text{ open}} \ \bigoplus_{\lambda \in \mathcal{F}(V)} \mathcal{O}_\lambda \Big) \twoheadrightarrow \mathcal{F},$$

and this epimorphism depends functorially on $\mathcal{F}$.

We deduce then, for each $\mathcal{F}$, a functorial flat resolution

$$\cdots \to P_2(\mathcal{F}) \to P_1(\mathcal{F}) \to P_0(\mathcal{F}) \twoheadrightarrow \mathcal{F}$$

with $P_1(\mathcal{F}) := P_0\big(\ker(P_0(\mathcal{F}) \twoheadrightarrow \mathcal{F})\big)$, etc. Set $P_n(\mathcal{F}) = 0$ if $n < 0$. Then to a complex C we associate the *flat* complex $P = P_C$ such that $P^r := \oplus_{m-n=r} P_n(C^m)$ and the restriction of the differential $P^r \to P^{r+1}$ to $P_n(C^m)$ is $P_n(C^m \to C^{m+1}) \oplus (-1)^m\big(P_n(C^m) \to P_{n-1}(C^m)\big)$, together with the natural map of complexes $P \to C$ induced by the epimorphisms $P_0(C^m) \twoheadrightarrow C^m$ $(m \in \mathbb{Z})$. Elementary arguments, with or without spectral sequences, show that for the truncations $\tau_{\le m}C$ of §1.10, the maps $P_{\tau_{\le m}C} \to \tau_{\le m}C$ are *quasi-isomorphisms.* Since homology commutes with direct limits, the resulting map

$$\psi_C\colon P_C = \varinjlim_m P_{\tau_{\le m}C} \to \varinjlim_m \tau_{\le m}C = C,$$

(which depends functorially on C) is a quasi-isomorphism; and by (2.5.4), P_C is q-flat. That $P_{C[1]} = P_C[1]$ and $\psi_{C[1]} = \psi_C[1]$ is immediate. Q.E.D.

Exercises **(2.5.6).** (a) Let P and Q be complexes in $\mathcal{A}$, the category of $\mathcal{O}$-modules, and suppose that for all integers s, t, u, v the complex $\tau_{\le s}\tau_{\ge t}P \otimes_{\mathcal{O}} \tau_{\le u}\tau_{\ge v}Q$ is exact. Then

$$P \otimes Q = \varinjlim_{s,u} \tau_{\le s}P \otimes \tau_{\le u}Q$$

is exact.

(b) If for all $n \in \mathbb{Z}$ the homology $H^n(P)$ is a flat $\mathcal{O}$-module and furthermore, for all n the kernel of $P^n \to P^{n+1}$ is a direct summand of P^n (or, for all n the image of $P^n \to P^{n+1}$ is a direct summand of P^{n+1}), then P is q-flat. (Use (a) to reduce to where P is bounded; then apply induction to the number of n such that $P^n \neq 0$.)

(2.5.7). Let $\mathcal{A}$ be, as above, the category of $\mathcal{O}$-modules, and let

$$\mathbf{J}' \subset \mathbf{K} := \mathbf{K}(\mathcal{A})$$

be the Δ-subcategory of $\mathbf{K}$ whose objects are all the q-flat complexes, see (2.5.3). Fix $B \in \mathbf{K}$ and consider the Δ-functor

$$F_B \colon \mathbf{K} \to \mathbf{D} := \mathbf{D}(\mathcal{A})$$

such that

$$F_B(A) = A \otimes B \qquad \text{(see (1.5.4))}.$$

If A is both q-flat and exact, then $A \otimes B$ is exact: to see this, we may replace B by any quasi-isomorphic complex B' (since A is q-flat), and by (2.5.5) we may assume that B' is q-flat, whence, by (2.5.1), $A \otimes B'$ is exact. Hence the restriction of F_B to $\mathbf{J}'$ transforms quasi-isomorphisms into isomorphisms.

There exists, by (2.5.5), a functorial family of quasi-isomorphisms

$$\psi_A \colon P_A \to A \qquad (A \in \mathbf{K},\ P_A \in \mathbf{J}').$$

with $P_{A[1]} = P_A[1]$. An argument dual to that in (2.2.4) (with $\mathbf{J}'' := \mathbf{K}$) shows then that F_B has a left-derived Δ-functor

(2.5.7.1) $$(\mathsf{L}F_B, \text{identity}) \colon \mathbf{D} \to \mathbf{D}$$

with

$$\mathsf{L}F_B(A) = P_A \otimes B \cong P_A \otimes P_B \cong A \otimes P_B,$$

the isomorphisms being the ones induced by ψ_A and ψ_B. Alternatively, P_A is left-F_B-acyclic for all A, B (see 2.5.10(d)), so one can apply (2.2.6).

For fixed A and variable B, $P_A \otimes B$ is a Δ-functor from $\mathbf{K}$ to $\mathbf{D}$ which takes quasi-isomorphisms to isomorphisms, so by (1.5.1) there results a Δ-functor from $\mathbf{D}$ to $\mathbf{D}$. Hence there is a functor of two variables, called a *derived tensor product,*

$$\underline{\underline{\otimes}} \colon \mathbf{D} \times \mathbf{D} \longrightarrow \mathbf{D}$$

which together with appropriate θ (see (1.5.4)) is a Δ-functor in each variable separately (i.e., it is a Δ-functor as defined in (2.4.3)).

Though the variables A and B have been treated differently in the foregoing, their roles are essentially equivalent. Indeed, there is a universal property analogous to (the dual of) that in (2.4.4), characterizing the natural composite map of Δ-functors from $\mathbf{K} \times \mathbf{K}$ to $\mathbf{D}$:

$$QA \underline{\underline{\otimes}} QB \xrightarrow{\ \sim\ } Q(P_A \otimes P_B) \longrightarrow Q(A \otimes B)\,.$$

Hence, in view of (1.5.4.1), there is a canonical Δ-bifunctorial isomorphism

$$B \underline{\underline{\otimes}} A \xrightarrow{\ \sim\ } A \underline{\underline{\otimes}} B.$$

This arises, in fact, from the natural isomorphism $P_B \otimes P_A \xrightarrow{\ \sim\ } P_A \otimes P_B$.

(2.5.8). The *local hypertor* sheaves are defined by

$$\mathcal{T}\mathit{or}_n(A,B) = H^{-n}(A \underset{=}{\otimes} B) \qquad (n \in \mathbb{Z};\; A, B \in \mathbf{D}).$$

As in (2.1.4), short exact sequences in either the A or B variable give rise to long exact hypertor sequences.

We remark that when U is a *scheme* and $\mathcal{O} = \mathcal{O}_U$, if the homology sheaves of the complexes A and B are all *quasi-coherent* then so are the sheaves $\mathcal{T}\mathit{or}_n(A,B)$. This is clear, by reduction to the affine case, if A and B are quasi-coherent $\mathcal{O}_X$-modules (i.e., complexes vanishing except in degree 0). In the general case, since

$$A \otimes B = \varinjlim_{s,u} \tau_{\le s} A \otimes \tau_{\le u} B\,,$$

we may assume that A and B lie in $\mathbf{D}^-$, and then argue as in [**H**, p. 98, Prop. 4.3], or alternatively, use the Künneth spectral sequence

$$E_{pq}^2 = \underset{i+j=q}{\oplus} \mathcal{T}\mathit{or}_p(H^{-i}(A),\, H^{-j}(B)) \Rightarrow \mathcal{T}\mathit{or}_\bullet(A,B)$$

(as described e.g., in [**B**, p. 186, Exercise 9(b)], with *flat* resolutions replacing projective ones). Thus, with notation as in (1.9), denoting by $\mathbf{D}_{\mathsf{qc}}$ the Δ-subcategory $\mathbf{D}_{\#} \subset \mathbf{D}$ with $\mathcal{A}^{\#} \subset \mathcal{A}$ the subcategory of *quasi-coherent* $\mathcal{O}_U$-modules (which is plump, see [**GD**, p. 217, (2.2.2) (iii)]), we have a Δ-functor

$$\underset{=}{\otimes}\colon \mathbf{D}_{\mathsf{qc}} \times \mathbf{D}_{\mathsf{qc}} \longrightarrow \mathbf{D}_{\mathsf{qc}}\,. \tag{2.5.8.1}$$

(2.5.9). The definitions in (1.5.4) can be extended to three (or more) variables, to give a Δ-functor $A \otimes B \otimes C$ from $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ to $\mathbf{K}$.

There exists a Δ-functor $T_3\colon \mathbf{D} \times \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ together with a Δ-functorial map

$$\eta\colon T_3(A,B,C) \longrightarrow A \otimes B \otimes C \qquad (A, B, C \in \mathbf{K})$$

such that for any Δ-functor $H\colon \mathbf{D} \times \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ and any Δ-functorial map $\mu\colon H(A,B,C) \longrightarrow A \otimes B \otimes C$ there is a unique Δ-functor map $\bar\mu\colon H \to T_3$ such that $\mu = \eta \circ \bar\mu$. (The reader can fill in the missing Q's.) In fact there is such a T_3 with

$$T_3(A,B,C) = P_A \otimes P_B \otimes P_C\,.$$

We usually write

$$T_3(A,B,C) = A \underset{=}{\otimes} B \underset{=}{\otimes} C\,.$$

There are canonical Δ-functorial isomorphisms

$$(A \underset{=}{\otimes} B) \underset{=}{\otimes} C \xrightarrow{\ \sim\ } A \underset{=}{\otimes} B \underset{=}{\otimes} C \xleftarrow{\ \sim\ } A \underset{=}{\otimes} (B \underset{=}{\otimes} C)\,.$$

Similar considerations hold for $n > 3$ variables. Details are left to the reader. (See, for example, (2.6.5) below.)

Exercises **(2.5.10).** (a) Show that if $A \in \mathbf{K}(\mathcal{A})$ is q-flat and $B \in \mathbf{K}(\mathcal{A})$ is q-injective then $\mathcal{H}om^\bullet(A,B)$ is q-injective.

(b) Let $\Gamma\colon \mathcal{A} \to \mathfrak{Ab}$ be the global section functor. Show that there is a natural isomorphism of Δ-functors (of two variables, see (2.4.3))

$$\mathbf{R}\mathrm{Hom}^\bullet(A,B) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma\mathbf{R}\mathcal{H}om^\bullet(A,B).$$

(Use (a) and (2.2.7), or [**Sp**, 5.14, 5.12, 5.17].)

(c) Let (A_α) be a (small, directed) inductive system of $\mathcal{A}$-complexes. Show that for any complex $B \in \mathbf{D}(\mathcal{A})$ there are natural isomorphisms

$$\varinjlim_{\alpha} \mathcal{T}or_n(A_\alpha,\, B) \xrightarrow{\ \sim\ } \mathcal{T}or_n((\varinjlim_{\alpha} A_\alpha),\, B) \qquad (n \in \mathbb{Z}).$$

(d) Show that for P to be q-flat it is necessary that P be left-F_B-acyclic for all B (F_B as in (2.5.7)), and sufficient that P be left-F_B-acyclic for all *exact* B. (For the last part, (2.2.6) could prove helpful.) Formulate and prove an analogous statement involving q-injectivity and $\mathrm{Hom}^\bullet$. (See (2.3.8).)

2.6. Adjoint associativity

Again let U be a topological space, $\mathcal{O}$ a sheaf of commutative rings, and $\mathcal{A}$ the abelian category of $\mathcal{O}$-modules. Set $\mathbf{K} := \mathbf{K}(\mathcal{A})$, $\mathbf{D} := \mathbf{D}(\mathcal{A})$. This section is devoted to (2.6.1)—or better, (2.6.1)* at the end—which expresses the basic adjointness relation between the Δ-functors $\mathbf{R}\mathcal{H}om^\bullet\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{D}$ and $\underline{\underline{\otimes}}\colon \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ defined in (2.4.5) and (2.5.7) respectively.

Proposition (2.6.1). *There is a natural isomorphism of* Δ*-functors* (*see* (2.4.3)):

$$\mathbf{R}\mathcal{H}om^\bullet(A \underline{\underline{\otimes}} B,\, C) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))\,.$$

Remarks. (i) Strictly speaking, the Δ-functors $\mathbf{R}\mathcal{H}om^\bullet$ and $\underline{\underline{\otimes}}$ are defined only up to canonical isomorphism by universal properties, for example, (2.5.9). We leave it to the reader to verify that the map in (2.6.1) (to be constructed below) is compatible, in the obvious sense, with such canonical isomorphisms.

(ii) A proof similar to the following one[20] yields a natural isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet(A \underline{\underline{\otimes}} B,\, C) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))\,.$$

Applying homology H^0 we have, by (2.4.2), the *adjunction isomorphism*

$$(2.6.1)' \qquad \mathrm{Hom}_{\mathbf{D}}(A \underline{\underline{\otimes}} B,\, C) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}}(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))\,.$$

(iii) Prop. (2.6.1) gives a derived-category upgrade of the standard sheaf isomorphism

$$(2.6.2) \qquad \mathcal{H}om(F \otimes G,\, H) \xrightarrow{\ \sim\ } \mathcal{H}om(F,\, \mathcal{H}om(G,\, H)) \qquad (F, G, H \in \mathcal{A}).$$

[20] or application of the functor $\mathbf{R}\Gamma$ to (2.6.1), see (2.5.10),

Proof of (2.6.1). We discuss the proof at several levels of pedantry, beginning with the argument, in full, given in [**I**, p. 151, Lemme 7.4] (see also [**Sp**, p. 147, Prop. 6.6]): "Resolve C injectively and B flatly."

This argument can be expanded as follows. Choose quasi-isomorphisms

$$C \to I_C, \qquad P_B \to B$$

where I_C is q-injective and P_B is q-flat. It follows from (2.3.8)(iv) that the complex of sheaves $\mathcal{H}om^\bullet(P_B, I_C)$ is *q-injective,* since for any exact complex $X \in \mathbf{K}$, the isomorphism of complexes

$$\mathrm{Hom}^\bullet(X \otimes P_B, I_C) \xrightarrow{\sim} \mathrm{Hom}^\bullet(X, \mathcal{H}om^\bullet(P_B, I_C))$$

coming out of (2.6.2) yields, upon application of homology H^0,

$$0 = \mathrm{Hom}_{\mathbf{K}}(X \otimes P_B, I_C) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}}(X, \mathcal{H}om^\bullet(P_B, I_C)).$$

Now consider the natural sequence of $\mathbf{D}$-maps

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om^\bullet(A \underline{\underline{\otimes}} B, C) \arrow[d] & \mathbf{R}\mathcal{H}om^\bullet(A, \mathbf{R}\mathcal{H}om^\bullet(B, C)) \arrow[d] \\
\mathbf{R}\mathcal{H}om^\bullet(A \underline{\underline{\otimes}} B, I_C) \arrow[d] & \mathbf{R}\mathcal{H}om^\bullet(A, \mathbf{R}\mathcal{H}om^\bullet(B, I_C)) \arrow[d] \\
\mathbf{R}\mathcal{H}om^\bullet(A \underline{\underline{\otimes}} P_B, I_C) & \mathbf{R}\mathcal{H}om^\bullet(A, \mathbf{R}\mathcal{H}om^\bullet(P_B, I_C)) \\
\mathbf{R}\mathcal{H}om^\bullet(A \otimes P_B, I_C) \arrow[u] & \mathbf{R}\mathcal{H}om^\bullet(A, \mathcal{H}om^\bullet(P_B, I_C)) \arrow[u] \\
\mathcal{H}om^\bullet(A \otimes P_B, I_C) \arrow[u] \arrow[r, "\text{from (2.6.2)}"'] & \mathcal{H}om^\bullet(A, \mathcal{H}om^\bullet(P_B, I_C)) \arrow[u]
\end{tikzcd}$$

Since P_B is q-flat, and I_C and $\mathcal{H}om^\bullet(P_B, I_C)$ are q-injective, all these maps are isomorphisms (as follows, e.g., from the last assertion of (2.2.6)); so we can compose to get the isomorphism (2.6.1).

But we really should check that this isomorphism does not depend on the chosen quasi-isomorphisms, and that it is in fact Δ-functorial. This can be quite tedious. The following remarks outline a method for managing such verifications. The basic point is (2.6.4) below.

Let M be a set. An *M-category* is an additive category $\mathbf{C}$ plus a map $t\colon M \to \mathrm{Hom}(\mathbf{C},\mathbf{C})$ from M into the set of additive functors from $\mathbf{C}$ to $\mathbf{C}$, such that with $T_m := t(m)$ it holds that $T_i \circ T_j = T_j \circ T_i$ for all $i, j \in M$. Such an M-category will be denoted $\mathbf{C}_M$, the map f—or equivalently, the commuting family $(T_m)_{m\in M}$—understood to have been specified; and when the context renders it superfluous, the subscript "M" may be omitted.

An *M-functor* $F\colon \mathbf{C}_M \to \mathbf{C}'_M$ is an additive functor $F\colon \mathbf{C} \to \mathbf{C}'$ together with isomorphisms of functors

$$\theta_i\colon F\circ T_i \xrightarrow{\sim} T'_i \circ F \qquad (i\in M)$$

(with $(T'_m)_{m\in M}$ the commuting family of functors defining the M-structure on $\mathbf{C}'$) such that for all $i \neq j$, the following diagram commutes:

$$\begin{tikzcd}
F\circ T_i\circ T_j \arrow[r, "\text{via } \theta_i"] \arrow[d, equal] & T'_i\circ F\circ T_j \arrow[r, "T'_i(\theta_j)"] & T'_i\circ T'_j\circ F \arrow[d, equal] \\
F\circ T_j\circ T_i \arrow[r, "\text{via } \theta_j"'] & T'_j\circ F\circ T_i \arrow[r, "-T'_j(\theta_i)"'] & T'_j\circ T'_i\circ F
\end{tikzcd}$$

where, for instance, $T'_j(\theta_i)$ is the isomorphism of functors such that for each object $X\in\mathbf{C}$, $[T'_j(\theta_i)](X)$ is the $\mathbf{C}'$-isomorphism

$$T'_j\bigl(\theta_i(X)\bigr)\colon T'_j\bigl(FT_i(X)\bigr) \xrightarrow{\sim} T'_j\bigl(T'_iF(X)\bigr).$$

A *morphism* $\eta\colon (F,\{\theta_i\}) \to (G,\{\psi_i\})$ of M-functors is a morphism of functors $\eta\colon F\to G$ such that for every $i\in M$ and every object X in $\mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd}
FT_i(X) \arrow[r, "\theta_i(X)"] \arrow[d, "\eta(T_i(X))"'] & T'_iF(X) \arrow[d, "T'_i(\eta(X))"] \\
GT_i(X) \arrow[r, "\psi_i(X)"'] & T'_iG(X)
\end{tikzcd}$$

Composition of such η being defined in the obvious way, the M-functors from $\mathbf{C}$ to $\mathbf{C}'$, and their morphisms, form a category $\mathbf{H} := \mathbf{Hom}_M(\mathbf{C},\,\mathbf{C}')$. If $M'\supset M$ and $\mathbf{C}'_{M'}$ is viewed as an M-category via "restriction of scalars" then $\mathbf{H}$ is itself an M'-category, with $j\in M'$ being sent to the functor $T_j^{\#}\colon \mathbf{H}\to\mathbf{H}$ such that on objects of $\mathbf{H}$,

$$T_j^{\#}\bigl(F,\,\{\theta_i\}\bigr) = \bigl(T'_j\circ F,\,\{-T'_j(\theta_i)\}\bigr)\,,$$

where the isomorphism of functors

$$T'_j(\theta_i)\colon (T'_j\circ F)\circ T_i \xrightarrow{\sim} T'_j\circ T'_i\circ F = T'_i\circ(T'_j\circ F)$$

is as above.[21] The definition of $T_j^{\#}\eta$ (η as above), and the verification that $\mathbf{H}$ is thus an M'-category, are straightforward.

[21] The reason for the minus sign in the definition of $T_j^{\#}$ is hidden in the details of the proof of Lemma (2.6.3) below.

Suppose given such categories $\mathbf{A}_M$, $\mathbf{B}_N$, and $\mathbf{C}_{M\cup N}$, where the sets M and N are disjoint. $\mathbf{A}\times\mathbf{B}$ is considered to be an $(M\cup N)$-category, with $i\in M$ going to the functor $T_i\times 1$ and $j\in N$ to the functor $1\times T_j$. Also, $\mathbf{Hom}_N(\mathbf{B},\,\mathbf{C})$ is considered, as above, to be an $(M\cup N)$-category

Lemma (2.6.3). *With preceding notation, there is a natural isomorphism of $M\cup N$-categories*

$$\mathbf{Hom}_{M\cup N}\big(\mathbf{A}\times\mathbf{B},\,\mathbf{C}\big)\ \xrightarrow{\ \sim\ }\ \mathbf{Hom}_M\big(\mathbf{A},\mathbf{Hom}_N(\mathbf{B},\mathbf{C})\big)$$

The *proof,* left to the reader, requires very little imagination, but a good deal of patience.

For any positive integer n, let $\triangle_n$ be the set $\{1,2,\dots,n\}$. From now on, we deal with Δ-categories, always considered to be $\triangle_1$-categories via their translation functors. If $\mathbf{C_1},\dots,\mathbf{C_n}$ are Δ-categories, then the product category $\mathbf{C}=\mathbf{C_1}\times\mathbf{C_2}\times\cdots\times\mathbf{C_n}$ becomes a $\triangle_n$-category by the product construction used in (2.6.3). A Δ-category $\mathbf{E}$ can also be made into an $\triangle_n$-category by sending each $i\in\triangle_n$ to the translation functor of $\mathbf{E}$. With these understandings, we see that the $\triangle_n$-functors from $\mathbf{C_1}\times\mathbf{C_2}\times\cdots\times\mathbf{C_n}$ to $\mathbf{E}$ are just the Δ-functors of (2.4.3) (categories of which we denote by $\mathbf{Hom}_\triangle$). For example, one checks that the source and target of the isomorphism in (2.6.1) are both $\triangle_3$-functors.

Now for $1\le i\le n$ fix abelian categories $\mathcal{A}_i$, and let $\mathbf{L_i}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A}_i)$, with corresponding derived category $\mathbf{D_i}$ and canonical functor $Q_i\colon\mathbf{L_i}\to\mathbf{D_i}$. Let $\mathbf{E}$ be any Δ-category. We can generalize (1.5.1) as follows:

Proposition (2.6.4). *The canonical functor*

$$\mathbf{L_1}\times\cdots\times\mathbf{L_n}\ \xrightarrow[Q_1\times\cdots\times Q_n]{}\ \mathbf{D_1}\times\cdots\times\mathbf{D_n}$$

induces an isomorphism from the category $\mathbf{Hom}_\triangle(\mathbf{D_1}\times\mathbf{D_2}\times\cdots\times\mathbf{D_n},\,\mathbf{E})$ onto the full subcategory of $\mathbf{Hom}_\triangle(\mathbf{L_1}\times\mathbf{L_2}\times\cdots\times\mathbf{L_n},\,\mathbf{E})$ whose objects are the Δ-functors F such that for any quasi-isomorphisms $\alpha_1,\dots,\alpha_n$ in $\mathbf{L_1},\dots,\mathbf{L_n}$ respectively, $F(\alpha_1,\dots,\alpha_n)$ is an isomorphism in $\mathbf{E}$.

Proof. The case $n=1$ is contained in (1.5.1). We can then proceed by induction on n, using the natural isomorphism

$$\begin{aligned}&\mathbf{Hom}_{\triangle_\mathbf{n}}\big(\mathbf{C_1}\times\mathbf{C_2}\times\cdots\times\mathbf{C_n},\,\mathbf{E}\big)\\ &\qquad\qquad\xrightarrow{\ \sim\ }\ \mathbf{Hom}_{\triangle_\mathbf{1}}\big(\mathbf{C_1},\,\mathbf{Hom}_{\triangle_\mathbf{n-1}}(\mathbf{C_2}\times\cdots\times\mathbf{C_n},\,\mathbf{E})\big)\end{aligned}$$

provided by (2.6.3) (with $\mathbf{C_i}:=\mathbf{D_i}$ or $\mathbf{L_i}$). Q.E.D.

Suppose next that we have pairs of Δ-subcategories $\mathbf{L'_i}\subset\mathbf{L''_i}$ in $\mathbf{K}(\mathcal{A}_i)$, with respective derived categories $\mathbf{D'_i}$, $\mathbf{D''_i}$, and canonical functors $Q'_i\colon\mathbf{L'_i}\to\mathbf{D'_i}$, $Q''_i\colon\mathbf{L''_i}\to\mathbf{D''_i}$ $(1\le i\le n)$. Suppose further that every complex $A \in \mathbf{L_i''}$ admits a quasi-isomorphism into a complex $I_A \in \mathbf{L_i'}$. Then as in (1.7.2) the natural Δ-functors $\tilde{\jmath}_i\colon \mathbf{D_i'} \to \mathbf{D_i''}$ are Δ-equivalences, having quasi-inverses ρ_i satisfying $\rho_i(A) = I_A$ $(A \in \mathbf{L_i''})$. There result functors

$$\begin{aligned}
\tilde{\jmath}^* &: \mathbf{Hom}_\Delta(\mathbf{D_1''} \times \cdots \times \mathbf{D_n''}, \mathbf{E}) \longrightarrow \mathbf{Hom}_\Delta(\mathbf{D_1'} \times \cdots \times \mathbf{D_n'}, \mathbf{E})\\
\rho^* &: \mathbf{Hom}_\Delta(\mathbf{D_1'} \times \cdots \times \mathbf{D_n'}, \mathbf{E}) \longrightarrow \mathbf{Hom}_\Delta(\mathbf{D_1''} \times \cdots \times \mathbf{D_n''}, \mathbf{E})
\end{aligned}$$

together with functorial isomorphisms

$$\tilde{\jmath}^*\rho^* \xrightarrow{\sim} \text{identity}, \qquad \rho^*\tilde{\jmath}^* \xrightarrow{\sim} \text{identity},$$

i.e., $\tilde{\jmath}^*$ and ρ^* are quasi-inverse equivalences of categories.

We deduce the following variation on the theme of (2.2.3), thereby arriving at a general method for specifying maps between Δ-functors on products of derived categories:[22]

Corollary (2.6.5). *With above notation let* $H\colon \mathbf{L_1'} \times \cdots \times \mathbf{L_n'} \to \mathbf{E}$, $F\colon \mathbf{D_1''} \times \cdots \times \mathbf{D_n''} \to \mathbf{E}$, *and* $G\colon \mathbf{D_1''} \times \cdots \times \mathbf{D_n''} \to \mathbf{E}$ *be Δ-functors. Let*

$$\begin{aligned}
\zeta\colon H &\xrightarrow{\sim} F\circ(\tilde{\jmath}_1Q_1' \times \cdots \times \tilde{\jmath}_nQ_n'),\\
\beta\colon H &\longrightarrow G\circ(\tilde{\jmath}_1Q_1' \times \cdots \times \tilde{\jmath}_nQ_n')
\end{aligned}$$

be Δ-functorial maps, with ζ an isomorphism. Then:

(i) *There exists a unique Δ-functorial map $\bar{\beta}\colon F \to G$ such that for all* $A_1 \in \mathbf{L_1'}, \ldots, A_n \in \mathbf{L_n'}$, $\beta(A_1,\ldots,A_n)$ *factors as*

(2.6.5.1) $$H(A_1,\ldots,A_n) \xrightarrow{\zeta} F(A_1,\ldots,A_n) \xrightarrow{\bar{\beta}} G(A_1,\ldots,A_n).$$

Moreover, if β is an isomorphism then so is $\bar{\beta}$.

(ii) *If H in* (i) *extends to a Δ-functor* $H\colon \mathbf{L_1''} \times \cdots \times \mathbf{L_n''} \to \mathbf{E}$, *and ζ (resp. β) to a Δ-functorial map* $\zeta\colon H \to F\circ(\tilde{\jmath}_1Q_1'' \times \cdots \times \tilde{\jmath}_nQ_n'')$ (*resp.* $\beta\colon H \to G\circ(\tilde{\jmath}_1Q_1'' \times \cdots \times \tilde{\jmath}_nQ_n'')$), *then the factorization* (2.6.5.1) *of* $\beta(A_1,\ldots,A_n)$ *holds for all* $A_1 \in \mathbf{L_1''}, \ldots, A_n \in \mathbf{L_n''}$.

Proof. (i) The assertion just means that $\bar{\beta}$ is the unique map (resp. isomorphism) $F \to G$ in the category $\mathbf{Hom}_\Delta(\mathbf{D_1''} \times \cdots \times \mathbf{D_n''}, \mathbf{E})$ corresponding via the above equivalence $\tilde{\jmath}^*$ and (2.6.4) to the map (resp. isomorphism) $\beta\zeta^{-1}$ in the category $\mathbf{Hom}_\Delta(\mathbf{L_1'} \times \cdots \times \mathbf{L_n'}, \mathbf{E})$.

(ii) Use quasi-isomorphisms $A_i \to I_{A_i}$ to map (2.6.5.1) into the corresponding diagram with $I_{A_i} \in \mathbf{L_i'}$ in place of A_i. To this latter diagram (i) applies; and as the resulting map $G(A_1,\ldots,A_n) \to G(I_{A_1},\ldots,I_{A_n})$ is an isomorphism, the rest is clear. Q.E.D.

[22] This is no more (or less) than a careful formulation of the method used, e.g., throughout [**H**, Chapter II].

We can now derive (2.6.1) as follows. Take $n = 3$, and set

$$\mathbf{L_1'} := \mathbf{K}$$
$$\mathbf{L_2'} := \begin{cases} \Delta\text{-subcategory of } \mathbf{K} \text{ whose objects are} \\ \text{the q-flat complexes (2.5.3).} \end{cases}$$
$$\mathbf{L_3'} := \begin{cases} \Delta\text{-subcategory of } \mathbf{K} \text{ whose objects are} \\ \text{the q-injective complexes (2.3.2.2).} \end{cases}$$

Let $\mathbf{D_1'}$, $\mathbf{D_2'}$, $\mathbf{D_3'}$ be the corresponding derived categories, and set

$$\mathbf{L_i''} := \mathbf{K}, \quad \mathbf{D_i''} := \mathbf{D} \qquad (i = 1, 2, 3),$$

so that the natural maps $j_i \colon \mathbf{D_i}' \to \mathbf{D_i''}$ are Δ-equivalences, with quasi-inverses obtained for $i = 2$ and $i = 3$ from q-flat (resp. q-injective) resolutions, i.e., from families of quasi-isomorphisms

$$P_B \to B \qquad (B \in \mathbf{K},\ P_B \in \mathbf{L_2'}),$$
$$C \to I_C \qquad (C \in \mathbf{K},\ I_C \in \mathbf{L_3'}).$$

In Corollary (2.6.5)(ii), let $H \colon \mathbf{L_1''} \times \mathbf{L_2''} \times \mathbf{L_3''} \to \mathbf{D}$ be the Δ-functor

$$H(A, B, C) := \mathcal{H}om^\bullet(A \otimes B,\, C),$$

let ζ be the natural composed Δ-functorial map

$$\mathcal{H}om^\bullet(A \otimes B,\, C) \to \mathbf{R}\mathcal{H}om^\bullet(A \otimes B, C) \to \mathbf{R}\mathcal{H}om^\bullet(A \underset{=}{\otimes} B, C),$$

and let β be the natural composed Δ-functorial map

$$\begin{aligned} \mathcal{H}om^\bullet(A \otimes B,\, C) \underset{(2.6.2)}{\xrightarrow{\ \sim\ }} & \mathcal{H}om^\bullet(A,\, \mathcal{H}om^\bullet(B,\, C)) \\ \longrightarrow\ & \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathcal{H}om^\bullet(B,\, C)) \\ \longrightarrow\ & \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C)). \end{aligned}$$

(Meticulous readers may wish to insert the missing Q's).

We saw earlier, near the beginning of the proof of (2.6.1), that for $(B, C) \in \mathbf{L_2'} \times \mathbf{L_3'}$, the complex $\mathcal{H}om^\bullet(B,\, C)$ is q-injective, and hence for such (B, C), *ζ and β are isomorphisms.* Modifying (2.6.5) in the obvious way to take contravariance into account, we deduce the following elaboration of (2.6.1):

Proposition (2.6.1)*. *There is a unique* Δ*-functorial isomorphism*

$$\alpha\colon \mathbf{R}\mathcal{H}om^\bullet(A \underset{=}{\otimes} B,\, C) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))$$

such that for all $A, B, C \in \mathbf{D}$, *the following natural diagram* (*in which* $\mathcal{H}^\bullet$ *stands for* $\mathcal{H}om^\bullet$) *commutes*:

$$\begin{array}{ccccc}
\mathcal{H}^\bullet(A\otimes B, C) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A\otimes B, C) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A \underset{=}{\otimes} B, C) \\
{\scriptstyle\text{via}}\big\downarrow{\scriptstyle(2.6.2)} & & & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\alpha} \\
\mathcal{H}^\bullet(A,\, \mathcal{H}^\bullet(B,C)) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A,\, \mathcal{H}^\bullet(B,C)) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A,\, \mathbf{R}\mathcal{H}^\bullet(B,C))
\end{array}$$

This Δ-functorial isomorphism is the same as the one described—non-canonically, via P_B and I_C—near the beginning of this section. See also exercise (3.5.3)(e) below.

From (2.5.7.1) and (3.3.8) below (dualized), we deduce:

Corollary (2.6.7). *For fixed* A *the* Δ*-functor* $F_A(-) := \mathrm{Hom}^\bullet(A, -)$ *of §2.4 has a right-derived* Δ*-functor of the form* ($\mathbf{R}F_A$, identity).

Exercise **(2.6.7)** (see [**De**, §1.2]). Define derived functors of several variables, and generalize the relevant results from §§2.2–2.3.

2.7. Acyclic objects; finite-dimensional derived functors

This section contains additional results about acyclicity, used to get some more ways to construct derived functors, further illustrating (2.2.6). It can be skipped on first reading.

Let $\mathcal{A}$, $\mathcal{A}'$ be abelian categories, and let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be an additive functor. We also denote by ϕ the composed Δ-functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(\phi)} \mathbf{K}(\mathcal{A}') \xrightarrow{\ Q\ } \mathbf{D}(\mathcal{A}')$$

where $\mathbf{K}(\phi)$ is the natural extension of the original ϕ to a Δ-functor. We say then that an object in $\mathcal{A}$ is right-(or left-)ϕ-acyclic if it is so when viewed as a complex vanishing outside degree zero (see (2.2.5) with $\mathbf{J} := \mathbf{K}(\mathcal{A})$). In this section we deal mainly with the "left" context, and so we abbreviate "left-ϕ-acyclic" to "ϕ-acyclic." (The corresponding—dual—results in the "right" context are left to the reader. They are perhaps marginally less important because of the abundance of injectives in situations that we will deal with.)

If $X \in \mathcal{A}$ and $Z \to X$ is a quasi-isomorphism in $\mathbf{K}(\mathcal{A})$, then the natural map $\tau_{\le 0}Z \to Z$ of §1.10 is a quasi-isomorphism. If furthermore the induced map $\phi(Z) \to \phi(X)$ is a quasi-isomorphism and the functor ϕ is either right exact or left exact, then, one checks, the natural composition $\phi(\tau_{\le 0}Z) \to \phi(Z) \to \phi(X)$ is also a quasi-isomorphism. One deduces the following characterization of ϕ-acyclicity:

Lemma (2.7.1). *If $X \in \mathcal{A}$ is such that every exact sequence*

$$\cdots \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow X \longrightarrow 0$$

embeds into a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
\cdots \arrow[r] & Z_2 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Z_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & Y_2 \arrow[r] & Y_1 \arrow[r] & Y_0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

with the top row and its image under ϕ both exact, then X is ϕ-acyclic; and the converse holds whenever ϕ is either right exact or left exact.

Proposition (2.7.2). *With preceding notation, let $\mathbf{P}$ be a class of objects in $\mathcal{A}$ such that*

(i) *every object in $\mathcal{A}$ is a quotient of (i.e., target of an epimorphism from) one in $\mathbf{P}$;*
(ii) *if A and B are in $\mathbf{P}$ then so is $A \oplus B$; and*
(iii) *for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, if B and C are in $\mathbf{P}$, then $A \in \mathbf{P}$ and the corresponding sequence $0 \to \phi A \to \phi B \to \phi C \to 0$ in $\mathcal{A}'$ is also exact.*

Then every bounded-above $\mathbf{P}$-complex (i.e., complex with all components in $\mathbf{P}$)—in particular every object in $\mathbf{P}$—is ϕ-acyclic; the restriction ϕ_- of ϕ to $\overline{\mathbf{K}}^-(\mathcal{A})$ has a left-derived functor $\mathbf{L}\phi_- \colon \overline{\mathbf{D}}^-(\mathcal{A}) \to \mathbf{D}(\mathcal{A}')$; and if $\phi \ncong 0$ then $\dim^+\mathbf{L}\phi_- = 0$ (see (1.11.1)).

Proof. Since $\mathbf{P}$ is nonempty—by (i)—therefore (iii) with $B = C \in \mathbf{P}$ shows that $0 \in \mathbf{P}$. Then (ii) implies that the $\mathbf{P}$-complexes in $\mathbf{K}^-(\mathcal{A})$ are the objects of a Δ-subcategory, see (1.6). Starting from (i), an inductive argument ([**H**, p. 42, 4.6, 1)], dualized—and with assistance, if desired, from [**Iv**, p. 34, Prop. 5.2]) shows that every complex in $\mathbf{K}^-(\mathcal{A})$—and so, via $(1.8.1)^-$, in $\overline{\mathbf{K}}^-(\mathcal{A})$—is the target of a quasi-isomorphism from a bounded-above $\mathbf{P}$-complex. Hence, for the first assertion it suffices to show that ϕ transforms quasi-isomorphisms between bounded-above $\mathbf{P}$-complexes into isomorphisms, i.e., that *for any bounded-above exact $\mathbf{P}$-complex $X^\bullet$, $\phi(X^\bullet) \cong 0$* (see (1.5.1)).

Using (iii), we find by descending induction (starting with i_0 such that $X^j = 0$ for all $j > i_0$) that for every i, the kernel K^i of $X^i \to X^{i+1}$ lies in $\mathbf{P}$ and the obvious sequence

$$0 \to \phi(K^i) \to \phi(X^i) \to \phi(K^{i+1}) \to 0$$

is exact. Consequently, the complex obtained by applying ϕ to $X^\bullet$ is exact, i.e., $\phi(X^\bullet) \cong 0$ in $\mathbf{D}(\mathcal{A}')$.

Now by (2.2.4) (dualized) we see that $\mathbf{L}\phi_-$ exists and $\dim^+\mathbf{L}\phi_- \le 0$, with equality if $\phi(A) \ncong 0$ for some $A \in \mathcal{A}$, because there is a natural epimorphism $H^0\mathbf{L}\phi_- A \twoheadrightarrow \phi(A)$. Q.E.D.

Exercise **(2.7.2.1).** Let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be as above. Let $(\Lambda_i)_{0\le i<\infty}$ be a "homological functor" [**Gr**, p. 140], with $\Lambda_0 = \phi$. Let **P** consist of all objects B in $\mathcal{A}$ such that $\Lambda_i(B) = 0$ for all $i > 0$, and suppose that every object $A \in \mathcal{A}$ is a quotient of one in **P**. Then $\mathbf{L}\phi_-$ exists, and the homological functors (Λ_i) and $(\Lambda'_i) := (H^{-i}\mathbf{L}\phi_-)$ are coeffaceable, hence universal [**Gr**, p. 141, Prop. 2.2.1], hence isomorphic to each other.

Examples **(2.7.3).** A *ringed space* is a pair $(X, \mathcal{O}_X)$ with X a topological space and $\mathcal{O}_X$ a sheaf of commutative rings on X; and a *morphism* of ringed spaces $(f,\theta)\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ together with a map $\theta\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings. Any such (f,θ) gives rise to a (left-exact) *direct image* functor

$$f_*\colon \{\mathcal{O}_X\text{-modules}\} \to \{\mathcal{O}_Y\text{-modules}\}$$

such that $[f_*M](U) = M(f^{-1}U)$ for any $\mathcal{O}_X$-module M and any open set $U \subset Y$, the $\mathcal{O}_Y$-module structure on f_*M arising via θ; and also to a (right-exact) *inverse image* functor

$$f^*\colon \{\mathcal{O}_Y\text{-modules}\} \to \{\mathcal{O}_X\text{-modules}\}$$

defined up to isomorphism as being left-adjoint to f_* [**GD**, Chap. 0, §4]. For every $\mathcal{O}_Y$-module N, the stalk $(f^*N)_x$ at $x \in X$ is $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} N_{f(x)}$.

An $\mathcal{O}_Y$-module F is *flat* if the stalk F_y is a flat $\mathcal{O}_{Y,y}$-module for all $y \in Y$. *The class* **P** *of flat* $\mathcal{O}_Y$*-modules satisfies the hypotheses of* (2.7.2) *when* $\phi = f^*$: (i) is given by [**H**, p. 86, Prop. 1.2], (ii) is easy, and for (iii) see [**B′**, Chap. 1, §2, no. 5]. Thus the restriction f^*_- of f^* to $\overline{\mathbf{K}}^-(Y)$ has a left-derived functor

$$\mathbf{L}f^*_-\colon \overline{\mathbf{D}}^-(Y) \to \mathbf{D}(X)$$

($\mathbf{D}(X)$ being the derived category of the category of $\mathcal{O}_X$-modules, etc.), defined via resolutions (on the left) by complexes of flat $\mathcal{O}_Y$-modules.

Using the family of quasi-isomorphisms $\psi_A\colon P_A \to A$ $(A \in \mathbf{D}(Y))$ with P_A q-flat (see (2.5.5)), we can, in view of (2.5.2) and (2.5.3), show as in (2.5.7) that $\mathbf{L}f^*_-$ extends to a derived Δ-functor

$$(2.7.3.1)\qquad (\mathbf{L}f^*, \text{identity})\colon \mathbf{D}(Y) \to \mathbf{D}(X)$$

satisfying $\mathbf{L}f^*(A) = f^*(P_A)$.

For any $\mathcal{O}_Y$-module N, the stalk of the homology

$$L_i f^*(N) := H^{-i}\mathbf{L}f^*(N) \qquad (i \ge 0)$$

at any $x \in X$ is $\mathrm{Tor}_i^{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x}, N_{f(x)})$. So by the last assertion in (2.2.6) (dualized), or in (2.7.4), N is f^*-acyclic iff $\mathrm{Tor}_i^{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x}, N_{f(x)}) = 0$ for all $x \in X$ and $i > 0$. (Note here that since f^* is right exact, the natural map is an isomorphism $L_0 f^*(N) \xrightarrow{\sim} f^*(N)$.) Thus—or by (2.7.2)—any flat $\mathcal{O}_Y$-module is f^*-acyclic.

Recall that an $\mathcal{O}_X$-module M is *flasque* (or *flabby*) if the restriction map $M(X) \to M(U)$ is surjective for every open subset U of X. For example, injective $\mathcal{O}_X$-modules are flasque [**G**, p. 264, 7.3.2] (with $\mathcal{L} = \mathcal{O}_X$). *The class of flasque $\mathcal{O}_X$-modules satisfies the hypotheses of* (2.7.2) (*dual version*) *when $\phi = f_*$:* for (i) see [**G**, p. 147], (ii) is easy, and (iii) follows from the fact that if

$$0 \to F \to G \to H \to 0$$

is an exact sequence of $\mathcal{O}_X$-modules, with F flasque, then for all open sets $V \subset X$ the sequence

$$0 \to F(V) \to G(V) \to H(V) \to 0$$

is still exact [**G**, p. 148, Thm. 3.1.2]. So the restriction f_*^+ of f^* to $\overline{\mathbf{K}}^+(X)$ has a right-derived functor

$$\mathbf{R}f_*^+ : \overline{\mathbf{D}}^+(X) \to \mathbf{D}(Y)$$

defined via resolutions (on the right) by complexes of flasque $\mathcal{O}_X$-modules.

Of course we already know from (2.3.4), via (somewhat less elementary) *injective* resolutions, that $\mathbf{R}f_*^+$ exists, and by (2.3.5) it extends to a derived functor $\mathbf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$. (See also (2.3.7).) In fact, in view of (2.7.3.1), it follows from (3.2.1) and (3.3.8) (dualized) that:

(2.7.3.2). *The Δ-functor* (f_*, identity) *has a derived Δ-functor of the form* ($\mathbf{R}f_*$, identity).

An $\mathcal{O}_X$-module M is f_*-acyclic iff the "higher direct image" sheaves

$$R^if_*(M) := H^i\mathbf{R}f_*(M) \qquad (i \ge 0)$$

vanish for all $i > 0$, see last assertion in (2.2.6) or in (2.7.4) (dualized). (Since f_* is left-exact, the natural map is an isomorphism $f_* \xrightarrow{\sim} R^0f_*$.) Flasque sheaves are f_*-acyclic.

For more examples involving flasque sheaves see [**H**, p. 225, Variations 6 and 7] ("cohomology with supports").

Proposition (2.7.4). *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, and let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be a* right-exact *additive functor. If C is ϕ-acyclic, then for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ the corresponding sequence $0 \to \phi A \to \phi B \to \phi C \to 0$ is also exact, and A is ϕ-acyclic iff B is. So if every object in $\mathcal{A}$ is a quotient of a ϕ-acyclic one, then the conclusions of* (2.7.2) *hold with* $\mathbf{P}$ *the class of ϕ-acyclic objects; and then*

$D \in \mathcal{A}$ is ϕ-acyclic iff the natural map $\mathbf{L}\phi_-(D) \to \phi(D)$ is an isomorphism in $\mathbf{D}(\mathcal{A}')$, i.e., iff $H^{-i}\mathbf{L}\phi_-(D) = 0$ for all $i > 0$.

Proof. For the first assertion, note that by (2.7.1) there exists a commutative diagram

$$\begin{tikzcd}
C_2 \arrow[r, "\delta"] \arrow[d] & C_1 \arrow[r, "\gamma"] \arrow[d] & C_0 \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\alpha"'] & C \arrow[r] & 0
\end{tikzcd}$$

such that the top row is exact and remains so after application of ϕ. There results a commutative diagram

$$\begin{tikzcd}
 & & C_2 \arrow[r, equal] \arrow[d, "\delta"'] & C_2 \arrow[d, "\delta"] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_1 \arrow[r, equal] \arrow[d, "\gamma'"'] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & C_0 \times_C B \arrow[r, "\pi"'] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
 & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact columns, in which the middle row is *split exact,* a right inverse for the projection π being given by the graph of the map β.[23] (The coordinates of γ' are γ and 0.) Applying ϕ preserves split-exactness; and then, since ϕ is right-exact, so that e.g., $\phi C = \operatorname{coker}(\phi\gamma)$, the "snake lemma" yields an exact sequence

$$0 \to \ker(\phi\gamma') \to \ker(\phi\gamma) \to \phi A \to \phi B \to \phi C \to 0\,.$$

Since

$$\ker(\phi\gamma) = \operatorname{im}(\phi\delta) \subset \ker(\phi\gamma')$$

we conclude that $0 \to \phi A \to \phi B \to \phi C \to 0$ is exact, as asserted in (2.7.4).

In other words, if Z is the complex which looks like $A \to B$ in degrees -1 and 0 and which vanishes elsewhere, then the quasi-isomorphism

[23] Recall that $C_0 \times_C B$ is the kernel of the map $C_0 \oplus B \to C$ whose restriction to C_0 is $\alpha\beta$ and to B is $-\alpha$.

$Z \to C$ given by the exact sequence $0 \to A \to B \to C \to 0$ becomes, upon application of ϕ, an isomorphism in $\mathbf{D}(\mathcal{A}')$; and hence, by (2.2.5.2) (dualized), Z is a ϕ-acyclic complex.

The natural semi-split sequence $0 \to B \to Z \to A[1] \to 0$ leads, as in (1.4.3), to a triangle

$$B \longrightarrow Z \longrightarrow A[1] \longrightarrow B[1]\,;$$

and since the ϕ-acyclic complexes are the objects of a Δ-subcategory, see (2.2.5.1), it follows that A is ϕ-acyclic iff B is.

Since Δ-subcategories are closed under direct sum, it is clear now that (ii) and (iii) in (2.7.2) hold when $\mathbf{P}$ is the class of ϕ-acyclic objects, whence the second-last assertion in (2.7.4). In view of (2.7.2) and its proof, the last assertion of (2.7.4) is contained in (2.2.6). Q.E.D.

The derived functor $\mathsf{L}\phi_{\_}$ of (2.7.4) satisfies $\dim^{+}\mathsf{L}\phi_{\_} = 0$ (unless $\phi \cong 0$, see (2.7.2)). When its lower dimension satisfies $\dim^{-}\mathsf{L}\phi_{\_} < \infty$, more can be said.

Proposition (2.7.5). *Let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be a right-exact functor such that every object in $\mathcal{A}$ is a quotient of a ϕ-acyclic one, and let $\mathsf{L}\phi_{\_}$ be a left-derived functor of $\phi|\overline{\mathbf{K}}^{-}(\mathcal{A})$, see* (2.7.4). *Then the following conditions on an integer $d \geq 0$ are equivalent:*

(i) $\dim^{-}\mathsf{L}\phi_{\_} \leq d$.

(ii) *For any $F \in \mathcal{A}$ we have*

$$L_j\phi(F) := H^{-j}\mathsf{L}\phi_{\_}(F) = 0 \quad \text{for all } j > d.$$

(iii) *In any exact sequence in $\mathcal{A}$*

$$0 \to 0 \to B_d \to B_{d-1} \to \cdots \to B_0\,,$$

if $B_0, B_1, \dots, B_{d-1}$ are all ϕ-acyclic then so is B_d.[24]

(iv) *For any $F \in \mathcal{A}$ there is an exact sequence*

$$0 \to B_d \to B_{d-1} \to \cdots \to B_0 \to F \to 0$$

in which every B_i is ϕ-acyclic.

(v) *For any complex $F^\bullet \in \mathbf{K}(\mathcal{A})$ and integers $m \leq n$, if $F^j = 0$ for all $j \notin [m,n]$ then there exists a quasi-isomorphism $B^\bullet \to F^\bullet$ where B^j is ϕ-acyclic for all j and $B^j = 0$ for $j \notin [m-d, n]$.*

(vi) *For any complex $F^\bullet \in \mathbf{K}(\mathcal{A})$ and any integer m, if $F^j = 0$ for all $j < m$ then there exists a quasi-isomorphism $B^\bullet \to F^\bullet$ where B^j is ϕ-acyclic for all j and $B^j = 0$ for all $j < m-d$.*

[24] For $d = 0$ this means that every $B \in \mathcal{A}$ is ϕ-acyclic, i.e., ϕ is an *exact* functor, see (2.7.4) (and then every $F^\bullet \in \mathbf{K}(\mathcal{A})$ is ϕ-acyclic, see (2.2.8(a)).

When there exists an integer $d \ge 0$ for which these conditions hold, then:

(a) *Every complex of ϕ-acyclic objects is a ϕ-acyclic complex.*

(b) *Every complex in $\mathcal{A}$ is the target of a quasi-isomorphism from a ϕ-acyclic complex.*

(c) *A left-derived functor $\mathbf{L}\phi\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A}')$ exists, $\dim^+\mathbf{L}\phi = 0$ (unless $\phi \cong 0$) and $\dim^-\mathbf{L}\phi \le d$.*

(d) *The restriction $\mathbf{L}\phi|_{\overline{\mathbf{D}}^*(\mathcal{A})}$ is a left-derived functor of $\phi|_{\overline{\mathbf{K}}^*(\mathcal{A})}$, and*

$$\mathbf{L}\phi(\overline{\mathbf{D}}^*(\mathcal{A})) \subset \overline{\mathbf{D}}^*(\mathcal{A}') \qquad (* = +,\ -,\ \text{or } b).$$

Proof. (i)$\Leftrightarrow$(ii). This is given by (iii) and (iv) in (1.11.2).

(iii)$\Rightarrow$(v)$\Rightarrow$(iv). Let $F^\bullet$ and $m \le n$ be as in (v). As in the proof of (2.7.2), there is a quasi-isomorphism $P^\bullet \to F^\bullet$ with P^j ϕ-acyclic for all j and $P^j = 0$ for $j > n$. Let B^{m-d} be the cokernel of $P^{m-d-1} \to P^{m-d}$. If (iii) holds, then B^{m-d} is ϕ-acyclic: this is trivial if $d = 0$, and otherwise follows from the exact sequence

$$0 \to B^{m-d} \to P^{m-d+1} \to \cdots \to P^{m-1} \to P^m.$$

So all components of the complex $B^\bullet = \tau_{\ge m-d}P^\bullet$ (see (1.10)) are ϕ-acyclic, and clearly $P^\bullet \to F^\bullet$ factors naturally as $P^\bullet \to B^\bullet \to F^\bullet = \tau_{\ge m-d}F^\bullet$ where both arrows represent quasi-isomorphisms. Thus (iii)$\Rightarrow$(v); and (v)$\Rightarrow$(iv) is obvious.

Recalling from (2.7.4) that $B \in \mathcal{A}$ is ϕ-acyclic iff $L_i\phi(B) = 0$ for all $i > 0$, we easily deduce the implications (iv)$\Rightarrow$(ii)$\Rightarrow$(iii) from:

Lemma (2.7.5.1). *Let*

$$0 = B_{d+1} \to B_d \to B_{d-1} \to \cdots \to B_0 \to F \to 0$$

be an exact sequence in $\mathcal{A}$ with $B_0, B_1, \dots, B_{d-1}$ all ϕ-acyclic, and let K_j be the cokernel of $B_{j+1} \to B_j$ $(0 \le j \le d)$. Then for any $i > 0$, there results a natural sequence of isomorphisms

$$\begin{aligned} L_{i+d}\phi(F) = L_{i+d}\phi(K_0) &\xrightarrow{\sim} L_{i+d-1}\phi(K_1) \xrightarrow{\sim} \cdots \\ \cdots \xrightarrow{\sim} L_{i+2}\phi(K_{d-2}) &\xrightarrow{\sim} L_{i+1}\phi(K_{d-1}) \xrightarrow{\sim} L_i\phi(K_d) = L_i\phi(B_d)\,. \end{aligned}$$

Proof. When $d = 0$, it's obvious. If $d > 0$, apply $(2.1.4)^{\mathrm{H}}$ (dualized) to the natural exact sequences

$$0 \to K_j \to B_{j-1} \to K_{j-1} \to 0 \qquad (0 < j \le d)$$

to obtain exact sequences

$$\begin{aligned} 0 = L_{i+d-j+1}\phi(B_{j-1}) &\to L_{i+d-j+1}\phi(K_{j-1}) \\ &\to L_{i+d-j}\phi(K_j) \to L_{i+d-j}\phi(B_{j-1}) = 0. \end{aligned}$$

Q.E.D.

(iii)$\Rightarrow$(vi). Condition (iii) coincides with condition (iii) of [**H**, p. 42, Lemma 4.6, 2)] (dualized, and with P the set of ϕ-acyclics in $\mathcal{A}$). Condition (i) of *loc. cit.* holds by assumption, and condition (ii) of *loc. cit.* is contained in (2.7.4). So if (iii) holds, *loc. cit.* gives the existence of a quasi-isomorphism $B^\bullet \to F^\bullet$ with B^j ϕ-acyclic for all j; and the recipe at the bottom of [**H**, p. 43] for constructing $B^\bullet$ allows us, when $F^j = 0$ for all $j < m$, to do so in such a way that $B^j = 0$ for all $j < m - d$.

(vi)$\Rightarrow$(ii). Assuming (vi), we can find for each object $F \in \mathcal{A}$ a quasi-isomorphism $B^\bullet \to F$ with all B^j ϕ-acyclic and $B^j = 0$ for $j < -d$. If K is the cokernel of $B^{-1} \to B^0$ then the natural composition

$$H^0(B^\bullet) \longrightarrow K \longrightarrow F$$

is an isomorphism, whence so are the functorially induced compositions

$$L_j\phi(H^0(B^\bullet)) \longrightarrow L_j\phi(K) \longrightarrow L_j\phi(F) \qquad (j \in \mathbb{Z}). \tag{2.7.5.2}$$

But for every $j > d$, (2.7.5.1) with K in place of F yields $L_j\phi(K) = 0$, so that the isomorphism (2.7.5.2) is the zero-map. Thus (ii) holds.

Now suppose that (i)–(vi) hold for some $d \ge 0$. We have just seen, in proving that (iii)$\Rightarrow$(vi), that then every complex in $\mathcal{A}$ receives a quasi-isomorphism from a complex $B^\bullet$ of ϕ-acyclics; and so, as in the proof of (2.7.2), assertion (2.7.5)(a)—and hence (b)—will result if we can show that whenever such a $B^\bullet$ is exact, then so is $\phi(B^\bullet)$. But condition (iii) guarantees that when $B^\bullet$ is exact, the kernel K^i of $B^i \to B^{i+1}$ is ϕ-acyclic for all i, whence by (2.7.4) we have exact sequences

$$0 \to \phi(K^{i-1}) \to \phi(B^{i-1}) \to \phi(K^i) \to 0 \qquad (i \in \mathbb{Z})$$

which together show that $\phi(B^\bullet)$ is indeed exact.

The existence of $\mathbf{L}\phi$, via resolutions by complexes of ϕ-acyclic objects, follows now from (2.2.6); and the dimension statements follow, after application of $(1.8.1)^+$ or $(1.8.1)^-$, from (v) with $m = -\infty$ (obvious interpretation, see beginning of above proof that (iii)$\Rightarrow$(v)) and from (vi). Similar considerations yield (d). Q.E.D.

Examples **(2.7.6).** The *dimension* $\dim f$ of a map $f\colon X \to Y$ of ringed spaces is defined to be the upper dimension (see (1.11)) of the functor $\mathbf{R}f_*^+ : \overline{\mathbf{D}}^{\mathbf{+}}(X) \to \mathbf{D}(Y)$ of (2.7.3):

$$\dim f := \dim^+ \mathbf{R}f_*^+ ,$$

a nonnegative integer unless $f_*\mathcal{O}_X \cong 0$, in which case $\dim f = -\infty$. When f has *finite* dimension, (2.7.5)(c) (dualized) gives the existence of a derived functor $\mathbf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ via resolutions (on the right) by complexes of f_*-acyclic objects, and we have $\infty > \dim f = \dim^+ \mathbf{R}f_*$.

The *tor-dimension* (or *flat dimension*) tor-dim f of a map $f\colon X \to Y$ of ringed spaces is defined to be the lower dimension (see (1.11)) of the functor $\mathsf{L}f^*_-\colon \overline{\mathbf{D}}^-(Y) \to \mathbf{D}(X)$ of (2.7.3):

$$\text{tor-dim}\, f := \dim^- \mathsf{L}f^*_-\,,$$

a nonnegative integer unless $\mathcal{O}_X \cong 0$, in which case tor-dim $f = -\infty$. When f has *finite* tor-dimension, (2.7.5)(c) gives the existence of a derived functor $\mathsf{L}f^*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ via resolutions (on the left) by complexes of f^*-acyclic objects, and we have $\infty > \text{tor-dim}\, f = \dim^- \mathsf{L}f^*$.

Following [**I**, p. 241, Définition 3.1] one says that an $\mathcal{O}_X$-complex E has *flat f-amplitude in* $[m, n]$ if for any $\mathcal{O}_Y$-module F,

$$H^i(E \underset{=}{\otimes} \mathsf{L}f^*F) = 0 \;\text{ for all } i \notin [m, n],$$

or equivalently, for the functor $L_E(F) := E \underset{=}{\otimes} \mathsf{L}f^*F$ of $\mathcal{O}_Y$-module F,

$$\dim^+ L \le m \;\text{ and }\; \dim^- L \le -n.$$

This means that the stalk E_x at each $x \in X$ is $\mathbf{D}(\mathcal{O}_{Y,f(x)})$-isomorphic to a flat complex vanishing in degrees outside $[m, n]$, see [**I**, p. 242, 3.3], or argue as in (2.7.6.4) below. E has *finite flat f-amplitude* if such m and n exist.

It follows from (2.7.6.4) below and [**I**, p. 131, 5.1] that *f has finite tor-dimension $\iff$ $\mathcal{O}_X$ has finite flat f-amplitude.*

(2.7.6.1). If X is a compact Hausdorff space of dimension $\le d$ (in the sense that each point has a neighborhood homeomorphic to a locally closed subspace of the Euclidean space $\mathbb{R}^d$), and $\mathcal{O}_X$ is the constant sheaf $\mathbb{Z}$, then $\dim f \le d$.

Indeed, if $I^\bullet$ is a flasque resolution of the abelian sheaf F, then for any open $U \subset Y$ the restriction $I^\bullet|f^{-1}(U)$ is a flasque resolution of $F|f^{-1}(U)$, and $R^jf_*(F)$ is, up to isomorphism, the sheaf associated to the presheaf taking any such U to the group $H^j(\Gamma(f^{-1}(U), I^\bullet|f^{-1}(U))$, a group isomorphic to $H^j(f^{-1}(U), F|f^{-1}(U))$ [**G**, p. 181, Thm. 4.7.1(a)], and hence vanishing for $j > d$, see [**Iv**, Chap. III, §9].

More generally, if X is locally compact and we assume only that the fibers $f^{-1}y$ ($y \in Y$) are compact and have dimension $\le d$, then $\dim f \le d$ (because the stalk $(R^jf_*F)_y$ is the cohomology $H^j(f^{-1}y, F|f^{-1}y)$, see [**Iv**, p. 315, Thm. 1.4], whose proof does not require any assumption on Y).

(2.7.6.2). (Grothendieck, see [**H**, p. 87]). If $(X, \mathcal{O}_X)$ is a noetherian scheme of finite Krull dimension d, then $\dim f \le d$.

(2.7.6.3). For a ringed-space map $f\colon X \to Y$ with $\mathcal{O}_X \ncong 0$, the following conditions are equivalent:

(i) tor-dim $f = 0$.

(i)′ Every $\mathcal{O}_Y$-module is f^*-acyclic.

(i)″ The functor f^* of $\mathcal{O}_Y$-modules is exact.

(ii) f is flat (i.e., $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$).

Proof. Since every $\mathcal{O}_X$-module is a quotient of a flat one, which is f^*-acyclic (see (2.7.3)), the equivalence of (i), (i)′, and (i)″ is given, e.g., by that of (i) and (iii) in (2.7.5) (for $d = 0$). The equivalence of (i) and (ii) is the case $d = 0$ of:

(2.7.6.4). *Let $f\colon X \to Y$ be a ringed-space map and $d \ge 0$ an integer. Then* tor-dim $f \le d \iff$ *for each $x \in X$ there exists an exact sequence of $\mathcal{O}_{Y,f(x)}$-modules*

$$(*) \qquad 0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \mathcal{O}_{X,x} \to 0$$

with P_i flat over $\mathcal{O}_{Y,f(x)}$ $(0 \le i \le d)$.

Proof. ("*if*") Let F be an $\mathcal{O}_Y$-module and let $Q^\bullet \to F$ be a quasi-isomorphism with $Q^\bullet$ a flat complex (1.8.3). Then for $j \ge 0$, the homology

$$L_j f^*(F) \cong H^{-j}(f^*Q^\bullet) \qquad \text{(see (2.7.3))}$$

vanishes iff for each $x \in X$, with $y = f(x)$, $R = \mathcal{O}_{Y,y}$, and $S = \mathcal{O}_{X,x}$ we have

$$0 = H^{-j}\big((f^*Q^\bullet)_x\big) = H^{-j}\big(S \otimes_R Q^\bullet_y\big) = \mathrm{Tor}^R_j(S, F_y)$$

(where the last equality holds since $Q^\bullet_y \to F_y$ is an R-flat resolution of F_y), whence the assertion.

("*only if*") Suppose only that $L_{d+1}f^*(F) = 0$ for all F, so that (see above) $\mathrm{Tor}^R_{d+1}(S, F_y) = 0$; and let

$$\cdots \to P'_2 \to P'_1 \to P'_0 \to S \to 0$$

be an R-flat resolution of S. Then, I claim, the module

$$P_d := \ \mathrm{coker}(P'_{d+1} \to P'_d)$$

is *R-flat,* whence we have $(*)$ with $P_i = P'_i$ for $0 \le i < d$.

Indeed, the flatness of P_d is equivalent to the vanishing of $\mathrm{Tor}^R_1(P_d, R/I)$ for all R-ideals I [**B**$'$, §4, Prop. 1]. But any such I is $\mathcal{I}_y$ where $\mathcal{I} \subset \mathcal{O}_Y$ is the $\mathcal{O}_Y$-ideal such that for any open $U \subset Y$,

$$\begin{aligned} \mathcal{I}(U) &= \{\, r \in \mathcal{O}_Y(U) \mid r_y \in I \,\} \qquad && \text{if } y \in U \\ &= 0 && \text{if } y \notin U; \end{aligned}$$

so that if $F = \mathcal{O}_Y/\mathcal{I}$, then $R/I = F_y$; and from the flat resolution

$$\cdots \to P'_{d+2} \to P'_{d+1} \to P'_d \to P_d \to 0$$

of P_d, we get the desired vanishing:

$$\mathrm{Tor}^R_1(P_d, R/I) = \mathrm{Tor}^R_1(P_d, F_y) = \mathrm{Tor}^R_{d+1}(S, F_y) = 0.$$

Exercise **(2.7.6.5)**. (For amusement only.) If Y is a quasi-separated scheme, then $f\colon X \to Y$ satisfies tor-dim $f \le d$ if (and only if) for every quasi-coherent $\mathcal{O}_Y$-ideal $\mathcal{I}$, we have

$$L_{d+1}f^*(\mathcal{O}_Y/\mathcal{I}) = 0.$$

If in addition Y is quasi-compact or locally noetherian, then we need only consider *finite-type* quasi-coherent $\mathcal{O}_Y$-ideals.

[The following facts in [**GD**] can be of use here: p. 111, (5.2.8); p. 313, (6.7.1); p. 294, (6.1.9) (i); p. 295, (6.1.10)(iii); p. 318, (6.9.7).]

Chapter 3

Derived Direct and Inverse Image

A *ringed space* is a pair $(X, \mathcal{O}_X)$ with X a topological space and $\mathcal{O}_X$ a sheaf of commutative rings on X; and a *morphism* (or *map*) of ringed spaces $(f,\theta)\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ together with a map $\theta\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings. (Usually we will just denote such a morphism by $f\colon X \to Y$, the accompanying θ understood to be standing by.) Associated with (f,θ) are the adjoint functors

$$\mathcal{A}_X := \{\mathcal{O}_X\text{-modules}\} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \{\mathcal{O}_Y\text{-modules}\} =: \mathcal{A}_Y$$

and their respective derived functors $\mathbf{R}f_*$, $\mathbf{L}f^*$, which are also adjoint—as Δ-functors, (3.2), (3.3). In this chapter we first review the definitions and basic formal (i.e., category-theoretic) properties of these adjoint derived functors, their interactions with $\underline{\underline{\otimes}}$ and $\mathbf{R}\mathcal{H}om^\bullet$, and their "pseudofunctorial" behavior with respect to composition of ringed-space maps (3.6), many of the main results being packaged in (3.6.10).

A basic objective, in the spirit of Grothendieck's philosophy of the "six operations," is the categorical formalization of relations among functorial maps involving the four operations $\mathbf{R}f_*$, $\mathbf{L}f^*$, $\underline{\underline{\otimes}}$ and $\mathbf{R}\mathcal{H}om^\bullet$.[25]

More explicitly (details in §§3.4, 3.5), if $f\colon X \to Y$ is a map of ringed spaces, then the derived categories $\mathbf{D}(\mathcal{A}_X)$, $\mathbf{D}(\mathcal{A}_Y)$ have natural structures of symmetric monoidal closed categories, given by $\underline{\underline{\otimes}}$ and $\mathbf{R}\mathcal{H}om^\bullet$; and the adjoint Δ-functors $\mathbf{R}f_*$ and $\mathbf{L}f^*$ respect these structures, as do the conjugate isomorphisms, arising from a second map $g\colon Y \to Z$, $\mathbf{R}(gf)_* \xrightarrow{\sim} \mathbf{R}g_*\mathbf{R}f_*$, $\mathbf{L}f^*\mathbf{L}g^* \xrightarrow{\sim} \mathbf{L}(gf)^*$. We express all this by saying that $\mathbf{R}-_*$ and $\mathbf{L}-^*$ are *adjoint monoidal Δ-pseudofunctors.*

Thus, relations among the four operations can be worked with as instances of category-theoretic relations involving adjoint monoidal functors between closed categories. This eliminates excess baggage of resolutions of complexes, which would otherwise cause intolerable tedium later on, where proofs of major results depend heavily on involved manipulations of such relations.[26] Even so, the situation is far from ideal—see the introductory

[25] A fifth operation, "twisted inverse image," is brought into play in Chapter 4, at least for schemes. The sixth, "direct image with proper supports" [**De′**, n° 3] will not appear here, except for proper scheme-maps, where it coincides with derived direct image.

[26] Cf. in this vein Hartshorne's remarks on "compatibilities" [**H**, pp. 117–119]. Note however that the formalization became fully feasible only after Spaltenstein's extension of the theory of derived functors in [**H**] to unbounded complexes [**Sp**].

remarks in §3.4, and, for example, the proof of Proposition (3.7.3), which addresses the interaction between the projection morphisms of (3.4.6) and "base change."

By way of illustration, consider the following basic functorial maps, with $A, B \in \mathbf{D}(\mathcal{A}_Y)$ and $E, F \in \mathbf{D}(\mathcal{A}_X)$:[27]

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*B,\, E) \to \mathbf{R}\mathcal{H}om^\bullet_Y(B,\, \mathbf{R}f_*E)\,, \tag{3.2.3.2}$$

$$\mathbf{L}f^*A \underset{=}{\otimes} \mathbf{L}f^*B \leftarrow \mathbf{L}f^*(A \underset{=}{\otimes} B)\,, \tag{3.2.4}$$

$$\mathbf{R}f_*(E) \underset{=}{\otimes} \mathbf{R}f_*(F) \to \mathbf{R}f_*(E \underset{=}{\otimes} F)\,, \tag{3.2.4.2}$$

$$\mathbf{R}f_*E \underset{=}{\otimes} B \to \mathbf{R}f_*(E \underset{=}{\otimes} \mathbf{L}f^*B)\,. \tag{3.4.6}$$

The first two can be defined at the level of complexes, after replacing the arguments by appropriate resolutions. (The reduction is straightforward for the second, but not quite so for the first.) At that level, one sees that they are both isomorphisms. For fixed B, the source and target of the first are left-adjoint, respectively, to the target and source of the second; and it turns out that the two maps are *conjugate* (3.3.5). This is shown by reduction to the analogous statement for the ordinary direct and inverse image functors for sheaves, which can be treated concretely (3.1.10) or formally (3.5.5). So each one of these isomorphisms determines the other from a purely categorical point of view.

The second and third maps determine each other via $\mathbf{L}f^*$–$\mathbf{R}f_*$ adjunction (3.4.5), as do the third and fourth (3.4.6). When the first map is given, the second and third maps also determine each other via $\mathbf{R}\mathcal{H}om^\bullet$–$\underset{=}{\otimes}$ adjunction. (This is not obvious, see Proposition (3.2.4).)

Thus, any three of the four maps can be deduced category-theoretically from the remaining one.

In (3.9) we consider the case when our ringed spaces are *schemes*. Under mild assumptions, we note that then $\mathbf{R}f_*$ and $\mathbf{L}f^*$ "respect quasi-coherence" (3.9.1), (3.9.2). We also show that some previously introduced functorial morphisms become isomorphisms: (3.9.4) treats variants of the projection morphisms, while (3.9.5) signifies that $\mathbf{R}f_*$ behaves well—even for *unbounded* complexes—with respect to flat base change.[28] More generally, in (3.10) we see that such good behavior of $\mathbf{R}f_*$ characterizes *tor-independent* base changes, as does a certain Künneth map's being an isomorphism; the precise statement is given in (3.10.3), a culminating result for the chapter.

[27] The first is a sheafified version of $\mathbf{L}f^*$–$\mathbf{R}f_*$ adjunction (3.2.5)(f), the second and third underly monoidality of $\mathbf{L}f^*$ and $\mathbf{R}f_*$, and the fourth is "projection."

[28] cf. [**I**, III, 3.7 and IV, 3.1].

3.1. Preliminaries

For any ringed space $(X, \mathcal{O}_X)$, let $\mathcal{A}_X$ be the category of (sheaves of) $\mathcal{O}_X$-modules—which is abelian, see e.g., [**G**, Chap. II, §2.2, §2.4, and §2.6], $\mathbf{C}(X)$ the category of $\mathcal{A}_X$-complexes, $\mathbf{K}(X)$ the category of $\mathcal{A}_X$-complexes with homotopy equivalence classes of maps of complexes as morphisms, and $\mathbf{D}(X)$ the derived category gotten by "localizing" $\mathbf{K}(X)$ with respect to quasi-isomorphisms (see §§(1.1), (1.2)).

To any ringed-space map $(f,\theta)\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ one can associate the additive *direct image* functor

$$f_* : \mathcal{A}_X \to \mathcal{A}_Y$$

such that $[f_*M](U) = M(f^{-1}U)$ for any $\mathcal{O}_X$-module M and any open set $U \subset Y$, the $\mathcal{O}_Y$-module structure on f_*M arising via θ; and also an *inverse image* functor

$$f^*\colon \mathcal{A}_Y \to \mathcal{A}_X$$

defined up to isomorphism as a left-adjoint of f_*, see [**GD**, p. 100, (4.4.3.1)] (where $\Psi^*(\mathcal{F})$ should be $\Psi_*(\mathcal{F})$). Such an adjoint exists with, e.g.,

$$f^*A := f^{-1}A \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \qquad (A \in \mathcal{A}_Y)$$

where $f^{-1}A$ is the sheaf associated to the presheaf taking an open $V \subset X$ to $\varinjlim A(U)$ with U running through all the open neighborhoods of $f(V)$ in Y. In particular, if X is an open subset of Y, $\mathcal{O}_X$ is the restriction of $\mathcal{O}_Y$, f is the inclusion, and θ is the obvious map, then the functor "restriction to X" is left-adjoint to f_*, so it is the natural choice for f^*. Being adjoint to an additive functor, f^* is also additive.[29] From adjointness, or directly, one sees that f_* is left-exact and f^* is right-exact. (The stalk $(f^*N)_x$ at $x \in X$ is functorially isomorphic to $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} N_{f(x)}$.)

Derived functors (see (2.1.1) and its complement)

$$\mathsf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y), \qquad \mathsf{L}f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X)$$

can be constructed by means of q-injective and q-flat resolutions, respectively, as follows.

Assume chosen once and for all, for each ringed space X, two families of quasi-isomorphisms

$$A \to I_A, \qquad P_A \to A \qquad (A \in \mathbf{K}(X)) \tag{3.1.1}$$

with each I_A a q-injective complex and each P_A q-flat, see (2.3.5), (2.5.5), with $A \to I_A$ the identity map when A is itself q-injective, and $P_A \to A$ the identity when A is q-flat.

---

[29] Additivity of f^* means that for any two maps $A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B$ in $\mathcal{A}_Y$ and any $E \in \mathcal{A}_X$, the sum of the induced maps $\mathrm{Hom}(f^*B, E) \rightrightarrows \mathrm{Hom}(f^*A, E)$ is the map induced by $\alpha+\beta$, a condition which follows from the additivity of f_* via the adjunction isomorphisms (of abelian groups) $\mathrm{Hom}(f^*-, E) \to \mathrm{Hom}(f_*f^*-, f_*E) \to \mathrm{Hom}(-, f_*E)$.

Then set

$$\mathbf{R}f_*(B) := f_*(I_B) \qquad \big(B \in \mathbf{D}(X)\big), \tag{3.1.2}$$

and for a map α in $\mathbf{D}(X)$ define $\mathbf{R}f_*(\alpha)$ as indicated in (2.3.2.3) (with $\mathbf{J} := \mathbf{K}(X)$). The Δ-structure on $\mathbf{R}f_*$ is specified at the end of (2.2.4). Similar considerations apply to $\mathbf{L}f^*$, once one verifies that f^* takes exact q-flat complexes to exact complexes (for which argue as in (2.5.7), keeping in mind (2.5.2)). Proceeding as in (2.2.4) (dualized, with $\mathbf{J}' \subset \mathbf{K}(Y)$ the Δ-subcategory whose objects are the q-flat complexes, and $\mathbf{J}'' := \mathbf{K}(Y)$), set

$$\mathbf{L}f^*(A) := f^*(P_A) \qquad \big(A \in \mathbf{D}(Y)\big), \tag{3.1.3}$$

etc. [See also (2.7.3).]

Proposition (3.2.1) below says in particular that *these derived functors are also adjoint.* Before getting into that we review some elementary functorial sheaf maps, and their interconnections.

For $\mathcal{O}_X$-modules E and F, there is a natural map of $\mathcal{O}_Y$-modules

$$\phi_{E,F} \colon f_*\mathcal{H}om_X(E,\,F) \to \mathcal{H}om_Y(f_*E,\,f_*F) \tag{3.1.4}$$

taking a section of $f_*\mathcal{H}om_X(E,\,F)$ over an open subset U of Y— i.e., a map $\alpha\colon E|_{f^{-1}U} \to F|_{f^{-1}U}$—to the section α_ϕ of $\mathcal{H}om_Y(f_*E,\,f_*F)$ given by the family of maps $\alpha_\phi(V)\colon (f_*E)(V) \to (f_*F)(V)$ (V open $\subset U$) with

$$\alpha_\phi(V) := \alpha(f^{-1}V)\colon E(f^{-1}V) \to F(f^{-1}V)\,.$$

Here is another description of $\phi_{E,F}(U)$: given the commutative diagram

$$\begin{tikzcd} f^{-1}U \arrow[r, "j"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ U \arrow[r, "i"'] & Y \end{tikzcd}$$

where i and j are inclusions and g is the restriction $f|_{f^{-1}U}$, and recalling that i^* and j^* are restriction functors, one verifies the functorial equalities

$$f_*j_*j^* = i_*g_*j^* = i_*i^*f_*$$

and checks then that $\phi_{E,F}(U)$ is the natural composition

$$\begin{aligned} f_*\mathcal{H}om_X(E,\,F)(U) &\overset{\text{def}}{=\!=} \operatorname{Hom}(j^*E,\,j^*F) \\ &\xrightarrow{\ \sim\ } \operatorname{Hom}(E,\,j_*j^*F) \\ &\longrightarrow \operatorname{Hom}(f_*E,\,f_*j_*j^*F) \\ &=\!=\, \operatorname{Hom}(f_*E,\,i_*i^*f_*F) \\ &\xrightarrow{\ \sim\ } \operatorname{Hom}(i^*f_*E,\,i^*f_*F) \overset{\text{def}}{=\!=} \mathcal{H}om_Y(f_*E,\,f_*F)(U)\,. \end{aligned}$$

Lemma (3.1.5). *Let* $f\colon X \to Y$ *be a ringed-space map,* $A \in \mathcal{A}_Y$, $B \in \mathcal{A}_X$, $\phi := \phi_{f^*A,\,B}$ *(see* (3.1.4)*). Let* $\eta_A\colon A \to f_*f^*A$ *be the map corresponding by adjunction to the identity map of* f^*A*. Then the composition*

$$f_*\mathcal{H}om_X(f^*A,\,B) \xrightarrow{\;\phi\;} \mathcal{H}om_Y(f_*f^*A,\,f_*B) \xrightarrow{\text{via }\eta_A} \mathcal{H}om_Y(A,\,f_*B)$$

is an isomorphism of additive bifunctors.

Proof. The preceding description of ϕ identifies (up to isomorphism) the sections over an open $U \subset Y$ of the composite map in (3.1.5) with the natural composition

$$\mathrm{Hom}(f^*A,\,j_*j^*B) \longrightarrow \mathrm{Hom}(f_*f^*A,\,f_*j_*j^*B) \xrightarrow{\text{via }\eta_A} \mathrm{Hom}(A,\,f_*j_*j^*B)$$

which is, by adjointness of f^* and f_*, an isomorphism. Additive bifunctoriality of this isomorphism is easily verified. Q.E.D.

(3.1.6). We leave it to the reader to elaborate the foregoing to get isomorphisms of *complexes,* functorial in $A^\bullet \in \mathbf{C}(Y)$, $B^\bullet \in \mathbf{C}(X)$,

$$\mathrm{Hom}^\bullet_X(f^*A^\bullet,\,B^\bullet) \xrightarrow{\;\sim\;} \mathrm{Hom}^\bullet_Y(A^\bullet,\,f_*B^\bullet)\,,$$
$$f_*\mathcal{H}om^\bullet_X(f^*A^\bullet,\,B^\bullet) \xrightarrow{\;\sim\;} \mathcal{H}om^\bullet_Y(A^\bullet,\,f_*B^\bullet)\,.$$

(See (1.5.3) and (2.4.5) for the definitions of $\mathrm{Hom}^\bullet$ and $\mathcal{H}om^\bullet$.)

Ditto for the maps in (3.1.7)–(3.1.9) below.

For any two $\mathcal{O}_X$-modules E, F, the tensor product $E \otimes_X F$ is by definition the sheaf associated to the presheaf $U \mapsto E(U) \otimes_{\mathcal{O}_X(U)} F(U)$ (U open $\subset X$), so there exist canonical maps

$$E(U) \otimes_{\mathcal{O}_X(U)} F(U) \to (E \otimes_X F)(U)$$

from which, taking $U = f^{-1}V$ (V open $\subset Y$), one gets a canonical map

$$f_*E \otimes_Y f_*F \to f_*(E \otimes_X F)\,. \tag{3.1.7}$$

(3.1.8). We will abbreviate by omitting the subscripts attached to $\otimes$, and by writing $\mathcal{H}_Z(-,-)$ for $\mathcal{H}om_{\mathcal{O}_Z}(-,-)$.

The maps (3.1.4) and (3.1.7) are related via $\mathcal{H}om$-$\otimes$ adjunction (2.6.2) as follows. After taking global sections of (2.6.2) (with F, G replaced by E, F respectively) one finds, corresponding to the identity map of $E\otimes F$, a canonical map

$$E \to \mathcal{H}_X(F,\,E \otimes F)\,. \tag{3.1.8.1}$$

Similarly, corresponding to the identity map of $\mathcal{H}_X(E,\,F)$ one has a map

$$\mathcal{H}_X(E,\,F) \otimes E \to F\,. \tag{3.1.8.2}$$

Verification of the following two assertions is left to the reader.

—The map (3.1.7) is $\mathcal{H}om$-$\otimes$ adjoint to the composition

$$f_*E \xrightarrow{(3.1.8.1)} f_*\mathcal{H}_X(F,\, E\otimes F) \xrightarrow{(3.1.4)} \mathcal{H}_Y(f_*F,\, f_*(E\otimes F))\,.$$

—The map (3.1.4) is $\mathcal{H}om$-$\otimes$ adjoint to the composition

$$f_*\mathcal{H}_X(E,\, F)\otimes f_*E \xrightarrow{(3.1.7)} f_*(\mathcal{H}_X(E,\, F)\otimes E) \xrightarrow{(3.1.8.2)} f_*F\,.$$

(3.1.9) Define the functorial map

$$f^*(A\otimes B) \xrightarrow{\alpha} f^*A\otimes f^*B \qquad (A, B\in \mathcal{A}_Y)$$

to be the adjoint of the composition

$$A\otimes B \xrightarrow{\text{natural}} f_*f^*A\otimes f_*f^*B \xrightarrow{(3.1.7)} f_*(f^*A\otimes f^*B).$$

Let $x\in X$, $y=f(x)$, so that f induces a map of local rings $\mathcal{O}_y\to\mathcal{O}_X$, where $\mathcal{O}_X$ is the stalk $\mathcal{O}_{X,x}$, and similarly for $\mathcal{O}_y$. One checks that the stalk map α_x is just the natural map

$$(A_y\otimes_{\mathcal{O}_y} B_y)\otimes_{\mathcal{O}_y}\mathcal{O}_X \to (A_y\otimes_{\mathcal{O}_y}\mathcal{O}_X)\otimes_{\mathcal{O}_X}(B_y\otimes_{\mathcal{O}_y}\mathcal{O}_X)\,,$$

whence α coincides with the standard *isomorphism* defined, e.g., in [**GD**, p. 97, (4.3.3.1)].

Exercise **(3.1.10).** Show that the source and target of the map α in (3.1.9) are, as functors in the variable A, left-adjoint to the target and source (respectively) of the composed isomorphism—call it β—in (3.1.5), considered as functors in B; and that α and β are *conjugate,* see (3.3.5). (See also (3.5.5).) Work out the analog for complexes.

3.2. Adjointness of derived direct and inverse image

We begin with a direct proof of adjointness of the derived direct and inverse image functors $\mathbf{R}f_*$ and $\mathbf{L}f^*$ associated to a ringed-space map $f\colon X\to Y$.[30] A more elaborate localized formulation is given in (3.2.3). Proposition (3.2.4) introduces the basic maps connecting $\mathbf{R}f_*$ and $\mathbf{L}f^*$ to $\underline{\underline{\otimes}}$. It includes derived-category versions of part of (3.1.8) and of (3.1.10), as an illustration of the basic strategy for understanding relations among maps of derived functors through purely formal considerations (see 3.5.4).

[30] An ultra-generalization of this "trivial duality formula" is given in [**De**, p. 298, Thm. 2.3.7].

Proposition (3.2.1). *For any ringed-space map $f\colon X\to Y$, there is a natural bifunctorial isomorphism,*

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathsf{L}f^*A,\,B) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(A,\,\mathsf{R}f_*B) \qquad \big(A\in\mathbf{D}(Y),\ B\in\mathbf{D}(X)\big).$$

Proof. There is a simple equivalence between giving the adjunction isomorphism (3.2.1) and giving functorial morphisms

$$\eta\colon 1\to\mathsf{R}f_*\mathsf{L}f^*,\qquad \epsilon\colon \mathsf{L}f^*\mathsf{R}f_*\to 1 \tag{3.2.1.0}$$

$(1:=\text{identity})$ such that the corresponding compositions

$$\begin{gathered}\mathsf{R}f_* \xrightarrow{\text{via }\eta} \mathsf{R}f_*\mathsf{L}f^*\mathsf{R}f_* \xrightarrow{\text{via }\epsilon} \mathsf{R}f_*\\ \mathsf{L}f^* \xrightarrow[\text{via }\eta]{} \mathsf{L}f^*\mathsf{R}f_*\mathsf{L}f^* \xrightarrow[\text{via }\epsilon]{} \mathsf{L}f^*\end{gathered} \tag{3.2.1.1}$$

are identity morphisms [**M**, p. 83, Thm. 2]. Indeed, $\eta(A)$ (resp. $\epsilon(B)$) corresponds under (3.2.1) to the identity map of $\mathsf{L}f^*A$ (resp. $\mathsf{R}f_*B$); and conversely, (3.2.1) can be recovered from η and ϵ thus: to a map $\alpha\colon \mathsf{L}f^*A\to B$ associate the composed map

$$A\xrightarrow{\eta(A)}\mathsf{R}f_*\mathsf{L}f^*A\xrightarrow{\mathsf{R}f_*\alpha}\mathsf{R}f_*B\,,$$

and inversely, to a map $\beta\colon A\to\mathsf{R}f_*B$ associate the composed map

$$\mathsf{L}f^*A\xrightarrow{\mathsf{L}f^*\beta}\mathsf{L}f^*\mathsf{R}f_*B\xrightarrow{\epsilon(B)}B.$$

Define ϵ to be the unique Δ-functorial map such that the following natural diagram in $\mathbf{D}(X)$ commutes for all $B\in\mathbf{K}(X)$:[31]

$$\begin{tikzcd} \mathsf{L}f^*f_*B \arrow[r] \arrow[d] & \mathsf{L}f^*\mathsf{R}f_*B \arrow[d, "\epsilon(B)"] \\ f^*f_*B \arrow[r] & B \end{tikzcd} \tag{3.2.1.2}$$

Such an ϵ exists because $\mathsf{L}f^*\mathsf{R}f_*$ is a right-derived functor of $\mathsf{L}f^*Q_Yf_*$ (where $Q_Y\colon\mathbf{K}(Y)\to\mathbf{D}(Y)$ is the canonical functor), and the natural composition $\mathsf{L}f^*Q_Yf_*\to Q_Xf^*f_*\to Q_X$ is Δ-functorial, see (2.1.1) and (2.2.6.1). (Alternatively, use (2.6.5), with $n=1$, $\mathbf{L}''=\mathbf{K}(X)$, $\mathbf{L}'\subset\mathbf{L}''$ the Δ-subcategory whose objects are the q-injective complexes, and β the preceding Δ-functorial composition.)

[31] Here, and elsewhere, we lighten notation by omitting Qs, so that, e.g., B sometimes denotes the (physically identical) image QB of B in $\mathbf{D}(X)$. This should not cause confusion.

Dually, define η to be the unique Δ-functorial map such that the following natural diagram commutes for all $A \in \mathbf{K}(Y)$:

(3.2.1.3)
$$\begin{tikzcd}
\mathbf{R}f_*f^*A & \mathbf{R}f_*\mathbf{L}f^*A \arrow[l] \\
f_*f^*A \arrow[u] & A \arrow[l] \arrow[u, "\eta(A)"']
\end{tikzcd}$$

To see then that the first row in (3.2.1.1) is the identity, i.e., that its composition with the canonical map $\zeta\colon f_* \to \mathbf{R}f_*$ is just ζ itself, consider the diagram (with obvious maps)

$$\begin{tikzcd}
f_* \arrow[rr] \arrow[d, equal] & & \mathbf{R}f_* \arrow[d] \\
f_* \arrow[r] \arrow[d] \arrow[dr, phantom, "①"] & \mathbf{R}f_*\mathbf{L}f^*f_* \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & \mathbf{R}f_*\mathbf{L}f^*\mathbf{R}f_* \arrow[d] \\
f_*f^*f_* \arrow[r] \arrow[d] & \mathbf{R}f_*f^*f_* \arrow[r] & \mathbf{R}f_* \arrow[d, equal] \\
f_* \arrow[rr] & & \mathbf{R}f_*
\end{tikzcd}$$

Subdiagrams ① and ② commute by the definitions of η and ϵ. The top and bottom rectangles clearly commute. Thus the whole diagram commutes, giving the desired conclusion.

A similar argument applies to the second row in (3.2.1.1). Q.E.D.

Corollary (3.2.2). *The adjunction isomorphism* (3.2.1) *is the unique functorial map* ρ *making the following natural diagram commute for all* $A \in \mathbf{K}(Y)$, $B \in \mathbf{K}(X)$:

(3.2.2.1)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{K}(X)}(f^*A,\, B) \arrow[r] \arrow[d, "H^0(3.1.6)"', "\simeq"] & \mathrm{Hom}_{\mathbf{D}(X)}(f^*A,\, B) \arrow[r] & \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{L}f^*A,\, B) \arrow[d, "\rho"] \\
\mathrm{Hom}_{\mathbf{K}(Y)}(A,\, f_*B) \arrow[r, "\nu"] & \mathrm{Hom}_{\mathbf{D}(Y)}(A,\, f_*B) \arrow[r] & \mathrm{Hom}_{\mathbf{D}(Y)}(A,\, \mathbf{R}f_*B)
\end{tikzcd}$$

Moreover, ν *is an isomorphism whenever* A *is left-*f^**-acyclic* (*e.g., q-flat*) *and* B *is q-injective.*

Proof. Suppose ρ is the adjunction isomorphism. To show (3.2.2.1) commutes, chase a $\mathbf{K}(X)$-map $\phi\colon f^*A \to B$ around it in both directions to reduce to showing that the following natural diagram commutes:

$$\begin{tikzcd}
\mathbf{R}f_*\mathbf{L}f^*A \arrow[r] & \mathbf{R}f_*f^*A \arrow[r, "\text{via } \phi"] & \mathbf{R}f_*B \\
A \arrow[r] \arrow[u, "\eta"] & f_*f^*A \arrow[r, "\text{via } \phi"] \arrow[u] & f_*B \arrow[u]
\end{tikzcd}$$

Here the left square commutes by the definition of η, and the right square commutes by functoriality of the natural map $f_* \to \mathbf{R}f_*$.

If, furthermore, A is left-f^*-acyclic (i.e., $\mathbf{L}f^*A \to f^*A$ is an isomorphism (2.2.6)) and B is q-injective, then all the maps in (3.2.2.1) other than ν are isomorphisms (see (2.3.8)(v)), so ν is an isomorphism too.

Finally, to prove the uniqueness of a functorial map $\rho(A,B)$ making (3.2.2.1) commute, use the canonical maps $P_A \to A$ and $B \to I_B$ to map (3.2.2.1) to the corresponding diagram with P_A in place of A and I_B in place of B. As we have just seen, all the maps in this last diagram other than $\rho(P_A, I_B)$ are isomorphisms, so that $\rho(P_A, I_B)$ is uniquely determined by the commutativity condition; and since the sources and targets of $\rho(P_A, I_B)$ and $\rho(A,B)$ are isomorphic, it follows that $\rho(A,B)$ is uniquely determined. Q.E.D.

Exercise. With $\psi_A\colon P_A \to A$ (resp. $\varphi_B\colon B \to I_B$) the canonical isomorphism in $\mathbf{D}(Y)$ (resp. $\mathbf{D}(X)$), see (3.1.1), $\eta(A)$ and $\epsilon(B)$ are the respective compositions

$$A \xrightarrow{\psi_A^{-1}} P_A \xrightarrow{\text{natural}} f_*(f^*P_A) \xrightarrow{f_*(\varphi_{f^*P_A})} f_*(I_{f^*P_A}) = \mathbf{R}f_*\mathbf{L}f^*A\,,$$

$$B \xleftarrow[\varphi_B^{-1}]{} I_B \xleftarrow[\text{natural}]{} f^*(f_*I_B) \xleftarrow[f^*(\psi_{f_*I_B})]{} f^*(P_{f_*I_B}) = \mathbf{L}f^*\mathbf{R}f_*B\,.$$

Recall from §2.4 the derived functors $\mathbf{R}\mathrm{Hom}^\bullet$ and $\mathbf{R}\mathcal{H}om^\bullet$. We write $\mathbf{R}\mathrm{Hom}^\bullet_X$ and $\mathbf{R}\mathcal{H}om^\bullet_X$ to specify that we are working on the ringed space X. For $E, F \in \mathbf{K}(X)$, and I_F as in (3.1.1), we have then, in $\mathbf{D}(X)$,

$$\mathbf{R}\mathrm{Hom}^\bullet_X(E,F) = \mathrm{Hom}^\bullet(E, I_F),$$
$$\mathbf{R}\mathcal{H}om^\bullet_X(E,F) = \mathcal{H}om^\bullet(E, I_F).$$

Proposition (3.2.3) (see [**Sp**, p. 147]). *Let $f\colon X \to Y$ be a ringed-space map.*

(i) *There is a unique Δ-functorial isomorphism*

$$\alpha\colon \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*A, B) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_Y(A, \mathbf{R}f_*B) \qquad \big(A \in \mathbf{K}(Y),\ B \in \mathbf{K}(X)\big) \tag{3.2.3.1}$$

such that the following natural diagram in $\mathbf{D}(X)$[32] *commutes:*

$$\begin{array}{ccccc}
\mathrm{Hom}^\bullet_X(f^*A,\, B) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet_X(f^*A,\, B) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*A,\, B)\\
{\scriptstyle (3.1.6)}\big\downarrow\simeq & & & & \simeq\big\downarrow\alpha\\
\mathrm{Hom}^\bullet_Y(A,\, f_*B) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet_Y(A,\, f_*B) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet_Y(A,\, \mathbf{R}f_*B).
\end{array}$$

Moreover, the induced homology map

$$H^0(\alpha)\colon \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{L}f^*A,\, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(A,\, \mathbf{R}f_*B)$$

(*see* (2.4.2)) *is just the adjunction isomorphism in* (3.2.1).

[32] with missing Q's left to the reader

(ii) *There is a unique* Δ*-functorial isomorphism*

$$\beta\colon \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*A,\, B) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_Y(A,\, \mathbf{R}f_*B) \qquad \big(A \in \mathbf{K}(Y),\ B \in \mathbf{K}(X)\big) \tag{3.2.3.2}$$

such that the following natural diagram commutes

$$\begin{tikzcd}
f_*\mathcal{H}om^\bullet_X(f^*A, B) \arrow[r] \arrow[d, "(3.1.6)"', "\simeq"] & \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(f^*A, B) \arrow[r] & \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*A, B) \arrow[d, "\simeq"', "\beta"] \\
\mathcal{H}om^\bullet_Y(A,\, f_*B) \arrow[r] & \mathbf{R}\mathcal{H}om^\bullet_Y(A,\, f_*B) \arrow[r] & \mathbf{R}\mathcal{H}om^\bullet_Y(A,\, \mathbf{R}f_*B)
\end{tikzcd}$$

Proof. (i) For the first assertion it suffices, as in (2.6.5), that in the derived category of abelian groups the natural compositions

$$\mathrm{Hom}^\bullet_X(f^*A,\, B) \xrightarrow{a} \mathbf{R}\mathrm{Hom}^\bullet_X(f^*A,\, B) \xrightarrow{b} \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*A,\, B)$$

$$\mathrm{Hom}^\bullet_Y(A,\, f_*B) \xrightarrow{c} \mathbf{R}\mathrm{Hom}^\bullet_Y(A,\, f_*B) \xrightarrow{d} \mathbf{R}\mathrm{Hom}^\bullet_Y(A,\, \mathbf{R}f_*B)$$

be isomorphisms whenever A is q-flat and B is q-injective. But in this case we have $A = P_A$ and $B = I_B$, so that a, b, and d are identity maps. As for c, we need only note that by the last assertion of (3.2.2), the induced homology maps

$$H^i(c)\colon \mathrm{Hom}_{\mathbf{K}(Y)}(A[-i],\, f_*B) \to \mathrm{Hom}_{\mathbf{D}(Y)}(A[-i],\, f_*B)$$

are isomorphisms, see (1.2.2) and (2.4.2).

Now apply the functor H^0 to the diagram and conclude by the uniqueness of ρ in (3.2.2) that $H^0(\alpha)$ is as asserted.

(ii) As above, it comes down to showing that the natural maps

$$f_*\mathcal{H}om^\bullet_X(f^*A, B) \xrightarrow{a'} \mathbf{R}f_*\mathcal{H}om^\bullet_X(f^*A, B)$$

$$\mathcal{H}om^\bullet_Y(A, f_*B) \xrightarrow{c'} \mathbf{R}\mathcal{H}om^\bullet_Y(A, f_*B) = \mathcal{H}om^\bullet_Y(A, I_{f_*B})$$

are isomorphisms (in $\mathbf{D}(X)$, $\mathbf{D}(Y)$ respectively) whenever A is q-flat and B is q-injective. The stalk $(f^*A)_x$ $(x \in X)$ being isomorphic to $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} A_{f(x)}$, (2.5.2) shows that f^*A is q-flat, and then (2.3.8)(iv) shows (via (2.6.2)) that $\mathcal{H} := \mathcal{H}om^\bullet_X(f^*A, B)$ is q-injective; so $\mathcal{H} = I_{\mathcal{H}}$ and $a'\colon f_*\mathcal{H} \to f_*I_{\mathcal{H}}$ is in fact an identity map.

For c', it is enough to check that we get an isomorphism after applying the functor Γ_U (sections over U) for arbitrary open $U \subset Y$, since then c' induces isomorphisms of the homology presheaves—and hence of the homology sheaves—of its source and target (see (1.2.2)). Let $i\colon U \to Y$, $j\colon f^{-1}U \to X$ be the inclusion maps, and let $g\colon f^{-1}U \to U$ be the map induced by f.

We have then by (2.3.1) a commutative diagram of quasi-isomorphisms

$$\begin{tikzcd} i^*f_*B \arrow[r] \arrow[d, equal] & i^*I_{f_*B} \arrow[d, "\gamma"] \\ i^*f_*B \arrow[r] & I_{i^*f_*B} \end{tikzcd}$$

Since $i^*I_{f_*B}$ is q-injective (2.4.5.2), γ is an *isomorphism* in $\mathbf{K}(U)$ (2.3.2.2). Keeping in mind that $i^*f_* = g_*j^*$, consider the commutative diagram

$$\begin{tikzcd} \Gamma_U\mathcal{H}om^\bullet_Y(A,\, f_*B) \arrow[r, "\Gamma_U(c')"] \arrow[d, equal] & \Gamma_U\mathcal{H}om^\bullet_Y(A,\, I_{f_*B}) \arrow[d, equal] \\ \mathrm{Hom}^\bullet_U(i^*A,\, i^*f_*B) \arrow[r] \arrow[d, equal] & \mathrm{Hom}^\bullet_U(i^*A,\, i^*I_{f_*B}) \arrow[d, "\simeq"', "\text{via } \gamma"] \\ \mathrm{Hom}^\bullet_U(i^*A,\, i^*f_*B) \arrow[r] \arrow[d, equal] & \mathrm{Hom}^\bullet_U(i^*A,\, I_{i^*f_*B}) \arrow[d, equal] \\ \mathrm{Hom}^\bullet_U(i^*A,\, g_*j^*B) \arrow[r, "c_U"'] & \mathbf{R}\mathrm{Hom}^\bullet_U(i^*A,\, g_*j^*B) \end{tikzcd}$$

As in the proof of (i), since j^*B is q-injective and i^*A is q-flat (see above), therefore c_U is an isomorphism; and hence so is $\Gamma_U(c')$. Q.E.D.

Corollary (3.2.3.3). *Let* $U \subset Y$ *be open and let* $\Gamma_U\colon \mathcal{A}_Y \to \mathfrak{Ab}$ *be the abelian functor "sections over U." Then for any q-injective* $B \in \mathbf{K}(X)$, f_*B *is right-*Γ_U*-acyclic. Consequently, by* (2.2.7) *or* (2.6.5), *there is a unique* Δ*-functorial isomorphism* $\mathbf{R}\Gamma_{f^{-1}U} \xrightarrow{\sim} \mathbf{R}\Gamma_U\mathbf{R}f_*$ *making the following natural diagram commute for all* $B \in \mathbf{K}(X)$:

$$\begin{tikzcd} \Gamma_{f^{-1}U}B \arrow[rr] \arrow[d, equal] & & \mathbf{R}\Gamma_{f^{-1}U}B \arrow[d, "\simeq"] \\ \Gamma_U f_*B \arrow[r] & \mathbf{R}\Gamma_U f_*B \arrow[r] & \mathbf{R}\Gamma_U\mathbf{R}f_*B \end{tikzcd}$$

PROOF. Let $\mathcal{O}'_U \in \mathcal{A}_Y$ be the "extension by zero" of $\mathcal{O}_U \in \mathcal{A}_U$, i.e., the sheaf associated to the presheaf taking an open $V \subset Y$ to $\mathcal{O}_U(V)$ if $V \subset U$, and to 0 otherwise. Then there is a natural functorial identification $\Gamma_U(-) = \mathrm{Hom}_Y(\mathcal{O}'_U, -)$. Since $\mathcal{O}'_U$ is flat, we have as in the proof of (3.2.3)(i) that the map $c\colon \mathrm{Hom}^\bullet(\mathcal{O}'_U,\, f_*B) \to \mathbf{R}\mathrm{Hom}^\bullet(\mathcal{O}'_U,\, f_*B)$ is an isomorphism, i.e., $\Gamma_U(f_*B) \to \mathbf{R}\Gamma_U(f_*B)$ is an isomorphism, whence the conclusion (see last assertion in (2.2.6)). Q.E.D.

Proposition (3.2.4). (i) *For any ringed-space map* $f\colon X \to Y$, *there is a unique* Δ*-bifunctorial isomorphism*

$$\mathbf{L}f^*(A \underset{=}{\otimes}_Y B) \xrightarrow{\ \sim\ } \mathbf{L}f^*A \underset{=}{\otimes}_X \mathbf{L}f^*B \qquad \big(A, B \in \mathbf{D}(Y)\big)$$

making the following natural diagram commute for all A, B :

$$\begin{tikzcd}
\mathbf{L}f^*(A \underset{=}{\otimes}_Y B) \arrow[r, "\sim"] \arrow[d] & \mathbf{L}f^*A \underset{=}{\otimes}_X \mathbf{L}f^*B \arrow[d] \\
f^*(A \otimes_Y B) \arrow[r, "\sim", "(3.1.9)"'] & f^*A \otimes_X f^*B
\end{tikzcd} \tag{3.2.4.1}$$

This isomorphism is conjugate (3.3.5) *to the isomorphism* β *in* (3.2.3.2).

(ii) *With* $\eta'\colon E \to \mathbf{R}\mathcal{H}om^\bullet_X(F, E \underset{=}{\otimes} F)$ *corresponding via* $(2.6.1)^*$ *to the identity map of* $E \underset{=}{\otimes} F$, *and* $\epsilon\colon \mathbf{L}f^*\mathbf{R}f_* \to 1$ *as in* (3.2.1.0), *the* (Δ*-functorial*) *map*

$$\gamma\colon \mathbf{R}f_*(E) \underset{=}{\otimes} \mathbf{R}f_*(F) \longrightarrow \mathbf{R}f_*(E \underset{=}{\otimes} F) \qquad \big(E, F \in \mathbf{D}(X)\big) \tag{3.2.4.2}$$

adjoint to the composed map

$$\mathbf{L}f^*\big(\mathbf{R}f_*E \underset{=}{\otimes} \mathbf{R}f_*F\big) \xrightarrow{\ \sim\ } \mathbf{L}f^*\mathbf{R}f_*E \underset{=}{\otimes} \mathbf{L}f^*\mathbf{R}f_*F \xrightarrow[\epsilon \underset{=}{\otimes} \epsilon]{} E \underset{=}{\otimes} F \tag{3.2.4.3}$$

corresponds via $(2.6.1)^*$ *to the composed map*

$$\begin{aligned}
\mathbf{R}f_*E &\xrightarrow{\mathbf{R}f_*\eta'} \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, E \underset{=}{\otimes} F) \\
&\xrightarrow{\text{via } \epsilon} \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*\mathbf{R}f_*F, E \underset{=}{\otimes} F) \\
&\xrightarrow[(3.2.3.2)]{\beta} \mathbf{R}\mathcal{H}om^\bullet_X\big(\mathbf{R}f_*F, \mathbf{R}f_*(E \underset{=}{\otimes} F)\big).
\end{aligned} \tag{3.2.4.4}$$

Proof. (i) For $x \in X$, the stalk $(f^*A)_x$ is $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} A_{f(x)}$, and so (2.5.2) shows that f^*A is q-flat whenever A is. Hence if A and B are both q-flat (whence so, clearly, is $A \otimes_Y B$), then the vertical arrows in (3.2.4.1) are isomorphisms, and the first assertion follows from (2.6.5) (dualized).

The second assertion amounts to commutativity, for any complexes $E, F, G \in \mathbf{D}(X)$, of the following diagram of natural isomorphisms:

(3.2.4.5)

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{D}(X)}\big(\mathsf{L}f^*E,\,\mathsf{R}\mathcal{H}om^\bullet_X(\mathsf{L}f^*F,G)\big) & \xrightarrow{(2.6.1)^*} & \mathrm{Hom}_{\mathbf{D}(X)}\big(\mathsf{L}f^*E\underset{=}{\otimes}\mathsf{L}f^*F,\,G\big)\\
{\scriptstyle(3.2.1)}\big\downarrow & & \big\downarrow{\scriptstyle\simeq}\\
\mathrm{Hom}_{\mathbf{D}(Y)}\big(E,\,\mathsf{R}f_*\mathsf{R}\mathcal{H}om^\bullet_X(\mathsf{L}f^*F,G)\big) & & \mathrm{Hom}_{\mathbf{D}(X)}\big(\mathsf{L}f^*(E\underset{=}{\otimes}F),\,G\big)\\
{\scriptstyle\text{via }\beta}\big\downarrow & & \big\downarrow{\scriptstyle(3.2.1)}\\
\mathrm{Hom}_{\mathbf{D}(Y)}\big(E,\,\mathsf{R}\mathcal{H}om^\bullet_Y(F,\mathsf{R}f_*G)\big) & \xrightarrow[(2.6.1)^*]{} & \mathrm{Hom}_{\mathbf{D}(Y)}\big(E\underset{=}{\otimes}F,\,\mathsf{R}f_*G\big)
\end{array}
$$

in proving which, we may replace E by P_E, F by p_f, and G by I_G, i.e., we may assume E and F to be q-flat and G to be q-injective. Using the commutativity in $(2.6.1)^*$ (after applying homology H^0), (3.2.2.1), (3.2.3.2), and (3.2.4.1), we find that (3.2.4.5) is the target of a natural map, in the category of diagrams of abelian groups, coming from the diagram of isomorphisms (see (3.1.6), and recall that $H^0\mathrm{Hom}^\bullet_X=\mathrm{Hom}_{\mathbf{K}(X)}$):

(3.2.4.6)

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{K}(X)}\big(f^*E,\,\mathcal{H}om^\bullet_X(f^*F,G)\big) & \longrightarrow & \mathrm{Hom}_{\mathbf{K}(X)}\big(f^*E\otimes f^*F,\,G\big)\\
\big\downarrow & & \big\downarrow\\
\mathrm{Hom}_{\mathbf{K}(Y)}\big(E,\,f_*\mathcal{H}om^\bullet_X(f^*F,G)\big) & & \mathrm{Hom}_{\mathbf{K}(X)}\big(f^*(E\otimes F),\,G\big)\\
\big\downarrow & & \big\downarrow\\
\mathrm{Hom}_{\mathbf{K}(Y)}\big(E,\,\mathcal{H}om^\bullet_Y(F,f_*G)\big) & \longrightarrow & \mathrm{Hom}_{\mathbf{K}(X)}\big(E\otimes F,\,f_*G\big)
\end{array}
$$

Also, E and F are q-flat (so that $\mathsf{L}f^*E\underset{=}{\otimes}\mathsf{L}f^*F \xrightarrow{\sim} f^*E\otimes f^*F$) and G is q-injective, so any $\mathbf{D}(X)$-map $\mathsf{L}f^*E\underset{=}{\otimes}\mathsf{L}f^*F\to G$ is represented by a map of complexes $f^*E\otimes f^*F\to G$, see (2.3.8)(v). Hence one need only show (3.2.4.6) commutative. This is exercise (3.1.10), left to the reader.

(ii) With $\eta\colon 1\to\mathsf{R}f_*\mathsf{L}f^*$ as in (3.2.1.0), the map (3.2.4.2) is the composition

$$
\mathsf{R}f_*(E)\underset{=}{\otimes}\mathsf{R}f_*(F)\xrightarrow{\eta}\mathsf{R}f_*\mathsf{L}f^*(\mathsf{R}f_*(E)\underset{=}{\otimes}\mathsf{R}f_*(F))\xrightarrow{\mathsf{R}f_*(3.2.4.3)}\mathsf{R}f_*(E\underset{=}{\otimes}F)
$$

which is clearly Δ-functorial. The rest of the statement is best understood in the formal context of closed categories, see (3.5.4). In the present instance of that context—see (3.5.2)(d) and (3.4.4)(b)—the map (3.4.2.1) is just γ, and hence the adjoint (3.5.4.1) of (3.4.2.1) is the map in (i) above. Commutativity of (3.5.5.1) says that (3.4.5.1) is conjugate to the map (3.5.4.2), which must then, by (i), be β. Hence (ii) follows from the sentence preceding (3.5.4.2) and the description of (3.5.4.1) immediately following (3.5.4.2). Q.E.D.

Remark. Commutativity of (3.2.4.5) yields another proof that β is an isomorphism, since the maps labeled (3.2.1) and $(2.6.1)^*$ are isomorphisms.

EXERCISES **(3.2.5).** $f\colon X \to Y$ is a ringed-space map, $A \in \mathbf{D}(A)$, $B \in \mathbf{D}(X)$.

(a) Show that the following two natural composed maps correspond under the adjunction isomorphism (3.2.1):

$$\mathbf{L}f^*\mathcal{O}_Y \to f^*\mathcal{O}_Y \to \mathcal{O}_X\,, \qquad \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_X\,.$$

(b) Write τ_n for the truncation functor $\tau_{\ge n}$ of §1.10. Also, write f_* (resp. f^*) for $\mathbf{R}f_*$ (resp. $\mathbf{L}f^*$). Define the functorial map

$$\psi\colon f^*\tau_n \longrightarrow \tau_n f^*$$

to be the adjoint of the natural composed map

$$\tau_n \longrightarrow \tau_n f_* f^* \longrightarrow \tau_n f_* \tau_n f^* \xrightarrow{\ \sim\ } f_*\tau_n f^*.$$

(The isomorphism obtains because $f_*\mathbf{D}_{\ge \mathbf{n}}(X) \subset \mathbf{D}_{\ge \mathbf{n}}(Y)$, see (2.3.4).) Show that the following natural diagram commutes:

$$\begin{tikzcd}
f^* \arrow[r] \arrow[d] & f^*\tau_n \arrow[r, equal] \arrow[d, "\psi"'] & f^*\tau_n \arrow[d] \\
\tau_n f^* \arrow[r, equal] & \tau_n f^* \arrow[r] & \tau_n f^*\tau_n
\end{tikzcd}$$

(One way is to check commutativity of the diagram whose columns are adjoint to those of the one in question. For this, (1.10.1.2) may be found useful.)

(c) The natural map $\mathrm{Hom}^\bullet_Y(A, f_*B) \to \mathbf{R}\mathrm{Hom}^\bullet_Y(A, \mathbf{R}f_*B)$ is an isomorphism for all q-injective $B \in \mathbf{K}(X)$ iff $\mathbf{L}f^*A \to f^*A$ is an isomorphism.

(d) Formulate and prove a statement to the effect that the map β in (3.2.3.2) is compatible with open immersions $U \hookrightarrow Y$.

(e) With Γ_Y as in (3.2.3.3), show that the natural map

$$\Gamma_Y f_*\mathcal{H}om^\bullet_X(f^*A, B) \to \mathbf{R}\Gamma_Y \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*A, B)$$

is an isomorphism if A is q-flat and B is q-injective.

(f) Show that there is a natural diagram of isomorphisms

$$\begin{tikzcd}
\mathbf{R}\Gamma_Y \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*A, B) \arrow[r, "\sim", "(3.2.3.2)"'] \arrow[d, "\simeq"'] & \mathbf{R}\Gamma_Y \mathbf{R}\mathcal{H}om^\bullet_Y(A, \mathbf{R}f_*B) \arrow[d, "\simeq"] \\
\mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*A,\ B) \arrow[r, "\sim", "(3.2.3.1)"'] & \mathbf{R}\mathrm{Hom}^\bullet_Y(A,\ \mathbf{R}f_*B)
\end{tikzcd}$$

see (2.5.10)(b) and (3.2.3.3).

(First show the same with all $\mathbf{R}$'s and $\mathbf{L}$'s dropped; then apply (e) and (2.6.5).)

3.3. Δ-adjoint functors

We now run through the sorites related to adjointness of Δ-functors. Later, we will be constructing numerous functorial maps between multivariate Δ-functors by purely formal (category-theoretic) methods. The results in this section, together with the Proposition in §1.5, will guarantee that the so-constructed maps are in fact Δ-functorial.

Let $\mathbf{K_1}$ and $\mathbf{K_2}$ be Δ-categories with respective translation functors T_1 and T_2, and let $(f_*, \theta_*)\colon \mathbf{K_1} \to \mathbf{K_2}$ and $(f^*, \theta^*)\colon \mathbf{K_2} \to \mathbf{K_1}$ be Δ-functors such that f^* is left-adjoint to f_*. (Recall from §1.5 that $\theta_*\colon f_*T_1 \xrightarrow{\ \sim\ } T_2 f_*$, and similarly $\theta^*\colon f^*T_2 \xrightarrow{\ \sim\ } T_1 f^*$.) Let $\eta\colon 1 \to f_*f^*$, $\epsilon\colon f^*f_* \to 1$ be the functorial maps corresponding by adjunction to the identity maps of f^*, f_* respectively.

Lemma-Definition (3.3.1). *In the above circumstances, the following conditions are equivalent:*

(i) *η is Δ-functorial.*
(i)′ *ϵ is Δ-functorial.*
(ii) *For all $A \in \mathbf{K_2}$ and $B \in \mathbf{K_1}$, the following natural diagram commutes:*

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbf{K_1}}(f^*A,\, B) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathbf{K_1}}(T_1f^*A,\, T_1B) \arrow[r, "\theta^*"] & \mathrm{Hom}_{\mathbf{K_1}}(f^*T_2A,\, T_1B) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathbf{K_2}}(A,\, f_*B) \arrow[r] & \mathrm{Hom}_{\mathbf{K_2}}(T_2A,\, T_2f_*B) \arrow[r, "\theta_*"'] & \mathrm{Hom}_{\mathbf{K_2}}(T_2A,\, f_*T_1B)
\end{tikzcd}
$$

When these conditions hold, we say that (f^*, θ^*) and (f_*, θ_*) are Δ-adjoint, *or—leaving θ^* and θ_* to the reader—that (f^*, f_*) is a Δ-adjoint pair.*

Proof. Suppose (i) holds. To prove (ii), chase a map $\xi\colon f^*A \to B$ around the diagram in both directions to reduce to showing that the following diagram commutes:

$$
\begin{tikzcd}
T_2A \arrow[r, "T_2\eta(A)"] \arrow[d, "\eta(T_2A)"'] & T_2f_*f^*A \arrow[r, "T_2f_*\xi"] \arrow[d, "\theta_*^{-1}(f^*A)"] & T_2f_*B \arrow[d, "\theta_*^{-1}(B)"] \\
f_*f^*T_2A \arrow[r, "f_*\theta^*(A)"'] & f_*T_1f^*A \arrow[r, "f_*T_1\xi"'] & f_*T_1B
\end{tikzcd}
\tag{3.3.1.1}
$$

The first square commutes by (i), and the second by functoriality of θ_*.

Conversely, (i) is just commutativity of (3.3.1.1) when $B := f^*A$ and ξ is the identity map.

Thus (i) ⇔ (ii); and a similar proof (starting with a map $\xi'\colon A \to f_*B$) yields (i)′ ⇔ (ii). Q.E.D.

Example **(3.3.2).** Quasi-inverse Δ-equivalences of categories (1.7.2) are Δ-adjoint pairs.

Example **(3.3.3).** The pair $(\mathbf{L}f^*, \mathbf{R}f_*)$ in (3.2.1) is Δ-adjoint. Indeed, in the proof of (3.2.1) the associated η and ϵ were defined to be certain Δ-functorial maps.

Example **(3.3.4).** With reference to (2.6.1)*, let $\mathbf{K_1} := \mathbf{D}(\mathcal{A}) =: \mathbf{K_2}$, fix $F \in \mathbf{D}(\mathcal{A})$, and for any $A, B \in \mathbf{D}(\mathcal{A})$ set

$$f^*A := A \underset{=}{\otimes} F, \qquad f_*B := \mathbf{R}\mathcal{H}om^\bullet(F,\, B)\,.$$

Then this pair (f^*, f_*) is Δ-adjoint. To verify condition (ii) in (3.3.1), consider the following diagram of natural isomorphisms, where $\mathrm{H}^\bullet$ stands for $\mathbf{R}\mathrm{Hom}^\bullet$ and $\mathcal{H}^\bullet$ stands for $\mathbf{R}\mathcal{H}om^\bullet$:

$$
\begin{tikzcd}
\mathrm{H}^\bullet(A \underset{=}{\otimes} F, B) \arrow[rr, "(2.6.1)^*"] \arrow[d] & \arrow[d, phantom, "\textcircled{1}"] & \mathrm{H}^\bullet(A, \mathcal{H}^\bullet(F,B)) \arrow[d] \\
\mathrm{H}^\bullet(A \underset{=}{\otimes} F, B[1])[-1] \arrow[r] \arrow[d] & \mathrm{H}^\bullet(A, \mathcal{H}^\bullet(F, B[1]))[-1] \arrow[dd] & \mathrm{H}^\bullet(A, \mathcal{H}^\bullet(F,B)[1])[-1] \arrow[l] \arrow[dd] \\
\mathrm{H}^\bullet((A \underset{=}{\otimes} F)[1], B[1]) \arrow[d] \arrow[r, phantom, "\textcircled{2}"] & \arrow[r, phantom, "\textcircled{3}"] & \\
\mathrm{H}^\bullet(A[1] \underset{=}{\otimes} F, B[1]) \arrow[r] & \mathrm{H}^\bullet(A[1], \mathcal{H}^\bullet(F, B[1])) & \mathrm{H}^\bullet(A[1], \mathcal{H}^\bullet(F,B)[1]) \arrow[l]
\end{tikzcd}
$$

Subdiagram ① commutes because $(2.6.1)^*$ is Δ-functorial in the last variable; ② commutes because $(2.6.1)^*$ is Δ-functorial in the first variable; and ③ commutes for obvious reasons. One checks that application of the functor H^0 to this big commutative diagram gives (ii) in (3.3.1). Q.E.D.

In particular, we have the canonical Δ-*functorial* maps

$$
\begin{aligned}
&\eta'\colon A \to \mathbf{R}\mathcal{H}om^\bullet(F, A \underset{=}{\otimes} F)\,, \\
&\epsilon'\colon \mathbf{R}\mathcal{H}om^\bullet(F,B) \underset{=}{\otimes} F \to B\,.
\end{aligned}
\tag{3.3.4.1}
$$

Lemma-Definition (3.3.5). *If* $f_*\colon \mathbf{X} \to \mathbf{Y}$, $g_*\colon \mathbf{X} \to \mathbf{Y}$ *are functors with respective left adjoints* $f^*\colon \mathbf{Y} \to \mathbf{X}$, $g^*\colon \mathbf{Y} \to \mathbf{X}$, *then with* "Hom" *denoting "functorial morphisms," the following natural compositions are inverse isomorphisms:*

$$\mathrm{Hom}(f_*,\, g_*) \longrightarrow \mathrm{Hom}(f_*f^*,\, g_*f^*) \longrightarrow \mathrm{Hom}(1,\, g_*f^*) \xrightarrow{\ \sim\ } \mathrm{Hom}(g^*,\, f^*)\,,$$

$$\mathrm{Hom}(f_*,\, g_*) \xleftarrow{\ \sim\ } \mathrm{Hom}(g^*f_*,\, 1) \longleftarrow \mathrm{Hom}(g^*f_*,\, f^*f_*) \longleftarrow \mathrm{Hom}(g^*,\, f^*)\,.$$

Functorial morphisms $f_* \to g_*$ *and* $g^* \to f^*$ *which correspond under these isomorphisms will be said to be* conjugate (*the first* right-conjugate *to the second, the second* left-conjugate *to the first*).

Proof. Exercise, or see [**M**, p. 100, Theorem 2].

Corollary (3.3.6). *Let* (f^*, f_*) *and* (g^*, g_*) *be* Δ*-adjoint pairs of* Δ*-functors between* $\mathbf{K_1}$ *and* $\mathbf{K_2}$. *Then a functorial morphism* $\alpha\colon f_* \to g_*$ *is* Δ*-functorial if and only if so is its conjugate* $\beta\colon g^* \to f^*$. *In particular,* f_* *and* g_* *are isomorphic* Δ*-functors* $\Leftrightarrow$ *so are* f^* *and* g^*.

The first assertion follows from (3.3.1) since, for example, α is the composition

$$f_* \xrightarrow{\eta} g_*g^*f_* \xrightarrow{\text{via } \beta} g_*f^*f_* \xrightarrow{\epsilon} g_*\,.$$

That the conjugate of a functorial isomorphism is an isomorphism follows from Exercise (3.3.7)(c) below.

Exercises **(3.3.7).** (a) Maps $\alpha\colon f_* \to g_*$ and $\beta\colon g^* \to f^*$ are conjugate $\Leftrightarrow$ (either one of) the following diagrams commute:

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[d, "\eta"'] & g_*g^* \arrow[d, "\text{via }\beta"] \\ f_*f^* \arrow[r, "\text{via }\alpha"'] & g_*f^* \end{tikzcd} \qquad \begin{tikzcd} 1 & g^*g_* \arrow[l, "\epsilon"'] \\ f^*f_* \arrow[u, "\epsilon"] & g^*f_* \arrow[l, "\text{via }\beta"] \arrow[u, "\text{via }\alpha"'] \end{tikzcd}$$

(b) The conditions in (a) are equivalent to commutativity, for all $X \in \mathbf{X}$, $Y \in \mathbf{Y}$ of the diagram

$$\begin{tikzcd} \mathrm{Hom}(Y, f_*X) \arrow[r, "\text{via }\alpha"] \arrow[d, "\simeq"'] & \mathrm{Hom}(Y, g_*X) \arrow[d, "\simeq"] \\ \mathrm{Hom}(f^*Y, X) \arrow[r, "\text{via }\beta"'] & \mathrm{Hom}(g^*Y, X) \end{tikzcd}$$

(c) Denoting the conjugate of a functorial map α by α' we have (with the obvious interpretation) $1' = 1$ and $(\alpha_2\alpha_1)' = \alpha_1'\alpha_2'$.

(d) The conditions in (3.3.1) are equivalent to either one of:

(iii) The functorial map $\theta^*\colon f^*T_2 \xrightarrow{\sim} T_1f^*$ is left-conjugate to

$$f_*T_1^{-1} = T_2^{-1}T_2f_*T_1^{-1} \xrightarrow{\theta_*^{-1}} T_2^{-1}f_*T_1T_1^{-1} = T_2^{-1}f_*\,.$$

(iii)′ The functorial map $\theta_*\colon f_*T_1 \xrightarrow{\sim} T_2f_*$ is right-conjugate to

$$f^*T_2^{-1} = T_1^{-1}T_1f^*T_2^{-1} \xrightarrow{\theta^{*\,-1}} T_1^{-1}f^*T_2T_2^{-1} = T_1^{-1}f^*.$$

The next Proposition, generalizing some of (1.7.2), says that a left adjoint of a Δ-functor can be made into a left Δ-adjoint, in a unique way.

Let $\mathbf{K_1}$, $\mathbf{K_2}$ be Δ-categories with respective translation functors T_1, T_2, and let $(f_*, \theta_*)\colon \mathbf{K_1} \to \mathbf{K_2}$ be a Δ-functor such that f_* has a left adjoint $f^*\colon \mathbf{K_2} \to \mathbf{K_1}$ (automatically additive, see first footnote in §3.1) .

Proposition (3.3.8). *There exists a unique functorial isomorphism*

$$\theta^*\colon f^*T_2 \xrightarrow{\sim} T_1f^*$$

such that

(i) (f^*, θ^*) *is a* Δ*-functor, and*
(ii) *the* Δ*-functors* (f^*, θ^*) *and* (f_*, θ_*) *are* Δ*-adjoint.*

Proof. The functors f^*T_2 and T_1f^* are left-adjoint to $T_2^{-1}f_*$ and $f_*T_1^{-1}$ respectively; and since the latter two are isomorphic (in the obvious way via θ_*), so are the former two, and one checks that the conjugate isomorphism θ^* between them is adjoint to the composite map

$$T_2 \xrightarrow{\eta} T_2f_*f^* \xrightarrow{\theta_*^{-1}} f_*T_1f^*,$$ [33]

i.e., θ^* is the *unique* map making the following diagram commute:

(3.3.8.1)
$$\begin{tikzcd} T_2 \arrow[rr, equal] \arrow[d, "\eta"'] & & T_2 \arrow[d, "\eta"] \\ f_*f^*T_2 \arrow[r, "f_*\theta^*"'] & f_*T_1f^* \arrow[r, "\theta_*"'] & T_2f_*f^* \end{tikzcd}$$

[33] whence, dually, θ_*^{-1} is adjoint to $T_1 \xleftarrow{\epsilon} T_1f^*f_* \xleftarrow{\theta^*} f^*T_2f_*$.

If (i) holds, then commutativity of (3.3.8.1) also expresses the condition that $\eta\colon 1 \to f_*f^*$ be Δ-functorial, i.e., that (ii) hold. Thus no other θ^* can satisfy (i) and (ii). (So far, the argument is just a variation on (3.3.7)(d).)

We still have to show that (i) holds for the θ^* we have specified. So let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T_2A$ be a triangle in $\mathbf{K_2}$. Apply $(\Delta 3)'$ in (1.4) to embed f^*u into a triangle $f^*A \xrightarrow{f^*u} f^*B \xrightarrow{p} C^* \xrightarrow{q} T_1f^*A$. I claim that

(a) *there is a map* $\gamma\colon f^*C \to C^*$ *making the following diagram commute:*

$$\begin{tikzcd}
f^*A \arrow[r, "f^*u"] \arrow[d, equal] & f^*B \arrow[r, "f^*v"] \arrow[d, equal] & f^*C \arrow[r, "\theta^*\circ f^*w"] \arrow[d, "\gamma"'] & T_1f^*A \arrow[d, equal] \\
f^*A \arrow[r, "f^*u"'] & f^*B \arrow[r, "p"'] & C^* \arrow[r, "q"'] & T_1f^*A
\end{tikzcd}$$

and that (b) *any such* γ *must be an isomorphism.*

Given (a) and (b), condition $(\Delta 1)''$ in (1.4) ensures that the top row in the preceding diagram is a triangle, so that (f^*, θ^*) is indeed a Δ-functor.

Assertion (a) results by adjunction from the map of triangles

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\eta"'] & B \arrow[r, "v"] \arrow[d, "\eta"'] & C \arrow[r, "w"] \arrow[d, "\gamma'"] & T_2A \arrow[d, "T_2\eta"] \\
f_*f^*A \arrow[r, "f_*f^*u"'] & f_*f^*B \arrow[r, "f_*p"'] & f_*C^* \arrow[r, "\theta_*\circ f_*q"'] & T_2f_*f^*A
\end{tikzcd}$$

where γ' is given by $(\Delta 3)''$ in (1.4).

For (b), consider the commutative diagram (with $D \in \mathbf{K_1}$, and obvious maps):

$$\begin{tikzcd}
\mathrm{Hom}(T_1f^*B, D) \arrow[r, equal] \arrow[d] & \mathrm{Hom}(T_1f^*B, D) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}(T_2B,\ f_*D) \arrow[d] \\
\mathrm{Hom}(T_1f^*A, D) \arrow[r, equal] \arrow[d] & \mathrm{Hom}(T_1f^*A, D) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}(T_2A,\ f_*D) \arrow[d] \\
\mathrm{Hom}(C^*, D) \arrow[r, "\text{via } \gamma"] \arrow[d] & \mathrm{Hom}(f^*C, D) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}(C,\ f_*D) \arrow[d] \\
\mathrm{Hom}(f^*B, D) \arrow[r, equal] \arrow[d] & \mathrm{Hom}(f^*B, D) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}(B,\ f_*D) \arrow[d] \\
\mathrm{Hom}(f^*A, D) \arrow[r, equal] & \mathrm{Hom}(f^*A, D) \arrow[r, "\sim"] & \mathrm{Hom}(A,\ f_*D)
\end{tikzcd}$$

The left and right columns are *exact* [**H**, p. 23, Prop. 1.1, b], hence the map "via γ" is an isomorphism for all D, i.e., γ is an isomorphism. Q.E.D.

3.4. Adjoint functors between monoidal categories

This section and the following one introduce some of the formalism arising from a pair of adjoint monoidal functors between closed categories. A simple example of such a pair occurs with respect to a map $R \to S$ of commutative rings, namely extension and restriction of scalars on the appropriate module categories. The module functors f^* and f_* associated with a map $f\colon X \to Y$ of ringed spaces form another such pair. The example which mosts interests us is that of the pair $(\mathbf{L}f^*, \mathbf{R}f_*)$ of §3.2. The point is to develop by purely categorical methods a host of relations, expressed by commutative functorial diagrams, among the four operations $\underline{\underline{\otimes}}$, $\mathbf{R}\mathcal{H}om^\bullet$, $\mathbf{L}f^*$ and $\mathbf{R}f_*$.

But even the purified categorical approach leads quickly to stultifying complexity—at which the exercises (3.5.6) merely hint. Ideally, we would like to have an implementable algorithm for deciding when a functorial diagram built up from the data given in the relevant categorical definitions (see (3.4.1), (3.4.2), (3.5.1)) commutes; or in other words, to prove a "constructive coherence theorem" for the generic context "monoidal functor between closed categories, together with left adjoint." (Lewis [**Lw**] does this, to some extent, without the left adjoint.) Though there exists a substantial body of results on "coherence in categories," see e.g., [**K′**], [**Sv**], and their references, it does not yet suffice; we will have to be content with subduing individual diagrams as needs dictate.

We treat symmetric monoidal categories in this section, leaving the additional "closed" structure to the next.

Definition (3.4.1). A *symmetric monoidal category*

$$\mathbf{M} = (\mathbf{M_0}, \otimes, \mathcal{O}_M, \alpha, \lambda, \rho, \gamma)$$

consists of a category $\mathbf{M_0}$, a "product" functor $\otimes\colon \mathbf{M_0} \times \mathbf{M_0} \to \mathbf{M_0}$, an object $\mathcal{O}_M$ of $\mathbf{M_0}$, and functorial isomorphisms

(associativity) $\qquad \alpha : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$

(units) $\qquad \lambda : \mathcal{O}_M \otimes A \xrightarrow{\sim} A \qquad \rho\colon A \otimes \mathcal{O}_M \xrightarrow{\sim} A$

(symmetry) $\qquad \gamma\colon A \otimes B \xrightarrow{\sim} B \otimes A$

(where A, B, C are objects in $\mathbf{M_0}$) such that $\gamma \circ \gamma = 1$ and the following diagrams (3.4.1.1) commute.

$$\begin{tikzcd}
(A \otimes \mathcal{O}_M) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes 1"'] & & A \otimes (\mathcal{O}_M \otimes B) \arrow[dl, "1\otimes\lambda"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes D \arrow[r, "\alpha"] \arrow[d, "\alpha\otimes 1"'] & (A\otimes B)\otimes(C\otimes D) \arrow[r, "\alpha"] & A\otimes(B\otimes(C\otimes D)) \arrow[d, "1\otimes\alpha"] \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "\alpha"'] & & A\otimes((B\otimes C)\otimes D)
\end{tikzcd}$$

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\alpha"] \arrow[d, "\gamma\otimes 1"'] & A\otimes(B\otimes C) \arrow[r, "\gamma"] & (B\otimes C)\otimes A \arrow[d, "\alpha"] \\
(B\otimes A)\otimes C \arrow[r, "\alpha"'] & B\otimes(A\otimes C) \arrow[r, "1\otimes\gamma"'] & B\otimes(C\otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_M\otimes A \arrow[rr, "\gamma"] \arrow[dr, "\lambda"'] & & A\otimes\mathcal{O}_M \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

(3.4.1.1)

Definition (3.4.2). A *symmetric monoidal functor* $f_*\colon \mathbf{X}\to\mathbf{Y}$ between symmetric monoidal categories $\mathbf{X},\mathbf{Y}$ is a functor $f_{*0}\colon \mathbf{X_0}\to\mathbf{Y_0}$ together with two functorial maps

(3.4.2.1)
$$\begin{gathered} f_*A\otimes f_*B \longrightarrow f_*(A\otimes B) \\ \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \end{gathered}$$

(where we have abused notation, *as we will henceforth,* by omitting the subscript "$_0$" and by not distinguishing notationally between $\otimes$ in $\mathbf{X}$ and $\otimes$ in $\mathbf{Y}$), such that the following natural diagrams (3.4.2.2) commute.

$$\begin{tikzcd}
f_*\mathcal{O}_X\otimes f_*A \arrow[r] & f_*(\mathcal{O}_X\otimes A) \arrow[d, "f_*(\lambda_X)"] \\
\mathcal{O}_Y\otimes f_*A \arrow[u] \arrow[r, "\lambda_Y"'] & f_*A
\end{tikzcd}
\qquad
\begin{tikzcd}
f_*A\otimes f_*B \arrow[r] \arrow[d, "\gamma_Y"'] & f_*(A\otimes B) \arrow[d, "f_*(\gamma_X)"] \\
f_*B\otimes f_*A \arrow[r] & f_*(B\otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(f_*A\otimes f_*B)\otimes f_*C \arrow[r] \arrow[d, "\alpha"'] & f_*(A\otimes B)\otimes f_*C \arrow[r] & f_*((A\otimes B)\otimes C) \arrow[d, "f_*(\alpha)"] \\
f_*A\otimes(f_*B\otimes f_*C) \arrow[r] & f_*A\otimes f_*(B\otimes C) \arrow[r] & f_*(A\otimes(B\otimes C))
\end{tikzcd}$$

(3.4.2.2)

(3.4.3). We assume further that the symmetric monoidal functor f_* has a *left adjoint* $f^*\colon \mathbf{Y} \to \mathbf{X}$. In other words we have functorial maps

$$\eta\colon 1 \to f_*f^* \qquad\qquad \epsilon\colon f^*f_* \to 1$$

such that the composites

$$f_* \xrightarrow{\text{via } \eta} f_*f^*f_* \xrightarrow{\text{via } \epsilon} f_* \qquad\qquad f^* \xrightarrow{\text{via } \eta} f^*f_*f^* \xrightarrow{\text{via } \epsilon} f^*$$

are identities, giving rise to a bifunctorial isomorphism

$$(3.4.3.1) \qquad \mathrm{Hom}_{\mathbf{X}}(f^*F,\, G) \xrightarrow{\;\sim\;} \mathrm{Hom}_{\mathbf{Y}}(F,\, f_*G) \qquad (F \in \mathbf{Y},\ G \in \mathbf{X}).$$

EXAMPLES **(3.4.4).** (a) Let $f\colon X \to Y$ be a map of ringed spaces, $\mathbf{X}$ (resp. $\mathbf{Y}$) the category of $\mathcal{O}_X$- (resp. $\mathcal{O}_Y$-)modules with its standard structure of symmetric monoidal category ($\otimes$ having its usual meaning, etc. etc.), f_* and f^* the usual direct and inverse image functors, see (3.1.7).

(b) Let $f\colon X \to Y$ be a ringed-space map, $\mathbf{X} := \mathbf{D}(X)$, $\mathbf{Y} := \mathbf{D}(Y)$, $\otimes := \underset{=}{\otimes}$, $f_* := \mathbf{R}f_*$, $f^* := \mathbf{L}f^*$ (see (3.2.1)). To establish symmetric monoidality of, e.g., $\mathbf{D}(X)$, one need only work with q-flat complexes, ... For (3.4.2.1), use the map γ from (3.2.4.2) and the natural composition $\mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_X$. One can then deduce via adjointness that $\mathbf{R}f_*$ is symmetric monoidal from the fact that $\mathbf{L}f^*$ is symmetric monoidal when considered as a functor from $\mathbf{Y}^{\mathsf{op}}$ to $\mathbf{X}^{\mathsf{op}}$, see (3.2.4). For this property of $\mathbf{L}f^*$, one can check the requisite commutativity in (3.4.2.2) after replacing each object A in $\mathbf{X}$ by an isomorphic q-flat complex, and recalling that if A is q-flat, then so is f^*A (see proof of (3.2.3)(ii)); in view of (3.1.3), the checking is thereby reduced to the context of (a) above, where one can use adjointness (see (3.1.9)) to deduce what needs to be known about f^* after showing directly that f_* is symmetric monoidal!

For example, to show commutativity of

$$\begin{tikzcd}
\mathbf{R}f_*(\mathcal{O}_X) \underset{=}{\otimes} \mathbf{R}f_*(A) \arrow[r, "\gamma"] & \mathbf{R}f_*(\mathcal{O}_X \underset{=}{\otimes} A) \arrow[d, "\lambda_X"] \\
\mathcal{O}_Y \underset{=}{\otimes} \mathbf{R}f_*(A) \arrow[u] \arrow[r, "\lambda_Y"'] & \mathbf{R}f_*(A)
\end{tikzcd}$$

consider the following natural diagram, in which we have written f^*, f_*, $\otimes$ for $\mathbf{L}f^*$, $\mathbf{R}f_*$, $\underset{=}{\otimes}$ respectively:

$$\begin{tikzcd}
& f^*f_*\mathcal{O}_X \otimes f^*f_*A \arrow[r] & \mathcal{O}_X \otimes A \arrow[dr] & \\
f^*(f_*\mathcal{O}_X \otimes f_*A) \arrow[ur] & f^*\mathcal{O}_Y \otimes f^*f_*A \arrow[u] \arrow[r] \arrow[ur, phantom, "\textcircled{1}"] & \mathcal{O}_X \otimes f^*f_*A \arrow[u] \arrow[d] & A \\
& f^*(\mathcal{O}_Y \otimes f_*A) \arrow[ul] \arrow[u] \arrow[r] \arrow[ur, phantom, "\textcircled{2}"] & f^*f_*A \arrow[ur] &
\end{tikzcd}$$

It will be enough to show that the outer border commutes, because it is "adjoint" to the preceding diagram, see (3.4.5.2). Subdiagram ① commutes by exercise (3.2.5)(a). For commutativity of ② replace f_*A by an isomorphic q-flat complex to reduce to showing commutativity of the corresponding diagram in context (a); then reduce via adjointness to checking (easily) that in that context the following natural diagram commutes:

$$\begin{tikzcd}
f_*(\mathcal{O}_X) \underset{=}{\otimes} f_*(A) \arrow[r, "(3.1.7)"] & f_*(\mathcal{O}_X \underset{=}{\otimes} A) \arrow[d] \\
\mathcal{O}_Y \underset{=}{\otimes} f_*(A) \arrow[u] \arrow[r] & f_*(A)
\end{tikzcd}$$

The rest is evident.

Exercise **(3.4.4.1).** Let R be a commutative ring, $Z := \operatorname{Spec}(R)$, T an indeterminate, $X := \operatorname{Spec}(R[T])$ with its obvious Z-scheme structure, $\delta\colon X \to Y := X \times_Z X$ the diagonal map, and $\sigma\colon Y \xrightarrow{\sim} Y$ the symmetry isomorphism, i.e., $\pi_1\sigma = \pi_2$ and $\pi_2\sigma = \pi_1$ where π_1 and π_2 are the canonical projections from Y to X.

Show that in the context of (3.4.4)(a) the natural composite $\mathcal{O}_X$-module map

$$\delta^*\delta_*F = (\sigma\delta)^*(\sigma\delta)_*F \xrightarrow{\sim} \delta^*\sigma^*\sigma_*\delta_*F \to \delta^*\delta_*F$$

is the identity map for all $\mathcal{O}_X$-modules F; but that in the context of (3.4.4)(b) the natural composite $\mathbf{D}(X)$-map

$$\mathbf{L}\delta^*\delta_*\mathcal{O}_X = \mathbf{L}(\sigma\delta)^*(\sigma\delta)_*\mathcal{O}_X \xrightarrow{\sim} \mathbf{L}\delta^*\sigma^*\sigma_*\delta_*\mathcal{O}_X \to \mathbf{L}\delta^*\delta_*\mathcal{O}_X$$

is not the identity map unless $2 = 0$ in R.

(More challenging.) Show: if $\iota\colon Z \to X$ is the closed immersion corresponding to the R-homomorphism $R[T] \twoheadrightarrow R$ taking T to 0, then the natural composite $\mathbf{D}(X)$-map

$$\mathbf{L}\delta^*\delta_*\iota_*\mathcal{O}_Z = \mathbf{L}(\sigma\delta)^*(\sigma\delta)_*\iota_*\mathcal{O}_Z \xrightarrow{\sim} \mathbf{L}\delta^*\sigma^*\sigma_*\delta_*\iota_*\mathcal{O}_Z \to \mathbf{L}\delta^*\delta_*\iota_*\mathcal{O}_Z$$

is an automorphism of order 2, inducing the identity map on homology.

(3.4.5) (Duality principle). From (3.4.2.1) we get, by adjunction, functorial maps

$$\begin{aligned} f^*C \otimes f^*D &\longleftarrow f^*(C \otimes D)\,, \\ \mathcal{O}_X &\longleftarrow f^*\mathcal{O}_Y\,. \end{aligned} \tag{3.4.5.1}$$

Specifically, the second of these maps is defined to be adjoint to the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ in (3.4.2.1) (i.e., the two maps correspond under the isomorphism (3.4.3.1)); and the first is defined to be adjoint to the composition

$$C \otimes D \xrightarrow{\eta\otimes\eta} f_*f^*C \otimes f_*f^*D \xrightarrow{(3.4.2.1)} f_*(f^*C \otimes f^*D)\,.$$

It follows that "dually,"

(3.4.5.2): $f_*A\otimes f_*B \xrightarrow{(3.4.2.1)} f_*(A\otimes B)$ *is adjoint to the composition*

$$A\otimes B \xleftarrow[\epsilon\otimes\epsilon]{} f^*f_*A\otimes f^*f_*B \xleftarrow[(3.4.5.1)]{} f^*(f_*A\otimes f_*B)\,.$$

To see this, it suffices to note that the following diagram, whose top row composes to the identity, commutes:

$$\begin{tikzcd}
f_*A\otimes f_*B \arrow[d, "(3.4.2.1)"'] \arrow[dr, phantom, "①"] & f_*f^*f_*A\otimes f_*f^*f_*B \arrow[l, "\epsilon\otimes\epsilon"'] \arrow[d, "(3.4.2.1)"] \arrow[dr, phantom, "②"] & f_*A\otimes f_*B \arrow[l, "\eta\otimes\eta"'] \arrow[d, "\eta"] \\
f_*(A\otimes B) & f_*(f^*f_*A\otimes f^*f_*B) \arrow[l, "\epsilon\otimes\epsilon"] & f_*f^*(f_*A\otimes f_*B) \arrow[l, "(3.4.5.1)"]
\end{tikzcd}$$

(Subdiagram ① commutes by functoriality of (3.4.2.1), and ② commutes by the above definition of (3.4.5.1).)

The maps (3.4.5.1) satisfy compatibility conditions with the associativity, unit, and symmetry isomorphisms in the symmetric monoidal categories $\mathbf{X}$, $\mathbf{Y}$, conditions which are dual to those expressed by the commutativity of the diagrams (3.4.2.2) (i.e., in (3.4.2.2) replace f_* by f^*, interchange $\mathcal{O}_X$ and $\mathcal{O}_Y$, and reverse all arrows). Proofs are left to the reader.

The maps (3.4.5.1) do not make f^* monoidal, since they point in the wrong direction (and we do *not* assume in general that they are isomorphisms, as happens to be the case in (3.4.4(a)) and (3.4.4(b)), so we cannot use their inverses).

However, to any symmetric monoidal category

$$\mathbf{M}=(\mathbf{M_0},\otimes,\mathcal{O}_M,\alpha,\lambda,\rho,\gamma)$$

we can associate the *dual* symmetric monoidal category

$$\mathbf{M}^{\mathsf{op}}=(\mathbf{M}_\mathbf{0}^{\mathsf{op}},\otimes^{\mathsf{op}},\mathcal{O}_M,\overline{\alpha},\overline{\lambda},\overline{\rho},\overline{\gamma})$$

where $\mathbf{M}_\mathbf{0}^{\mathsf{op}}$ is the dual category of $\mathbf{M_0}$ (same objects; arrows reversed), $\otimes^{\mathsf{op}}$ is the functor

$$\mathbf{M}_\mathbf{0}^{\mathsf{op}}\times\mathbf{M}_\mathbf{0}^{\mathsf{op}}=(\mathbf{M_0}\times\mathbf{M_0})^{\mathsf{op}} \xrightarrow{\otimes^{\mathsf{op}}} \mathbf{M}_\mathbf{0}^{\mathsf{op}}$$

(so that $A\otimes^{\mathsf{op}}B=A\otimes B$ for all objects $A,B\in\mathbf{M_0}$),

$$\overline{\alpha}=(\alpha^{\mathsf{op}})^{-1}=(\alpha^{-1})^{\mathsf{op}}:(A\otimes B)\otimes C \xrightarrow{\sim} A\otimes(B\otimes C) \qquad (\text{in } \mathbf{M}_\mathbf{0}^{\mathsf{op}})$$

and similarly for $\overline{\lambda},\overline{\rho},\overline{\gamma}$.

Then one checks that the functor

$$(f^*)^{\mathsf{op}}\colon \mathbf{Y}^{\mathsf{op}} \to \mathbf{X}^{\mathsf{op}}$$

together with the maps (3.4.5.1) is indeed symmetric monoidal;[34] and it has a left adjoint

$$(f_*)^{\mathsf{op}}\colon \mathbf{X}^{\mathsf{op}} \to \mathbf{Y}^{\mathsf{op}}$$

(which need no longer be monoidal, because, for example, there may be no good map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ in $\mathbf{Y}^{\mathsf{op}}$). Thus to every pair f_*, f^* as in (3.4.3), we can associate a "dual" such pair $(f^*)^{\mathsf{op}}$, $(f_*)^{\mathsf{op}}$.

This gives rise to a **duality principle**, which we now state rather imprecisely, but whose meaning should be clarified by the illustrations which follow (in connection with projection morphisms). We will be considering numerous diagrams whose vertices are functors build up from the constant functors $\mathcal{O}_X$ and $\mathcal{O}_Y$ (on $\mathbf{X}$, $\mathbf{Y}$ respectively), identity functors, f_*, f^*, and $\otimes$, and whose arrows are morphisms of functors built up from those which express the "monoidality" of f_*, and from the adjunction isomorphism (3.4.3.1). (For example the above-mentioned "compatibility conditions" state that certain such diagrams commute.) By interpreting any such diagram in the dual context, we get another such diagram: specifically, in the original diagram, *interchange*

- $\mathcal{O}_X$ and $\mathcal{O}_Y$
- the identity functors of $\mathbf{X}$ and $\mathbf{Y}$
- the adjunction maps η and ϵ
- the functors f^* and f_*
- the maps in (3.4.2.1) and (3.4.5.1).

If the original diagram commutes solely by virtue of the fact that f_* is a monoidal functor with left adjoint f^*, then the second diagram must also commute (because $(f^*)^{\mathsf{op}}$ is a monoidal functor with left adjoint $(f_*)^{\mathsf{op}}$).

Example **(3.4.6)** (Projection morphisms). With preceding notation, and $F \in \mathbf{X}$, $G \in \mathbf{Y}$, the bifunctorial *projection morphisms*

$$p_1 = p_1(F,G)\colon f_*F \otimes G \longrightarrow f_*(F \otimes f^*G)$$
$$p_2 = p_2(G,F)\colon G \otimes f_*F \longrightarrow f_*(f^*G \otimes F)$$

are the respective compositions

$$f_*F \otimes G \xrightarrow{1\otimes\eta} f_*F \otimes f_*f^*G \xrightarrow{(3.4.2.1)} f_*(F \otimes f^*G)$$

$$G \otimes f_*F \xrightarrow{\eta\otimes 1} f_*f^*G \otimes f_*F \xrightarrow{(3.4.2.1)} f_*(f^*G \otimes F)\,.$$

[34] f^* may then be said to be "op-monoidal" or "co-monoidal."

Remarks **(3.4.6.1).** p_1 and p_2 determine each other via the following commutative diagram, in which γ_X, γ_Y are the respective symmetry isomorphisms in $\mathbf{X}$, $\mathbf{Y}$:

$$\begin{tikzcd} f_*F \otimes G \arrow[r, "p_1"] \arrow[d, "\gamma_Y"'] & f_*(F \otimes f^*G) \arrow[d, "f_*(\gamma_X)"] \\ G \otimes f_*F \arrow[r, "p_2"'] & f_*(f^*G \otimes F) \end{tikzcd}$$

The commutativity of this diagram follows from that of

$$\begin{tikzcd} f_*F \otimes f_*f^*G \arrow[r, "(3.4.2.1)"] \arrow[d, "\gamma_Y"'] & f_*(F \otimes f^*G) \arrow[d, "f_*(\gamma_X)"] \\ f_*f^*G \otimes f_*F \arrow[r, "(3.4.2.1)"'] & f_*(f^*G \otimes F) \end{tikzcd}$$

which holds as part of the definition of "symmetric monoidal functor" (see (3.4.2.2)).

(3.4.6.2). The map $p_1(F,G)$ is adjoint to the composed map

$$f^*(f_*F \otimes G) \xrightarrow{(3.4.5.1)} f^*f_*F \otimes f^*G \xrightarrow{\epsilon \otimes 1} F \otimes f^*G$$

(a map which is *dual* (3.4.5) to $p_2(F,G)$): this follows from commutativity of the natural diagram

$$\begin{tikzcd} f^*(f_*F \otimes G) \arrow[r, "\text{via } \eta"] \arrow[d, "(3.4.5.1)"'] \arrow[dr, phantom, "①"] & f^*(f_*F \otimes f_*f^*G) \arrow[r, "(3.4.2.1)"] \arrow[d, "(3.4.5.1)"'] \arrow[dr, phantom, "②"] & f^*f_*(F \otimes f^*G) \arrow[d, "\epsilon"] \\ f^*f_*F \otimes f^*G \arrow[r, "\text{via } \eta"'] & f^*f_*F \otimes f^*f_*f^*G \arrow[r, "\epsilon\otimes\epsilon"'] & F \otimes f^*G. \end{tikzcd}$$

(Here commutativity of ① is clear, and that of ② results from (3.4.5.2).) Similarly $p_2(G,F)$ is adjoint to the dual of $p_1(G,F)$.

Lemma (3.4.7). *The following diagrams commute:*

(i)
$$\begin{tikzcd} A \otimes B \arrow[r, "\eta"] \arrow[d, "1\otimes\eta"'] & f_*f^*(A \otimes B) \arrow[d, "(3.4.5.1)"] \\ A \otimes f_*f^*B \arrow[r, "p_2"'] & f_*(f^*A \otimes f^*B) \end{tikzcd}$$

(ii)
$$\begin{tikzcd} A \otimes \mathcal{O}_Y \arrow[r, "(3.4.2.1)"] \arrow[d, "\rho"'] & A \otimes f_*\mathcal{O}_X \arrow[r, "p_2"] & f_*(f^*A \otimes \mathcal{O}_X) \arrow[d, "f_*(\rho)"] \\ A \arrow[rr, "\eta"'] & & f_*f^*A \end{tikzcd}$$

(iii)
$$\begin{tikzcd}
f_*B \otimes \mathcal{O}_Y \arrow[r, "p_1"] \arrow[d, "\rho"'] & f_*(B \otimes f^*\mathcal{O}_Y) \arrow[d, "(3.4.5.1)"] \\
f_*B & f_*(B \otimes \mathcal{O}_X) \arrow[l, "f_*(\rho)"]
\end{tikzcd}$$

(iv)
$$\begin{tikzcd}
(A \otimes B) \otimes f_*C \arrow[r, "\alpha"] \arrow[d, "p_2"'] & A \otimes (B \otimes f_*C) \arrow[r, "1\otimes p_2"] & A \otimes f_*(f^*B \otimes C) \arrow[d, "p_2"] \\
f_*(f^*(A \otimes B) \otimes C) \arrow[r, "(3.4.5.1)"'] & f_*((f^*A \otimes f^*B) \otimes C) \arrow[r, "\alpha"'] & f_*(f^*A \otimes (f^*B \otimes C))
\end{tikzcd}$$

Proof. (i) The commutativity of this diagram simply states that the first map in (3.4.5.1) is adjoint to the composition

$$A \otimes B \xrightarrow{1\otimes\eta} A \otimes f_*f^*B \xrightarrow{\eta\otimes 1} f_*f^*A \otimes f_*f^*B \xrightarrow{(3.4.2.1)} f_*(f^*A \otimes f^*B)$$

which is so by definition (see beginning of (3.4.5)).

(ii) We expand the diagram in question as follows:

$$\begin{tikzcd}
A \otimes f_*\mathcal{O}_X \arrow[r, "\eta\otimes 1"] & f_*f^*A \otimes f_*\mathcal{O}_X \arrow[r, "(3.4.2.1)"] & f_*(f^*A \otimes \mathcal{O}_X) \arrow[d, "f_*(\rho)"] \\
A \otimes \mathcal{O}_Y \arrow[u, "(3.4.2.1)"] \arrow[r, "\eta\otimes 1"'] \arrow[ur, phantom, "\textcircled{1}"] \arrow[d, "\rho"'] \arrow[drr, phantom, "\textcircled{3}"] & f_*f^*A \otimes \mathcal{O}_Y \arrow[u] \arrow[r, "\rho"'] \arrow[ur, phantom, "\textcircled{2}"] & f_*f^*A \arrow[d, equal] \\
A \arrow[rr, "\eta"'] & & f_*f^*A
\end{tikzcd}$$

Subdiagrams ① and ③ clearly commute; and so does ② because of the compatibility of (3.4.2.1) and ρ, which can be deduced from the two top diagrams in (3.4.2.2) (the first of which expresses the compatibility of (3.4.2.1) and λ) and the bottom diagram in (3.4.1.1).

(iii) The diagram expands as

$$\begin{tikzcd}
f_*B \otimes \mathcal{O}_Y \arrow[rrrr, "1\otimes\eta"] \arrow[ddd, "\rho"'] \arrow[ddrr] & & & & f_*B \otimes f_*f^*\mathcal{O}_Y \arrow[ddd, "(3.4.2.1)"] \arrow[ddll] \\
& & \textcircled{1} & & \\
& \textcircled{2} & f_*B \otimes f_*\mathcal{O}_X \arrow[d, "(3.4.2.1)"'] & \textcircled{3} & \\
f_*B & & f_*(B \otimes \mathcal{O}_X) \arrow[ll, "f_*(\rho)"] & & f_*(B \otimes f^*\mathcal{O}_Y) \arrow[ll, "(3.4.5.1)"]
\end{tikzcd}$$

Subdiagram ① commutes by the definition of the map $f^*\mathcal{O}_Y \to \mathcal{O}_X$ in (3.4.5.1), ② by the compatibility of (3.4.2.1) and ρ (see preceding proof of (ii)), and ③ by functoriality of (3.4.2.1).

(iv) An expanded version of this diagram can be obtained by fitting together the following two diagrams (whose maps are the obvious ones):

$$
\begin{tikzcd}
(A\otimes B)\otimes f_*C \arrow[r] \arrow[d, equal] & A\otimes(B\otimes f_*C) \arrow[r] \arrow[d, phantom, "①"] & A\otimes(f_*f^*B\otimes f_*C) \arrow[d, "a"] \\
(A\otimes B)\otimes f_*C \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & (f_*f^*A\otimes f_*f^*B)\otimes f_*C \arrow[r, "b"'] \arrow[d, "c"] & f_*f^*A\otimes(f_*f^*B\otimes f_*C) \\
f_*f^*(A\otimes B)\otimes f_*C \arrow[r] \arrow[d] \arrow[dr, phantom, "③"] & f_*(f^*A\otimes f^*B)\otimes f_*C \arrow[d, "d"] & \\
f_*(f^*(A\otimes B)\otimes C) \arrow[r] & f_*((f^*A\otimes f^*B)\otimes C) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & A\otimes(f_*f^*B\otimes f_*C) \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "⑤"] & A\otimes f_*(f^*B\otimes C) \arrow[d] \\
(f_*f^*A\otimes f_*f^*B)\otimes f_*C \arrow[r, "b"] \arrow[d, "c"'] & f_*f^*A\otimes(f_*f^*B\otimes f_*C) \arrow[r] \arrow[d, phantom, "④"] & f_*f^*A\otimes f_*(f^*B\otimes C) \arrow[dd] \\
f_*(f^*A\otimes f^*B)\otimes f_*C \arrow[d, "d"'] & & \\
f_*((f^*A\otimes f^*B)\otimes C) \arrow[rr] & & f_*(f^*A\otimes(f^*B\otimes C))
\end{tikzcd}
$$

Subdiagram ① commutes by functoriality of a; ② by the definition of (3.4.5.1); ③ by functoriality of (3.4.2.1); ④ by commutativity of the bottom diagram in (3.4.2.2); and ⑤ for obvious reasons. Q.E.D.

Remark **(3.4.7.1)**. By duality (3.4.5) we get four other commutative diagrams out of (3.4.7). For example, the dual of (ii) is

$$
\begin{tikzcd}
A\otimes\mathcal{O}_X \arrow[d, "\rho"'] & A\otimes f^*\mathcal{O}_Y \arrow[l, "(3.4.5.1)"'] & f^*(f_*A\otimes\mathcal{O}_Y) \arrow[l, "(3.4.6.2)"'] \arrow[d, "f^*(\rho)"] \\
A & & f^*f_*A \arrow[ll, "\epsilon"]
\end{tikzcd}
$$

Using the symmetry isomorphism γ, Remark (3.4.6.1), the bottom diagram in (3.4.1.1), etc., we can also transform the commutative diagrams in (3.4.7) into similar ones with p_2 (resp. p_1) replaced by p_1 (resp. p_2), and with ρ replaced by λ.

3.5. Adjoint functors between closed categories

The adjoint symmetric functors f_*, f^*, remain as in (3.4.3). Additional structure comes into play when the monoidal categories $\mathbf{X}$ and $\mathbf{Y}$ are *closed,* in the following sense.

Definition (3.5.1). A *symmetric monoidal closed category* (briefly, a *closed category*) is a symmetric monoidal category

$$\mathbf{M} = (\mathbf{M_0}, \otimes, \mathcal{O}_M, \alpha, \lambda, \rho, \gamma)$$

as in (3.4.1), together with a functor, called "internal hom,"

$$[-,-]\colon \mathbf{M_0^{op}} \times \mathbf{M_0} \to \mathbf{M_0} \tag{3.5.1.1}$$

(where $\mathbf{M_0^{op}}$ is the dual category of $\mathbf{M_0}$) and a functorial isomorphism

$$\pi\colon \mathrm{Hom}(A \otimes B,\, C) \xrightarrow{\sim} \mathrm{Hom}(A,\, [B, C])\,. \tag{3.5.1.2}$$

The notion of closed category reduces myriad relations among, and maps involving, "tensor" and "hom" to the few basic ones appearing in the definition. (See, e.g., the following exercises (3.5.3).)[35] The original treatise on closed categories is [**EK**], in particular Chap. III, (p. 512 *ff*). Some more recent theory can be found starting with [**Sv**] and its references.

Examples **(3.5.2).**

(a) $\mathbf{M_0}$ is the category of modules over a given commutative ring R. Let $\otimes$ be the usual tensor product, $\mathcal{O}_M := R$, and $[B, C] := \mathrm{Hom}_R(B,\, C)$. Fill in the rest.

(b) $\mathbf{M_0}$ is the category of $\mathcal{O}_X$-modules on a ringed space X. Let $\otimes$ be the usual tensor product, $\mathcal{O}_M := \mathcal{O}_X$, and $[B, C] := \mathcal{H}om_X(B,\, C)$ . . . .

(c) $\mathbf{M}'_\mathbf{0} := \mathbf{K}(X)$ is the homotopy category of complexes in the category $\mathbf{M_0}$ of (b). Let $\otimes$ be the tensor product in (1.5.4), set $\mathcal{O}_{M'} := \mathcal{O}_X$ (considered as a complex vanishing in all nonzero degrees), and set $[B, C] := \mathcal{H}om^\bullet_X(B,\, C)$, see (2.4.5), (2.6.7), ...

(d) $\mathbf{M}''_\mathbf{0} := \mathbf{D}(X)$, the derived category of $\mathbf{M_0}$ in (b), $\otimes := \underline{\underline{\otimes}}$ (2.5.7), $\mathcal{O}_{M''} := \mathcal{O}_X$, $[B, C] := \mathbf{R}\mathcal{H}om^\bullet_X(B,\, C)$, see (2.6.1)′, (3.4.4)(b), ...

[35] When $\mathbf{M_0}$ has direct sums, π gives rise to a distributivity isomorphism

$$(A' \oplus A'') \otimes B \xrightarrow{\sim} (A' \otimes B) \oplus (A'' \otimes B)$$

whose consequences we will not follow up here—but see [**L**], [**L′**], [**K′**].

Exercises **(3.5.3).** Let $(\mathbf{M}, [-,-], \pi)$ as above be a closed category. Write (A,B) for $\mathrm{Hom}_{\mathbf{M_0}}(A,B)$.

(a) Define the set-valued functor Γ on $\mathbf{M_0}$ to be the usual functor $(\mathcal{O}_M, -)$. Establish a bifunctorial isomorphism

$$\Gamma[A,B] \xrightarrow{\sim} (A,B).$$

(b) Let $t_{AB}\colon [A,B]\otimes A \to B$ correspond under π to the identity map of $[A,B]$. Use t_{AB} and π to obtain a natural map $[A,B] \to [A\otimes C, B\otimes C]$.

(c) Use π, t_{CA}, and t_{AB} (see (b)) to get a natural "internal composition" map

$$c\colon [A,B]\otimes[C,A] \to [C,B].$$

Prove associativity (up to canonical isomorphism) for this c.

(d) Show that the map

$$\ell = \ell_{A,B,C}\colon [A,B] \to [[C,A],\,[C,B]]$$

corresponding under π to internal composition (see (c)) is compatible with ordinary composition in $\mathbf{M_0}$ in that the following natural diagram (with Γ as in (a) and "Hom" meaning "set maps") commutes:

$$\begin{tikzcd}
\Gamma[A,B] \arrow[r, "\sim"] \arrow[d, "\Gamma(\ell)"'] & (A,B) \arrow[r, "\text{composition}"] \arrow[d, "\text{functoriality}"', "\text{of } {[C,-]}"] & \mathrm{Hom}\big((C,A),\,(C,B)\big) \arrow[d, "\simeq"] \\
\Gamma[[C,A],\,[C,B]] \arrow[r, "\sim"] & \big([C,A],\,[C,B]\big) \arrow[r, "\text{functoriality}", "\text{of }\Gamma"'] & \mathrm{Hom}\big(\Gamma[C,A],\,\Gamma[C,B]\big)
\end{tikzcd}$$

(e) From the sequence of functorial isomorphisms

$$\begin{aligned}(D,\,[A\otimes B, C]) &\xrightarrow{\pi} (D\otimes(A\otimes B),\,C) \xrightarrow{\alpha} ((D\otimes A)\otimes B,\,C)\\ &\xrightarrow{\pi} (D\otimes A,\,[B,C]) \xrightarrow{\pi} (D,\,[A,\,[B,C]])\end{aligned}$$

deduce a functorial isomorphism

$$p = p_{A,B,C}\colon [A\otimes B,\,C] \xrightarrow{\sim} [A,\,[B,C]]\,.$$

(Take $D := [A\otimes B,\,C]$.) Referring to (a), show that $\Gamma(p) = \pi$. In example (3.5.2)(d), does this p coincide with the isomorphism in $(2.6.1)^*$?

(f) Let $u_{AB}\colon A \to [B,\,A\otimes B]$ correspond under π to the identity map of $A\otimes B$. Show that the map $p_{A,B,C}$ in (e) factors as

$$[A\otimes B,\,C] \xrightarrow{\ell_{A\otimes B,C,B}} [[B,A\otimes B],\,[B,C]] \xrightarrow{\text{via } u_{AB}} [A,\,[B,C]].$$

with ℓ as in (d).

Let $t_{AB}\colon [A,B]\otimes A \to B$ correspond under π to the identity map of $[A,B]$. Show that $\ell_{A,B,C}$ factors as

$$[A,B] \xrightarrow{\text{via } t_{AC}} [[C,A]\otimes C,\,B] \xrightarrow{p_{[C,A],C,B}} [[C,A],\,[C,B]]\,.$$

(g) The preceding exercises make no use of the symmetry isomorphism γ, but this one does. Construct functorial maps

$$[A,B]\otimes[C,D] \to [[B,C],\,[A,D]]\,,$$

$$[A,B]\otimes[C,D] \to [A\otimes C,\,B\otimes D]\,.$$

using π, c and γ for the first (see (c)), π, t and γ for the second (see (b)).

(h) Let $\alpha\colon B \to A$ be an $\mathbf{M}$-map. Show that the following diagrams—in which unlabeled maps correspond under π to identity maps—commute for any C:

$$
\begin{tikzcd}
{[A,C]\otimes B} \arrow[r, "\text{via }\alpha"] \arrow[d, "\text{via }\alpha"'] & {[A,C]\otimes A} \arrow[d] \\
{[B,C]\otimes B} \arrow[r] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
{[B,C\otimes A]} & {[A,C\otimes A]} \arrow[l, "\text{via }\alpha"'] \\
{[B,C\otimes B]} \arrow[u, "\text{via }\alpha"] & C \arrow[l] \arrow[u]
\end{tikzcd}
$$

Hint. For the first diagram, consider the adjoint (via π) diagram, with D arbitrary,

$$
\begin{tikzcd}
\mathrm{Hom}([A,C]\otimes B,D) & \mathrm{Hom}([A,C]\otimes A,D) \arrow[l] \\
\mathrm{Hom}([B,C]\otimes B,D) \arrow[u] & \mathrm{Hom}(C,D) \arrow[l] \arrow[u]
\end{tikzcd}
$$

Commutativity of the second diagram can be deduced from that of the first (and vice-versa), or proved independently.

(3.5.4). Now let us see how f_* and f^* interact with closed structures (assumed given) on $\mathbf{X}$ and $\mathbf{Y}$.

First we have a functorial map, with $A,\ B \in \mathbf{X}$,

$$f_*[A,B] \longrightarrow [f_*A,\ f_*B] \tag{3.5.4.1}$$

corresponding under π (3.5.1.2) to the composed map

$$f_*[A,B]\otimes f_*A \xrightarrow[(3.4.2.1)]{} f_*\big([A,B]\otimes A\big) \xrightarrow[(3.5.3)(\mathrm{b})]{f_*t_{AB}} f_*B\,.$$

Conversely (verify!), the functorial map

$$f_*(A\otimes B) \longleftarrow f_*A\otimes f_*B$$

in (3.4.2.1) corresponds to the composition

$$\big[f_*B,\ f_*(A\otimes B)\big] \xleftarrow[(3.5.4.1)]{} f_*[B,\ A\otimes B] \xleftarrow[(3.5.3)(\mathrm{f})]{f_*u_{AB}} f_*A\,.$$

There results a functorial composition

$$f_*[f^*A,B] \xrightarrow[(3.5.4.1)]{} [f_*f^*A,\ f_*B] \xrightarrow[(3.4.3)]{\text{via }\eta} [A,\ f_*B]\,, \tag{3.5.4.2}$$

from which (verify!) (3.5.4.1) can be recovered as the composition

$$f_*[A,B] \xrightarrow[(3.4.3)]{\text{via }\epsilon} f_*[f^*f_*A,B] \xrightarrow[(3.5.4.2)]{} [f_*A,\ f_*B]\,.$$

The functors $C \mapsto f^*(C\otimes A)$ and $C \mapsto f^*C\otimes f^*A$ (from $\mathbf{Y}$ to $\mathbf{X}$) both have right adjoints, namely $B \mapsto [A,\ f_*B]$ and $B \mapsto f_*[f^*A,B]$. Hence there is a functorial map

$$[A,\ f_*B] \longleftarrow f_*[f^*A,B] \tag{3.5.4.3}$$

right-conjugate (see (3.3.5)) to the functorial map $f^*(C\otimes A) \to f^*C\otimes f^*A$ in (3.4.5.1).

Similarly, there is a functorial map

$$f_*[B, A] \longrightarrow [f_*B,\, f_*A] \tag{3.5.4.4}$$

right-conjugate to the adjoint $f^*C \otimes B \leftarrow f^*(C \otimes f_*B)$ of $p_2(C, B)$ (3.4.6).

If $f^*(C \otimes A) \to f^*C \otimes f^*A$—and hence its conjugate (3.5.4.3)—is a functorial *isomorphism,* then we have the functorial map

$$f^*[A, B] \longrightarrow [f^*A,\, f^*B] \tag{3.5.4.5}$$

which is adjoint to the composition

$$[A, B] \xrightarrow{\quad\eta\quad} [A,\, f_*f^*B] \xrightarrow{(3.5.4.3)^{-1}} f_*[f^*A,\, f^*B]\,;$$

and (verify!) $(3.5.4.3)^{-1}$ is the map adjoint to the composition

$$f^*[A,\, f_*B] \xrightarrow{(3.5.4.5)} [f^*A,\, f^*f_*B] \xrightarrow{\text{via } \epsilon} [f^*A,\, B]\,,$$

from which (3.5.4.5) can be recovered as the composition

$$f^*[A, B] \xrightarrow{\text{via } \eta} f^*[A,\, f_*f^*B] \longrightarrow [f^*A,\, f^*B]\,.$$

This all holds in the most relevant (for us) cases, see e.g., (3.4.4)(a), (b), and (3.5.2).

Does the map in (3.5.4.3) (resp. (3.5.4.4)) coincide with that in (3.5.4.2) (resp. (3.5.4.1))? Of course, but it's not entirely obvious: it amounts to commutativity of the respective diagrams in (3.5.5) below. (Cf. (3.2.4)(i), but recall that in proving (3.2.4)(i), we used (3.1.10), for whose last assertion, given (3.1.8), (3.5.5) provides a formal proof.)[36]

Proposition (3.5.5). *The following functorial diagrams—in which* $A, B, G \in \mathbf{X_0}$, $E, F, C \in \mathbf{Y_0}$, H_X, H_Y *stand for* $\mathrm{Hom}_{\mathbf{X_0}}$, $\mathrm{Hom}_{\mathbf{Y_0}}$ *respectively, and with maps arising naturally from those defined above—commute:*

$$\begin{tikzcd}
H_X\big(f^*E,\, [f^*F, G]\big) \arrow[r, "\sim"] \arrow[d, "\simeq"'] & H_X\big(f^*E \otimes f^*F,\, G\big) \arrow[d, "(3.4.5.1)"] \\
H_Y\big(E,\, f_*[f^*F, G]\big) \arrow[d, "(3.5.4.2)"'] & H_X\big(f^*(E \otimes F),\, G\big) \arrow[d, "\simeq"] \\
H_Y\big(E,\, [F, f_*G]\big) \arrow[r, "\sim"] & H_Y\big(E \otimes F,\, f_*G\big)
\end{tikzcd} \tag{3.5.5.1}$$

---

[36] Diagram (3.2.4.6) is, in view of (3.1.8), an instance of (3.5.5.1). So is (3.2.4.5); but we don't know that *a priori,* because we don't know that the maps in (3.2.3.2) and (3.5.4.2) coincide until after proving either (3.2.4)(i) or the derived-category analog of (3.1.8), viz. (3.2.4)(ii)—in whose proof (3.5.5) was used.

$$
(3.5.5.2)\qquad
\begin{tikzcd}
H_Y\big(C\otimes f_*B,\ f_*A\big) & H_Y\big(C,\ [f_*B,\ f_*A]\big) \arrow[l, "\sim"'] \\
H_X\big(f^*(C\otimes f_*B),\ A\big) \arrow[u, "\simeq"] & H_Y\big(C,\ f_*[B,A]\big) \arrow[u, "(3.5.4.1)"'] \\
H_X\big(f^*C\otimes B,\ A\big) \arrow[u, "(3.4.6.2)"] & H_X\big(f^*C,\ [B,A]\big) \arrow[l, "\sim"'] \arrow[u, "\simeq"']
\end{tikzcd}
$$

The *proof* will be based on:

Lemma (3.5.5.3). *The following diagram* (*with preceding notation*) *commutes:*

$$
\begin{tikzcd}
H_X(A,\,[B,G]) \arrow[r, "\text{natural}"] \arrow[d, "\simeq"'] & H_Y(f_*A,\ f_*[B,G]) \arrow[r, "(3.5.4.1)"] & H_Y(f_*A,\ [f_*B,\ f_*G]) \arrow[d, "\simeq"] \\
H_X(A\otimes B,\ G) \arrow[r, "\text{natural}"'] & H_Y(f_*(A\otimes B),\ f_*G) \arrow[r, "(3.4.2.1)"'] & H_Y(f_*A\otimes f_*B,\ f_*G)
\end{tikzcd}
$$

Proof. Chasing a map $\varphi\colon A\to[B,G]$ around the diagram both clockwise and counterclockwise from upper left to lower right, one comes down to showing commutativity of the following diagram (with t as in (3.5.3(b)):

$$
\begin{tikzcd}
f_*A\otimes f_*B \arrow[r, "f_*\varphi\otimes 1_{f_*B}"] \arrow[d, "(3.4.2.1)"'] & f_*[B,G]\otimes f_*B \arrow[r, "(3.5.4.1)"] \arrow[d, "(3.4.2.1)"'] & {[f_*B,f_*G]\otimes f_*B} \arrow[d, "t_{f_*B,f_*G}"] \\
f_*(A\otimes B) \arrow[r, "f_*(\varphi\otimes 1_B)"'] & f_*\big([B,G]\otimes B\big) \arrow[r, "f_*(t_{BG})"'] & f_*G
\end{tikzcd}
$$

The left square commutes by functoriality, and the right one by the definition of (3.5.4.1). Q.E.D.

Proof of (3.5.5). Expand (3.5.5.1) to (3.5.5.1.)*, shown on the next page, where the map ξ is induced by the map $\xi'\colon E\otimes F\to f_*(f^*E\otimes f^*F)$ adjoint to $f^*(E\otimes F)\to f^*E\otimes f^*F$, see (3.4.5.1); and the other maps are the obvious ones. The outer border of (3.5.5.1)* commutes, by (3.5.5.3) (with $A:=f^*E,\ B:=f^*F$). Hence if all the subdiagrams other than (3.5.5.1) commute, then so does (3.5.5.1), as desired.

Commutativity of ① follows from adjointness of f_* and f^*.

Commutativity of ② follows from the definition (3.5.4.2) of the map $f_*[f^*F,G]\to[F,f_*G]$.

Commutativity of ③ follows from functoriality of π (3.5.1.2).

Commutativity of ④ and of ⑤ result respectively from the following two factorizations of the map ξ':

$$E\otimes F\xrightarrow{\eta} f_*f^*(E\otimes F)\xrightarrow{(3.4.5.1)} f_*(f^*E\otimes f^*F)\,,$$

$$E\otimes F\xrightarrow{\eta\otimes\eta} f_*f^*E\otimes f_*f^*F\xrightarrow{(3.4.2.1)} f_*(f^*E\otimes f^*F)\,.$$

Thus (3.5.5.1) does commute.

$$
\begin{tikzcd}
H_X\big(f^*E,\,[f^*F,G]\big) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "1"] & H_Y\big(f_*f^*E,\,f_*[f^*F,G]\big) \arrow[rr] \arrow[d] & \arrow[d, phantom, "2"] & H_Y\big(f_*f^*E,\,[f_*f^*F,\,f_*G]\big) \arrow[d, equal] \\
H_X\big(f^*E,\,[f^*F,G]\big) \arrow[r] \arrow[d, "\simeq"'] & H_Y\big(E,\,f_*[f^*F,G]\big) \arrow[r] \arrow[d, phantom, "(3.5.5.1)"] & H_Y\big(E,\,[F,\,f_*G]\big) \arrow[d, "\simeq"'] \arrow[dr, phantom, "3"] & H_Y\big(f_*f^*E,\,[f_*f^*F,\,f_*G]\big) \arrow[l] \arrow[d, "\simeq"] \\
H_X\big(f^*E\otimes f^*F,\,G\big) \arrow[r] \arrow[d, equal] & H_X\big(f^*(E\otimes F),\,G\big) \arrow[r] \arrow[d, phantom, "5"] & H_Y\big(E\otimes F,\,f_*G\big) \arrow[dr, phantom, "4"] & H_Y\big(f_*f^*E\otimes f_*f^*F,\,f_*G\big) \arrow[l] \arrow[d, equal] \\
H_X\big(f^*E\otimes f^*F,\,G\big) \arrow[rr] & & H_Y\big(f_*(f^*E\otimes f^*F),\,f_*G\big) \arrow[r] \arrow[u, "\xi"] & H_Y\big(f_*f^*E\otimes f_*f^*F,\,f_*G\big)
\end{tikzcd}
$$

(3.5.5.1)*

$$
\begin{tikzcd}
H_X\big(f^*C,\,[B,A]\big) \arrow[rr] \arrow[d, equal] & \arrow[d, phantom, "1"] & H_Y\big(C,\,f_*[B,A]\big) \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "2"] & H_Y\big(C,\,f_*[B,A]\big) \arrow[d] \\
H_X\big(f^*C,\,[B,A]\big) \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "3"] & H_X\big(f^*C,\,[f^*f_*B,\,A]\big) \arrow[r] \arrow[d, "\simeq"'] & H_Y\big(C,\,f_*[f^*f_*B,\,A]\big) \arrow[r] \arrow[d, phantom, "4"] & H_Y\big(C,\,[f_*B,\,f_*A]\big) \arrow[d, "\simeq"'] \\
H_X\big(f^*C\otimes B,\,A\big) \arrow[r] & H_X\big(f^*C\otimes f^*f_*B,\,A\big) \arrow[r] & H_X\big(f^*(C\otimes f_*B),\,A\big) \arrow[r] & H_Y\big(C\otimes f_*B,\,f_*A\big)
\end{tikzcd}
$$

(3.5.5.2)*

Now look at $(3.5.5.2)^*$, whose outer border is identical with (3.5.5.2). Subdiagrams ① and ③ commute by functoriality. Commutativity of ② comes from the statement immediately following (3.5.4.2). Subdiagram ④ is just (3.5.5.1) with $E := C$, $F := f_*B$, $G := A$; so it commutes too. Thus $(3.5.5.2)^*$ commutes. Q.E.D.

Exercises **(3.5.6).** (a) Show that if the natural map $f^*(C \otimes A) \to f^*C \otimes f^*A$ is an isomorphism for all C and A, then (3.5.4.5) corresponds under π (see (3.5.1.2)) to the natural composite map $f^*[A,B] \otimes f^*A \xrightarrow{\sim} f^*([A,B] \otimes A) \longrightarrow f^*B$.

(b) Given a fixed map $e\colon B' \to B$, show that the functorial maps

$$f_*[B,A] \xrightarrow{\text{via } e} f_*[B',A] \qquad \text{and} \qquad f^*C \otimes B \xleftarrow{\text{via } e} f^*C \otimes B'$$

are conjugate; and then deduce the equality of the maps (3.5.4.1) and (3.5.4.4) from that of (3.5.4.2) and (3.5.4.3).

(c) In $(3.5.5.i)$ $(i = 1,2,3)$, replace $H_X(-,-)$ by $f_*[-,-]$, and $H_Y(-,-)$ by $[-,-]$. Show that the resulting diagrams commute. (For example, reduce to commutativity of $(3.5.5.i)$, by applying the functor $H_Y(D,-)$ to the diagram in question.)

Show that $(3.5.5.i)$ can be recovered from "the resulting diagram" by application of the functor $\Gamma_Y := H_Y(\mathcal{O}_Y,-)$ of (3.5.3)(a).

(d) By Yoneda's principle, commutativity of (3.5.5.1) can be proved by taking $E = f_*[f^*F,G]$ and chasing the identity map of $f_*[f^*F,G]$ around the diagram in both directions. Deduce that commutativity of (3.5.5.1) is equivalent to that of the diagram

$$\begin{tikzcd}
f^*\big(f_*[f^*F,G]\otimes F\big) \arrow[r, "(3.5.4.2)"'] \arrow[d, "(3.4.5.1)"'] & f^*\big([F,f_*G]\otimes F\big) \arrow[r, "t_{F,f_*G}", "(3.5.3)(\mathrm{b})"'] & f^*f_*G \arrow[d, "\epsilon"] \\
f^*f_*[f^*F,G]\otimes f^*F \arrow[r, "(3.4.3)", "\text{via }\epsilon"'] & {[f^*F,G]\otimes f^*F} \arrow[r, "t_{f^*F,G}"'] & G
\end{tikzcd}$$

(e) In a closed category $\mathbf{X}$ the natural composite functorial map

$$\mathrm{Hom}(F,G) \xrightarrow{\sim} \mathrm{Hom}(F\otimes\mathcal{O}_X, G) \xrightarrow{\sim} \mathrm{Hom}(F,[\mathcal{O}_X,G]),$$

being an isomorphism, takes (when $F = G$) the identity map of G to an *isomorphism* $G \xrightarrow{\sim} [\mathcal{O}_X, G]$. Let $\mathbf{Y}$ be another closed category, and (f^*, f_*) be as in (3.4.3). Show that for $G \in \mathbf{X}$ and $E \in \mathbf{Y}$ the following natural diagrams commute:

$$\begin{tikzcd}
f_*[\mathcal{O}_X,G] \arrow[r] \arrow[d, "\simeq"'] & {[f_*\mathcal{O}_X, f_*G]} \arrow[d] \\
f_*G \arrow[r, "\sim"] & {[\mathcal{O}_Y, f_*G]}
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*[\mathcal{O}_Y,E] \arrow[r] \arrow[d, "\simeq"'] & {[f^*\mathcal{O}_Y, f^*E]} \arrow[d, "\simeq"] \\
f^*E \arrow[r, "\sim"] & {[\mathcal{O}_X, f^*E]}
\end{tikzcd}$$

Hint. The first diagram is right-conjugate to the *dual* (3.4.5) of (3.4.7)(iii). For the second diagram, use, e.g., (a) above.

(f) With notation as in (e), and $\pi_{\mathbf{X}}$, $\pi_{\mathbf{Y}}$ as in (3.5.1.2), and assuming the functorial map $f^*(C\otimes D) \to f^*C \otimes f^*D$ in (3.4.5.1) to be a functorial isomorphism, show that $\pi_{\mathbf{X}}$ takes the inverse of the isomorphism $f^*(G\otimes B) \to f^*G \otimes f^*B$ to the composite map

$$f^*G \xrightarrow{\text{natural}} f^*[B, G\otimes B] \xrightarrow{(3.5.4.5)} [f^*B, f^*(G\otimes B)],$$

or, equivalently, that the following diagram commutes.

$$\begin{tikzcd}
{[f^*B, f^*(G\otimes B)]} & {[f^*B, f^*G\otimes f^*B]} \arrow[l, "(3.4.5.1)^{-1}"'] \\
f^*[B, G\otimes B] \arrow[u, "(3.5.4.5)"] & f^*G \arrow[l, "\text{via } \pi_{\mathbf{Y}}"] \arrow[u, "\text{via } \pi_{\mathbf{X}}"']
\end{tikzcd}$$

(g) With assumptions as in (f), and using the commutative diagram in (f)—or otherwise—show that for any $\mathbf{Y}$-map $\alpha\colon C\otimes D\to E$, and α^f the composite map

$$f^*C\otimes f^*D \xrightarrow{(3.4.5.1)^{-1}} f^*(C\otimes D) \xrightarrow{f^*\alpha} f^*E,$$

it holds that $\pi_{\mathbf{X}}(\alpha^f)$ is the composite map

$$f^*C \xrightarrow{f^*(\pi_{\mathbf{Y}}\alpha)} f^*[D,E] \xrightarrow{(3.5.4.5)} [f^*D, f^*E].$$

3.6. Adjoint monoidal Δ-pseudofunctors

We review next the behavior of derived direct and inverse image functors vis-à-vis a pair of ringed-space maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.

First, relative to the categories of $\mathcal{O}_X$- ($\mathcal{O}_Y$-, $\mathcal{O}_Z$-) modules we have the functorial *isomorphism* (in fact *equality*)

$$(gf)_* \xrightarrow{\sim} g_*f_* \tag{3.6.1$_*$}$$

and hence, since f^*g^* is left-adjoint to g_*f_* and $(fg)^*$ is left-adjoint to $(gf)_*$ there is a unique functorial isomorphism

$$f^*g^* \xrightarrow{\sim} (gf)^* \tag{3.6.1*}$$

such that the following natural diagram of functors commutes:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & g_*g^* \arrow[r] & g_*(f_*f^*g^*) \arrow[d, equal] \\
(gf)_*(gf)^* \arrow[r, "\sim"] & g_*f_*(gf)^* & g_*f_*f^*g^* \arrow[l, "\sim"']
\end{tikzcd} \tag{3.6.2}$$

or, equivalently, such that the “dual” diagram

$$\begin{tikzcd}
1 & f^*f_* \arrow[l] & f^*(g^*g_*f_*) \arrow[l] \\
(gf)^*(gf)_* \arrow[u] & f^*g^*(gf)_* \arrow[l, "\sim"'] \arrow[r, "\sim"] & f^*g^*g_*f_* \arrow[u, equal]
\end{tikzcd} \tag{3.6.2$^{\mathsf{op}}$}$$

commutes. (This statement follows from (3.3.5), see also (3.3.7)(a)).

Given a third map $Z \xrightarrow{h} W$, we have the commutative diagram of functorial isomorphisms (actually equalities)

$$
\text{(3.6.3)}_* \qquad
\begin{array}{ccc}
(hgf)_* & \longrightarrow & (hg)_*f_* \\
\big\downarrow & & \big\downarrow \\
h_*(gf)_* & \longrightarrow & h_*g_*f_*
\end{array}
$$

from which we deduce formally, via adjunction, a commutative diagram of functorial isomorphisms

$$
\text{(3.6.3)}^* \qquad
\begin{array}{ccc}
(hgf)^* & \longleftarrow & f^*(hg)^* \\
\big\uparrow & & \big\uparrow \\
(gf)^*h^* & \longleftarrow & f^*g^*h^*
\end{array}
$$

From these observations we can derive similar ones involving the corresponding derived functors.

Indeed, taking $U := g^{-1}V$ (V open $\subset Z$) in (3.2.3.3), we find that f_*B is g_*-acyclic for any q-injective $B \in \mathbf{K}(X)$, whence, by (2.2.7), there is a unique Δ-functorial isomorphism

$$
\text{(3.6.4)}_* \qquad \mathsf{R}(gf)_* \xrightarrow{\sim} \mathsf{R}g_*\mathsf{R}f_*
$$

making the following natural diagram commute:

$$
\text{(3.6.4.1)} \qquad
\begin{array}{ccccc}
(gf)_* & \xrightarrow{\sim} & g_*f_* & \longrightarrow & (\mathsf{R}g_*)f_* \\
\big\downarrow & & & & \big\downarrow \\
\mathsf{R}(gf)_* & & \xrightarrow{\qquad\sim\qquad} & & \mathsf{R}g_*\mathsf{R}f_*
\end{array}
$$

This allows us to build a diagram analogous to $(3.6.3)_*$, with $\mathsf{R}e_*$ in place of e_* for each map e involved. The resulting derived functor diagram still commutes, as can be seen by reduction (via suitable quasi-isomorphisms) to the case of q-injective complexes in $\mathbf{D}(X)$, for which the diagram in question is essentially $(3.6.3)_*$.

In a parallel fashion, using q-flat instead of q-injective complexes, and recalling that f^* transforms q-flat complexes into q-flat complexes (see proof of (3.2.4)(i)), etc., we get a natural Δ-functorial isomorphism

$$
\text{(3.6.4)}^* \qquad \mathsf{L}f^*\mathsf{L}g^* \xrightarrow{\sim} \mathsf{L}(gf)^*,
$$

and a commutative diagram analogous to $(3.6.3)^*$, with $\mathsf{L}e^*$ in place of e^*. By (3.3.5), we also have commutative diagrams like (3.6.2) and $(3.6.2)^{\mathsf{op}}$, with f_*, f^* etc. replaced by their respective derived functors.

It is helpful to conceptualize some of the foregoing, as follows, leading up to (3.6.10). We begin with some standard terminology.[37]

(3.6.5). Let $\mathbf{S}$ be a category. A *covariant pseudofunctor* $\#$ on $\mathbf{S}$ assigns to each object $X \in \mathbf{S}$ a category $\mathbf{X}_\#$, to each map $f\colon X \to Y$ in $\mathbf{S}$ a functor $f_\# : \mathbf{X}_\# \to \mathbf{Y}_\#$, with $f_\#$ the identity functor if $X = Y$ and $f = 1_X$, and to each pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{S}$ an isomorphism of functors

$$c_{f,g}\colon (gf)_\# \xrightarrow{\sim} g_\# f_\#$$

such that

1) $c_{1,g} = c_{f,1} =$ identity, and
2) for any triple of maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ the following diagram commutes:

(3.6.5.1)
$$\begin{array}{ccc}
(hgf)_\# & \xrightarrow{c_{f,hg}} & (hg)_\# f_\# \\
{\scriptstyle c_{gf,h}}\Big\downarrow & & \Big\downarrow{\scriptstyle c_{g,h}} \\
h_\#(gf)_\# & \xrightarrow[c_{f,g}]{} & h_\# g_\# f_\#
\end{array}$$

Similarly, a *contravariant pseudofunctor* on $\mathbf{S}$ assigns to each $X \in \mathbf{S}$ a category $\mathbf{X}^\#$, to each map $f\colon X \to Y$ a functor $f^\#\colon \mathbf{Y}^\# \to \mathbf{X}^\#$ (with $1^\# = 1$), and to each map-pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ a functorial isomorphism

$$d_{f,g}\colon f^\# g^\# \to (gf)^\#$$

satisfying $d_{1,g} = d_{g,1} =$ identity, and such that for each triple of maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ the following diagram commutes:

(3.6.5.2)
$$\begin{array}{ccc}
(hgf)^\# & \xleftarrow{d_{f,hg}} & f^\#(hg)^\# \\
{\scriptstyle d_{gf,h}}\Big\uparrow & & \Big\uparrow{\scriptstyle d_{g,h}} \\
(gf)^\# h^\# & \xleftarrow[d_{f,g}]{} & f^\# g^\# h^\#
\end{array}$$

There is an obvious way of identifying contravariant pseudofunctors on $\mathbf{S}$ with pseudofunctors on the dual category $\mathbf{S}^{\mathsf{op}}$.

[37] Pseudofunctors can also be interpreted as 2-functors.

(3.6.6). Given covariant pseudofunctors $*$ and $\#$ with $\mathbf{X}_* = \mathbf{X}_\#$ for all $X \in \mathbf{S}$, a *morphism of pseudofunctors* $* \to \#$ is a family of morphisms of functors

$$\alpha_f \colon f_* \to f_\#$$

(one for each map f in $\mathbf{S}$) such that for any pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ the following diagram commutes:

$$\begin{tikzcd}
(gf)_* \arrow[rr, "\alpha_{gf}"] \arrow[d, "c_{f,g}"'] & & (gf)_\# \arrow[d, "c_{f,g}"] \\
g_* f_* \arrow[r, "g_*\alpha_f"'] & g_* f_\# \arrow[r, "\alpha_g"'] & g_\# f_\#
\end{tikzcd}$$

and such that for all $X \in \mathbf{S}$, with identity map 1_X, $\alpha_{1_X} \colon (1_X)_* \to (1_X)_\#$ is the identity automorphism of $\mathbf{X}_* = \mathbf{X}_\#$. Morphisms of contravariant pseudofunctors are defined analogously.

Suppose we are given a pseudofunctor $*$, and a family of functors $f_\# \colon \mathbf{X}_* \to \mathbf{Y}_*$, one for each $\mathbf{S}$-morphism $f \colon X \to Y$, such that $f_\#$ is an identity functor whenever f is an identity map, and a family of functorial isomorphisms $\alpha_f \colon f_* \to f_\#$. It is left as an exercise to show that then *there is a unique family of isomorphisms of functors $c_{f,g} \colon (gf)_\# \xrightarrow{\sim} g_\# f_\#$ which together with the family $(f_\#)$ constitute a pseudofunctor such that the family (α_f) is an isomorphism of pseudofunctors.*

(3.6.7). Various refinements of these notions can be made.

(a). Assume that each category $\mathbf{X}_\#$ is a Δ-category, that each $f_\#$ (resp. $f^\#$) is a Δ-functor, and that each $c_{f,g}$ (resp. $d_{f,g}$) is an isomorphism of Δ-functors. We say then that $\#$ is a covariant (resp. contravariant) Δ-pseudofunctor.

A morphism of Δ-pseudofunctors is then a family α_f as in (3.6.6), with each α_f a morphism of Δ-functors.

(b). Assume that each category $\mathbf{X}_\#$ is a symmetric monoidal category, see (3.4.1), that each $f_\#$ is a symmetric monoidal functor (3.4.2), and that each $c_{f,g}$ is a morphism of symmetric monoidal functors [**EK**, p. 474], i.e., that the following natural diagrams commute (where $\otimes$ denotes the appropriate product functor, and $\mathcal{O}$ the unit; and $A, B \in \mathbf{X}_\#$):

(3.6.7.1)
$$\begin{tikzcd}
\mathcal{O}_Z \arrow[r] \arrow[d] & (gf)_\# \mathcal{O}_X \arrow[d] \\
g_\# \mathcal{O}_Y \arrow[r] & g_\# f_\# \mathcal{O}_X
\end{tikzcd}$$

(3.6.7.2)
$$\begin{tikzcd}
(gf)_\# A \otimes (gf)_\# B \arrow[rr] \arrow[d] & & (gf)_\# (A \otimes B) \arrow[d] \\
g_\# f_\# A \otimes g_\# f_\# B \arrow[r] & g_\# (f_\# A \otimes f_\# B) \arrow[r] & g_\# f_\# (A \otimes B)
\end{tikzcd}$$

We say then that $\#$ is a monoidal pseudofunctor.

We say that a contravariant pseudofunctor $\#$ is monoidal if for each map $f\colon X \to Y$ in $\mathbf{S}$, the *opposite* functor $(f^{\#})^{\mathsf{op}}\colon (\mathbf{Y}^{\#})^{\mathsf{op}} \to (\mathbf{X}^{\#})^{\mathsf{op}}$ is monoidal. In other words, we have functorial maps

$$f^{\#}(A \otimes B) \to f^{\#}A \otimes f^{\#}B$$

and a map

$$f^{\#}\mathcal{O}_Y \to \mathcal{O}_X$$

satisfying the obvious conditions.

A morphism of monoidal pseudofunctors is a family α_f as in (3.6.6) such that each α_f is a morphism of symmetric monoidal functors (i.e., α_f is compatible, in an obvious sense, with the maps (3.4.2.1).

(c). If every $\mathbf{X}_{\#}$ is both a Δ-category and a symmetric monoidal category, and if the multiplication $\mathbf{X}_{\#} \times \mathbf{X}_{\#} \to \mathbf{X}_{\#}$ is a Δ-functor (see (2.4.3)), then we say that $\mathbf{X}_{\#}$ is a monoidal Δ-category; and we speak correspondingly of monoidal Δ-pseudofunctors and their morphisms.

(d). A pair $({}^{*}, {}_{*})$ with ${}_{*}$ a pseudofunctor and * a contravariant pseudofunctor on $\mathbf{S}$ are said to be *adjoint* if the following conditions hold:

(i) $\mathbf{X}_{*} = \mathbf{X}^{*}$ *for all objects X in* $\mathbf{S}$.

(ii) *For every $f\colon X \to Y$ in $\mathbf{S}$ there are bifunctorial isomorphisms*

$$\mathrm{Hom}_{\mathbf{X}^*}(f^*C,\, D) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{Y}_*}(C,\, f_*D) \qquad (C \in \mathbf{Y}^*,\ D \in \mathbf{X}_*),$$

i.e., the functor $f_*\colon \mathbf{X}_* \to \mathbf{Y}_*$ is right adjoint *to $f^*\colon \mathbf{Y}^* \to \mathbf{X}^*$.*

(iii) *The resulting functorial diagrams* (3.6.2) (*or* $(3.6.2)^{\mathsf{op}}$) *commute.*

In the monoidal case, we also require:

(iv) *The natural maps*

$$f_*(A) \otimes f_*(B) \to f_*(A \otimes B),$$
$$f^*(f_*A \otimes f_*B) \to f^*f_*A \otimes f^*f_*B \to A \otimes B$$

correspond under the adjunction isomorphism of (ii) *above, as do the natural maps $f^*\mathcal{O}_Y \to \mathcal{O}_X$, $\mathcal{O}_Y \to f_*\mathcal{O}_X$.*

In the Δ-case, we also require that f^* and f_* be Δ-adjoint (3.3.1), i.e.,

(v) *The natural functorial morphisms*

$$1 \to f_*f^* \quad \textit{and} \quad f^*f_* \to 1$$

are both morphisms of Δ-functors.

(3.6.8). We add some remarks on existence and uniqueness, some of which are relevant to the subsequent construction and understanding of specific adjoint pairs of pseudofunctors.

(3.6.8.1). If $_*$ is a monoidal pseudofunctor on $\mathbf{S}$, and if for each map $f\colon X \to Y$ in $\mathbf{S}$ the functor $f_*\colon \mathbf{X}_* \to \mathbf{Y}_*$ has a left adjoint f^*, then *there is a unique contravariant monoidal pseudofunctor* * *on* $\mathbf{S}$ *such that* $\mathbf{X}^* = \mathbf{X}_*$ *for all objects* $X \in \mathbf{S}$*,* f^* *is the said left adjoint for each* $f\colon X \to Y$*, and the pair* $(^*, _*)$ *is adjoint.*

Indeed, condition (iii) in (d) above forces $d_{f,g}\colon f^*g^* \to (gf)^*$ to be the left conjugate of the given $c_{f,g}\colon (gf)_* \to g_*f_*$ (see beginning of this section, up to $(3.6.3)^*$). Similarly, (iv) imposes a unique monoidal structure on $(f^*)^{\mathsf{op}}$: given (ii), we see as in (3.4.5) that (iv) is equivalent to the following dual statement:

(iv)$'$ *The natural maps*

$$f^*(A) \otimes f^*(B) \leftarrow f^*(A \otimes B),$$

$$f_*(f^*A \otimes f^*B) \leftarrow f_*f^*A \otimes f_*f^*B \leftarrow A \otimes B$$

correspond under the above adjunction isomorphism (ii), *as do the natural maps*

$$f_*\mathcal{O}_X \leftarrow \mathcal{O}_Y, \qquad \mathcal{O}_X \leftarrow f^*\mathcal{O}_Y.$$

The rest of the proof is left to the reader.

(3.6.8.2). If $_*$ is a Δ-pseudofunctor on $\mathbf{S}$, and if for each map $f\colon X \to Y$ in $\mathbf{S}$ the functor $f_*\colon \mathbf{X}_* \to \mathbf{Y}_*$ has a left adjoint f^*, then *there is a unique contravariant* Δ*-pseudofunctor* * *on* $\mathbf{S}$ *such that* $\mathbf{X}^* = \mathbf{X}_*$ *for all objects* $X \in \mathbf{S}$*,* f^* *is the said left adjoint for each* $f\colon X \to Y$*, and the pair* $(^*, _*)$ *is adjoint.*

Indeed, by (3.3.8), each f^* carries a unique structure of Δ-functor such that (v) above holds; and for every $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{S}$, the isomorphism $d_{f,g}$—forced by (iii) to be the conjugate of the given Δ-functorial isomorphism $c_{f,g}$—is Δ-functorial, by (3.3.6).

(3.6.8.3). Here is another form of uniqueness:

If $(^*, _*)$ *and* $(^\#, _*)$ *are adjoint pairs of monoidal* (*or* Δ-)*pseudofunctors, and if for each* $f\colon X \to Y$ *we define the morphism* $\alpha_f\colon f^* \to f^\#$ *to be adjoint to the natural morphism* $1 \to f_*f^\#$*, then the family* α_f *is an isomorphism of monoidal* (*or* Δ-)*pseudofunctors.*

Remark **(3.6.9)** (Duality principle II). To each adjoint pair of monoidal pseudofunctors $(^*, _*)$ on $\mathbf{S}$, (3.6.7)(d), associate a *dual pair* $(^\#, _\#)$ of monoidal pseudofunctors on the dual category $\mathbf{S}^{\mathsf{op}}$ as follows:

$$\mathbf{X}^{\#} := (\mathbf{X}_{*})^{\mathsf{op}}, \qquad \mathbf{X}_{\#} := (\mathbf{X}^{*})^{\mathsf{op}}$$

for objects $X \in \mathbf{S}^{\mathsf{op}}$, and

$$f^{\#} := (f_*)^{\mathsf{op}} \colon (\mathbf{X}_*)^{\mathsf{op}} \to (\mathbf{Y}_*)^{\mathsf{op}}, \qquad f_{\#} := (f^*)^{\mathsf{op}} \colon (\mathbf{Y}^*)^{\mathsf{op}} \to (\mathbf{X}^*)^{\mathsf{op}}$$

for each map $f\colon Y \to X$ in $\mathbf{S}^{\mathsf{op}}$ (i.e., for each map $f\colon X \to Y$ in $\mathbf{S}$), the isomorphisms $f^{\#}g^{\#} \xrightarrow{\,\sim\,} (gf)^{\#}$ and $(gf)_{\#} \xrightarrow{\,\sim\,} g_{\#}f_{\#}$ being the obvious ones. The monoidal structure on the category $\mathbf{X}_{\#} = \mathbf{X}^{\#}$ is defined to be dual to that on $\mathbf{X}^{*} = \mathbf{X}_{*}$ see (3.4.5), and then each functor $f_{\#}$ is monoidal, with left adjoint $f^{\#}$. It follows that:

Each diagram built up from the basic data defining adjoint monoidal pairs can be interpreted in the dual context, giving rise to a "dual" diagram, obtained by interchanging $_*$ and * and reversing arrows, etc., etc.

This somewhat imprecise statement will be illustrated in Ex. (3.7.1.1) and in the proof of Prop. (3.7.2) below.

(3.6.10). With the terminology of (3.6.7), and with (3.5.2)(d) in mind, we can formally summarize many preceding results as follows.

Scholium. *Let* $\mathbf{S}$ *be the category of ringed spaces. For each object* $X \in \mathbf{S}$, *set* $\mathbf{X}^{*} = \mathbf{X}_{*} := \mathbf{D}(X)$ (*the derived category of the category of* $\mathcal{O}_X$*-modules*), *a closed* Δ*-category with product* $\underline{\otimes}$, *unit* $\mathcal{O}_X$, *and internal hom* $\mathsf{R}\mathcal{H}\mathrm{om}$. *For* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *in* $\mathbf{S}$, *write*

$$\begin{aligned} &f^* \ \textit{for}\ \ \mathsf{L}f^*\colon \mathbf{Y}^* \to \mathbf{X}^*, &\qquad &d_{f,g}\ \textit{for the map}\ \ (3.6.4)^*,\\ &f_* \ \textit{for}\ \ \mathsf{R}f_*\colon \mathbf{X}_* \to \mathbf{Y}_*, &\qquad &c_{f,g}\ \textit{for the map}\ \ (3.6.4)_*. \end{aligned}$$

This defines an adjoint pair $(^*, _*)$ *of monoidal* Δ*-pseudofunctors on* $\mathbf{S}$.

Proof. Essentially everything has already been proved, in (3.4.4)(b) and at the beginning of this §3.6, except for the commutativity of (3.6.7.1) and (3.6.7.2) (with $_*$ in place of $_\#$).

Commutativity of (3.6.7.1) is left to the reader.

To show that (3.6.7.2) commutes, first do it in the context of sheaves of modules—with the ordinary direct image functors see (3.1.7)—where it follows easily from definitions. A formal argument, using (iv) or (iv)′ above (details left to the reader), then yields the commutativity of the corresponding (dual) sheaf diagram with * in place of $_*$, and all arrows reversed. In this latter diagram, we can then replace f^* etc. by $\mathsf{L}f^*$, etc., and commutativity is preserved since the resulting derived functor diagram need only be checked when A and B are q-flat complexes, in which case it does not differ essentially from the original sheaf diagram.

Finally, the preceding formal (adjunction) argument, applied this time to derived functors, gives us commutativity in (3.6.7.2).

3.7. More formal consequences: projection, base change

We give some additional consequences, to be used later, of the formalism in §3.6. Again, the introductory remarks in §3.4, suitably modified, are relevant.

We consider an adjoint monoidal pair $(^{\boldsymbol{*}}, {}_{\boldsymbol{*}})$ as in (d) of (3.6.7).

Condition (ii) there means that for $f\colon X \to Y$ in $\mathbf{S}$, we have functorial maps

$$\eta\colon 1 \to f_*f^*, \qquad \epsilon\colon f^*f_* \to 1$$

such that the resulting compositions

$$f^* \xrightarrow{\eta} f^*f_*f^* \xrightarrow{\epsilon} f^*, \qquad f_* \xrightarrow{\eta} f_*f^*f_* \xrightarrow{\epsilon} f_*$$

are both identities.

For $X \in \mathbf{S}$, the product functor on the monoidal category $\mathbf{X}^{\boldsymbol{*}} = \mathbf{X}_{\boldsymbol{*}}$ will be denoted by $\otimes$.

For a map $f\colon X \to Y$ in $\mathbf{S}$, the functorial "projection" map

$$p_f\colon G \otimes f_*F \to f_*(f^*G \otimes F) \qquad (G \in \mathbf{Y}^{\boldsymbol{*}},\, F \in \mathbf{X}_{\boldsymbol{*}})$$

is defined as in (3.4.6). It is compatible with pseudofunctoriality, in the following sense.

Proposition (3.7.1). *For any* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *in* $\mathbf{S}$, *the following diagram, with* $F \in \mathbf{X}_{\boldsymbol{*}}$, $G \in \mathbf{Z}^{\boldsymbol{*}}$, *commutes.*

$$\begin{tikzcd}
G \otimes g_*f_*F \arrow[r, "p_g"] & g_*(g^*G \otimes f_*F) \arrow[r, "g_*(p_f)"] & g_*f_*(f^*g^*G \otimes F) \arrow[d, "\text{via } d_{f,g}"] \\
G \otimes (gf)_*F \arrow[u, "\text{via } c_{f,g}", "\simeq"'] \arrow[r, "p_{gf}"'] & (gf)_*((gf)^*G \otimes F) \arrow[r, "\sim", "c_{f,g}"'] & g_*f_*((gf)^*G \otimes F)
\end{tikzcd}$$

Proof. An expanded form of the diagram is obtained by pasting the first of the following diagrams, along its right edge, to the second, along its left edge. (All the arrows have an obvious interpretation.)

$$
\begin{tikzcd}
G\otimes g_*f_*F \arrow[r] & g_*g^*G\otimes g_*f_*F \arrow[r, equal] & g_*g^*G\otimes g_*f_*F \arrow[dd] \\
G\otimes (gf)_*F \arrow[u] \arrow[dd] \arrow[r] & g_*g^*G\otimes (gf)_*F \arrow[u] \arrow[d] & \\
\textcircled{1} & g_*f_*f^*g^*G\otimes (gf)_*F \arrow[r] \arrow[d] & g_*f_*f^*g^*G\otimes g_*f_*F \arrow[d] \\
(gf)_*(gf)^*G\otimes (gf)_*F \arrow[r] \arrow[dd] & g_*f_*(gf)^*G\otimes (gf)_*F \arrow[r] & g_*f_*(gf)^*G\otimes g_*f_*F \arrow[d] \\
 & \textcircled{2} & g_*(f_*(gf)^*G\otimes f_*F) \arrow[d] \\
(gf)_*((gf)^*G\otimes F) \arrow[rr] & & g_*f_*((gf)^*G\otimes F)
\end{tikzcd}
$$

$$
\begin{tikzcd}
g_*g^*G\otimes g_*f_*F \arrow[r] \arrow[d] & g_*(g^*G\otimes f_*F) \arrow[d] \\
g_*f_*f^*g^*G\otimes g_*f_*F \arrow[r] \arrow[d] & g_*(f_*f^*g^*G\otimes f_*F) \arrow[dd, equal] \\
g_*f_*(gf)^*G\otimes g_*f_*F \arrow[d] & \\
g_*(f_*(gf)^*G\otimes f_*F) \arrow[d] & g_*(f_*f^*g^*G\otimes f_*F) \arrow[l] \arrow[d] \\
g_*f_*((gf)^*G\otimes F) & g_*f_*(f^*g^*G\otimes F) \arrow[l]
\end{tikzcd}
$$

Subdiagram ① commutes because of commutativity of (3.6.2) (see condition (iii) in (3.6.7)(d)), Subdiagram ② commutes because of the commutativity of (3.6.7.2) (which is part of the definition of *monoidal* pseudofunctor); and commutativity of the remaining subdiagrams is clear. The conclusion follows.

Exercise **(3.7.1.1).** The preceding Proposition expresses the compatibility of the projection map with the structure "adjoint pair of monoidal pseudofunctors." One can ask about similar compatibilities for any of the maps introduced in §3.5. Here are some examples which will be needed later.

(*Challenge*: Establish metaresults of which such examples would be instances.)

With notation as in (3.7.1), ϱ as in (3.5.4.1), and $\beta_f\colon f_*[f^*-,-]\to[-,f_*-]$ as in (3.5.4.2) or (3.5.4.3), show that the following diagrams commute:

$$
\begin{tikzcd}
(gf)_*[(gf)^*G,F] \arrow[rr, "\beta_{gf}"] \arrow[d, "\text{via } c_{f,g}"', "\text{and } d_{f,g}"] & & {[G,(gf)_*F]} \arrow[d, "\text{via } c_{f,g}"] \\
g_*f_*[f^*g^*G,F] \arrow[r, "g_*\beta_f"'] & g_*[g^*G,f_*F] \arrow[r, "\beta_g"'] & {[G,g_*f_*F]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
f_*g_*[F,F'] \arrow[r, "f_*\varrho_g"] \arrow[d, "c^{-1}_{f,g}"'] & f_*[g_*F,g_*F'] \arrow[r, "\varrho_f"] & {[f_*g_*F,f_*g_*F']} \arrow[d, "\text{via } c^{-1}_{f,g}"] \\
(gf)_*[F,F'] \arrow[r, "\varrho_{gf}"'] & {[(gf)_*F,(gf)_*F']} \arrow[r, "\text{via } c^{-1}_{f,g}"'] & {[f_*g_*F,(gf)_*F']}
\end{tikzcd}
$$

Deduce from the first diagram that with $\rho_f\colon f^*[-,-]\to[f^*-,f^*-]$ as in (3.5.4.5), the next diagram commutes:

$$
\begin{tikzcd}
f^*g^*[G,G'] \arrow[r, "f^*\rho_g"] \arrow[d, "d_{f,g}"', "\simeq"] & f^*[g^*G,g^*G'] \arrow[r, "\rho_f"] & {[f^*g^*G,f^*g^*G']} \arrow[d, "\simeq"', "\text{via } d_{f,g}"] \\
(gf)^*[G,G'] \arrow[r, "\rho_{gf}"'] & {[(gf)^*G,(gf)^*G']} \arrow[r, "\sim", "\text{via } d_{f,g}"'] & {[f^*g^*G,(gf)^*G']}
\end{tikzcd}
$$

Hints. Apply (3.6.9) to the diagram in (3.6.7.2), resp. Prop. (3.7.1), and compare the result with the diagram left-conjugate to the first, resp. second, one above. The third diagram expands naturally as follows.

$$
\begin{tikzcd}[column sep=small]
f^*g^*[G,G'] \arrow[r] \arrow[dd, equal] & f^*g^*[G,g_*g^*G'] \arrow[r] \arrow[d] & f^*g^*g_*[g^*G,g^*G'] \arrow[r] \arrow[d] & f^*[g^*G,g^*G'] \arrow[d] \\
& f^*g^*[G,g_*f_*f^*g^*G'] \arrow[r] \arrow[d] & f^*g^*g_*[g^*G,f_*f^*g^*G'] \arrow[r] \arrow[d] & f^*[g^*G,f_*f^*g^*G'] \arrow[d] \\
f^*g^*[G,G'] \arrow[r] \arrow[dd] & f^*g^*[G,(gf)_*(gf)^*G'] \arrow[dd] & f^*g^*g_*f_*[f^*g^*G,f^*g^*G'] \arrow[r] \arrow[d] & f^*f_*[f^*g^*G,f^*g^*G'] \arrow[d] \\
& & (gf)^*(gf)_*[f^*g^*G,f^*g^*G'] \arrow[r] \arrow[d] & {[f^*g^*G,f^*g^*G']} \arrow[d] \\
(gf)^*[G,G'] \arrow[r] & (gf)^*[G,(gf)_*(gf)^*G'] \arrow[r] & (gf)^*(gf)_*[(gf)^*G,(gf)^*G'] \arrow[r] & {[(gf)^*G,(gf)^*G']}
\end{tikzcd}
$$

In this diagram, all but three subdiagrams clearly commute, and those three are taken care of by (3.6.2), $(3.6.2)^{\mathsf{op}}$, and the first diagram above.

Next, we introduce an oft-to-be-encountered "base change" morphism.

Proposition (3.7.2). (i) *To each commutative square σ in* $\mathbf{S}$:

$$
\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}
$$

there is associated a natural map of functors

$$\theta=\theta_\sigma : g^*f_* \longrightarrow f'_*g'^*\,,$$

equal to each of the following four compositions (with $h=fg'=gf'$):

(a) $$g^*f_* \xrightarrow{\eta} g^*f_*g'_*g'^* \xrightarrow{(c_{f',g})(c^{-1}_{g',f})} g^*g_*f'_*g'^* \xrightarrow{\epsilon} f'_*g'^*$$

(b) $$g^*f_* \xrightarrow{\eta(\ )\eta} f'_*f'^*g^*f_*g'_*g'^* \xrightarrow{(d_{f',g})(c^{-1}_{g',f})} f'_*h^*h_*g'^* \xrightarrow{\epsilon} f'_*g'^*$$

(c) $$g^*f_* \xrightarrow{\eta} f'_*f'^*g^*f_* \xrightarrow{(d^{-1}_{g',f})(d_{f',g})} f'_*g'^*f^*f_* \xrightarrow{\epsilon} f'_*g'^*$$

(d) $$g^*f_* \xrightarrow{\eta} g^*h_*h^*f_* \xrightarrow{(c_{f',g})(d^{-1}_{g',f})} g^*g_*f'_*g'^*f^*f_* \xrightarrow{\epsilon(\ )\epsilon} f'_*g'^*$$

(ii) *Given a pair of commutative squares*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] \arrow[d, "h'"'] & Y \arrow[d, "h"] \\
Z' \arrow[r, "g''"'] & Z
\end{tikzcd}$$

the following resulting diagram commutes:

$$\begin{tikzcd}
g''^*(hf)_* \arrow[rr, "\theta"] \arrow[d, "c_{f,h}"'] & & (h'f')_*g'^* \arrow[d, "c_{f',h'}"] \\
g''^*h_*f_* \arrow[r, "\theta"'] & h'_*g^*f_* \arrow[r, "\theta"'] & h'_*f'_*g'^*
\end{tikzcd}$$

(iii) *Given a pair of commutative squares*

$$\begin{tikzcd}
X'' \arrow[r, "h"] \arrow[d, "g''"'] & X' \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "g'"] \\
Y'' \arrow[r, "h'"'] & Y' \arrow[r, "f'"'] & Y
\end{tikzcd}$$

the following resulting diagram commutes:

$$\begin{tikzcd}
g''_*(fh)^* & & (f'h')^*g'_* \arrow[ll, "\theta"'] \\
g''_*h^*f^* \arrow[u, "d_{h,f}"] & h'^*g_*f^* \arrow[l, "\theta"] & h'^*f'^*g'_* \arrow[l, "\theta"] \arrow[u, "d_{h',f'}"']
\end{tikzcd}$$

Proof. (i) To get convinced that (a), (b) and (c) are the same, contemplate the following commutative diagram, noting that $\epsilon\circ\eta$ on the right (resp. bottom) edge is the identity map, and recalling for subdiagrams ① and ② the condition (iii) in the definition (3.6.7)(d) of "adjoint pair."

$$
\begin{tikzcd}
g^*f_* \arrow[r] \arrow[d, "\eta"'] & g^*f_*g'_*g'^* \arrow[r] \arrow[d, "\eta"'] & g^*g_*f'_*g'^* \arrow[r] \arrow[d, "\eta"'] & f'_*g'^* \arrow[d, "\eta"] \\
f'_*f'^*g^*f_* \arrow[r] \arrow[d] & f'_*f'^*g^*f_*g'_*g'^* \arrow[r] \arrow[d] & f'_*f'^*g^*g_*f'_*g'^* \arrow[r] \arrow[d] \arrow[r, phantom, shift right=6, "①"] & f'_*f'^*f'_*g'^* \arrow[dd, "\epsilon"] \\
f'_*g'^*f^*f_* \arrow[r] \arrow[d] & f'_*g'^*f^*f_*g'_*g'^* \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & f'_*h^*h_*g'^* \arrow[dr, "\epsilon"] & \\
f'_*g'^* \arrow[r, "\eta"'] & f'_*g'^*g'_*g'^* \arrow[rr, "\epsilon"'] & & f'_*g'^*
\end{tikzcd}
$$

The equality (c) = (d) is obtained from (a) = (b) by *duality* (3.6.9).

(ii) Consider the expanded diagram (3.7.2.2) on the following page.

Recall that the composition $\epsilon \circ \eta$ of the adjacent arrows in the middle is the identity. Commutativity of subdiagram ① is an easy consequence of the commutativity of (3.6.5.1) (axiom for pseudofunctors). Commutativity of the other subdiagrams is straightforward, and the conclusion follows.

(iii) is simply the *dual* of (ii) (see (3.6.9)). Q.E.D.

Proposition (3.7.3) (Base change and projection). *Let*

$$
\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}
$$

be a commutative $\mathbf{S}$*-diagram,* $P \in \mathbf{Y}_{*}$*,* $Q \in \mathbf{X}_{*}$*. Then with* θ *as in* (3.7.2)*,* $h = fg' = gf'$*, and* p *the projection map, the following diagram commutes*:

$$
\begin{tikzcd}
g^*P \otimes g^*f_*Q \arrow[d, "1\otimes\theta"'] & g^*(P \otimes f_*Q) \arrow[l, "(3.4.5.1)"'] \arrow[r, "g^*(p_f)"] & g^*f_*(f^*P \otimes Q) \arrow[d, "\theta"] \\
g^*P \otimes f'_*g'^*Q \arrow[d, "p_{f'}"'] & & f'_*g'^*(f^*P \otimes Q) \arrow[d, "(3.4.5.1)"] \\
f'_*(f'^*g^*P \otimes g'^*Q) \arrow[r, "d_{f',g}"'] & f'_*(h^*P \otimes g'^*Q) & f'_*(g'^*f^*P \otimes g'^*Q) \arrow[l, "d_{g',f}"]
\end{tikzcd}
$$

Proof. Consider the expanded diagram (3.7.3.1) on the following page (a diagram in which the arrows are self-explanatory). With a bit of patience, one checks that it suffices to show its commutativity.

(3.7.2.2)

$$\begin{tikzcd}
g''^*(hf)_* \arrow[r] \arrow[d] & g''^*(hf)_*g'_*g'^* \arrow[rr] \arrow[d] & & g''^*g''_*(h'f')_*g'^* \arrow[r] \arrow[d] & (h'f')_*g'^* \arrow[d] \\
g''^*h_*f_* \arrow[r] \arrow[d, "\eta"'] & g''^*h_*f_*g'_*g'^* \arrow[r] \arrow[d, "\eta"'] & g''^*h_*g_*f'_*g'^* \arrow[r] \arrow[d, shift right, "\eta"'] \arrow[u, phantom, "1"] & g''^*g''_*h'_*f'_*g'^* \arrow[r] & h'_*f'_*g'^* \\
g''^*h_*g_*g^*f_* \arrow[r] \arrow[d] & g''^*h_*g_*g^*f_*g'_*g'^* \arrow[r] & g''^*h_*g_*g^*g_*f'_*g'^* \arrow[r] \arrow[u, shift right, "\epsilon"'] & g''^*g''_*h'_*g^*g_*f'_*g'^* \arrow[r] \arrow[u, "\epsilon"'] & h'_*g^*g_*f'_*g'^* \arrow[u, "\epsilon"'] \\
g''^*g''_*h'_*g^*f_* \arrow[rr, "\epsilon"] & & h'_*g^*f_* \arrow[rr, "\eta"] & & h'_*g^*f_*g'_*g'^* \arrow[u]
\end{tikzcd}$$

(3.7.3.1)

$$\begin{tikzcd}
g^*P \otimes g^*f_*Q \arrow[d] & g^*(P \otimes f_*Q) \arrow[l] \arrow[r] \arrow[d] & g^*f_*(f^*P \otimes Q) \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & g^*f_*g'_*g'^*(f^*P \otimes Q) \arrow[r] \arrow[d] & g^*g_*f'_*g'^*(f^*P \otimes Q) \arrow[dd] \\
g^*P \otimes g^*f_*g'_*g'^*Q \arrow[d] & g^*(P \otimes f_*g'_*g'^*Q) \arrow[l] \arrow[r] \arrow[d] & g^*f_*(f^*P \otimes g'_*g'^*Q) \arrow[r] \arrow[d, phantom, "2"] & g^*f_*g'_*(g'^*f^*P \otimes g'^*Q) \arrow[d] & \\
g^*P \otimes g^*h_*g'^*Q \arrow[d] & g^*(P \otimes h_*g'^*Q) \arrow[l] \arrow[rr] \arrow[d] & & g^*h_*(h^*P \otimes g'^*Q) \arrow[dd] & f'_*g'^*(f^*P \otimes Q) \arrow[dd] \\
g^*P \otimes g^*g_*f'_*g'^*Q \arrow[d] \arrow[dr, phantom, "4"] & g^*(P \otimes g_*f'_*g'^*Q) \arrow[l] \arrow[d] & 3 & & \\
g^*P \otimes f'_*g'^*Q \arrow[d, equal] & g^*g_*(g^*P \otimes f'_*g'^*Q) \arrow[l] \arrow[rr] \arrow[d] & & g^*g_*f'_*(f'^*g^*P \otimes g'^*Q) \arrow[d] & f'_*(g'^*f^*P \otimes g'^*Q) \arrow[d] \\
g^*P \otimes f'_*g'^*Q \arrow[r, equal] & g^*P \otimes f'_*g'^*Q \arrow[rr] & & f'_*(f'^*g^*P \otimes g'^*Q) \arrow[r] & f'_*(h^*P \otimes g'^*Q)
\end{tikzcd}$$

Subdiagram ① commutes by (3.4.7)(i), subdiagrams ② and ③ by (3.7.1), and ④ by the last sentence in (3.4.6.2). Commutativity of the other subdiagrams is straightforward to check.

REMARK **(3.7.3.1).** In the case of ringed spaces (3.6.10), the unlabeled arrows in the preceding diagram represent isomorphisms. So if θ is an isomorphism too, then the maps $g^*(p_f)$ and $p_{f'}$ are isomorphic. For such diagrams we can say then that "projection commutes with base change."

For example, when g is an open immersion, then θ is an isomorphism. That amounts to *compatibility of* $\mathsf{R}f_*$ *with open immersions,* which is also an immediate consequence of (2.4.5.2).

For other situations in which θ is an isomorphism, see (3.9.5) and its generalization (3.10.3).

3.8. Direct Sums

Proposition (3.8.1). *Let X be a ringed space. Then arbitrary* (*small*) *direct sums exist in* $\mathbf{K}(X)$ *and in* $\mathbf{D}(X)$; *and the canonical functor* $Q\colon \mathbf{K}(X) \to \mathbf{D}(X)$ *preserves them. In both* $\mathbf{K}(X)$ *and* $\mathbf{D}(X)$, *natural maps of the type* $\oplus_{\alpha\in A}\big(C_\alpha[1]\big) \to \big(\oplus_{\alpha\in A} C_\alpha\big)[1]$ *are always isomorphisms—direct sums commute with translation; and any direct sum of triangles is a triangle.*

PROOF. Let $(C_\alpha)_{\alpha\in A}$ (A small) be a family of complexes of $\mathcal{O}_X$-modules. The usual direct sum C of the family (C_α)—together with the homotopy classes of the canonical maps $C_\alpha \to C$—is also a direct sum in the category $\mathbf{K}(X)$. Since any complex in $\mathbf{D}(X)$ is isomorphic to a q-injective one, and since $\mathrm{Hom}_{\mathbf{D}(X)}(-, I) = \mathrm{Hom}_{\mathbf{K}(X)}(-, I)$ for any q-injective I, see (2.3.8(v)), it follows that C is also a direct sum in $\mathbf{D}(X)$.[38] The remaining assertions are easily checked for $\mathbf{K}(X)$, where we need only consider standard triangles, see (1.4.3); and they follow for $\mathbf{D}(X)$ upon application of Q, see (1.4.4). Q.E.D.

Proposition (3.8.2). *Let Y be a ringed space, and let $(C_\alpha)_{\alpha\in A}$ be a small family of complexes of $\mathcal{O}_Y$-modules. Then*:

(i) *For any* $D \in \mathbf{D}(Y)$, *the canonical map is an isomorphism*

$$\oplus_\alpha(C_\alpha \underset{=}{\otimes} D) \xrightarrow{\sim} (\oplus_\alpha C_\alpha) \underset{=}{\otimes} D.$$

(ii) *For any ringed-space map* $f\colon X \to Y$, *the canonical map is an isomorphism*

$$\oplus_\alpha \mathsf{L}f^*C_\alpha \xrightarrow{\sim} \mathsf{L}f^*(\oplus_\alpha C_\alpha).$$

[38] A more elementary proof, not using q-injective resolutions, is given in [**BN**, §1].

PROOF. Each C_α is isomorphic to a q-flat complex; and any direct sum of q-flat complexes is still q-flat, see §2.5. Hence the assertions reduce to the corresponding ones for ordinary complexes, with $\otimes$ in place of $\underline{\underline{\otimes}}$ and f^* in place of $\mathbf{L}f^*$.

Alternatively, in view of (2.6.1)* and (3.2.1) one can use the fact that any functor having a right adjoint respects direct sums. Q.E.D.

Proposition (3.8.3) (See [**N**$'$, p. 38, Remark 1.2.2].) *Let Y be a ringed space and*

$$C'_\alpha \longrightarrow C_\alpha \longrightarrow C''_\alpha \longrightarrow TC'_\alpha \qquad (\alpha \in A)$$

a small family of $\mathbf{D}(Y)$-triangles. Then the naturally resulting sequence

$$\oplus_\alpha C'_\alpha \longrightarrow \oplus_\alpha C_\alpha \longrightarrow \oplus_\alpha C''_\alpha \longrightarrow \oplus_\alpha TC'_\alpha \cong T(\oplus_\alpha C'_\alpha) \qquad (\alpha \in A)$$

is also a $\mathbf{D}(Y)$-triangle.

EXERCISE. Deduce (3.8.2)(i) from (2.5.10)(c). Using, e.g., (2.5.5), prove an analogous generalization of (3.8.2)(ii), i.e., show that if (C_α) is a (small, directed) inductive system of complexes of $\mathcal{O}_Y$-modules, then there are natural isomorphisms

$$\varinjlim_\alpha H^n\mathbf{L}f^*C_\alpha \xrightarrow{\ \sim\ } H^n\mathbf{L}f^*(\varinjlim_\alpha C_\alpha) \qquad (n \in \mathbb{Z}).$$

3.9. Concentrated scheme-maps

This section contains some refinements of preceding considerations as applied to a map $f\colon X \to Y$ of *schemes,* see (3.4.4)(b). Except in (3.9.1), which does not involve $\mathbf{R}f_*$, we need f to be *concentrated* (= quasi-compact and quasi-separated). The main result (3.9.4) asserts that under mild restrictions on f or on the $\mathcal{O}_X$-complex F, the projection map

$$p\colon \mathbf{R}f_*F \underline{\underline{\otimes}} G \to \mathbf{R}f_*(F \underline{\underline{\otimes}} \mathbf{L}f^*G) \qquad \big(\text{see (3.4.6)}\big)$$

is an *isomorphism* for any $\mathcal{O}_Y$-complex G having quasi-coherent homology. The results of (3.9.1) and (3.9.2) on good behavior, vis-à-vis quasi-coherence, of the derived direct and inverse image functors of a concentrated map allow "way-out" reasoning to reduce (3.9.4) essentially to the trivial case $G = \mathcal{O}_Y$, provided that F and G are bounded above; homological compatibility of $\mathbf{R}f_*$ and $\varinjlim$ (proved in (3.9.3)) then gets rid of the boundedness.

Another Proposition, (3.9.5), says that for concentrated f the map θ associated as in (3.7.2) to certain *flat* base changes is an isomorphism. A stronger result will be given in Theorem (3.10.3), which contains (3.9.4) as well. (But (3.9.4) is used in the proof of (3.10.3)).

Proposition (3.9.6) takes note of, among other things, the fact that on a quasi-compact separated scheme, complexes with quasi-coherent homology are $\mathbf{D}$-isomorphic to quasi-coherent complexes.

We begin with some notation and terminology relative to any ringed space X, with $\mathbf{K}(X)$ and $\mathbf{D}(X)$ as in §3.1.

As in (1.6)–(1.8), we have various triangulated (i.e., Δ-)subcategories of $\mathbf{K}(X)$, denoted $\mathbf{K}^{*}(X)$, $\overline{\mathbf{K}}^{*}(X)$ (with "*" indicating a boundedness condition—below ($* = +$), above ($* = -$), or both above and below ($* = \mathsf{b}$)—and "$\overline{}$" indicating application of the boundedness condition to the homology of a complex rather than to the complex itself); and we have the corresponding derived categories $\mathbf{D}^{*}(X)$, $\overline{\mathbf{D}}^{*}(X)$, which are Δ-subcategories of $\mathbf{D}(X)$. For example, $\mathbf{K}^{+}(X)$ is the full subcategory of $\mathbf{K}(X)$ whose objects are complexes $A^\bullet$ of $\mathcal{O}_X$-modules such that $A^n = 0$ for all $n \le n_0(A^\bullet)$ (where $n_0(A^\bullet)$ is some integer depending on $A^\bullet$); and $\overline{\mathbf{D}}^{-}(X)$ is the full subcategory of $\mathbf{D}(X)$ whose objects are complexes $A^\bullet$ such that $H^n(A^\bullet) = 0$ for all $n \ge n_1(A^\bullet)$.

The subscript "qc" indicates collections of $\mathcal{O}_X$-complexes whose homology sheaves are all *quasi-coherent* (see (1.9), with $\mathcal{A}^{\#}$ the category of quasi-coherent $\mathcal{O}_X$-modules, which is a plump subcategory of the category of all $\mathcal{O}_X$-modules [**GD**, p. 217, (2.2.2) (iii)]). For example $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}(X)$ is the Δ-subcategory of $\mathbf{D}(X)$ whose objects are complexes $A^\bullet$ such that $H^n(A^\bullet)$ is quasi-coherent for all $n \in \mathbb{Z}$, and $H^n(A^\bullet) = 0$ for $n \le n_0(A^\bullet)$.

Proposition (3.9.1). *For any scheme-map* $f\colon X \to Y$ *we have*

$$\mathsf{L}f^{*}\big(\mathbf{D}_{\mathsf{qc}}(Y)\big) \subset \mathbf{D}_{\mathsf{qc}}(X).$$

Proof. For $C \in \mathbf{D}_{\mathsf{qc}}(Y)$ and $C_m := \tau_{\le m}C$ (1.10), there exists a q-flat resolution

$$\varinjlim Q_m = Q \to C = \varinjlim C_m \qquad (m \ge 0)$$

where for each i, Q_m is a bounded-above flat resolution of C_m, see (2.5.5). The resulting maps

$$\varinjlim f^{*}Q_m \to f^{*}Q \leftarrow \mathsf{L}f^{*}Q \to \mathsf{L}f^{*}C$$

are all *isomorphisms* in $\mathbf{D}(X)$ (recall that, as indicated just before (3.1.3), q-flat $\Rightarrow$ left-f^{*}-acyclic, and dualize the last assertion in (2.2.6)); and it follows that

$$H^n(\mathsf{L}f^{*}C) \cong \varinjlim H^n(f^{*}Q_m) \cong \varinjlim H^n(\mathsf{L}f^{*}C_m) \qquad (n \in \mathbb{Z}).$$

Since $\varinjlim$ preserves quasi-coherence, we need only deal with the case where $C = C_m \in \mathbf{D}^{-}_{\mathsf{qc}}(Y)$; and then way-out reasoning [**H**, p. 73, (ii) (dualized)] reduces us further to showing that *for any quasi-coherent* $\mathcal{O}_Y$*-module* F *and any* $i \in \mathbb{Z}$, *the* $\mathcal{O}_X$*-modules* $L_if^{*}(F) := H^{-i}\mathsf{L}f^{*}(F) \quad (i \ge 0)$ *are also quasi-coherent.*

For this, note that the restriction of a flat resolution of F to an open subset $U \subset Y$ is a flat resolution of the restriction $F|_U$, whence formation of $L_i f^*(F)$ "commutes" (in an obvious sense) with open immersions on Y; so we can assume X and Y to be affine, say $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$, and $F = \widetilde{G}$, the quasi-coherent $\mathcal{O}_Y$-module associated to some A-module G; and then if $G_\bullet \to G$ is an A-free resolution of G, it is easily seen (since $M \mapsto \widetilde{M}$ is an exact functor of A-modules M [**GD**, p. 198, (1.3.5)], and since $f^*\widetilde{M} = (B \otimes_A M)\widetilde{\ }$ [*ibid.,* p. 213, (1.7.7)]) that $L_i f^*(F)$ is the quasi-coherent $\mathcal{O}_X$-module $\widetilde{H_i}$, where H_i is the homology $H_i := H_i(B \otimes_A G_\bullet) = \operatorname{Tor}_i^A(B, G)$. Q.E.D.

We will use the adjective *concentrated* as a less cumbersome synonym for *quasi-compact and quasi-separated.* Elementary properties of concentrated schemes and scheme-maps can be found in [**GD**, pp. 290 *ff*]. In particular, if $f\colon X \to Y$ is a scheme-map with Y concentrated, then X is concentrated iff f is a concentrated map [*ibid.,* p. 295, (6.1.10)].

Proposition (3.9.2). *Let $f\colon X \to Y$ be a concentrated map of schemes. Then*

$$\mathbf{R}f_*\big(\mathbf{D}_{\mathsf{qc}}(X)\big) \subset \mathbf{D}_{\mathsf{qc}}(Y). \tag{3.9.2.1}$$

Moreover, with notation as in §1.10, *for all $n \in \mathbb{Z}$ it holds that*

$$\mathbf{R}f_*\big(\mathbf{D}_{\mathsf{qc}}(X)_{\geq \mathbf{n}}\big) \subset \mathbf{D}_{\mathsf{qc}}(Y)_{\geq \mathbf{n}}; \tag{3.9.2.2}$$

and if Y is quasi-compact, then there exists an integer d such that for every $n \in \mathbb{Z}$,

$$\mathbf{R}f_*\big(\mathbf{D}_{\mathsf{qc}}(X)_{\leq \mathbf{n}}\big) \subset \mathbf{D}_{\mathsf{qc}}(Y)_{\leq \mathbf{n+d}}. \tag{3.9.2.3}$$

Proof. That $\mathbf{R}f_*(\mathbf{D}(X)_{\geq \mathbf{n}}) \subset \mathbf{D}(Y)_{\geq \mathbf{n}}$ is, implicitly, in (2.7.3): any $F \in \mathbf{D}(X)_{\geq \mathbf{n}}$ admits the natural quasi-isomorphism $(1.8.1)^+ : F \to \tau^+ F$, and there is a quasi-isomorphism $\tau^+ F \to I$ where I is a flasque complex with $I^m = 0$ for all $m < n$, so that $\mathbf{R}f_* F \cong f_* I \in \mathbf{D}(Y)_{\geq \mathbf{n}}$.

To finish proving (3.9.2.2), i.e., to show that if I has quasi-coherent homology then so does $f_* I$, use the standard spectral sequence

$$R^p f_*\big(H^q(I)\big) \Rightarrow H^\bullet\big(f_* I\big) \qquad (R^p f_* := H^p \mathbf{R} f_*)$$

and the fact (proved in [**AHK**, p. 33, Thm. (5.6)] or [**Kf**, p. 643, Cor. 11]) that $R^p f_*$ preserves quasi-coherence of sheaves. Or, reduce to this fact by "way-out" reasoning, see [**H**, p. 88, Prop. 2.1].

For the rest, we need:

Lemma (3.9.2.4). *If Y is quasi-compact then there is an integer d such that for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any $i > d$, $R^i f_*\mathcal{F} = 0$.*

Proof. Since Y is covered by finitely many affine open subschemes Y_k and since for each k the restriction $R^i f_*\mathcal{F}|_{Y_k}$ is the quasi-coherent sheaf associated to the $\Gamma(Y_k, \mathcal{O}_Y)$-module $H^i(f^{-1}(Y_k), \mathcal{F})$ [**Kf**, p. 643, Cor. 11], we need only show that *if Y is affine then there is an integer d such that $H^i(X, \mathcal{F}) = 0$ for all $i > d$.*

Note that X is now a concentrated scheme. We proceed by induction on the unique integer $n = n(X)$ such that X can be covered by n quasi-compact separated open subschemes, but not by any $n-1$ such subschemes. (This integer exists because X is quasi-compact and its affine open subschemes are quasi-compact and separated.)

If $n = 1$, i.e., X is separated, then $H^i(X, \mathcal{F})$ is the Čech cohomology with respect to a finite cover $X = \cup_{j=0}^d X_j$ by affine open subschemes, so it vanishes for $i > d$.

Suppose next that

$$X = X_1 \cup X_2 \cup \cdots \cup X_n \qquad (n = n(X) > 1)$$

with each X_j a quasi-compact separated open subscheme of X. Since X is quasi-separated therefore $X_j \cap X_1$ is quasi-compact and separated,[39] so setting

$$X_0 := X_2 \cup \cdots \cup X_n$$

we have $n(X_0) < n$ and $n(X_0 \cap X_1) < n$. The desired conclusion follows then from the inductive hypothesis and from the long exact sequence

$$\cdots \to H^{i-1}(X_0 \cap X_1, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(X_0, \mathcal{F}) \oplus H^i(X_1, \mathcal{F}) \to \ldots$$

associated to the obvious short exact sequence of complexes

$$0 \to \Gamma(X, \mathcal{I}^\bullet) \to \Gamma(X_0, \mathcal{I}^\bullet) \oplus \Gamma(X_1, \mathcal{I}^\bullet) \to \Gamma(X_0 \cap X_1, \mathcal{I}^\bullet) \to 0$$

where $\mathcal{I}^\bullet$ is a flasque resolution of $\mathcal{F}$. Q.E.D.

[39] Quasi-compactness holds by [**GD**, p. 296, (6.1.12)], where (U_α) should be a *base* of the topology.

Now let $F \in \mathbf{D}_{\mathsf{qc}}(X)$ and $N \in \mathbb{Z}$. Starting with an injective resolution $\tau_{\geq N}F \to I_N$, and using (3.9.2.5)(ii) below (with $\mathbf{J}$ the category of bounded-below injective complexes), we build inductively a commutative ladder

$$\begin{tikzcd} \dots \arrow[r] & \tau_{\geq n}F \arrow[r, "\alpha_n"] \arrow[d, "\beta_n"'] & \tau_{\geq n+1}F \arrow[r] \arrow[d, "\beta_{n+1}"] & \dots \arrow[r] & \tau_{\geq N}F \arrow[d] \\ \dots \arrow[r] & I_n \arrow[r, "\gamma_n"'] & I_{n+1} \arrow[r] & \dots \arrow[r] & I_N \end{tikzcd}$$

where for $-\infty < n < N$, α_n is the natural map, β_n is a quasi-isomorphism, I_{n+1} is a bounded-below injective (hence, by (2.3.4), q-injective) complex, and γ_n is split-surjective in each degree. Then $I := \varprojlim I_n$ is q-injective [**Sp**, p. 130, 2.5]; and the natural map $\varprojlim \tau_{\geq n}F = F \to I$ is a quasi-isomorphism [**Sp**, p. 134, 3.13]. So we have an isomorphism $\mathbf{R}f_*F \xrightarrow{\sim} f_*I$.

It follows from (2.4.5.2) that $\mathbf{R}f_*$ is compatible with open immersions on Y, and hence if (3.9.2.1) holds whenever Y is quasi-compact (indeed, affine) then it holds always. Assuming Y to be quasi-compact, we argue further as in *loc. cit.* Since γ_n is split surjective in each degree m, its kernel C_n is a bounded-below injective complex, and for any affine open $U \subset Y$, γ_n induces a surjection $\Gamma(f^{-1}U, I_n^m) \twoheadrightarrow \Gamma(f^{-1}U, I_{n+1}^m)$ with kernel $\Gamma(f^{-1}U, C_n^m)$. The five-lemma yields that β_n induces a quasi-isomorphism to C_n from the kernel A_n of the surjection α_n; and in $\mathbf{D}(X)$, $A_n \cong H^n(F)[-n]$. Thus $C_n[n]$ is an injective resolution of $H^n(F)$, and so if d is the integer in (3.9.2.4) then for any $m > n + d$,

$$H^m\big(\Gamma(f^{-1}U, C_n)\big) \cong H^{m-n}\big(f^{-1}U, H^n(F)\big) \cong \Gamma\big(U, R^{m-n}f_*H^n(F)\big) = 0,$$

so that the sequence

$$\Gamma(f^{-1}U, C_n^{m-1}) \to \Gamma(f^{-1}U, C_n^m) \to \Gamma(f^{-1}U, C_n^{m+1}) \to \Gamma(f^{-1}U, C_n^{m+2})$$

is *exact.* A Mittag-Leffler-like diagram chase ([**Sp**, p. 126, Lemma], applied to the inverse system of diagrams

$$\Gamma(f^{-1}U, I_n^{m-1}) \to \Gamma(f^{-1}U, I_n^m) \to \Gamma(f^{-1}U, I_n^{m+1}) \to \Gamma(f^{-1}U, I_n^{m+2})$$

where n runs through $\mathbb{Z}$ and $I_n := I_N$ for all $n > N$) shows then that if $m \geq N + d$ then the natural map

$$\begin{aligned} H^m\big(\Gamma(U, f_*I)\big) &= H^m\big(\varprojlim \Gamma(f^{-1}U, I_n)\big) \\ &\to H^m\big(\Gamma(f^{-1}U, I_N)\big) = H^m\big(\Gamma(U, f_*I_N)\big) \end{aligned}$$

is an isomorphism. Sheafifying on Y, we get that *for any* $m \geq N + d$, *the natural composition*

$$R^mf_*F = H^m(\mathbf{R}f_*F) \xrightarrow{\sim} H^m(f_*I) \longrightarrow H^m(f_*I_N) \xrightarrow{\sim} R^mf_*(\tau_{\geq N}F)$$

is an isomorphism. From (3.9.2.2) we conclude then that R^mf_*F is quasi-coherent, which gives (3.9.2.1) (since N is arbitrary); and furthermore if $\tau_{\geq N}F \cong 0$, then $\tau_{\geq N+d}\mathbf{R}f_*F \cong 0$, proving (3.9.2.3). Q.E.D.

Lemma (3.9.2.5). *Let $\mathcal{A}$ be an abelian category, and let $\mathbf{J}$ be a full subcategory of the category $\mathbf{C}$ of $\mathcal{A}$-complexes such that* (1): *a complex B is in $\mathbf{J}$ iff $B[1]$ is, and* (2): *for any map f in $\mathbf{J}$, the cone C_f* (§1.3) *is in $\mathbf{J}$.*

(i) *Let $u\colon P \to C$ be a map in $\mathbf{C}$ with $P \in \mathbf{J}$ and such that there exists a quasi-isomorphism $h\colon Q \to C_u$ with $Q \in \mathbf{J}$. Then u factors as $P \xrightarrow{v} P_1 \xrightarrow{u_1} C$ where $P_1 \in \mathbf{J}$, u_1 is a quasi-isomorphism, and in each degree m, $v^m\colon P^m \to P_1^m$ is a split monomorphism, i.e., has a left inverse.*

(ii) *Let $s\colon C \to I$ be a map in $\mathbf{C}$ with $I \in \mathbf{J}$ and such that there exists a quasi-isomorphism $C_s \to J$ with $J \in \mathbf{J}$. Then s factors as $C \xrightarrow{s_1} I_1 \xrightarrow{t} I$ where $I_1 \in \mathbf{J}$, s_1 is a quasi-isomorphism, and in each degree m, $t^m\colon I_1^m \to I^m$ is a split epimorphism, i.e., has a right inverse.*

Proof. (i) We have a diagram in $\mathbf{C}$

$$\begin{tikzcd}
P \arrow[r, "v"] \arrow[d, equal] & C_{wh}[-1] \arrow[r] \arrow[d, "g"'] & Q \arrow[r, "wh"] \arrow[d, "h"] & P[1] \arrow[d, equal] \\
P \arrow[r] \arrow[d, equal] & C_w[-1] \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "①"] & C_u \arrow[r, "w"'] \arrow[d, equal] & P[1] \arrow[d, equal] \\
P \arrow[r, "u"'] & C \arrow[r] & C_u \arrow[r, "w"'] & P[1]
\end{tikzcd}$$

where the bottom row is the standard triangle associated to u, the top two rows are made up of natural maps, φ is as in (1.4.3.1), and g is given in degree m by the map

$$g^m = 1 \oplus h^m\colon C_{wh}[-1]^m = P^m \oplus Q^m \to P^m \oplus C_u^m = C_w[-1]^m\,.$$

Here all the subdiagrams other than ① commute, and ① is homotopy-commutative (see (1.4.3.1)). By ($\Delta 2$) in §1.4, the rows of the diagram become triangles in $\mathbf{K}(\mathcal{A})$. Since h is a quasi-isomorphism, we see, using the exact homology sequences $(1.4.5)^{\mathrm{H}}$ of these triangles, that the composed map $\varphi \circ g$ is also a quasi-isomorphism. Since P and Q are in $\mathbf{J}$, so is $C_{wh}[-1]$. Thus we can take $P_1 := C_{wh}[-1]$ and $u_1 := \varphi \circ g$.

(ii) A proof resembling that of (i) (with arrows reversed) is left to the reader. See also the following exercise (a), or [**Sp**, p. 132, proof of 3.3]. Q.E.D.

Exercises **(3.9.2.6).** (a) Convince yourself that (i) and (ii) in (3.9.2.5) are *dual*, i.e., (ii) is essentially the statement about $\mathcal{A}$ obtained by replacing $\mathcal{A}$ in (i) by its opposite category $\mathcal{A}^{\mathsf{op}}$.

(b) (Cf. (1.11.2)(iv).) Let X be a scheme and let $\mathcal{A}_X$ (resp. $\mathcal{A}_X^{\mathsf{qc}}$) be the category of all $\mathcal{O}_X$-modules (resp. quasi-coherent $\mathcal{O}_X$-modules). Let $\phi\colon \mathcal{A}_X \to \mathfrak{Ab}$ be an additive functor satisfying $\phi(\varprojlim I_n) = \varprojlim \phi(I_n)$ for any inverse system $(I_n)_{n<0}$ of $\mathcal{A}_X$-injectives in which all the maps $I_n \to I_{n+1}$ are *split surjective.* Then

$$\dim^+\big(\mathbf{R}\phi|_{\mathbf{D}_{\mathsf{qc}}(X)}\big) = \dim^+\big(\mathbf{R}\phi|_{\mathcal{A}_X^{\mathsf{qc}}}\big)\,.$$

(c) Show: for any proper map $f\colon X \to Y$ of noetherian schemes, $\mathbf{R}f_*\mathbf{D}_{\mathsf{c}}(X) \subset \mathbf{D}_{\mathsf{c}}(Y)$.
Hint. (3.9.2), [**H**, p. 74, (iii)], [**EGA**, III, (3.2.1)].

(3.9.3). Henceforth, index sets A for inductive systems are assumed to be (small and) filtered: $\alpha, \beta \in A \Rightarrow \exists\gamma \in A$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$. (More generally, the results will be valid for limits over filtered—or even pseudo-filtered—categories [**GV**, pp. 14–15], [**M**, p. 211].)

Lemma (3.9.3.1). *Let $f\colon X \to Y$ be a concentrated scheme-map. Fix $n \in \mathbb{Z}$, let $(C_\alpha, \varphi_{\beta\alpha})_{\alpha,\beta\in A}$ be an inductive system of $\mathcal{O}_X$-complexes all of whose homology vanishes in degree $< n$, and set $C := \varinjlim_\alpha C_\alpha$. Then we have natural isomorphisms*

$$\varinjlim_\alpha \mathbf{R}^i f_*(C_\alpha) \xrightarrow{\ \sim\ } \mathbf{R}^i f_*(C) \qquad (\mathbf{R}^i f_* := H^i\mathbf{R}f_*,\ i \in \mathbb{Z}).$$

Proof.[40] In the category of bounded-below $\mathcal{O}_X$-complexes D, we can choose flasque resolutions $D \to F$ *functorially,* as follows: for each $q \in \mathbb{Z}$, let $0 \to D^q \to F^{0q} \to F^{1q} \to F^{2q} \to \dots$ be the (flasque) Godement resolution of D^q [**G**, p. 167, 4.3], set $F^{pq} := 0$ if $p < 0$, and let F be the complex coming from the double complex F^{pq}, i.e., $F^m := \oplus_{p+q=m} F^{pq}$, etc; then F^m is flasque, and diagram chasing, or a simple spectral sequence argument, shows that the family of natural maps $D^m \to F^{0m} \subset F^m$ gives a quasi-isomorphism $g_D\colon D \to F$. We will refer to this g_D (or simply F) as the *Godement resolution* of D.

With C_α and n as above, the truncation operator $\tau_{\geq n}$ as in §1.10, and F_α the Godement resolution of $\tau_{\geq n} C_\alpha$, we have an inductive system of quasi-isomorphisms $C_\alpha \to \tau_{\geq n} C_\alpha \to F_\alpha$, and hence a quasi-isomorphism $C \to F := \varinjlim F_\alpha$. Each F_α is flasque, hence f_*-acyclic (2.7.3). By [**Kf**, p. 641, Cor. 5 and 7], F is a complex of f_*-acyclic sheaves, and so, being bounded below, F itself is f_*-acyclic, see (2.7.4) (dualized). The last assertion in (2.2.6) shows then that the (obvious) map in (3.9.3.1) is isomorphic to the natural map

$$\varinjlim H^i(f_*F_\alpha) = H^i(\varinjlim f_*F_\alpha) \to H^i(f_* \varinjlim F_\alpha) = H^i(f_*F),$$

which is an isomorphism since f_* commutes with $\varinjlim$ [**Kf**, p. 641, Prop. 6]. Q.E.D.

Corollary (3.9.3.2). *Let $f\colon X \to Y$ be a concentrated scheme-map. With notation as in §1.9, let $\mathcal{A}^\#$ be a plump subcategory of the category $\mathcal{A}_X$ of $\mathcal{O}_X$-modules, such that any $\varinjlim$ of objects in $\mathcal{A}^\#$ is itself in $\mathcal{A}^\#$ and such that the restriction of $\mathbf{R}f_*$ to $\mathbf{D}_\#(X)$ is bounded above (§1.11). Let $(C_\alpha, \varphi_{\beta\alpha})_{\alpha,\beta\in A}$ be an inductive system of complexes all of whose homology lies in $\mathcal{A}^\#$, and set $C := \varinjlim_\alpha C_\alpha$. Then we have natural isomorphisms*

$$\varinjlim_\alpha \mathbf{R}^i f_*(C_\alpha) \xrightarrow{\ \sim\ } \mathbf{R}^i f_*(C) \qquad (\mathbf{R}^i f_* := H^i\mathbf{R}f_*,\ i \in \mathbb{Z}).$$

[40] cf. [**EGA**, III, Chap. 0, p. 36, (11.5.1)].

Remarks. (a) If the map f is *finite-dimensional* (2.7.6), (e.g., if X is noetherian, of finite Krull dimension (2.7.6.2)), then all the hypotheses in (3.9.3.2) are satisfied when $\mathcal{A}^{\#} = \mathcal{A}_X$.

(b) By (3.9.2.3), if Y is quasi-compact then all the hypotheses in (3.9.3.2) are satisfied when $\mathcal{A}^{\#} = \mathcal{A}^{\mathsf{qc}}$, the category of quasi-coherent $\mathcal{O}_X$-modules. Even if Y is not quasi-compact, the conclusion of (3.9.3.2) still holds, because $\mathbf{R}f_*$ and $\varinjlim$ "commute" with open immersions on Y (see (2.4.5.2)), so it suffices to check over affine open subsets of Y.

Proof of (3.9.3.2). By (1.11.2)(ii) we have natural isomorphisms

$$\mathbf{R}^i f_*(D) \xrightarrow{\sim} \mathbf{R}^i f_*(\tau_{\geq i-d} D) \qquad \big(D \in \mathbf{D}_{\#}(X),\ d := \dim^+(\mathbf{R}f_*|_{\mathbf{D}_{\#}(X)})\big).$$

Note that $C \in \mathbf{D}_{\#}(X)$ since homology commutes with $\varinjlim$; and clearly $\tau_{\geq i-d} C = \varinjlim \tau_{\geq i-d} C_\alpha$. Fixing i, we conclude by applying (3.9.3.1) to the inductive system $\tau_{\geq i-d} C_\alpha$. Q.E.D.

Corollary (3.9.3.3). *Let $(C_\beta)_{\beta \in B}$ be a small family of complexes in $\mathbf{D}_{\geq \mathbf{n}}$ (n fixed, see (1.10)) or in $\mathbf{D}_{\#}(X)$ ($\mathcal{A}^{\#}$ as in (3.9.3.2)). Then the natural map $\oplus_\beta \mathbf{R}f_* C_\beta \to \mathbf{R}f_*\big(\oplus_\beta C_\beta\big)$ (see (3.8.1)) is an isomorphism.*

Proof. We need only check that the induced homology maps are isomorphisms, which follows from (3.9.3.1) or (3.9.3.2), a direct sum over B being $\varinjlim$ of the family of direct sums over finite subsets of B. Q.E.D.

Corollary (3.9.3.4). *Under the hypotheses of (3.9.3.1) or (3.9.3.2), if each C_α is f_*-acyclic then so is C.*

Proof. The assertion is that the natural map $f_*C \to \mathbf{R}f_*C$ is an isomorphism in $\mathbf{D}(Y)$, i.e., that the induced maps $H^i(f_*C) \to H^i(\mathbf{R}f_*C)$ are all isomorphisms. By assumption, this holds with C_α in place of C; and since H^i and f_* commute with $\varinjlim$ [**Kf**, p. 641, Prop. 6], it also holds, by (3.9.3.1) or (3.9.3.2), for C. Q.E.D.

Corollary (3.9.3.5). *With $\mathcal{A}^{\#}$ as in (3.9.3.2), any complex C of f_*-acyclic $\mathcal{A}^{\#}$-objects is itself f_*-acyclic.*

Proof. The complexes $\cdots \to 0 \to 0 \to C^{-n} \to C^{-n+1} \to \cdots$ ($n \in \mathbb{Z}$) form an inductive system of f_*-acyclic complexes (see (2.7.2), dualized), whose $\varinjlim$ is C. Conclude by (3.9.3.4). Q.E.D.

Proposition (3.9.4). *Let $f\colon X \to Y$ be a concentrated scheme-map, and let $F \in \mathbf{D}(X)$, $G \in \mathbf{D}_{\mathsf{qc}}(Y)$. If f is finite-dimensional (2.7.6), or if $F \in \mathbf{D}_{\mathsf{qc}}(X)$, then the projection maps*

$$p_1\colon (\mathbf{R}f_*F) \underline{\otimes} G \to \mathbf{R}f_*(F \underline{\otimes} \mathbf{L}f^*G), \qquad p_2\colon G \underline{\otimes} \mathbf{R}f_*F \to \mathbf{R}f_*(\mathbf{L}f^*G \underline{\otimes} F)$$

(see (3.4.6)) are isomorphisms.

Proof. We treat only p_1 (p_2 can be handled similarly; or (3.4.6.1) can be applied). The question is local on Y (check directly, or see (3.7.3.1)), so we may assume Y affine.

Suppose first that both F and G are bounded-above complexes. Then the source and target of p_1 are, for fixed F, bounded-above functors of G: this is clear when f is finite-dimensional, and if $F \in \mathbf{D}_{\mathsf{qc}}(X)$ then it follows from (3.9.2.3) since $F \underset{=}{\otimes} \mathsf{L}f^*G \in \mathbf{D}_{\mathsf{qc}}(X)$, see (3.9.1) and (2.5.8). By (1.11.3.1), with $\mathcal{A}^{\#}$ the category of quasi-coherent $\mathcal{O}_Y$-modules on the affine scheme Y, we reduce the question to where G is a single *free* $\mathcal{O}_Y$-module G^0, whence $\mathsf{L}f^*G$ is isomorphic to the free $\mathcal{O}_X$-module f^*G^0. After verifying via (3.8.2) and (3.9.3.3) that everything in sight commutes with direct sums, we have a further reduction to the case $G = \mathcal{O}_Y$. We check then, via (3.2.5)(a) and commutativity of the upper diagrams in (3.4.2.2), that p_1 is isomorphic to the identity map of $\mathsf{R}f_*F$.

Next, drop the assumption that F is bounded above. For any integer i and any triangle in $\mathbf{D}(X)$ based on the natural map $F \to \tau_{\geq i}F$, the vertex C_i (depending, up to isomorphism, only on F) lies in $\mathbf{D}_{<\mathbf{i}}(X)$, see §§1.4, 1.10. We are still assuming that $G \in \mathbf{D}_{\leq \mathbf{e}}(Y)$ for some e, so that $C_i \underset{=}{\otimes} \mathsf{L}f^*G \in \mathbf{D}_{<\mathbf{i+e}}(X)$ (as one sees upon replacing C_i and G, via $(1.8.1)^-$, by quasi-isomorphic flat complexes vanishing in degrees above $i-1$ and e respectively). As above, $C \in \mathbf{D}_{\mathsf{qc}}(X) \Rightarrow C \underset{=}{\otimes} \mathsf{L}f^*G \in \mathbf{D}_{\mathsf{qc}}(X)$. The finite dimensionality of $\mathsf{R}f_*|_{\mathbf{D}_{\mathsf{qc}}(X)}$ (3.9.2.3), or of $\mathsf{R}f_*$ itself when f is finite-dimensional, then gives $\mathsf{R}f_*(C_i \underset{=}{\otimes} \mathsf{L}f^*G) \in \mathbf{D}_{<\mathbf{i+e+d}}(Y)$ for some integer d depending only on f, and so from the homology sequence $(1.4.5)^{\mathrm{H}}$ of the triangle

$$\mathsf{R}f_*(F \underset{=}{\otimes} \mathsf{L}f^*G) \to \mathsf{R}f_*(\tau_{\geq i}F \underset{=}{\otimes} \mathsf{L}f^*G) \to \mathsf{R}f_*(C_i \underset{=}{\otimes} \mathsf{L}f^*G) \to \mathsf{R}f_*(F \underset{=}{\otimes} \mathsf{L}f^*G)[1]$$

we get isomorphisms

$$H^j\big(\mathsf{R}f_*(F \underset{=}{\otimes} \mathsf{L}f^*G)\big) \xrightarrow{\ \sim\ } H^j\big(\mathsf{R}f_*(\tau_{\geq i}F \underset{=}{\otimes} \mathsf{L}f^*G)\big)$$

for all $j > i+e+d$. Similarly, we have natural isomorphisms

$$H^j\big(\mathsf{R}f_*F \underset{=}{\otimes} G\big) \xrightarrow{\ \sim\ } H^j\big(\mathsf{R}f_*\tau_{\geq i}F \underset{=}{\otimes} G\big).$$

Therefore, to show for any given j that the homology map $H^j(p_1)$ is an isomorphism—which suffices, by (1.2.2)—we can replace F by $\tau_{\geq j-1-e-d}F$. Thus we may assume that F is bounded below. Also, as above, we may assume that G is flat, whence so is $f^*G \cong \mathsf{L}f^*G$.

Let F_m be the Godement resolution of $\tau_{\leq m}F$ $(m \in \mathbb{Z})$, see proof of (3.9.3.1), so that the canonical map

$$F = \varinjlim_m \tau_{\leq m}F \to \varinjlim_m F_m$$

is the Godement resolution of F.

By the first part of this proof, there is a natural isomorphism

$$
\begin{aligned}
H^j\big(f_*F_m \otimes G\big) &\cong H^j\big(\mathbf{R}f_*\tau_{\le m}F \underset{=}{\otimes} G\big) \\
&\xrightarrow{\sim} H^j\big(\mathbf{R}f_*(\tau_{\le m}F \underset{=}{\otimes} \mathbf{L}f^*G)\big) \cong H^j\big(\mathbf{R}f_*(F_m \otimes f^*G)\big)\,.
\end{aligned}
$$

As before, if $F \in \mathbf{D}_{\mathsf{qc}}(X)$ then $(F_m \otimes f^*G) \cong (\tau_{\le m}F \underset{=}{\otimes} \mathbf{L}f^*G) \in \mathbf{D}_{\mathsf{qc}}(X)$. Using (3.9.3.2) and—as in the proof of (3.9.3.1)—commutativity of $\varinjlim$ with f_*, $\otimes$, and H^j, we find then that $H^j(p_1)$ factors as the composition of the natural isomorphisms

$$
\begin{aligned}
H^j\big(\mathbf{R}f_*F \underset{=}{\otimes} G\big) &\xrightarrow{\sim} H^j\big(f_*\varinjlim F_m \otimes G\big) \\
&\xrightarrow{\sim} \varinjlim H^j\big(f_*F_m \otimes G\big) \\
&\xrightarrow{\sim} \varinjlim H^j\big(\mathbf{R}f_*(F_m \otimes f^*G)\big) \\
&\xrightarrow{\sim} H^j\big(\mathbf{R}f_*\varinjlim (F_m \otimes f^*G)\big) \xrightarrow{\sim} H^j\big(\mathbf{R}f_*(F \underset{=}{\otimes} \mathbf{L}f^*G)\big),
\end{aligned}
$$

proving (3.9.3) whenever G is bounded above.

Finally, to extend the assertion to any $G \in \mathbf{D}_{\mathsf{qc}}(Y)$, use a quasi-isomorphism $Q \to G$ where $Q = \varinjlim Q_m$ with $Q_m \in \mathbf{D}^-_{\mathsf{qc}}(Y)$ bounded-above and flat, so that $\mathbf{L}f^*G \cong f^*Q$, see proof of (3.9.1). As in (3.1.2), $\mathbf{R}f_*F = f_*I_F$; and, again, if $F \in \mathbf{D}_{\mathsf{qc}}(X)$ then $I_F \otimes f^*Q_m \in \mathbf{D}_{\mathsf{qc}}(X)$. Applying $\varinjlim_m$ to the system of natural maps

$$
\begin{aligned}
H^j\big(f_*I_F \otimes Q_m\big) &\cong H^j\big(\mathbf{R}f_*F \underset{=}{\otimes} Q_m\big) \\
&\longrightarrow H^j\big(\mathbf{R}f_*(F \underset{=}{\otimes} \mathbf{L}f^*Q_m)\big) \cong H^j\big(\mathbf{R}f_*(I_F \otimes f^*Q_m)\big),
\end{aligned}
$$

maps which we have already seen to be isomorphisms, we find, via (3.9.3.2) and commutativity of $\varinjlim$ with H^j, with $\otimes$, and with f^*, that the maps

$$
H^j(p_1)\colon H^j\big(\mathbf{R}f_*F \underset{=}{\otimes} Q\big) \longrightarrow H^j\big(\mathbf{R}f_*(F \underset{=}{\otimes} \mathbf{L}f^*Q)\big) \qquad (j \in \mathbb{Z})
$$

are all isomorphisms, whence the conclusion. Q.E.D.

Remark **(3.9.4.1).** The projection map p_1 need not be an isomorphism for non-quasi-coherent $\mathcal{O}_Y$-modules G. For example, let R be a two-dimensional noetherian local ring with maximal ideal $\mathfrak{m}$, $Y = \operatorname{Spec}(R)$, $X = \operatorname{Spec}(R) - \{\mathfrak{m}\}$, $f\colon X \to Y$ the inclusion, $F = \mathcal{O}_Y$, and $G = \mathcal{O}_X$ extended by 0 (so that G is a *flat* $\mathcal{O}_Y$-module). Then the stalk of $R^1f_*(F)\otimes G$ at $\mathfrak{m}$ is 0, whereas the stalk of $R^1f_*(F\otimes f^*G) = R^1f_*(\mathcal{O}_X)$ is $H^1(X,\mathcal{O}_X) = H^2_{\mathfrak{m}}(R) \neq 0$ (where $H_{\mathfrak{m}}$ denotes local cohomology supported at $\mathfrak{m}$).

Exercises **(3.9.4.2).** Let X be a ringed space.

(a) Show that an $\mathcal{O}_X$-module F is flat iff $\mathcal{T}or_i(F,G) := H^{-i}(F \underline{\underline{\otimes}} G) = 0$ for all $\mathcal{O}_X$-modules G and all $i \neq 0$. (One need only consider $i = 1$, see proof of (2.7.6.4).)

(b) [**I**, p. 131]. A complex F of $\mathcal{O}_X$-modules has *finite flat amplitude* (or *finite tor-dimension*) if for some integers $d_1 \le d_2$, $\mathcal{T}or_i(F,G) = 0$ for all $\mathcal{O}_X$-modules G and all i outside the interval $[d_1, d_2]$. Show that this condition is equivalent to there being a $\mathbf{D}(X)$-isomorphism $F \xrightarrow{\sim} P$ with P *flat* and $P^i = 0$ for all $i \notin [-d_2, -d_1]$. (See (2.7.6), with f the identity map of X.)

(c) [**I**, p. 249]. Suppose further in (3.9.4) that f has finite tor-dimension (2.7.6) and that F has finite flat amplitude (b). Show that then $\mathbf{R}f_*F$ also has finite flat amplitude.

(d) Show: if X is an affine scheme and if $F \in \mathbf{D}_{\mathsf{qc}}(X)$ has finite flat amplitude, then the complex P in (b) may be assumed to be quasi-coherent. (Use (3.9.6) below.)

(e) *Let $f\colon X \to Y$ be a concentrated scheme-map. Let $F \in \overline{\mathbf{D}}^{+}(X)$ and let $G \in \mathbf{D}_{\mathsf{qc}}(Y)$ have finite flat amplitude. Then the projection map p_1 in* (3.9.4) *is an isomorphism.*

Hint. We may assume Y to be affine. Induction on the number of non-zero terms of a bounded flat quasi-coherent complex $P \cong G$ (see (d)) reduces the question to where G is a single flat quasi-coherent $\mathcal{O}_Y$-module. Then by a theorem of Lazard [**GD**, p. 163, Prop. (6.6.24)], G is a direct limit of finite-rank free $\mathcal{O}_Y$-modules, and so (3.9.3.1) gives a reduction to the trivial case $G = \mathcal{O}_Y$.

(f) Let Y be a ringed space. Show that the following conditions on a complex G of $\mathcal{O}_Y$-modules are equivalent:

(i) For some $d \in \mathbb{Z}$, $\mathcal{T}or_i(F,G) = 0$ for all $\mathcal{O}_Y$-modules F and all $i > d$.

(ii) The functor $E \mapsto E \underline{\underline{\otimes}} G$ ($E \in \mathbf{D}(Y)$ is bounded below (1.11.1).

(iii) In $\mathbf{D}(Y)$, $G \cong P$ with P bounded-below and q-flat.

(iv) In $\mathbf{D}(Y)$, $G \cong P$ with P bounded-below, *flat,* and q-flat.

When these conditions hold we say that G has *bounded-below flat amplitude.*

(g) Do exercise (e) assuming only that G has bounded-below flat amplitude.

Hint. Assuming G to be bounded-below, flat, and q-flat, show that it suffices to apply (e) to each of the complexes $\cdots \to G^{n-1} \to G^n \to 0 \to 0 \to \dots$ $(n \in \mathbb{Z})$.

The following result will be generalized in (3.10.3).

Proposition (3.9.5). *Given a commutative square σ of scheme-maps*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

suppose that f is concentrated, that u is flat, and that σ is a fiber square (i.e., that the associated map $X' \to X \times_Y Y'$ is an isomorphism). Then for any $F \in \mathbf{D}_{\mathsf{qc}}(X)$, the natural composed map (see (3.7.2)(a)*)*

$$\begin{aligned} \theta_\sigma(F)\colon u^*\mathbf{R}f_*F &\xrightarrow{\eta} u^*\mathbf{R}f_*\mathbf{R}v_*v^*F \\ &\xrightarrow{\sim} u^*\mathbf{R}u_*\mathbf{R}g_*v^*F \xrightarrow{\epsilon} \mathbf{R}g_*v^*F \end{aligned}$$

is an isomorphism.

Proof. It should be noted that since u, and hence v, is flat, we have functorial isomorphisms $\mathbf{L}u^* \xrightarrow{\sim} u^*$ and $\mathbf{L}v^* \xrightarrow{\sim} v^*$. (This follows from (2.2.6)(dualized), since the exactness of (e.g.) u^* implies at once that every $\mathcal{O}_X$-complex is u^*-acyclic.)

In view of (3.9.2.2) and (3.9.2.3), (1.11.3)(iv) allows us to assume that F is a single quasi-coherent $\mathcal{O}_X$-module. It will suffice then, by (1.2.2), to show that application of the homology functors H^n to $\theta_\sigma(F)$ produces (what else?) the "base change" isomorphisms $\alpha^n(F)$ of [**AHK**, p. 35, Theorem (6.7)].

For this purpose, we need to express θ_σ in terms of canonical flasque (Godement) resolutions—which we denote by $\mathcal{C}^\bullet$. In [**AHK**, p. 28, §3] there is defined a map

$$\varphi\colon \mathcal{C}^\bullet(F) \to v_*\mathcal{C}^\bullet(v^*F)$$

(denoted there by $\theta_v^\bullet(F)$) which, as easily checked, makes the following natural diagram commute:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & v_*v^*F \arrow[d] \\ \mathcal{C}^\bullet(F) \arrow[r, "\varphi"'] & v_*\mathcal{C}^\bullet(v^*F) \end{tikzcd}$$

With the definitions of ϵ and η in §3.2, and the fact that the direct image of a flasque sheaf is still flasque, it is a straightforward exercise to verify that *the map $\theta_\sigma(F)$ is isomorphic to the derived category map given by the natural composition*

$$u^*f_*\mathcal{C}^\bullet(F) \xrightarrow{\varphi} u^*f_*v_*\mathcal{C}^\bullet(v^*F) \xrightarrow{\sim} u^*u_*g_*\mathcal{C}^\bullet(v^*F) \longrightarrow g_*\mathcal{C}^\bullet(v^*F).$$

Now applying H^n, and recalling that u is flat, we get a composed map

$$\begin{aligned}\alpha'^n\colon u^*H^n\big(f_*\mathcal{C}^\bullet(F)\big) \xrightarrow{\varphi} u^*H^n\big(f_*v_*\mathcal{C}^\bullet(v^*F)\big) &\xrightarrow{\sim} u^*H^n\big(u_*g_*\mathcal{C}^\bullet(v^*F)\big) \\ &\xrightarrow{\gamma} H^n\big(g_*\mathcal{C}^\bullet(v^*F)\big).\end{aligned}$$

Let's look more closely at γ. Setting $g_*\mathcal{C}^\bullet(v^*F) = E^\bullet$, let K^n be the kernel of the differential $E^n \to E^{n+1}$, and let $\delta\colon E^{n-1} \to K^n$ be the obvious map. Then γ can be identified with the map

$$\operatorname{coker}(u^*u_*\delta) = u^*\operatorname{coker}(u_*\delta) \to u^*u_*\operatorname{coker}(\delta) \to \operatorname{coker}(\delta)$$

which is adjoint to the natural map

$$\gamma'\colon H^n(u_*E^\bullet) = \operatorname{coker}(u_*\delta) \to u_*\operatorname{coker}(\delta) = u_*H^n(E^\bullet).$$

Note that $\operatorname{coker}(u_*\delta)$ is the sheaf associated to the presheaf

$$U \mapsto \operatorname{coker}\big(\delta(u^{-1}U)\big) = H^n\big(E^\bullet(u^{-1}U)\big) \qquad (U \text{ open in } Y)$$

and that γ' is the sheafification of the natural presheaf map

$$H^n\big(E^\bullet(u^{-1}U)\big) \to \Gamma\big(u^{-1}U,\, H^n(E^\bullet)\big).$$

It is then readily verified that the adjoint of α'^n, viz. the composed map

$$\begin{aligned}H^n\big(f_*\mathcal{C}^\bullet(F)\big) \xrightarrow{\varphi} H^n\big(f_*v_*\mathcal{C}^\bullet(v^*F)\big) &\xrightarrow{\sim} H^n\big(u_*g_*\mathcal{C}^\bullet(v^*F)\big) \\ &\xrightarrow{\gamma'} u_*H^n\big(g_*\mathcal{C}^\bullet(v^*F)\big),\end{aligned}$$

is the map $\beta^n(f,g,u,v,F)$ near the top of p. 34 of [**AHK**]. But by definition the adjoint of this β^n is $\alpha^n(F)$; thus $\alpha'^n = \alpha^n(F)$, and we are done. Q.E.D.

Here are two important results about quasi-coherence on quasi-compact *separated* schemes. Proofs can be found in the indicated references.

Proposition (3.9.6). *Let X be a quasi-compact separated scheme and $\mathcal{A}_X^{\mathsf{qc}}$ the category of quasi-coherent $\mathcal{O}_X$-modules. Then:*

(a) [**BN**, p. 230, Corollary 5.5.] *The natural functor $\mathbf{D}(\mathcal{A}_X^{\mathsf{qc}}) \to \mathbf{D}_{\mathsf{qc}}(X)$ is an equivalence of categories.*

(b) [**AJL**, p. 10, Proposition 1.1.] *Every complex in $\mathbf{D}_{\mathsf{qc}}(X)$ is $\mathbf{D}(X)$-isomorphic to a* quasi-coherent *q-flat complex.*

3.10. Independent squares; Künneth isomorphism

Throughout this section, $({}^{\boldsymbol{*}}, {}_{\boldsymbol{*}})$ will be the adjoint monoidal pair in (3.6.10), but with $\mathbf{S}$ restricted to be the category of quasi-separated schemes and concentrated (= quasi-compact and quasi-separated) maps between them [**GD**, p. 291, (6.1.5) and p. 294, (6.1.9)], and with the further restriction $\mathbf{X}_{\boldsymbol{*}} = \mathbf{X}^{\boldsymbol{*}} = \mathbf{D}_{\mathsf{qc}}(X)$ for all $X \in \mathbf{S}$ (see (3.9.1), (3.9.2)). Note that any subscheme of a quasi-separated scheme is quasi-separated; and that the category $\mathbf{S}$ is closed under fiber product. Note also that if X and Y are quasi-separated then *any* scheme-map $f : X \to Y$ is quasi-separated, and further, quasi-compact if X is [**GD**, p. 295, (6.1.10)].

Accordingly (*except* in (3.10.1) and the proof of (3.10.2.2), where we need to distinguish between ordinary and derived functors), for any scheme-map α we write α_* for $\mathsf{R}\alpha_*$, and α^* for $\mathsf{L}\alpha^*$. We also write $\otimes$ for $\underline{\underline{\otimes}}$. These abbreviations should not be allowed to obscure the fact that we are working throughout with derived categories and derived functors.

After discussing some basic maps we define, in (3.10.2), various notions of *independence* of commutative $\mathbf{S}$-squares. The main result, (3.10.3), is that *all these independence conditions are equivalent.*[41] This implies, e.g., that the isomorphism in (3.9.5) holds for any tor-independent $\mathbf{S}$-square, as does a certain Künneth isomorphism, which subsumes the projection isomorphisms of (3.9.4).

Independent squares are important in Grothendieck duality theory, where they support base-change maps (Remark (3.10.2.1)(c)).

An *orientation* of a commutative $\mathbf{S}$-square σ

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

is an ordering of the pair (u, f).

[41] The knowledgeable reader might wish to place this result in the context of the *Künneth spectral sequences* of [**EGA**, III, (6.7.5)].

In this section, unless otherwise indicated, all commutative **S**-squares will be understood to be equipped with the orientation for which the bottom arrow precedes the right vertical one.

To such an oriented σ associate the functorial maps

$$\theta = \theta_\sigma \colon u^*f_* \to g_*v^* \qquad \text{(see Proposition (3.7.2))}$$

and

$$\theta' = \theta'_\sigma := \theta_{\sigma'} \colon f^*u_* \to v_*g^*$$

where σ' is σ with its orientation reversed.

Setting $h := fv = ug$, define the functorial *Künneth map*

$$\eta = \eta_\sigma \colon u_*E \otimes f_*F \to h_*(g^*E \otimes v^*F) \qquad \big(E \in \mathbf{Y}'^{\boldsymbol{*}},\ F \in \mathbf{X}^{\boldsymbol{*}}\big)$$

to be the natural composition

$$\begin{aligned} u_*E \otimes f_*F &\to h_*h^*(u_*E \otimes f_*F) \\ &\xrightarrow[(3.6.1)^*]{(3.4.5.1)} h_*(g^*u^*u_*E \otimes v^*f^*f_*F) \to h_*(g^*E \otimes v^*F). \end{aligned}$$

The map η generalizes (3.4.2.1): let $X' = Y' = X$, let $v = g$ be the identity map, let $u = f$, so that $h = f$, and see (3.4.5.2) and 1) in (3.6.5).

The map η also generalizes the projection maps p_1 and p_2 in (3.4.6): for p_1, let f be the identity map of $X = Y$, let g be the identity map of $X' = Y'$, so that $h = v = u$, and see (3.4.6.2); and similarly for p_2 let u and v be identity maps, ...

Examples **(3.10.1).** Let us see what the above θ_σ and η_σ look like in a concrete situation, when σ is a diagram of affine schemes. The results are hardly surprising, but do need proof.

(a) We deal first with θ. On **S** there is a second adjoint pair $(^\star, {}_\star)$ such that for each ringed space X, $\mathbf{X}^\star = \mathbf{X}_\star := \mathbf{K}(X)$, the homotopy category of $\mathcal{O}_X$-complexes, with monoidal structure given by the ordinary tensor product, and such that for each **S**-map $f\colon X \to Y$ the associated adjoint functors are the standard (sheaf-theoretic) inverse- and direct-image functors, $f^\star := f^*$ and $f_\star := f_*$. So, as above, for each commutative **S**-square σ one gets functorial maps

$$\begin{aligned} &\boldsymbol{\theta} = \boldsymbol{\theta}_\sigma \colon \mathsf{L}u^*\mathsf{R}f_* \to \mathsf{R}g_*\mathsf{L}v^*, \\ &\theta = \theta_\sigma \colon u^*f_* \to g_*v^*, \end{aligned} \tag{3.10.1.0}$$

related as follows.

Lemma (3.10.1.1). *With $Q\colon \mathbf{K}\to\mathbf{D}$ as usual, the following natural diagram of functors from $\mathbf{K}(X)$ to $\mathbf{D}(Y')$ commutes.*

$$\begin{tikzcd}
\mathbf{L}u^*\mathbf{R}f_*Q \arrow[d, "\boldsymbol{\theta}"'] & \mathbf{L}u^*Qf_* \arrow[l, "\alpha"'] \arrow[r] & Qu^*f_* \arrow[d, "Q\theta"] \\
\mathbf{R}g_*\mathbf{L}v^*Q \arrow[r, "\beta"'] & \mathbf{R}g_*Qv^* & Qg_*v^* \arrow[l, "\gamma"]
\end{tikzcd}$$

PROOF. Expand the diagram (all maps being the obvious ones):

$$\begin{tikzcd}
\mathbf{L}u^*\mathbf{R}f_*Q \arrow[ddd] & \mathbf{L}u^*Qf_* \arrow[l] \arrow[r] \arrow[d] & Qu^*f_* \arrow[r, equal] \arrow[d] & Qu^*f_* \arrow[ddd] \\
& \mathbf{R}g_*\mathbf{L}g^*\mathbf{L}u^*Qf_* \arrow[r] \arrow[d, equal] & \mathbf{R}g_*\mathbf{L}g^*Qu^*f_* \arrow[d] & \\
\mathbf{R}g_*\mathbf{L}g^*\mathbf{L}u^*\mathbf{R}f_*Q \arrow[d, "\simeq"'] & \mathbf{R}g_*\mathbf{L}g^*\mathbf{L}u^*Qf_* \arrow[l] \arrow[r] \arrow[d, "\simeq"'] & \mathbf{R}g_*Qg^*u^*f_* \arrow[d, "\simeq"] & Qg_*g^*u^*f_* \arrow[l] \arrow[d, "\simeq"] \\
\mathbf{R}g_*\mathbf{L}v^*\mathbf{L}f^*\mathbf{R}f_*Q \arrow[ddd] & \mathbf{R}g_*\mathbf{L}v^*\mathbf{L}f^*Qf_* \arrow[l] \arrow[r] \arrow[d] & \mathbf{R}g_*Qv^*f^*f_* \arrow[d, equal] & Qg_*v^*f^*f_* \arrow[l] \arrow[ddd] \\
& \mathbf{R}g_*\mathbf{L}v^*Qf^*f_* \arrow[r] \arrow[d] & \mathbf{R}g_*Qv^*f^*f_* \arrow[d] & \\
\mathbf{R}g_*\mathbf{L}v^*Q \arrow[r, equal] & \mathbf{R}g_*\mathbf{L}v^*Q \arrow[r] & \mathbf{R}g_*Qv^* & Qg_*v^* \arrow[l]
\end{tikzcd}$$

The upper right (resp. lower left) subdiagram commutes by (3.2.1.3) (resp. (3.2.1.2)). Commutativity of the rest is easy to verify. Q.E.D.

Next, we make the map θ in (3.10.1.0) more explicit, at least locally.

Lemma (3.10.1.2). *Let*

$$\begin{tikzcd}
V & S \arrow[l] \\
U \arrow[u] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

be a commutative diagram of commutative-ring homomorphisms, let σ as above be the corresponding diagram of affine schemes ($Y := \operatorname{Spec}(R)$, etc.), and let $\theta = \theta_\sigma\colon u^*f_* \to g_*v^*$ be as in (3.10.1.0). For any S-complex E, let $\theta_0(E)$ be the natural composition $U\otimes_R E \to V\otimes_R E \to V\otimes_S E$, i.e., the U-homomorphism taking $1\otimes_R e$ to $1\otimes_S e$ for all $e\in E^n$ $(n\in\mathbb{Z})$.

Then there is a natural commutative diagram of $\mathcal{O}_{Y'}$-modules

$$\begin{tikzcd}
u^*f_*\widetilde{E} \arrow[r, "\sim"] \arrow[d, "\theta(\widetilde{E})"'] & (U\otimes_R E)^\sim \arrow[d, "\widetilde{\theta_0(E)}"] \\
g_*v^*\widetilde{E} \arrow[r, "\sim"] & (V\otimes_S E)^\sim
\end{tikzcd}$$

where $^\sim$ *denotes the usual functor from modules to quasi-coherent sheaves* [**GD**, p. 197ff, §1.3], *and where the horizontal arrows are isomorphisms.*

PROOF. The horizontal isomorphisms come from [**GD**, p. 213, (1.7.7)]. To check commutativity, expand the diagram as follows, where in the right hand column, the complexes to which $^\sim$ is applied are all regarded as U-complexes, and the maps are sheafifications of natural U-complex homomorphisms:

$$
\begin{tikzcd}
u^*f_*\widetilde{E} \arrow[rr, "\sim"] \arrow[d] \arrow[drr, phantom, "①"] & & (U\otimes_R E)^\sim \arrow[d] \\
g_*g^*u^*f_*\widetilde{E} \arrow[rr, "\sim"] \arrow[d, "\simeq"'] & & \big(V\otimes_U (U\otimes_R E)\big)^\sim \arrow[d, "\simeq"] \\
g_*v^*f^*f_*\widetilde{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & g_*v^*(S\otimes_R E)^\sim \arrow[r] \arrow[d] & \big(V\otimes_S (S\otimes_R E)\big)^\sim \arrow[d] \\
g_*v^*\widetilde{E} \arrow[r, equal] & g_*v^*\widetilde{E} \arrow[r] & (V\otimes_S E)^\sim
\end{tikzcd}
$$

Commutativity of subdiagrams ① and ② is given by [**GD**, p. 214, (1.7.9)]. The rest is straightforward. Q.E.D.

Under the hypotheses of (3.10.1.2), for any $G\in\mathbf{D}_{\mathsf{qc}}(X)$ the map $\boldsymbol{\theta}(G)\colon \mathsf{L}u^*\mathsf{R}f_*G\to\mathsf{R}g_*\mathsf{L}v^*G$ can now be described as follows.

By (3.9.6)(a), G is $\mathbf{D}$-isomorphic to a quasi-coherent complex, which is $\widetilde{E}$ for some S-complex E. Arguing as in (2.5.5)—using that any S-module F is naturally a homomorphic image of the free S-module $P_0(F)$ with basis F—one sees that there exists a quasi-isomorphism $P\to E$ with P a $\varinjlim$ of bounded-above complexes of free S-modules. There results a quasi-isomorphism $\widetilde{P}\to\widetilde{E}$; and $\widetilde{P}$, being a $\varinjlim$ of bounded-above complexes of free $\mathcal{O}_X$-modules, is q-flat, as is $v^*\widetilde{P}$. One can replace $\widetilde{E}$ by $\widetilde{P}$, i.e., one may assume that there exists a $\mathbf{D}$-isomorphism $\lambda\colon G\xrightarrow{\sim}\widetilde{E}$ such that both $\widetilde{E}$ and $v^*\widetilde{E}$ are q-flat as well as quasi-coherent.

Since f_* is an exact functor on the category of quasi-coherent $\mathcal{O}_X$-modules [**GD**, p. 214, (1.7.8)], therefore the natural map $f_*\widetilde{E}\to\mathsf{R}f_*\widetilde{E}$ is a $\mathbf{D}(Y)$-isomorphism. Also, the natural map $\mathsf{L}v^*\widetilde{E}\to v^*\widetilde{E}$ is a $\mathbf{D}(Y')$-isomorphism. So the maps $\alpha(\widetilde{E})$ and $\beta(\widetilde{E})$ in (3.10.1.1) are isomorphisms. Moreover, the map $\theta(\widetilde{E})$ can be identified as in (3.10.1.2) with $\widetilde{\theta_0(E)}$. The map $\boldsymbol{\theta}(\widetilde{E})$ is thereby determined by (3.10.1.1) and (3.10.1.2); and via λ (a "quasi-coherent q-flat resolution"), so is the map $\boldsymbol{\theta}(G)$.

(b) We turn now to η. With σ, $({}^{\boldsymbol *}, {}_{\boldsymbol *})$ and $({}^{\star}, {}_{\star})$ as in (a), and $h = fv = gu$, one has for $\mathcal{O}_{Y'}$-complexes E and $\mathcal{O}_X$-complexes F the functorial maps

$$
\begin{aligned}
\boldsymbol{\eta} = \boldsymbol{\eta}_\sigma(E,F)\colon \mathsf{R}u_*E \underset{=}{\otimes} \mathsf{R}f_*F &\to \mathsf{R}h_*(\mathsf{L}g^*E \underset{=}{\otimes} \mathsf{L}v^*F),\\
\eta = \eta_\sigma(E,F)\colon u_*E \otimes f_*F &\to h_*(g^*E \otimes v^*F),
\end{aligned}
$$

related as follows.

Lemma (3.10.1.3). *For all E and F as above, the following natural bifunctorial diagram—where appropriate insertions of "Q" are left to the reader—commutes.*

$$
\begin{tikzcd}
\mathsf{R}u_*E \underset{=}{\otimes} \mathsf{R}f_*F \arrow[d, "\boldsymbol{\eta}"'] & u_*E \underset{=}{\otimes} f_*F \arrow[l, "\alpha'"'] \arrow[r] & u_*E \otimes f_*F \arrow[d, "\eta"] \\
\mathsf{R}h_*(\mathsf{L}g^*E \underset{=}{\otimes} \mathsf{L}v^*F) \arrow[r, "\beta'"] & \mathsf{R}h_*(g^*E \otimes v^*F) & h_*(g^*E \otimes v^*F) \arrow[l]
\end{tikzcd}
$$

Proof. Paste the following two diagrams along their common edge:

$$
\begin{tikzcd}
\mathsf{R}u_*E \underset{=}{\otimes} \mathsf{R}f_*F \arrow[d] & & u_*E \underset{=}{\otimes} f_*F \arrow[ll, "\alpha'"'] \arrow[d] \\
\mathsf{R}h_*\mathsf{L}h^*(\mathsf{R}u_*E \underset{=}{\otimes} \mathsf{R}f_*F) \arrow[d, "\simeq"'] & & \mathsf{R}h_*\mathsf{L}h^*(u_*E \underset{=}{\otimes} f_*F) \arrow[ll] \arrow[d, "\simeq"] \\
\mathsf{R}h_*(\mathsf{L}h^*\mathsf{R}u_*E \underset{=}{\otimes} \mathsf{L}h^*\mathsf{R}f_*F) \arrow[d, "\simeq"'] & & \mathsf{R}h_*(\mathsf{L}h^*u_*E \underset{=}{\otimes} \mathsf{L}h^*f_*F) \arrow[ll] \arrow[d, "\simeq"] \\
\mathsf{R}h_*(\mathsf{L}(g^*u^*)\mathsf{R}u_*E \underset{=}{\otimes} \mathsf{L}(v^*f^*)\mathsf{R}f_*F) \arrow[d, "\simeq"'] & & \mathsf{R}h_*(\mathsf{L}(g^*u^*)u_*E \underset{=}{\otimes} \mathsf{L}(v^*f^*)f_*F) \arrow[ll] \arrow[d, "\simeq"] \\
\mathsf{R}h_*(\mathsf{L}g^*\mathsf{L}u^*\mathsf{R}u_*E \underset{=}{\otimes} \mathsf{L}v^*\mathsf{L}f^*\mathsf{R}f_*F) \arrow[dd] & & \mathsf{R}h_*(\mathsf{L}g^*\mathsf{L}u^*u_*E \underset{=}{\otimes} \mathsf{L}v^*\mathsf{L}f^*f_*F) \arrow[ll] \arrow[d] \\
 & ③ & \mathsf{R}h_*(\mathsf{L}g^*u^*u_*E \underset{=}{\otimes} \mathsf{L}v^*f^*f_*F) \arrow[d] \\
\mathsf{R}h_*(\mathsf{L}g^*E \underset{=}{\otimes} \mathsf{L}v^*F) \arrow[rr, equal] & & \mathsf{R}h_*(\mathsf{L}g^*E \underset{=}{\otimes} \mathsf{L}v^*F)
\end{tikzcd}
$$

$$
\begin{tikzcd}
u_*E \underline{\otimes} f_*F \arrow[r] \arrow[d] & u_*E \otimes f_*F \arrow[d] \arrow[r, equal] & u_*E \otimes f_*F \arrow[dd] \\
\mathsf{R}h_*\mathsf{L}h^*(u_*E \underline{\otimes} f_*F) \arrow[r] \arrow[dd, "\simeq"'] & \mathsf{R}h_*\mathsf{L}h^*(u_*E \otimes f_*F) \arrow[d] \arrow[r, phantom, "4"] & \\
\arrow[r, phantom, "5"] & \mathsf{R}h_*h^*(u_*E \otimes f_*F) \arrow[d] & h_*h^*(u_*E \otimes f_*F) \arrow[l] \arrow[d, "\simeq"] \\
\mathsf{R}h_*(\mathsf{L}h^*u_*E \underline{\otimes} \mathsf{L}h^*f_*F) \arrow[r] \arrow[d, "\simeq"'] & \mathsf{R}h_*(h^*u_*E \otimes h^*f_*F) \arrow[dd, "\simeq"] & h_*(h^*u_*E \otimes h^*f_*F) \arrow[l] \arrow[dd, "\simeq"] \\
\mathsf{R}h_*(\mathsf{L}g^*\mathsf{L}u^*u_*E \underline{\otimes} \mathsf{L}v^*\mathsf{L}f^*f_*F) \arrow[d] \arrow[r, phantom, "6"] & & \\
\mathsf{R}h_*(\mathsf{L}g^*u^*u_*E \underline{\otimes} \mathsf{L}v^*f^*f_*F) \arrow[r] \arrow[d] & \mathsf{R}h_*(g^*u^*u_*E \underline{\otimes} v^*f^*f_*F) \arrow[d] & h_*(g^*u^*u_*E \underline{\otimes} v^*f^*f_*F) \arrow[l] \arrow[d] \\
\mathsf{R}h_*(\mathsf{L}g^*E \underline{\otimes} \mathsf{L}v^*F) \arrow[r, "\beta'"'] & \mathsf{R}h_*(g^*E \otimes v^*F) & h_*(g^*E \otimes v^*F) \arrow[l]
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams of the preceding diagrams is pretty clear.

Commutativity of subdiagram 3 follows from that of (3.2.1.2), of 4 from (3.2.1.3), of 5 from (3.2.4.1), and of 6 from the dual of the commutative diagram (3.6.4.1) (see the remarks surrounding (3.6.4)*).

Lemma (3.10.1.3) results. Q.E.D.

Lemma (3.10.1.4). *With notation as in* (3.10.1.2), *for any U-complex E and any S-complex F let $\eta = \eta_\sigma(\widetilde{E}, \widetilde{F})$ be as above, and let $\eta_0 = \eta_0(E, F)$ be the natural composition*

$$E \otimes_R F \to V \otimes_R (E \otimes_R F) \xrightarrow{\sim} (V \otimes_R E) \otimes_V (V \otimes_R F) \to (V \otimes_U E) \otimes_V (V \otimes_S F).$$

Then there is a natural commutative diagram of $\mathcal{O}_Y$-modules

$$
\begin{tikzcd}
u_*\widetilde{E} \otimes f_*\widetilde{F} \arrow[r, "\sim"] \arrow[d, "\eta"'] & (E \otimes_R F)^{\sim} \arrow[d, "\widetilde{\eta}_0"] \\
h_*(g^*\widetilde{E} \otimes v^*\widetilde{F}) \arrow[r, "\sim"] & \big((V \otimes_U E) \otimes_V (V \otimes_S F)\big)^{\sim}
\end{tikzcd}
$$

in which the horizontal arrows are isomorphisms.

Proof. The horizontal isomorphisms in the diagram are given by [**GD**, p. 213, (1.7.7) and p. 202, (1.3.12)(i)].

For commutativity, expand the diagram naturally as follows:

$$\begin{array}{ccc}
u_*\widetilde{E}\otimes f_*\widetilde{F} & \longrightarrow & \big(E\otimes_R F\big)^\sim \\
\big\downarrow & & \big\downarrow \\
h_*h^*(u_*\widetilde{E}\otimes f_*\widetilde{F}) & \longrightarrow & \big(V\otimes_R (E\otimes_R F)\big)^\sim \\
\big\downarrow & & \big\downarrow \\
h_*(h^*u_*\widetilde{E}\otimes h^*f_*\widetilde{F}) & \longrightarrow & \big((V\otimes_R E)\otimes_V (V\otimes_R F)\big)^\sim \\
\big\downarrow & & \big\downarrow \\
h_*(g^*u^*u_*\widetilde{E}\otimes v^*f^*f_*\widetilde{F}) & \longrightarrow & \big((V\otimes_U U\otimes_R E)\otimes_V (V\otimes_S S\otimes_R F)\big)^\sim \\
\big\downarrow & & \big\downarrow \\
h_*(g^*\widetilde{E}\otimes v^*\widetilde{F}) & \longrightarrow & (V\otimes_U E)\otimes_V (V\otimes_S F)\big)^\sim
\end{array}$$

Verification of commutativity of the subdiagrams is left as an exercise. (Suggestion: recall (3.1.9), and use [**GD**, p. 214, (1.7.9)(ii)].) Q.E.D.

As in (a), Lemmas (3.10.1.3) and (3.10.1.4) determine (via quasi-coherent q-flat resolutions) the map $\boldsymbol{\eta}(G_1, G_2)$ for any $G_1\in\mathbf{D}_{\mathsf{qc}}(Y')$ and $G_2\in\mathbf{D}_{\mathsf{qc}}(X)$, in terms of the concrete functorial map η_0.

Definition (3.10.2). A commutative oriented $\mathbf{S}$-square

$$\begin{array}{ccc}
X' & \xrightarrow{\;v\;} & X \\
{\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\;u\;]{} & Y
\end{array}$$

is said to be

- *independent* if θ_σ is a functorial isomorphism;
- $'$*-independent* if θ'_σ is a functorial isomorphism;
- *Künneth-independent* if η_σ is a bifunctorial isomorphism;
- *tor-independent* if σ is a fiber square (i.e., the map $X'\to X\times_Y Y'$ associated to σ is an isomorphism) and if the following equivalent conditions hold for all pairs of points $y'\in Y'$, $x\in X$ such that $y := u(y') = f(x)$:

(i) $$\operatorname{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y',y'},\mathcal{O}_{X,x}) = 0 \quad \text{for all } i>0.$$

(ii) There exist an affine open neighborhood $\operatorname{Spec}(A)$ of y and affine open sets $\operatorname{Spec}(A')\subset u^{-1}\operatorname{Spec}(A)$, $\operatorname{Spec}(B)\subset f^{-1}\operatorname{Spec}(A)$ such that

$$\operatorname{Tor}_i^{A}(A', B) = 0 \quad \text{for all } i>0.$$

(ii)′ For any affine open neighborhood $\mathrm{Spec}(A)$ of y and affine open sets $\mathrm{Spec}(A') \subset u^{-1}\mathrm{Spec}(A)$, $\mathrm{Spec}(B) \subset f^{-1}\mathrm{Spec}(A)$,

$$\mathrm{Tor}_i^A(A', B) = 0 \quad \text{for all } i > 0.$$

Remarks **(3.10.2.1).** (a) The conditions of Künneth-independence and tor-independence do not depend on an orientation of σ.

(b) Condition (ii)′ in (3.10.2) implies condition (ii); and (ii) implies (i) because if $p \subset A$, $q \subset A'$, and $r \subset B$ are the prime ideals corresponding to y, y' and x respectively, then there are natural isomorphisms

$$\begin{aligned}\mathrm{Tor}_i^{A_p}(A'_q, B_r) \cong \mathrm{Tor}_i^{A}(A'_q, B_r) &\cong A'_q \otimes_{A'} \mathrm{Tor}_i^{A}(A', B_r)\\ &\cong A'_q \otimes_{A'} \mathrm{Tor}_i^{A}(A', B) \otimes_B B_r\,.\end{aligned}$$

These isomorphisms also show that, conversely, (i) implies (ii′): for if $m \subset A' \otimes_A B$ were a prime ideal in the support of $\mathrm{Tor}_i^A(A', B)$ and p, q and r were its inverse images in A, A' and B respectively, then $0 \neq \mathrm{Tor}_i^A(A', B)_m$ would be a localization of $\mathrm{Tor}_i^{A_p}(A'_q, B_r) = 0$.

(c) Let σ, as above, be an independent square; and suppose that the functors f_* and g_* have right adjoints $f^\times$ and $g^\times$ respectively. Then one can associate to σ a functorial *base-change map* (for $f^\times$ rather than f_*):

$$\beta_\sigma \colon v^*f^\times \to g^\times u^*,$$

adjoint to the natural composition $g_*v^*f^\times \xrightarrow{\theta^{-1}} u^*f_*f^\times \to u^*$.

This map plays a crucial role in Grothendieck duality theory on, say, the full subcategory of $\mathbf{S}$ whose objects are all the concentrated schemes, in which situation the right adjoints $f^\times$ and $g^\times$ exist, see (4.1.1) below.

(d) We call an $\mathbf{S}$-map $f\colon X \to Y$ *isofaithful* if any $\mathbf{X}^*$-map α such that $f_*\alpha$ is a $\mathbf{Y}^*$-isomorphism is itself an isomorphism.

For example, if f is an open immersion then f is isofaithful because of the natural functorial isomorphism $G \xrightarrow{\sim} \mathbf{L}f^*\mathbf{R}f_*G \ \ (G \in \mathbf{D}(Y))$.

Lemma (3.10.2.2). *If the* $\mathbf{S}$*-map* $f\colon X \to Y$ *is affine* $\big($[**GD**, p. 357, (9.1.10)]: *for each affine open* $U \subset Y$, $f^{-1}U$ *is affine*$\big)$ *then* f *is isofaithful.*

Proof. In this proof only, $f_*\colon \mathbf{K}(X) \to \mathbf{K}(Y)$ will be the ordinary direct-image functor, and $\mathbf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ its derived functor.

From (2.4.5.2) it follows that $\mathbf{R}f_*$ "commutes" with open immersions, so the question is local, and we may assume that X and Y are affine, say $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$.

By (3.9.6)(a), every complex in $\mathbf{D}_{\mathsf{qc}}(X)$ is $\mathbf{D}$-isomorphic to a quasi-coherent complex. Therefore—and since a $\mathbf{D}$-map α is an isomorphism iff the vertex of a triangle based on α is exact—we need only show: *if* C *is a quasi-coherent* $\mathcal{O}_X$*-complex such that* $\mathbf{R}f_*(C)$ *is exact then* C *is exact.*

Since the functor f_* of quasi-coherent $\mathcal{O}_X$-modules is exact, therefore, by (3.9.2.3) and the dual of (2.7.4), C is f_*-acyclic, so that $f_*C \cong \mathbf{R}f_*C$ is exact, and for all i, $f_*H^iC \cong H^if_*C = 0$.

Finally, $C = \widetilde{E}$ for some B-complex E, so $H^iC = (H^iE)^\sim$, and when H^iE is regarded as an A-module, $f_*H^iC = (H^iE)^\sim$ (see [**GD**, p. 214, (1.7.7.2)]), whence $H^iE = 0$. The desired conclusion results. Q.E.D.

The following assertions result at once from commutativity (to be shown) of diagram (3.10.2.3) below, for any $E \in \mathbf{Y'^*}$ and $F \in \mathbf{X^*}$.

- Independence or $'$-independence of σ implies Künneth independence.
- If u (resp. f) is isofaithful then Künneth independence of σ implies independence (resp. $'$-independence). (Take E (resp. F) to be $\mathcal{O}_{Y'}$ (resp. $\mathcal{O}_X$).) Thus:
- *If u and f are isofaithful then independence, $'$-independence and Künneth independence are equivalent conditions on σ.*

This applies, for instance, if the schemes Y', Y and X are affine.

(3.10.2.3)
$$
\begin{tikzcd}
u_*(E\otimes u^*f_*F) \arrow[d, "\text{via }\theta"'] & u_*E\otimes f_*F \arrow[l, "\sim"', "(3.9.4)"] \arrow[r, "\sim", "(3.9.4)"'] \arrow[dd, "\eta"] & f_*(f^*u_*E\otimes F) \arrow[d, "\text{via }\theta'"] \\
u_*(E\otimes g_*v^*F) \arrow[d, "\simeq"', "(3.9.4)"] & & f_*(v_*g^*E\otimes F) \arrow[d, "(3.9.4)"', "\simeq"] \\
u_*g_*(g^*E\otimes v^*F) \arrow[r, "\sim", "(3.6.4)_*"'] & h_*(g^*E\otimes v^*F) & f_*v_*(g^*E\otimes v^*F) \arrow[l, "\sim"', "(3.6.4)_*"]
\end{tikzcd}
$$

Proving commutativity of (3.10.2.3) is a formal exercise on adjoint monoidal pseudofunctors. For example, in view of the definition of $\theta_\sigma(F)$ in (3.7.2)(c), commutativity of the left half follows from commutativity of the natural diagram
$$
\begin{tikzcd}
u_*E\otimes f_*F \arrow[r] \arrow[d, "(3.9.4)"', "\simeq"] \arrow[dr, phantom, "①"] & u_*u^*(u_*E\otimes f_*F) \arrow[r] \arrow[d] & u_*g_*g^*u^*(u_*E\otimes f_*F) \arrow[d] \\
u_*(E\otimes u^*f_*F) \arrow[d] & u_*(u^*u_*E\otimes u^*f_*F) \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & u_*g_*g^*(u^*u_*E\otimes u^*f_*F) \arrow[d] \\
u_*(E\otimes g_*g^*u^*f_*U) \arrow[d, "\simeq"] & u_*(u^*u_*E\otimes g_*g^*u^*f_*F) \arrow[l] \arrow[r, "\sim", "(3.9.4)"'] \arrow[d, "\simeq"] & u_*g_*(g^*u^*u_*E\otimes g^*u^*f_*F) \arrow[d, "\simeq"'] \\
u_*(E\otimes g_*v^*f^*f_*U) \arrow[d] & u_*(u^*u_*E\otimes g_*v^*f^*f_*F) \arrow[l] \arrow[r, "\sim", "(3.9.4)"'] \arrow[d] & u_*g_*(g^*u^*u_*E\otimes v^*f^*f_*F) \arrow[d] \\
u_*(E\otimes g_*v^*F) \arrow[r, equal] & u_*(E\otimes g_*v^*F) \arrow[r, "\sim", "(3.9.4)"'] & u_*g_*(g^*E\otimes v^*F)
\end{tikzcd}
$$
Commutativity of subsquare ① is given by 3.4.6.2, and of ② by (3.4.7)(i). Commutativity of the other subsquares is straightforward to check.

Commutativity of the right half of (3.10.3.2) is shown similarly.

Theorem (3.10.3). *For any fiber square of concentrated maps of quasi-separated schemes*

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

(σ *commutes and the associated map* $X' \to Y' \times_Y X$ *is an isomorphism*), *the four independence conditions in Definition* (3.10.2) *are equivalent.*

Proof. We first prove a special case.

Lemma (3.10.3.1). *Theorem* (3.10.3) *holds when all the schemes appearing in* σ *are affine.*

Proof. We saw above (just before (3.10.2.3)) that the first three independence conditions are equivalent. From (3.10.2.2) and (3.10.2.3) with $F = \mathcal{O}_X$, it follows that if $\theta(\mathcal{O}_X)$ is an isomorphism then $\theta'(E)$ is an isomorphism for all E, i.e., σ is $'$-independent. Thus it will suffice to show that $\theta(\mathcal{O}_X)$ *is an isomorphism iff* σ *is tor-independent.*

From (3.10.1.2) with $E = S$, and the assumption that σ is a fiber square, one sees that when applied to $\mathcal{O}_X$ the right column in (3.10.1.1) becomes an isomorphism. As $\mathcal{O}_X$ is flat and quasi-coherent, the maps $\alpha(\mathcal{O}_X)$, $\beta(\mathcal{O}_X)$ and $\gamma(\mathcal{O}_X)$ in (3.10.1.1) are isomorphisms, and hence the left column—which is what we are now denoting by $\theta(\mathcal{O}_X)$—is an isomorphism iff so is the canonical map $\psi\colon \mathbf{L}u^*f_*\mathcal{O}_X \to u^*f_*\mathcal{O}_X$. Since sheafification is exact and preserves flatness (flatness of a sheaf being guaranteed by flatness of its stalks), using [**GD**, p. 214, (1.7.7.2)] one finds that ψ is $\mathbf{D}(Y')$-isomorphic to the sheafification $\widetilde{\phi}$ of the natural U-homomorphism $\phi\colon U \otimes_R P^\bullet \to U \otimes_R S$, where U, R and S are as in (3.10.1.2) and $P^\bullet \to S$ is an R-flat resolution of S. Since ϕ is a quasi-isomorphism precisely when $\mathrm{Tor}_i^R(U,S) = 0$ for all $i > 0$, that is, when σ is tor-independent, the desired conclusion results. Q.E.D.

The *strategy* now is to show that:

(A) *Independence is a local condition,* i.e., it holds for σ iff it holds for every induced fiber square

$$\begin{array}{ccc} X'_0 & \xrightarrow{\ v\ } & X_0 \\ {\scriptstyle g}\big\downarrow & \sigma_0 & \big\downarrow{\scriptstyle f} \\ Y'_0 & \xrightarrow[\ u\ ]{} & Y_0 \end{array}$$

such that Y_0 is an affine open subscheme of Y, and Y'_0, X_0 are affine open subschemes of $u^{-1}Y_0$, $f^{-1}Y_0$ respectively. (See first paragraph of §3.10.)

It follows then from (3.10.3.1) that tor-independence for σ in (3.10.3) implies independence and, by symmetry, $'$-independence.

It has already been noted (before (3.10.2.3)) that independence or $'$-independence implies Künneth independence. To finish proving (3.10.3) it will therefore suffice to show that:

(B) *Künneth independence for σ implies the same for any σ_0 as above.*

For then it will follow from (3.10.3.1) that Künneth-independence implies tor-independence.

Finally, (A) and (B) result at once from the first assertion in (3.10.3.3) and the last assertion in (3.10.3.4) below.

Lemma (3.10.3.2) (Independence and concatenation). *For each one of the following* $\mathbf{S}$*-diagrams, assumed commutative,*

$$\begin{tikzcd} X'' \arrow[r, "v_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma_1"] & X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "u_1"'] & Y' \arrow[r, "u"'] & Y \end{tikzcd} \qquad \begin{tikzcd} Z' \arrow[r, "w"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma_1"] & Z \arrow[d, "f_1"] \\ X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

if σ and σ_1 are independent (resp. $'$-independent, Künneth-independent) then so is the rectangle $\sigma_0 := \sigma\sigma_1$ enclosed by the outer border.

Proof. As in (3.7.2)(iii), the following natural diagram commutes for any $G \in \mathbf{X}^*$:

(3.10.3.2.1)
$$\begin{tikzcd} (uu_1)^*f_*G \arrow[rr, "\theta_{\sigma_0}(G)"] \arrow[d, "\simeq"'] & & h_*(vv_1)^*G \arrow[d, "\simeq"] \\ u_1^*u^*f_*G \arrow[r, "u_1^*\theta_\sigma(G)"'] & u_1^*g_*v^*G \arrow[r, "\theta_{\sigma_1}(v^*G)"'] & h_*v_1^*v^*G \end{tikzcd}$$

whence the independence assertion for the first of the diagrams in (3.10.3.2). The second is dealt with similarly via (3.7.2)(ii).

The assertion for $'$-independence follows by symmetry. (Reflection in the appropriate diagonal interchanges independence and $'$-independence.)

Künneth independence for the first diagram in (3.10.3.2)—and hence, since Künneth independence does not depend on orientation, for the second diagram too—is treated via commutativity of the following natural diagram (with $E \in \mathbf{Y}''^{*}$ and $F \in \mathbf{X}^{*}$):

(3.10.3.2.2)

$$
\begin{tikzcd}
(uu_1)_*E \otimes f_*F \arrow[rr, "\eta_{\sigma_0}(E,F)"] \arrow[d, "\simeq"'] & & (uu_1h)_*(h^*E \otimes (vv_1)^*F) \arrow[d, "\simeq"] \\
u_*(u_{1*}E) \otimes f_*F \arrow[d, "\eta_\sigma(u_{1*}E,F)"'] & & u_*(u_1h)_*(h^*E \otimes v_1^*v^*F) \\
(ug)_*(g^*u_{1*}E \otimes v^*F) \arrow[r, "\sim"] & u_*g_*(g^*u_{1*}E \otimes v^*F) \arrow[r, "\sim", "(3.9.4)"'] & u_*(u_{1*}E \otimes g_*v^*F) \arrow[u, "u_*\eta_{\sigma_1}(E,v^*F)"']
\end{tikzcd}
$$

Commutativity can be verified, e.g., by using the left half of the commutative diagram (3.10.2.3) to reduce the question to commutativity of the natural diagram:

$$
\begin{tikzcd}
(uu_1)_*E \otimes f_*F \arrow[r, "\sim", "(3.9.4)"'] \arrow[d, "\simeq"] & (uu_1)_*(E \otimes (uu_1)^*f_*F) \arrow[r, "\theta_{\sigma_0}"] \arrow[d, "\simeq"] & (uu_1)_*(E \otimes h_*(vv_1)^*F) \arrow[d, "\simeq"', "(3.9.4)\text{ etc.}"] \\
u_*(u_{1*}E) \otimes f_*F \arrow[d, "(3.9.4)"', "\simeq"] \arrow[r, phantom, "①"] & u_*u_{1*}(E \otimes (uu_1)^*f_*F) \arrow[d, "\simeq"] & u_*u_{1*}h_*(h^*E \otimes v_1^*v^*F) \\
u_*(u_{1*}E \otimes u^*f_*F) \arrow[r, "\sim", "(3.9.4)"'] \arrow[d, "\theta_\sigma"'] & u_*u_{1*}(E \otimes u_1^*u^*f_*F) \arrow[r, "\theta_{\sigma_0}"] \arrow[d, "\theta_\sigma"] \arrow[dr, phantom, "②"] & u_*u_{1*}(E \otimes h_*v_1^*v^*F) \arrow[u, "\simeq", "(3.9.4)"'] \arrow[d, equal] \\
u_*(u_{1*}E \otimes g_*v^*F) \arrow[r, "\sim", "(3.9.4)"'] & u_*u_{1*}(E \otimes u_1^*g_*v^*F) \arrow[r, "\theta_{\sigma_1}"'] & u_*u_{1*}(E \otimes h_*v_1^*v^*F)
\end{tikzcd}
$$

Commutativity of subdiagram ① follows from (3.7.1), and of subdiagram ② from (3.7.2)(iii). The rest is straightforward. Q.E.D.

Corollary (3.10.3.3). *For σ as in* (3.10.3):

(i) *σ is independent if and only if for every diagram as in* (3.10.3.2) *with Y'' affine, $u_1\colon Y'' \to Y'$ an open immersion and σ_1 a fiber square, $\sigma_0 := \sigma \circ \sigma_1$ is independent.*

(i)$'$ *σ is $'$-independent if and only if for every diagram as in* (3.10.3.2) *with Z affine, $f_1\colon Z \to X_1$ an open immersion and σ_1 a fiber square, $\sigma_0 := \sigma \circ \sigma_1$ is $'$-independent.*

Proof. It follows from (1.2.2) that θ_σ is an isomorphism iff so is $u_1^*\theta_\sigma$ for all open immersions $u_1\colon Y'' \to Y'$ with Y'' affine. For such a u_1 the fiber square σ_1 is independent (as follows readily from (2.4.5.2)), so the commutative diagram (3.10.3.2.1) shows that $u_1^*\theta_\sigma$ is isomorphic to θ_{σ_0}, and (i) results.

Up to reversal of orientation, (i)$'$ is the same statement as (i). Q.E.D.

Lemma (3.10.3.4) (Independence and base change). *Given σ as in* (3.10.3) *let $i\colon U \to Y$ be an open immersion, let $i^*\sigma$ be the fiber square*

$$\begin{tikzcd} U \times_Y X' =: V' \arrow[r, "v_1"] \arrow[d, "g_1"'] & V := U \times_Y X \arrow[d, "f_1"] \\ U \times_Y Y' =: U' \arrow[r, "u_1"'] & U \end{tikzcd}$$

(*with obvious maps*) *and let $j\colon V \to X$ and $i'\colon U' \to Y'$ be the projections. Then $i^*\sigma$ is an* **S**-*square, and for any $G \in \mathbf{D}_{\mathsf{qc}}(X)$ the map*

$$\theta_{i^*\sigma}(j^*G)\colon u_1^*f_{1*}j^*G \to g_{1*}v_1^*j^*G$$

is isomorphic to the map

$$i'^*\theta_\sigma(G)\colon i'^*u^*f_*G \to i'^*g_*v^*G\,.$$

Moreover, for any $E \in \mathbf{D}_{\mathsf{qc}}(U')$ and $F \in \mathbf{D}_{\mathsf{qc}}(X)$ the map

$$i_*\eta_{i^*\sigma}(E, j^*F)\colon i_*(u_{1*}E \otimes f_{1*}j^*F) \to i_*(u_1g_1)_*(g_1^*E \otimes v_1^*j^*F)$$

is isomorphic to the map

$$\eta_\sigma(i'_*E, F)\colon u_*(i'_*E) \otimes f_*F \to (ug)_*(g^*i'_*E \otimes v^*F).$$

Consequently, σ is independent if and only if $i^*\sigma$ is independent for every open immersion $i\colon U \hookrightarrow Y$ with U affine; and if σ is Künneth-independent then so is $i^*\sigma$ for all such i.

Proof. That U, U', V and V' are quasi-separated is given by [**GD**, p. 294, (6.1.9)(i) and (ii)]; and that u_1, f_1, g_1 and v_1 are quasi-compact by [**GD**, p. 291, (6.1.5)(iii)]. By (3.7.2)(iii), the diagrams

$$\begin{tikzcd} V' \arrow[r, "v_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "i^*\sigma"] & V \arrow[r, "j"] \arrow[d, "f_1"] \arrow[dr, phantom, "\sigma'"] & X \arrow[d, "f"] \\ U' \arrow[r, "u_1"'] & U \arrow[r, "i"'] & Y \end{tikzcd} \qquad \begin{tikzcd} V' \arrow[r, "j'"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma''"] & X' \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ U' \arrow[r, "i'"'] & Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

which are two decompositions of the same square—call it τ—give rise to a commutative diagram of functorial maps (cf. (3.10.3.2.1)):

$$\begin{tikzcd} u_1^*i^*f_*G \arrow[r, "u_1^*\theta_{\sigma'}(G)"] \arrow[d, "\simeq"'] & u_1^*f_{1*}j^*G \arrow[r, "\theta_{i^*\sigma}(j^*G)"] & g_{1*}v_1^*j^*G \arrow[d, "\simeq"] \\ (iu_1)^*f_*G \arrow[rr, "\theta_\tau(G)"] \arrow[d, equal] & & g_{1*}(jv_1)^*G \arrow[d, equal] \\ (ui')^*f_*G \arrow[rr, "\theta_\tau(G)"'] \arrow[d, "\simeq"'] & & g_{1*}(vj')^*G \arrow[d, "\simeq"] \\ i'^*u^*f_*G \arrow[r, "i'^*\theta_\sigma(G)"'] & i'^*g_*v^*G \arrow[r, "\theta_{\sigma''}(v^*G)"'] & g_{1*}j'^*v^*G \end{tikzcd}$$

Since i and i' are open immersions, the maps $\theta_{\sigma'}$ and $\theta_{\sigma''}$ are isomorphisms (see proof of (3.10.3.3)), and the first isomorphism assertion in the Lemma results.

A similar argument using (3.10.3.2.2) proves the second isomorphism assertion.

The independence consequence for θ then follows from (1.2.2) and the fact that since j is an open immersion therefore $F \cong j^*j_*F$ for every $F \in \mathbf{D}(V)$.

The Künneth-independence consequence is proved similarly, with the additional observation that i is isofaithful (see (3.10.2.1)(d)). Q.E.D.

Exercise **(3.10.4)** (Conjugate base change). Let σ be a fiber square as in (3.10.3), and assume the schemes in σ are concentrated, so that by (4.1.1) below, f_* and g_* have right adjoints $f^\times$ and $g^\times$ respectively.

(a) Show that the map

$$\phi_\sigma\colon v_*g^\times \to f^\times u_*$$

(between functors from $\mathbf{D}_{\mathsf{qc}}(Y')$ to $\mathbf{D}_{\mathsf{qc}}(X)$) corresponding by adjunction to the natural composition $f_*v_*g^\times \xrightarrow{\sim} u_*g_*g^\times \to u_*$ is right-conjugate to θ_σ.

Deduce that σ *is independent iff* ϕ_σ *(or* $\phi_{\sigma'}$*) is an isomorphism.*

Hint. The first assertion is that $\phi_\sigma(E)$ is the image of the identity map under the sequence of natural isomorphisms

$$\begin{aligned}\mathrm{Hom}(v_*g^\times E, v_*g^\times E) &\xrightarrow{\sim} \mathrm{Hom}(v^*v_*g^\times E, g^\times E) \xrightarrow{\sim} \mathrm{Hom}(g_*v^*v_*g^\times E, E)\\ &\xrightarrow{\sim} \mathrm{Hom}(u^*f_*v_*g^\times E, E) \xrightarrow{\sim} \mathrm{Hom}(f_*v_*g^\times E, u_*E)\\ &\xrightarrow{\sim} \mathrm{Hom}(v_*g^\times E, f^\times u_*E).\end{aligned}$$

(b) Show that when σ is independent the map ϕ_σ^{-1}—right-conjugate to θ_σ^{-1}, see (a)—corresponds to the composition

$$v^*f^\times u_* \xrightarrow{\text{via } \beta_\sigma} g^\times u^*u_* \xrightarrow{\text{natural}} g^\times$$

with β_σ as in (3.10.2.1)(c).

(b)$'$ Show that when σ is independent the map β_σ corresponds to the composition

$$f^\times \xrightarrow{\text{natural}} f^\times u_*u^* \xrightarrow{\text{via } \phi_\sigma^{-1}} v_*g^\times u^*.$$

Hint. To deduce (b)$'$ from (b), use the natural diagram (whose bottom row and right column both compose to the identity):

$$\begin{tikzcd}
f^\times \arrow[r] \arrow[d] & f^\times u_*u^* \arrow[r, "\phi_\sigma^{-1}"] \arrow[d] & v^*g^\times u^* \arrow[d]\\
v_*v^*f^\times \arrow[r] \arrow[d, "v_*\beta_\sigma"'] & v_*v^*f^\times u_*u^* \arrow[r, "\phi_\sigma^{-1}"] \arrow[d, "v_*\beta_\sigma"] & v_*v^*v^*g^\times u^* \arrow[d]\\
v_*g^\times u^* \arrow[r] & v_*g^\times u^*u_*u^* \arrow[r] & v_*g^\times u^*
\end{tikzcd}$$

Similarly, (b)$' \Rightarrow$ (b).

(c) Show that ϕ_σ corresponds to the natural composition

$$g^\times \longrightarrow g^\times u^\times u_* \xrightarrow{\sim} v^\times f^\times u_*.$$

Chapter 4

Abstract Grothendieck Duality for schemes

In this chapter we review and elaborate on—with proofs and/or references—some basic abstract features of Grothendieck Duality for schemes with Zariski topology, a theory initially developed by Grothendieck [**Gr′**], [**H**], [**C**], Deligne [**De′**], and Verdier [**V′**].[42] The principal actor in this Chapter is the *twisted inverse image pseudofunctor,* described in the Introduction. The basic facts about this pseudofunctor—which may be seen as the main results in these Notes—are existence and flat base change, Theorems (4.8.1) and (4.8.3).

The abstract theory begins with Theorem (4.1) (Global Duality), asserting for any map $f\colon X \to Y$ of concentrated schemes the existence of a right adjoint $f^{\times}$ for the functor $\mathsf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}_{\mathsf{qc}}(Y)$. In order to sheafify this result, or, more generally, to prove tor-independent base change for $f^{\times}$—see (4.4.2) and (4.4.3), we need f to be *quasi-proper,* a condition which coincides with properness when the schemes involved are noetherian. This condition is discussed in section 4.3. The proofs of (4.4.2) and (4.4.3) are given in sections (4.5) and (4.6). That prepares the ground for the above main results.

Section (4.7) is concerned with *quasi-perfect* (= quasi-proper plus finite tor-dimension) maps of concentrated schemes. These maps have a number of especially nice properties with respect to $f^{\times}$.

Analogously, section (4.9) deals with *perfect* (= finite tor-dimension) finite-type separated maps of noetherian schemes. These maps behave nicely with respect to the twisted inverse image. For example, if $f\colon X \to Y$ is a finite-type separated map of noetherian schemes, and $f^!$ is the associated twisted inverse image functor, perfectness of f is characterized by boundedness of $f^!\mathcal{O}_Y$ plus the existence of a functorial isomorphism

$$f^!\mathcal{O}_Y \underset{=}{\otimes} \mathsf{L}f^*F \xrightarrow{\ \sim\ } f^!F \qquad \big(F \in \overline{\mathbf{D}}{}^{+}_{\mathsf{qc}}(Y)\big).$$

This, and other characterizations, are in Theorem (4.9.4). Theorem (4.7.1) contains the corresponding result for the functor $f^{\times}$ associated to a quasi-perfect map f.

[42] As regards these Notes, see the Introduction for some comments on "abstract" vis-à-vis "concrete" duality. Exercise (4.8.12)(b) is an example of the latter.

In an appendix, section (4.10), we say something about the role of dualizing complexes in duality theory. This is an important topic, but not a central one in these Notes.

Throughout, all schemes are assumed to be concentrated, i.e., quasi-separated and quasi-compact.

4.1. Global Duality

Fix once and for all a universe $\mathfrak{U}$ [**M**, p. 22]. Henceforth, any category is understood to have all its arrows and objects in $\mathfrak{U}$. Call a set *small* if it is a member of $\mathfrak{U}$. A *small category* is one whose arrows—and hence objects—form a small set. Every topological space X is understood to be small; and any sheaf E on X is understood to be such that for every open $U \subset X$, $\Gamma(U, E)$ is a small set.

For any scheme $(X, \mathcal{O}_X)$, $\mathcal{A}_X$ is, as before, the abelian category of $\mathcal{O}_X$-modules and their homomorphisms, and $\mathcal{A}_X^{\mathsf{qc}}$ is the full abelian subcategory whose objects are all the quasi-coherent $\mathcal{O}_X$-modules. Though these two categories are not small, they are *well-powered,* i.e., for each object E there is a small set J_E such that every subobject (or every quotient) of E is isomorphic to a member of J_E; and they have *small hom-sets,* i.e., for any objects E, F, the set $\mathrm{Hom}(E, F)$ is small.

"Global Duality" means:

Theorem (4.1). *Let X be a concentrated* (= *quasi-compact, quasi-separated*) *scheme and $f\colon X \to Y$ a concentrated scheme-map. Then the Δ-functor $\mathbf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}(Y)$ has a bounded-below right Δ-adjoint.*

By (1.2.2), (2.4.2), and the description of θ^* in (3.3.8) (where it may be assumed that θ_* is the identity, see (2.7.3.2)), the following statement is equivalent to (4.1).

Theorem (4.1.1). *Let X be a concentrated* (= *quasi-compact, quasi-separated*) *scheme and $f\colon X \to Y$ a concentrated scheme-map. Then there is a bounded-below Δ-functor* $(f^\times, \text{identity})\colon \mathbf{D}(Y) \to \mathbf{D}_{\mathsf{qc}}(X)$ *and a map of Δ-functors $\tau\colon \mathbf{R}f_*f^\times \to \mathbf{1}$ such that for all $F \in \mathbf{D}_{\mathsf{qc}}(X)$ and $G \in \mathbf{D}(Y)$, the composite Δ-functorial map* (*in the derived category of abelian groups*)

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}^\bullet_X(F, f^\times G) &\xrightarrow{(3.2.1.0)} \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*\mathbf{R}f_*F, f^\times G)\\
&\xrightarrow{(3.2.3.1)} \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^\times G)\\
&\xrightarrow{\text{via } \tau} \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, G)
\end{aligned}$$

is a Δ-functorial isomorphism.

Corollary (4.1.2). *When restricted to concentrated schemes, the* $\mathbf{D}_{\mathsf{qc}}$*-valued pseudofunctor "derived direct image"* (*see* (3.9.2)) *has a pseudofunctorial right* Δ*-adjoint* $^{\times}$ (*see* (3.6.7)(d))*.*

Proofs. To get (4.1.2) from (4.1.1), recalling that a map $f\colon X\to Y$ of concentrated schemes is itself concentrated [**GD**, §6.1, pp. 290*ff*], choose for each such f a functor $f^{\times}$ right-Δ-adjoint to $\mathbf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X)\to\mathbf{D}_{\mathsf{qc}}(Y)$, with $f^{\times}$ the identity functor whenever f is an identity map. For another such $g\colon Y\to Z$, define $d_{f,g}\colon f^{\times}g^{\times}\to(gf)^{\times}$ to be the functorial map adjoint to the natural composition

$$\mathbf{R}(gf)_*f^{\times}g^{\times}\ \xrightarrow{\ \sim\ }\ \mathbf{R}g_*\mathbf{R}f_*f^{\times}g^{\times}\to\mathbf{R}g_*g^{\times}\to\mathbf{1}.$$[43]

This $d_{f,g}$ is an isomorphism, its inverse $(gf)^{\times}\to f^{\times}g^{\times}$ being the map adjoint to the natural composition

$$\mathbf{R}g_*\mathbf{R}f_*(gf)^{\times}\ \xrightarrow{\ \sim\ }\ \mathbf{R}(gf)_*(gf)^{\times}\to\mathbf{1}.$$

The verification of (4.1.2) is then straightforward (see (3.6.5)).

As for (4.1), the classical abstract method was introduced by Verdier in his treatment of duality for locally compact spaces, then adapted to schemes by Deligne [**De′**] to show that with $\boldsymbol{j}\colon\mathbf{D}(\mathcal{A}_X^{\mathsf{qc}})\to\mathbf{D}_{\mathsf{qc}}(X)$ the natural functor, $\mathbf{R}f_*\circ\boldsymbol{j}$ has a right adjoint. This suffices only when f is *separated,* see (3.9.6). The proof given below (for historical reasons, because of the compactness of Deligne's original presentation) is just an elaboration of Deligne's arguments.

The reader may prefer to look up in [**N**] the more modern, lucidly exposed, approach of Neeman, who uses Brown Representability instead of, as below, the Special Adjoint Functor Theorem applied via injective resolutions. This is conceptually more elegant in that it gives a direct criterion for the existence of a right adjoint for a triangulated functor $\mathcal{F}$ on any compactly generated triangulated category, such as $\mathbf{D}_{\mathsf{qc}}(X)$. In analogy with the "cocontinuity" used in Deligne's method (see below), the condition on $\mathcal{F}$ is that it commute with small direct sums, a condition which follows for $\mathcal{F}=\mathbf{R}f_*$ from (3.9.3.3). The (nontrivial) proof in [**N**] that $\mathbf{D}_{\mathsf{qc}}(X)$ is compactly generated ostensibly requires X to be separated; but essentially the same proof shows that $\mathbf{D}_{\mathsf{qc}}(X)$ is compactly generated for any concentrated X, see [**BB**, §3], and this gives Theorem (4.1) in full generality.[44]

Proof of (4.1) (when X is separated, see above).

1. First, we review some terminology and basic results about abelian categories. Let $\mathcal{A}$ be an abelian category with small direct sums (i.e., every

[43] This definition makes the property TRA 1 in [**H**, p. 207] tautologous.

[44] Arguments much like Deligne's or Neeman's apply also to *noetherian formal schemes,* see [**AJL′**, §4, pp. 42–46] resp. [**AJL′**, p. 41, 3.5.2] and [**AJS**, p. 245, Cor. 5.9].

family of objects in $\mathcal{A}$ indexed by a small set has a direct sum). Any two arrows in $\mathcal{A}$ with the same source and target have a *coequalizer,* namely the cokernel of their difference [**M**, p. 70]. Hence $\mathcal{A}$ is *small-cocomplete,* i.e., any functor from a small category into $\mathcal{A}$ has a colimit, see [**M**, p. 113, Cor. 2] (dualized). An additive functor $\mathcal{F}$ from $\mathcal{A}$ to an abelian category $\mathcal{A}'$ is *cocontinuous* if $\mathcal{F}$ commutes with small colimits, in the sense that if $\mathcal{G}$ is any functor from a small category $\mathcal{C}$ into $\mathcal{A}$ and $\big(G, (g_c\colon \mathcal{G}c \to G)_{c\in\mathcal{C}}\big)$ is a colimit of $\mathcal{G}$ then $\big(\mathcal{F}G, (\mathcal{F}g_c)_{c\in\mathcal{C}}\big)$ is a colimit of $\mathcal{F}\mathcal{G}$. It follows from [**M**, p. 113, Thm. 2] that $\mathcal{F}$ *is cocontinuous iff it is right-exact and transforms small direct sums in* $\mathcal{A}$ *into small direct sums in* $\mathcal{A}'$.

We reserve the symbol $\varinjlim$ for denoting direct limits of small directed systems in $\mathcal{A}$, i.e., colimits of functors $\mathcal{G}\colon \mathcal{C} \to \mathcal{A}$ where $\mathcal{C}$ is the category associated to a small preordered set in which any two elements have an upper bound [**M**, p. 11, p. 211]. All such $\varinjlim$'s exist in an abelian category $\mathcal{A}$ iff $\mathcal{A}$ is small-cocomplete [**M**, p. 212, Theorem 1]. Similarly, an additive functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{A}'$ is cocontinuous iff it is right-exact and commutes with all $\varinjlim$'s.

2. An essential ingredient of the proof of Theorem (4.1) is the following consequence of the Special Adjoint Functor Theorem [**M**, p. 130, Corollary]. (See also [**De**$'$, p. 408, Cor. 1]).

Proposition (4.1.3). *For a concentrated scheme* X, *an additive functor* $\mathcal{F}$ *from* $\mathcal{A}_X^{\mathsf{qc}}$ *to an abelian category* $\mathcal{A}'$ *with small hom-sets has a right adjoint if and* (*clearly*) *only if it is cocontinuous.*

(4.1.3.1). For the Special Adjoint Functor Theorem to be applicable here, the category $\mathcal{A}_X^{\mathsf{qc}}$—which, as above, is well-powered and has small hom-sets, and which is also small-cocomplete [**GD**, p. 217, (2.2.2)(iv)]—must have a small set of generators. Recall that an $\mathcal{O}_X$-module E on a ringed space X is *locally finitely presentable* (lfp for short) if X is covered by open subsets U such that for each U the restriction $E|_U$ is isomorphic to the cokernel of a map $\mathcal{O}_U^m \to \mathcal{O}_U^n$ with finite m and n. Since every quasi-coherent $\mathcal{O}_X$-module is the $\varinjlim$ of its lfp submodules [**GD**, p. 319, (6.9.9)], the small-generated property follows from the fact that *for any scheme* X *there exists a small set* S *of lfp* $\mathcal{O}_X$*-modules such that every lfp* $\mathcal{O}_X$*-module is isomorphic to a member of* S.

Proof. With U ranging over the small set of affine open subschemes of X, and $i_U\colon U \hookrightarrow X$ the inclusion, any $\mathcal{O}_X$-module E is isomorphic to a submodule of $\prod_U i_{U*}i_U^*E$. If E is lfp then so is the $\mathcal{O}_U$-module i_U^*E, so that i_U^*E is a quotient of $\mathcal{O}_U^n$ for some finite n [**GD**, p. 207, (1.4.3)]. Thus every lfp E is isomorphic to a subsheaf of a sheaf of the form $\prod_U i_{U*}E_U$ where for each U, E_U ranges over a fixed small set of $\mathcal{O}_U$-modules, whence the conclusion. Q.E.D.

(For another argument see [**Kn**, pp. 43–44, proof of Thm. 4.])

3. The basic idea for proving (4.1) is to show that *there is a functorial exact $\mathcal{A}_X$-sequence* (*i.e., a finite resolution of the inclusion $\mathcal{A}_X^{\mathsf{qc}} \hookrightarrow \mathcal{A}_X$*)

(4.1.4)

$$0 \to M \xrightarrow{\delta(M)} \mathcal{D}^0(M) \xrightarrow{\delta^0(M)} \mathcal{D}^1(M) \xrightarrow{\delta^1(M)} \cdots \xrightarrow{\delta^{d-1}(M)} \mathcal{D}^d(M) \to 0 \qquad \big(M \in \mathcal{A}_X^{\mathsf{qc}}\big)$$

such that the functors $\mathcal{D}^i\colon \mathcal{A}_X^{\mathsf{qc}} \to \mathcal{A}_X$ $(0 \le i \le d)$ are additive and co-continuous, such that for all M, $\mathcal{D}^i(M)$ is f_*-acyclic, and such that the functors $f_*\mathcal{D}^i$ are right-exact.

Here is one way to do this. Recall the Godement resolution

$$0 \to M \to \mathcal{G}^0(M) \to \mathcal{G}^1(M) \to \cdots$$

where, with $\mathcal{G}^{-2}(M) := 0$, $\mathcal{G}^{-1}(M) := M$, and $\mathcal{K}^i(M)$ $(i \ge 0)$ the cokernel of $\mathcal{G}^{i-2}(M) \to \mathcal{G}^{i-1}(M)$, the sheaf $\mathcal{G}^i(M)$ is defined inductively by

$$\mathcal{G}^i(M)\big(U\big) := \prod_{x \in U} \mathcal{K}^i(M)_x \qquad (U \text{ open in } X).$$

One shows by induction on i that all the functors $\mathcal{G}^i$ and $\mathcal{K}^i$ (from $\mathcal{A}_X$ to itself) are *exact.* Moreover, for $i \ge 0$, $\mathcal{G}^i(M)$ is flasque, hence f_*-acyclic. With d as in (3.9.2.4), the dual version of (2.7.5)(iii) shows that $\mathcal{K}^d(M)$ is f_*-acyclic. So, setting

$$\mathcal{D}^i(M) := \begin{cases} \mathcal{G}^i(M) & (0 \le i < d) \\ \mathcal{K}^d(M) & (i = d) \\ 0 & (i > d) \end{cases}$$

we get a finite resolution (4.1.4) having all the desired properties except for commutativity of the $\mathcal{D}^i$ with $\varinjlim$.

To get commutativity with $\varinjlim$ we use the next Lemma, proved below.

Lemma (4.1.5). *Let $\mathcal{A}'$ be a small-cocomplete abelian category in which $\varinjlim$ preserves exactness of sequences. Then with $\mathbf{F}$ the category of additive functors from $\mathcal{A}_X^{\mathsf{qc}}$ to $\mathcal{A}'$, there is a functor $(-)_{\mathrm{cts}}\colon \mathbf{F} \to \mathbf{F}$ and a functorial map $i_{\mathcal{D}}\colon \mathcal{D}_{\mathrm{cts}} \to \mathcal{D}$ $(\mathcal{D} \in \mathbf{F})$ such that:*

(i) *For all lfp $M \in \mathcal{A}_X^{\mathsf{qc}}$, $i_{\mathcal{D}}(M)$ is an isomorphism $\mathcal{D}_{\mathrm{cts}}(M) \xrightarrow{\sim} \mathcal{D}(M)$.*

(ii) *For any $\mathcal{D} \in \mathbf{F}$, $\mathcal{D}_{\mathrm{cts}}$ commutes with $\varinjlim$.*

(iii) *If $\mathcal{D}$ commutes with $\varinjlim$ then $i_{\mathcal{D}}$ is a functorial isomorphism.*

(iv) *If $\mathcal{D}$ is right-exact then so is $\mathcal{D}_{\mathrm{cts}}$.*

(v) *For any exact sequence $\mathcal{D}' \to \mathcal{D} \to \mathcal{D}''$ in $\mathbf{F}$ (i.e., the $\mathcal{A}'$-sequence $\mathcal{D}'(M) \to \mathcal{D}(M) \to \mathcal{D}''(M)$ is exact for all $M \in \mathcal{A}_X^{\mathsf{qc}}$), the corresponding sequence $\mathcal{D}'_{\mathrm{cts}} \to \mathcal{D}_{\mathrm{cts}} \to \mathcal{D}''_{\mathrm{cts}}$ is exact.*

(vi) *When $\mathcal{A}' = \mathcal{A}_X$, if $\mathcal{D}(M)$ is f_*-acyclic for all $M \in \mathcal{A}_X^{\mathsf{qc}}$ then $\mathcal{D}_{\mathrm{cts}}(M)$ is f_*-acyclic for all $M \in \mathcal{A}_X^{\mathsf{qc}}$; and if, further, $\mathcal{D}$ is exact, then the functor $f_*\mathcal{D}_{\mathrm{cts}}\colon \mathcal{A}_X^{\mathsf{qc}} \to \mathcal{A}_Y$ is right-exact.*

Indeed, one can apply any such $(-)_{\text{cts}}$ for $\mathcal{A}' = \mathcal{A}_X$ to the just-constructed truncated Godement resolution, to produce a resolution with all the desired properties. (For this, condition (4.1.5)(iii) is needed only when $\mathcal{D} =$ identity functor.)

From (4.1.4) there results a Δ-functor

$$(\mathcal{D}, \text{Identity})\colon \mathbf{K}(\mathcal{A}_X^{\mathsf{qc}}) \to \mathbf{K}(\mathcal{A}_X) =: \mathbf{K}(X)$$

taking each $\mathcal{A}_X^{\mathsf{qc}}$-complex (M,d) to the f_*-acyclic $\mathcal{A}_X$-complex $\mathcal{D}(M)$ with

$$\mathcal{D}(M)^m := \oplus_{p+q=m}\, \mathcal{D}^q(M^p) \qquad (m\in\mathbb{Z},\ 0\le q\le d)$$

and with differential $\mathcal{D}(M)^m \to \mathcal{D}(M)^{m+1}$ defined on $\mathcal{D}^q(M^p)$ $(p+q=m)$ to be $\mathcal{D}^q(d^p) + (-1)^p\delta^q(M^p)$. One checks by elementary diagram chasing—or spectral sequences—that *the natural* $\mathbf{K}(X)$*-map* $\delta(M)\colon M \to \mathcal{D}(M)$ *is a quasi-isomorphism.*

It follows that the the natural maps are $\mathbf{D}(Y)$-isomorphisms

$$(4.1.6)\qquad f_*\mathcal{D}(M) \xrightarrow{\ \sim\ } \mathbf{R}f_*\mathcal{D}(M) \underset{\mathbf{R}f_*\delta(M)}{\xleftarrow{\ \sim\ }} \mathbf{R}f_*\boldsymbol{j}M, \qquad \big(M\in\mathbf{K}(\mathcal{A}_X^{\mathsf{qc}})\big)$$

the first, in view of (3.9.2.4), by the dual version of (2.7.5)(a). Thus we have realized $\mathbf{R}f_*\circ\boldsymbol{j}$ (up to isomorphism) at the homotopy level, as the functor $\mathcal{C}^\bullet := f_*\mathcal{D}$. Let us find a right adjoint at this level.

4. Each functor $\mathcal{C}^q := f_*\mathcal{D}^q\colon \mathcal{A}_X^{\mathsf{qc}} \to \mathcal{A}_Y$ $(0\le q\le d)$ is right-exact. Also, $\mathcal{C}^q$ commutes with $\varinjlim$ since both $\mathcal{D}^q$ and f_* do. (For f_* see [**Kf**, p. 641, Prop. 6], or imitate the proof on p. 163 of [**G**]). Thus $\mathcal{C}^q$ is cocontinuous, and so by (4.1.3), $\mathcal{C}^q$ *has a right adjoint* $\mathcal{C}_q\colon \mathcal{A}_Y \to \mathcal{A}_X^{\mathsf{qc}}$.

There are then functorial maps $\delta_s\colon \mathcal{C}_{s+1} \to \mathcal{C}_s$ right-conjugate to $f_*(\delta^s)\colon \mathcal{C}^s \to \mathcal{C}^{s+1}$, see (3.3.5).

For each $\mathcal{A}_Y$-complex $(F, d_\prime)$, let $\mathcal{C}_\bullet F$ be the $\mathcal{A}_X^{\mathsf{qc}}$-complex with

$$(\mathcal{C}_\bullet F)^m := \prod_{p-q\,=\,m} \mathcal{C}_q F^p \qquad (m\in\mathbb{Z},\ 0\le q\le d),$$

and whose differential $(\mathcal{C}_\bullet F)^m \to (\mathcal{C}_\bullet F)^{m+1}$ is the unique map making the following diagram (with vertical arrows coming from projections) commute for all r, s with $r-s = m+1$:

$$\begin{array}{ccc}
\displaystyle\prod_{p-q\,=\,m} \mathcal{C}_q F^p = (\mathcal{C}_\bullet F)^m & \longrightarrow & (\mathcal{C}_\bullet F)^{m+1} = \displaystyle\prod_{p-q\,=\,m+1} \mathcal{C}_q F^p \\
\big\downarrow & & \big\downarrow \\
\mathcal{C}_s F^{r-1} \oplus \mathcal{C}_{s+1} F^r & \xrightarrow[\mathcal{C}_s d_\prime^{r-1} + (-1)^{r+s}\delta_s(F^r)]{} & \mathcal{C}_s F^r
\end{array}$$

There results naturally a Δ-functor $(\mathcal{C}_\bullet, \text{Identity})\colon \mathbf{K}(Y) \to \mathbf{K}(\mathcal{A}_X^{\mathsf{qc}})$.

One checks that, applied componentwise, the adjunction isomorphism

$$\operatorname{Hom}_{\mathcal{A}_X^{\mathsf{qc}}}(M, \mathcal{C}_p N) \xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathcal{A}_Y}(\mathcal{C}^p M, N) \qquad \bigl(M \in \mathcal{A}_X^{\mathsf{qc}},\ N \in \mathcal{A}_Y\bigr)$$

produces an isomorphism of complexes of abelian groups

$$\operatorname{Hom}^{\bullet}_{\mathcal{A}_X^{\mathsf{qc}}}(G, \mathcal{C}_{\bullet} F) \xrightarrow{\ \sim\ } \operatorname{Hom}^{\bullet}_{\mathcal{A}_Y}(\mathcal{C}^{\bullet} G, F) \tag{4.1.7}$$

for all $\mathcal{A}_X^{\mathsf{qc}}$-complexes G and $\mathcal{A}_Y$-complexes F.

5. The isomorphism (4.1.7) suggests using $\mathcal{C}_{\bullet}$ to construct $f^{\times}$, as follows. Recall that a complex $J \in \mathbf{K}(\mathcal{A}_X^{\mathsf{qc}})$ is K-injective iff for each exact $G \in \mathbf{K}(\mathcal{A}_X^{\mathsf{qc}})$, the complex $\operatorname{Hom}^{\bullet}_{\mathcal{A}_X^{\mathsf{qc}}}(G, J)$ is exact too. The isomorphisms (4.1.6) show that $\mathcal{C}^{\bullet} G$ is exact if G is; so it follows from (4.1.7) that *if F is K-injective in $\mathbf{K}(Y)$ then $\mathcal{C}_{\bullet} F$ is K-injective in $\mathbf{K}(\mathcal{A}_X^{\mathsf{qc}})$.* Thus if $\mathbf{K_I}(-) \subset \mathbf{K}(-)$ is the full subcategory whose objects are all the K-injective complexes, then we have a Δ-functor $(\mathcal{C}_{\bullet}, \mathrm{Id})\colon \mathbf{K_I}(Y) \to \mathbf{K_I}(\mathcal{A}_X^{\mathsf{qc}})$.

Associating a K-injective resolution to each complex in $\mathcal{A}_Y$ leads to a Δ-functor $(\rho, \theta)\colon \mathbf{D}(Y) \to \mathbf{K_I}(Y)$. In fact (ρ, θ) is an equivalence of Δ-categories, see §1.7. This ρ is *bounded below*: an $\mathcal{A}_Y$-complex E such that $H^i(E) = 0$ for all $i < n$ is quasi-isomorphic to its truncation $\tau_{\ge n} E$, which is quasi-isomorphic to an injective complex F vanishing in all degrees below n; and such an F is K-injective.

Finally, one defines $f^{\times}$ to be the composition of the functors

$$\mathbf{D}(Y) \xrightarrow{\ \rho\ } \mathbf{K_I}(Y) \xrightarrow{\ \mathcal{C}_{\bullet}\ } \mathbf{K_I}(\mathcal{A}_X^{\mathsf{qc}}) \xrightarrow{\ \text{natural}\ } \mathbf{D}(\mathcal{A}_X^{\mathsf{qc}}),$$

and checks, via (4.1.6), (4.1.7), (2.3.8.1) and (2.3.8)(v), that $(f^{\times}, \text{identity})$ is indeed a bounded-below right Δ-adjoint of $\mathbf{R}f_* \circ \boldsymbol{j}$. (Checking the Δ-details can be tedious. Note that by (2.7.3.2) and (3.3.8), we can at least assume that $f^{\times}$ commutes with translation of complexes.)

That $f^{\times}$ is bounded below results from (3.9.2.3) and the following general fact.

Lemma (4.1.8). *Let $\mathfrak{A}^{\#}$, $\mathfrak{B}^{\#}$ be plump subcategories of the abelian categories $\mathfrak{A}$, $\mathfrak{B}$ respectively, let $\mathbf{E} = \mathbf{D}_{\#}(\mathfrak{A})$, $\mathbf{D}^{*}_{\#}(\mathfrak{A})$, or $\overline{\mathbf{D}}^{*}_{\#}(\mathfrak{A})$, see* (1.9), *and let $\mathbf{E}' = \mathbf{D}_{\#}(\mathfrak{B})$, $\mathbf{D}^{*}_{\#}(\mathfrak{B})$, or $\overline{\mathbf{D}}^{*}_{\#}(\mathfrak{B})$. If the functor $F\colon \mathbf{E} \to \mathbf{E}'$ has a right adjoint G, then for any $n, d \in \mathbb{Z}$:*

$$F(\mathbf{E}_{\le \mathbf{n}}) \subset \mathbf{E}'_{\le \mathbf{n+d}} \iff G(\mathbf{E}'_{\ge \mathbf{n}}) \subset \mathbf{E}_{\ge \mathbf{n-d}}\,.$$

Proof. Let $B \in \mathbf{E}'_{\ge \mathbf{n}}$. For $A = \tau_{\le n-d-1} G(B)$, the natural map $\alpha\colon A \to G(B)$ induces homology isomorphisms in all degrees $< n - d$, see (1.10). But since $F(A) \in \mathbf{E}'_{\le \mathbf{n-1}}$ and $\tau_{\le n-1} B \cong 0$, we have by adjointness and by (1.10.1.1):

$$\alpha \in \operatorname{Hom}_{\mathbf{E}}\bigl(A, G(B)\bigr) \cong \operatorname{Hom}_{\mathbf{E}'}\bigl(F(A), B\bigr) \cong \operatorname{Hom}_{\mathbf{E}'}\bigl(F(A), \tau_{\le n-1} B\bigr) = 0.$$

Hence $H^j G(B) = 0$ for all $j < n - d$, i.e., $G(B) \in \mathbf{E}_{\ge \mathbf{n-d}}$.

A dual argument gives the opposite implication. Q.E.D.

This completes the proof of Theorem (4.1), except for Lemma (4.1.5).

Proof of (4.1.5). For constructing $(-)_{\mathrm{cts}}$ let S be a small set of lfp $\mathcal{O}_X$-modules such that every lfp $\mathcal{O}_X$-module is isomophic to a member of S, see (4.1.3.1). For any $M \in \mathcal{A}_X^{\mathsf{qc}}$ let $S{\downarrow}M$ be the *small category* whose objects are all the maps $s \to M$ $(s \in S)$, a morphism from $\alpha\colon s \to M$ to $\beta\colon s' \to M$ being an $\mathcal{A}_X^{\mathsf{qc}}$-map $\mu\colon s \to s'$ with $\beta\mu = \alpha$. Sending each $\alpha\colon s \to M$ in $S{\downarrow}M$ to its source $s_\alpha := s$, we get a functor $\mathbf{s}_M\colon S{\downarrow}M \to \mathcal{A}_X^{\mathsf{qc}}$.

For any $\mathcal{D} \in \mathbf{F}$, the additive functor $\mathcal{D}_{\mathrm{cts}} \in \mathbf{F}$ is defined as follows:

$$\mathcal{D}_{\mathrm{cts}}(M) := \operatorname*{colim}_{S\downarrow M} \mathcal{D}\circ \mathbf{s}_M \qquad (M \in \mathcal{A}_X^{\mathsf{qc}});$$

and for any $\mathcal{A}_X^{\mathsf{qc}}$-map $\phi\colon M \to M'$, $\mathcal{D}_{\mathrm{cts}}(\phi)$ is the $\mathcal{A}'$-map induced by the functorial map $\mathbf{s}_M \to \mathbf{s}_{M'}$ given by composition with ϕ.[45] The functorial map $i_{\mathcal{D}}\colon \mathcal{D}_{\mathrm{cts}}(M) \to \mathcal{D}(M)$ is the one whose composition with the canonical map $\mathcal{D}(s_\alpha) = \mathcal{D}\mathbf{s}_M(\alpha) \to \mathcal{D}_{\mathrm{cts}}(M)$ is $\mathcal{D}(\alpha)\colon \mathcal{D}(s_\alpha) \to \mathcal{D}(M)$ for each object $\alpha\colon s_\alpha \to M$ in $S{\downarrow}M$.

Condition (4.1.5)(i) follows easily from the observation that when M is lfp, the identity map of M is a final object in the category $S{\downarrow}M$.

To prove (ii) we need:

$(*)$: *For any lfp E and directed system N_σ of quasi-coherent $\mathcal{O}_X$-modules the natural map is an isomorphism*

$$\varinjlim_{\sigma} \mathrm{Hom}_{\mathcal{O}_X}(E, N_\sigma) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_X}(E, \varinjlim_{\sigma} N_\sigma).$$

(*Proof*: Since X is concentrated, therefore $\Gamma(X,-)$ commutes with $\varinjlim$ [**Kf**, p. 641, Prop. 6], so it suffices to prove the statement with $\mathcal{H}om$ in place of Hom. Thus the statement is local, and so equivalent to the analogous well-known—and easily verifiable—one for modules over rings.)

Given a small directed system $\big(M_\gamma, (\phi_{\delta\gamma}\colon M_\gamma \to M_\delta)_{\delta\ge\gamma}\big)$ in $\mathcal{A}_X^{\mathsf{qc}}$, $(*)$ shows that each map $s \to M := \varinjlim M_\gamma$ with $s \in S$ is determined by a unique equivalence class of maps $s \to M_\gamma$ (s fixed, γ variable), where $[s \to M_{\gamma'}] \equiv [s \to M_{\gamma''}]$ if and only if there exists a commutative diagram

$$\begin{tikzcd} s \arrow[r] \arrow[d] & M_{\gamma'} \arrow[d, "\phi_{\gamma\gamma'}"] \\ M_{\gamma''} \arrow[r, "\phi_{\gamma\gamma''}"'] & M_\gamma \end{tikzcd}$$

This is the least equivalence relation such that $[s \to M_\gamma] \equiv [s \to M_\gamma \xrightarrow{\phi_{\delta\gamma}} M_\delta]$ for all $\delta \ge \gamma$. Moreover, $\mathcal{A}'$-maps $f\colon \mathcal{D}_{\mathrm{cts}}(M) \to A$ correspond naturally to families of maps $\big(f_\alpha\colon \mathcal{D}(s_\alpha) \to A\big)_{\alpha\in\mathcal{S}\downarrow M}$ such that for any $\mathcal{O}_X$-homomorphism $\mu\colon s' \to s_\alpha$ $(s' \in S)$, $f_{\alpha\circ\mu} = f_\alpha \circ \mathcal{D}(\mu)$. Hence an $\mathcal{A}'$-map $g\colon \mathcal{D}_{\mathrm{cts}}(M) \to A$ corresponds to a family of maps $g_\alpha\colon \mathcal{D}(s_\alpha) \to A$ indexed by $\mathcal{O}_X$-homomorphisms $\alpha\colon s \to M_\gamma$ with variable $s \in S$ and γ, such that for any $\phi = \phi_{\delta\gamma}$ $(\delta \ge \gamma)$,

$$g_{s\to M_\gamma \xrightarrow{\phi} M_\delta} = g_{s \to M_\gamma}$$

and such that for any $\mathcal{O}_X$-homomorphism $\mu\colon s' \to s_\alpha$ with $s' \in S$,

$$g_{\alpha\circ\mu} = g_\alpha \circ \mathcal{D}(\mu).$$

One checks that an $\mathcal{A}'$-map $\varinjlim \mathcal{D}_{\mathrm{cts}}(M_\gamma) \to A$ is specified by a family g_α subject to exactly the same conditions, whence the natural map is an isomorphism

$$\varinjlim \mathcal{D}_{\mathrm{cts}}(M_\gamma) \xrightarrow{\ \sim\ } \mathcal{D}_{\mathrm{cts}}(M) = \mathcal{D}_{\mathrm{cts}}(\varinjlim M_\gamma),$$

proving (ii).

Then (iii) results by application of $\varinjlim$ to (i), since by [**GD**, p. 320, (6.9.12)] every $M \in \mathcal{A}_X^{\mathsf{qc}}$ is a $\varinjlim$ of lfp $\mathcal{O}_X$-modules.

[45] For example, if X is noetherian then $\mathcal{D}_{\mathrm{cts}}(M) \cong \varinjlim \mathcal{D}(N)$ where N runs through all finite-type $\mathcal{O}_X$-submodules of M.

Again, [**GD**, p. 320, (6.9.12)] allows each $M \in \mathcal{A}_X^{\mathsf{qc}}$ to be represented in the form $M = \varinjlim\,(M_\lambda)$ with each M_λ lfp. From $(*)$ above we get a natural isomorphism

$$\mathcal{D}_{\text{cts}}(M) \cong \varinjlim \mathcal{D}(M_\lambda).$$

Since $\varinjlim$ preserves both exactness and f_*-acyclicity in $\mathcal{A}_X^{\mathsf{qc}}$ (see [**Kf**, p. 641, Thm. 8] for acyclicity), assertion (v) and the first part of (vi) follow.

As for (iv), for any exact $\mathcal{A}_X^{\mathsf{qc}}$-sequence $(\sharp)\colon 0 \to M' \to M \xrightarrow{\rho} M'' \to 0$ we must show exactness of the resulting sequence $\mathcal{D}_{\text{cts}}(M') \to \mathcal{D}_{\text{cts}}(M) \to \mathcal{D}_{\text{cts}}(M'') \to 0$. As in the preceding paragraph, write $M = \varinjlim\,(M_\lambda)$ with each M_λ lfp, and let $\phi_\lambda\colon M_\lambda \to M$ be the natural maps. Then $(\sharp)$ is the $\varinjlim$ of the exact $\mathcal{A}_X^{\mathsf{qc}}$-sequences

$$(\sharp)_\lambda\colon 0 \to \ker(\rho\phi_\lambda) \to M_\lambda \to \operatorname{im}(\rho\phi_\lambda) \to 0.$$

Since $\mathcal{D}_{\text{cts}}$ commutes with $\varinjlim$ and $\varinjlim$ preserves exactness, we can replace $(\sharp)$ by $(\sharp)_\lambda$, i.e., *we may assume that M is lfp.*

Now write $M' = \varinjlim\,(M'_\mu)$ with lfp M'_μ, so that as above, $\mathcal{D}_{\text{cts}}(M') \cong \varinjlim \mathcal{D}(M'_\mu)$. If M''_μ is the cokernel of the natural composition $M'_\mu \to M' \to M$, then, M''_μ is lfp; and since $\varinjlim$ preserves exactness, $M'' \cong \varinjlim M''_\mu$ and $\mathcal{D}_{\text{cts}}(M'') \cong \varinjlim \mathcal{D}(M''_\mu)$. Applying $\varinjlim$ to the exact sequences $\mathcal{D}(M'_\mu) \to \mathcal{D}(M) \to \mathcal{D}(M''_\mu) \to 0$, we conclude that $\mathcal{D}_{\text{cts}}$ is right-exact.

Finally, for the last part of (vi), note that if $\mathcal{D}$ is exact then since $R^1 f_*\mathcal{D}(M) = 0$ for all $M \in \mathcal{A}_X^{\mathsf{qc}}$ (because $\mathcal{D}(M)$ is f_*-acyclic), therefore $f_*\mathcal{D}$ is exact, and hence by (iv), $(f_*\mathcal{D})_{\text{cts}}$ is right-exact. But since, as above, f_* commutes with $\varinjlim$, there are functorial isomorphisms

$$(f_*\mathcal{D})_{\text{cts}}(M) \cong \varinjlim f_*\mathcal{D}(M_\lambda) \cong f_*\varinjlim \mathcal{D}(M_\lambda) \cong f_*\mathcal{D}_{\text{cts}}(M),$$

and so $f_*\mathcal{D}_{\text{cts}}$ is right-exact, as asserted. Q.E.D.

Exercises **(4.1.9).** (a) In (4.1.1), suppose only that X is noetherian as a topological space (resp. that both X and Y are concentrated). Then the conclusion is valid for *any* scheme-map $f\colon X \to Y$.

Hint. See the remarks just before the proof of (4.1), resp. [**GD**, p. 295, (6.1.10(i) and (iii))]).

(b) If $f\colon X \to Y$ is a concentrated scheme-map and Y is a finite union of open subschemes Y_i with $f^{-1}Y_i$ concentrated, then the conclusion of Theorem (4.1.1) holds.

Hint. Arguing as in [**AJL**′, p. 60, 6.1.1], by induction on the least possible number of Y_i, one reduces via [**GD**, p. 296, (6.1.12), a) ⇒ c)] to where X itself is concentrated; and then the remarks just before the proof of (4.1) apply.

(c) Let $f\colon X \hookrightarrow Y$ be an open-and-closed immersion of concentrated schemes (i.e., an isomorphism of X onto a union of connected components of Y). Then the sheaf-functors f_* and f^* are exact, so may also be regarded as derived functors.

Establish, for $E \in \mathbf{D}(Y)$, $F \in \mathbf{D}(X)$, natural *bifunctorial isomorphisms*

$$\operatorname{Hom}_{\mathbf{D}(X)}(f_*E, F) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(X)}(f^*f_*E, f^*f) \xleftarrow{\sim} \operatorname{Hom}_{\mathbf{D}(Y)}(E, f^*F),$$

whence, with $f^\times$ as in (b), for $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ there is a functorial isomorphism

$$\xi(F)\colon f^\times F \xrightarrow{\sim} f^*F,$$

corresponding under the preceding isomorphism (with $E = f^\times F$) to the natural map $f_*f^\times F \to F$, and with inverse adjoint to the natural map $f_*f^*F \to F = f_*f^*F \oplus g_*g^*F$ where g is the inclusion $(Y \setminus X) \hookrightarrow Y$.

Verify that for the independent square

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\tau"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

the associated map $\theta_\tau\colon f^*f_* \to 1_*1^* = \mathbf{1}$ is the *identity,* and hence the functorial base-change map from (3.10.2.1)(c)

$$\beta_\tau\colon 1^*f^\times = f^\times \to f^* = 1^\times f^*$$

is just the above isomorphism ξ.

Deduce (or prove directly) that ξ is a *pseudofunctorial* isomorphism. (Cf. (4.6.8), (4.8.1) and (4.8.7) below.)

(d) (Cf. [**Kn**, p. 43, Thm. 4].) Let $f\colon X \to Y$ be as in Theorem (4.1.1), with Y quasi-compact, and let d be an integer as in (3.9.2.3). Deduce from (4.1.1) a natural bifunctorial isomorphism

$$\mathrm{Hom}_X\big(A,\, H^{-d}f^\times(B)\big) \xrightarrow{\;\sim\;} \mathrm{Hom}_Y\big(R^df_*(A),\, B\big)$$

for all quasi-coherent $\mathcal{O}_X$-modules A and all $\mathcal{O}_Y$-modules B.

For the smallest such d, i.e., $\dim^+ \mathbf{R}f_*|_{\mathbf{D}_{\mathsf{qc}(X)}}$, the quasi-coherent $\mathcal{O}_X$-module $D_f := H^{-d}f^\times\mathcal{O}_Y$ is the lowest-degree nonvanishing homology of $f^\times\mathcal{O}_Y$. When f is proper, D_f is often called a *relative dualizing sheaf for* f. (But certain features of the duality theory for sheaves do not just come out of the abstract theory—see [**Kn**], [**S**].)

(e) Show that the inclusion $\mathcal{A}_X^{\mathsf{qc}} \hookrightarrow \mathcal{A}_X$ has a right inverse. Deduce that every $M \in \mathcal{A}_X^{\mathsf{qc}}$ admits a monomorphism into an $\mathcal{A}_X^{\mathsf{qc}}$-injective $\mathcal{O}_X$-module.

(f) Show that the functor $(-)_{\mathrm{cts}}\colon \mathbf{F} \to \mathbf{F}$ constructed in the proof of (4.1.5) is right-adjoint to the inclusion into $\mathbf{F}$ of the full subcategory of functors that commute with *filtered* colimits (see [**M**, p. 212]). Also, the restriction of $(-)_{\mathrm{cts}}$ to the full subcategory of right-exact functors is right adjoint to the inclusion of the full subcategory of cocontinuous functors.

4.2. Sheafified Duality—preliminary form

Theorem (4.2). *Let $f\colon X \to Y$, $f^\times$ and τ be as in Theorem* (4.1.1). *Then with* $\mathrm{Hom} := \mathrm{Hom}_{\mathbf{D}(Y)}$, *for any $E \in \mathbf{D}_{\mathsf{qc}}(Y)$, $F \in \mathbf{D}_{\mathsf{qc}}(X)$ and $G \in \mathbf{D}(Y)$, the composite map*

$$\begin{aligned} &\mathrm{Hom}\big(E,\, \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^\times G)\big) \\ &\qquad\xrightarrow{(3.2.1.0)} \mathrm{Hom}\big(E,\, \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*\mathbf{R}f_*F, f^\times G)\big) \\ &\qquad\xrightarrow{(3.2.3.2)} \mathrm{Hom}\big(E,\, \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^\times G)\big) \\ &\qquad\xrightarrow{\text{via } \tau} \mathrm{Hom}\big(E,\, \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G)\big) \end{aligned}$$

is an isomorphism.

Proof.[46] Using $(2.6.2)^*$ and (3.2.3), and checking all the requisite commutativities, one shows for fixed $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ that the composite *duality map*

$$
\begin{aligned}
\mathsf{R}f_*\mathsf{R}\mathcal{H}om^\bullet_X(F, f^\times G) &\xrightarrow{(3.2.1.0)} \mathsf{R}f_*\mathsf{R}\mathcal{H}om^\bullet_X(\mathsf{L}f^*\mathsf{R}f_*F, f^\times G)\\
&\xrightarrow{(3.2.3.2)} \mathsf{R}\mathcal{H}om^\bullet_Y(\mathsf{R}f_*F, \mathsf{R}f_*f^\times G)\\
&\xrightarrow{\text{via } \tau} \mathsf{R}\mathcal{H}om^\bullet_Y(\mathsf{R}f_*F, G)
\end{aligned}
\tag{4.2.1}
$$

(functorial in G) is right-conjugate (see (3.3.5)) to the functorial (in E) projection map $p_2\colon E \underset{=}{\otimes} \mathsf{R}f_*F \to \mathsf{R}f_*(\mathsf{L}f^*E \underset{=}{\otimes} F)$, which, by (3.9.4), is an isomorphism when $E \in \mathbf{D}_{\mathsf{qc}}(Y)$. Now apply Exercise (3.3.7)(b) (with $Y = E$ and $X = G$). Q.E.D.

For *proper* maps $f\colon X \to Y$ one writes $f^!$ instead of $f^\times$. When Y is noetherian and f is proper, it holds that $\mathsf{R}f_*\overline{\mathbf{D}}^-_{\mathsf{c}}(X) \subset \overline{\mathbf{D}}^-_{\mathsf{c}}(Y)$ (where the subscript c indicates "coherent homology")—see [**H**, p. 89, Prop. 2.2] in which, owing to (3.9.2.3) above, it is not necessary to assume that X has finite Krull dimension. So if $F \in \overline{\mathbf{D}}^-_{\mathsf{c}}(X)$ and $G \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$, then $\mathsf{R}f_*F \in \overline{\mathbf{D}}^-_{\mathsf{c}}(Y)$ and $f^!G \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(X)$, whence both $\mathsf{R}f_*\mathsf{R}\mathcal{H}om^\bullet_X(F, f^!G)$ and $\mathsf{R}\mathcal{H}om^\bullet_Y(\mathsf{R}f_*F, G)$ are in $\overline{\mathbf{D}}^+_{\mathsf{qc}}(X)$, see [**H**, p. 92, 3.3] or [**AJL**′, p. 35, 3.2.4]. One concludes that:

Corollary (4.2.2). *If $f\colon X \to Y$ is a proper map of noetherian schemes then for all $F \in \overline{\mathbf{D}}^-_{\mathsf{c}}(X)$ and $G \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$, the duality map* (4.2.1) *is an isomorphism*

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om^\bullet_X(F, f^!G) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om^\bullet_Y(\mathsf{R}f_*F, G).$$

One of our goals is to prove this Corollary under considerably weaker hypotheses—see (4.4.2) below. For this purpose we need some facts about *pseudo-coherence,* reviewed in the following section.

Exercises **(4.2.3).** Let X be a concentrated scheme. Ex. (4.1.9)(e) says that the inclusion $\mathcal{A}^{\mathsf{qc}}_X \hookrightarrow \mathcal{A}_X$ has a right adjoint Q_X, the "quasi-coherator." (Cf. [**I**, p. 186, §3].)

(a) Show that $\mathsf{R}Q_X$ is right-adjoint to the natural functor $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}^{\mathsf{qc}}_X) \to \mathbf{D}(\mathcal{A}_X)$; in other words, $\mathsf{R}Q_X = (1_X)^\times$. (Cf. [**AJL**′, p. 49, 5.2.2], where "let" in the second line should be "let $\boldsymbol{j}$ be the".)

In the rest of these exercises, assume all schemes to be quasi-compact and *separated*, so that by (3.9.6), $\boldsymbol{j}$ induces an equivalence $\boldsymbol{j}_{\mathsf{qc}}\colon \mathbf{D}(\mathcal{A}^{\mathsf{qc}}) \xrightarrow{\approx} \mathbf{D}_{\mathsf{qc}}$. Also, $\mathbf{Q}$ denotes the functor $\boldsymbol{j}_{\mathsf{qc}} \circ \mathsf{R}Q$, right-adjoint (from (a)) to the inclusion $\mathbf{D}_{\mathsf{qc}} \hookrightarrow \mathbf{D}$; and $[-,-]$ denotes the functor $\mathbf{Q} \circ \mathsf{R}\mathcal{H}om^\bullet(-,-)\colon \mathbf{D} \times \mathbf{D} \to \mathbf{D}_{\mathsf{qc}}$.

(b) Redo 3.6.10 with $\mathbf{S}$ the category of quasi-compact separated schemes and with $\mathbf{X}^* = \mathbf{X}_* := \mathbf{D}_{\mathsf{qc}}(X)$. (Recall (2.5.8.1), (3.9.1), (3.9.2); and use the preceding $[-,-]$.)

(c) For any scheme-map $f\colon X \to Y$ there are natural functorial isomorphisms

$$\mathsf{R}\Gamma(X, \mathbf{Q}_X-) \xrightarrow{\ \sim\ } \mathsf{R}\Gamma(X,-), \qquad \mathsf{R}f_*\mathbf{Q}_X \xrightarrow{\ \sim\ } \mathbf{Q}_Y\mathsf{R}f_*, \qquad f^\times\mathbf{Q}_Y \xrightarrow{\ \sim\ } f^\times.$$

[46] Cf. [**V**, p. 404, Proof of Prop. 3].

(d) Deduce from Theorem (4.2) a functorial isomorphism

$$\mathbf{R}f_*[F, f^\times G]_X \xrightarrow{\sim} [\mathbf{R}f_*F, G]_Y$$

to which application of the functor $\mathrm{H}^0\mathbf{R}\Gamma(Y,-)$ produces the adjunction isomorphism $\mathrm{Hom}_{\mathbf{D}_{\mathsf{qc}}(X)}(F, f^\times G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*F, G)$.

In particular, if f is an open immersion then there is a functorial isomorphism

$$f^\times G \xrightarrow{\sim} f^*[\mathbf{R}f_*\mathcal{O}_X, G]_Y \qquad (G \in \mathbf{D}(Y)).$$

(e) Under the conditions of Theorem (4.1.1), show that the map right-conjugate to $p_1\colon \mathbf{R}f_*E \underline{\underline{\otimes}} F \to \mathbf{R}f_*(E \underline{\underline{\otimes}} \mathbf{L}f^*F)$ (where $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ is fixed, and both functors of $E \in \mathbf{D}_{\mathsf{qc}}(X)$ take values in $\mathbf{D}(Y)$) is a functorial isomorphism

$$[\mathbf{L}f^*F, f^\times G]_X \xrightarrow{\sim} f^\times[F, G]_Y \qquad (G \in \mathbf{D}(Y)),$$

adjoint to the natural composition $\mathbf{R}f_*[\mathbf{L}f^*F, f^\times G]_X \xrightarrow{\text{(d)}} [\mathbf{R}f_*\mathbf{L}f^*F, G]_Y \to [F, G]_Y$.

(f) Establish a natural commutative diagram, for $F \in \mathbf{D}_{\mathsf{qc}}(Y)$, $G \in \mathbf{D}(Y)$:

$$\begin{tikzcd}
\mathbf{R}f_*[\mathbf{L}f^*F, f^\times G]_X \arrow[r, "\sim", "\text{(d)}"'] \arrow[d] & {[\mathbf{R}f_*\mathbf{L}f^*F, G]_Y} \arrow[d] \\
\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*F, f^\times G) \arrow[r] \arrow[d, "\text{(3.2.3.2)}"', "\simeq"] & \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*\mathbf{L}f^*F, G) \arrow[d] \\
\mathbf{R}\mathcal{H}om^\bullet_Y(F, \mathbf{R}f_*f^\times G) \arrow[r, "\text{via } \tau"'] & \mathbf{R}\mathcal{H}om^\bullet_Y(F, G),
\end{tikzcd}$$

and show that the isomorphism in (e) is adjoint to the map obtained by going from the upper left to the lower right corner of this diagram.

(g) Show, via the lower square in (f), or via (3.5.6)(e), or otherwise, that the following natural diagram commutes:

$$\begin{tikzcd}
\mathbf{R}f_*f^\times G \arrow[r, "\text{(4.2.1)}"] \arrow[d, "\tau"'] & \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*\mathcal{O}_X, G) \arrow[d] \\
G \arrow[r, "\sim"] & \mathbf{R}\mathcal{H}om^\bullet_Y(\mathcal{O}_Y, G)
\end{tikzcd}$$

In the next three exercises, for a scheme-map h we use the abbreviations $h_* := \mathbf{R}h_*$ and $h^* := \mathbf{L}h^*$.

(h) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of concentrated schemes. Referring to (e), show that for any $E, F \in \mathbf{D}_{\mathsf{qc}}(Z)$, the following diagram of natural isomorphisms commutes.

$$\begin{tikzcd}
{[(gf)^*E, (gf)^\times F]_X} \arrow[r] \arrow[d] & {[f^*g^*E, g^\times f^\times F]_X} \arrow[r] & f^\times[g^*E, g^\times F]_Y \arrow[d] \\
(gf)^\times[E, F]_Z \arrow[rr] & & f^\times g^\times[E, F]_Z
\end{tikzcd}$$

(i) Let $\beta_\sigma\colon v_*g^\times \to f^\times u_*$ be as in (3.10.2.1)(c). Taking into account (3.9.1), show that for any $E, F \in \mathbf{D}_{\mathsf{qc}}(Z)$ the following diagram commutes.

$$\begin{tikzcd}
v^*f^\times[E, F]_Y \arrow[d, "\beta_\sigma"'] & v^*[f^*E, f^\times F]_X \arrow[l, "(e)"'] \arrow[r, "\text{(3.2.4)}"] & {[v^*f^*E, v^*f^\times F]_{X'}} \arrow[d, "\text{via (3.6.4)}^*"', "\text{and } \beta_\sigma"] \\
g^\times u_*[E, F]_Y \arrow[r, "\text{(3.2.4)}"'] & g^\times[u^*E, u^*F]_{Y'} & {[g^*u^*E, g^\times u^*F]_{X'}} \arrow[l, "(e)"]
\end{tikzcd}$$

(j) Let $\phi_\sigma\colon v_*g^\times \to f^\times u_*$ be as in (3.10.4). Taking into account (3.9.2.1), show that for any $E, F \in \mathbf{D}_{\mathsf{qc}}(Z)$ the following diagram, with θ' as near the beginning of §3.10, commutes.

$$\begin{CD}
v_*g^\times[E,F]_{Y'} @<{(e)}<< v_*[g^*E, g^\times F]_{X'} @>{(3.5.4.1)}>> [v_*g^*E, v_*g^\times F]_X\\
@V{\phi_\sigma}VV @. @V{\text{via } \theta'_\sigma}V{\text{and } \phi_\sigma}V\\
f^\times u_*[E,F]_{Y'} @>>{(3.5.4.1)}> f^\times[u_*E, u_*F]_Y @<<{(e)}< [f^*u_*E, f^\times u_*F]_X
\end{CD}$$

4.3. Pseudo-coherence and quasi-properness

(4.3.1). Let us recall briefly some relevant definitions and results concerning *pseudo-coherence.* Details can be found in [**I**], as indicated, or, perhaps more accessibly, in [**TT**, pp. 283*ff,* §2].[47]

Let X be a scheme. A complex $F \in \overline{\mathbf{D}}^{\mathsf{b}}(X)$ is pseudo-coherent if each $x \in X$ has a neighborhood in which F is $\mathbf{D}$-isomorphic to a bounded-above complex of finite-rank free $\mathcal{O}_X$-modules [**I**, p. 175, 2.2.10]. If X is divisorial, and either separated or noetherian, such an F is (globally) $\mathbf{D}(X)$-isomorphic to a bounded-above complex of finite-rank locally free $\mathcal{O}_X$-modules [*ibid.,* p. 174, Cor. 2.2.9]. If $\mathcal{O}_X$ is coherent, pseudo-coherence of F means simply that F has coherent homology [*ibid.,* p. 115, Cor. 3.5 b)]. If X is noetherian, pseudo-coherence means that F is $\mathbf{D}(X)$-isomorphic to a bounded complex of coherent $\mathcal{O}_X$-modules [*ibid.,* p. 168, Cor. 2.2.2.1].

A scheme-map $f\colon X \to Y$ is pseudo-coherent if it factors locally as $f = p \circ i$ where $i\colon U \to Z$ (U open in X) is a closed immersion such that $i_*\mathcal{O}_U$ is pseudo-coherent on Z, and $p\colon Z \to Y$ is smooth [*ibid.,* p. 228, Déf. 1.2]. Pseudo-coherent maps are locally finitely-presentable (smooth maps being so by definition).

For example, any smooth map is pseudo-coherent, any regular immersion (= closed immersion corresponding to a quasi-coherent ideal generated locally by a regular sequence) is pseudo-coherent, and any composition of pseudo-coherent maps is still pseudo-coherent [*ibid.,* p. 236, Cor. 1.14].[48]

If $f\colon X \to Y$ is a proper map, and $\mathcal{L}$ is an f-ample invertible sheaf, then f is pseudo-coherent if and only if the $\mathcal{O}_Y$-complex $\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$ is pseudo-coherent for all $n \gg 0$. (The proof is indicated below, in (4.3.8)). In particular, a finite map $f\colon X \to Y$ is pseudo-coherent if and only if $f_*\mathcal{O}_X$ is a pseudo-coherent $\mathcal{O}_Y$-module.

For noetherian Y, any finite-type map $f\colon X \to Y$ is pseudo-coherent. Pseudo-coherence persists under tor-independent base change [**I**, p. 233, Cor. 1.10]. Hence, by descent to the noetherian case [**EGA**, IV, (11.2.7) and its proof], *any flat finitely-presentable scheme-map is pseudo-coherent.*

[47] Though [**I**] is written in the language of ringed topoi, the reader who, like me, is uncomfortable with that level of generality, ought with sufficient patience to be able to translate whatever's needed into the language of ringed spaces. A good starting point is 2.2.1 on p. 167 of *loc. cit.*, with examples b) on p. 88 and 2.15 on p. 108 kept in mind.

[48] In the triangle at the top of [*ibid.,* p. 234], the map $X \to Z$ should be labeled h.

Kiehl's Finiteness Theorem [**Kl**, p. 315, Thm. 2.9′] (due to Illusie for projective maps [**I**, p. 236, Thm. 2.2]) generalizes preservation of coherence by higher direct images under proper maps of noetherian schemes:

If $f\colon X \to Y$ is a proper pseudo-coherent map of quasi-compact schemes, and if $F \in \overline{\mathbf{D}}^{\mathsf{b}}(X)$ is pseudo-coherent, then so is $\mathbf{R}f_*F \in \overline{\mathbf{D}}^{\mathsf{b}}(Y)$.[49]

(4.3.2). For simplicity, we introduced pseudo-coherence only for complexes in $\overline{\mathbf{D}}^{\mathsf{b}}$, but that won't be enough. So let us recall [**I**, p. 98, Déf. 2.3]:

Let X be a ringed space, and let $n \in \mathbb{Z}$. A complex $F \in \mathbf{D}(X)$ is said to be *n-pseudo-coherent* if locally it is $\mathbf{D}$-isomorphic to a bounded-above complex E such that E^i is free of finite rank for all $i \geq n$. It is equivalent to say that each $x \in X$ has a neighborhood U over which there exists such an $E = E_U$ together with a quasi-isomorphism $E_U \to F|U$.

If $\mathcal{O}_X$ is coherent, then $F \in \overline{\mathbf{D}}^{-}(X)$ is n-pseudo-coherent $\Leftrightarrow H^i(F)$ is coherent for all $i > n$ and $H^n(F)$ is of finite type [**I**, p. 115, Cor. 3.5 b)].

F is called pseudo-coherent if F is n-pseudo-coherent for all $n \in \mathbb{Z}$. For $F \in \overline{\mathbf{D}}^{\mathsf{b}}(X)$, this defining condition is equivalent to the one given in (4.3.1). Moreover, when X is a *quasi-compact separated scheme,* then in view of (3.9.6)(a), [**I**, p. 173, 2.2.8] shows the same for *any* $F \in \mathbf{D}(X)$.

(4.3.3). Now the above Finiteness Theorem can be put more precisely (as can be seen from the statement of [**Kl**, p. 308, Satz 2.8] and the proof of [*ibid.,* p. 310, Thm. 2.9]):

For any proper pseudo-coherent map $f\colon X \to Y$ of quasi-compact schemes, there is an integer k such that for any $n \in \mathbb{Z}$ and any n-pseudo-coherent complex $F \in \overline{\mathbf{D}}^{\mathsf{b}}(X)$, the complex $\mathbf{R}f_*F$ is $(n+k)$-pseudo-coherent.

Definition (4.3.3.1). A map $f\colon X \to Y$ is *quasi-proper* if $\mathbf{R}f_*$ takes pseudo-coherent $\mathcal{O}_X$-complexes to pseudo-coherent $\mathcal{O}_Y$-complexes.

Corollary (4.3.3.2). *Proper pseudo-coherent maps are quasi-proper. In particular, flat finitely-presentable proper maps are quasi-proper.*

Proof. The question is easily seen to be local on Y, so we may assume that both X and Y are quasi-compact. Let F be a pseudo-coherent $\mathcal{O}_X$-complex. It follows from [**I**, p. 96, Prop. 2.2, b)(ii′)] that for each n, the truncation $\tau_{\geq n}F \in \overline{\mathbf{D}}^{\mathsf{b}}(X)$ (see §1.10) is n-pseudo-coherent, and so there exists an integer k depending only on f such that $\mathbf{R}f_*\tau_{\geq n}F$ is $(n+k)$-pseudo-coherent.

Let $C \in (\mathbf{D}_{\mathsf{qc}})_{\leq \mathbf{n-1}}$ be the summit of a triangle whose base is the natural map $F \to \tau_{\geq n}F$. With d be as in (3.9.2), application of $\mathbf{R}f_*$ to this triangle shows that $\mathbf{R}f_*(C)$ is exact in all degrees $\geq n+d-1$, so the natural map is an isomorphism $\tau_{\geq n+d}\mathbf{R}f_*F \xrightarrow{\sim} \tau_{\geq n+d}\mathbf{R}f_*\tau_{\geq n}F$

[49] The theorem actually involves a notion of pseudo-coherence of a complex *relative to a map f*; but when f itself is pseudo-coherent, relative pseudo-coherence coincides with pseudo-coherence [**I**, p. 236, Cor. 1.12].

(see (1.4.5), (1.2.2)). Hence by [**I**, p. 96, Prop. 2.2, b)(ii′)], $\tau_{\geq n+d}\mathbf{R}f_*F$ is $(n+d+k)$-pseudo-coherent for all n, whence $\mathbf{R}f_*F$ is pseudo-coherent. Q.E.D.

Remark. A *projective* map is quasi-proper iff it is pseudo-coherent, see the Remark following (4.7.3.3) below. See also Example (4.3.8).

As noted above, finite-type maps of noetherian schemes are pseudo-coherent. Using Exercise (4.3.9) below, one concludes that:

Corollary (4.3.3.3). *If Y is noetherian then a map $f\colon X \to Y$ is proper iff it is finite-type, separated and quasi-proper.*

The next two Lemmas are elementary.

Lemma (4.3.4). *For any scheme-map $f\colon X \to Y$, if $G \in \mathbf{D}(Y)$ is n-pseudo-coherent then so is $\mathbf{L}f^*G$.*

This is proved by reduction to the simple case where G is a bounded-above complex of finite-rank free $\mathcal{O}_Y$-modules, vanishing in all degrees $< n$, cf. [**I**, p. 106, proof of 2.13 and p. 130, 4.19.2].

Lemma (4.3.5). *If $F \in \mathbf{D}(X)$ is n-pseudo-coherent and if the complex $G \in \mathbf{D}_{\mathsf{qc}}(X)$ is such that $H^m(G) = 0$ for all $m < r$ then $H^j\mathbf{R}\mathcal{H}om^\bullet_X(F, G)$ is quasi-coherent for all $j < r-n$.*

Thus if F is pseudo-coherent then $\mathbf{R}\mathcal{H}om^\bullet_X(F, G) \in \mathbf{D}_{\mathsf{qc}}(X)$.

Proof. Replacing G by τ^+G (1.8.1), we may assume that $G^m = 0$ for $m < r$. Also, the question being local, we may assume that F is bounded above and that F^i is free of finite rank for $i \geq n$. If $F' \subset F$ is the bounded free complex which vanishes in degree $< n$ and agrees with F in degree $\geq n$, then by (1.4.4) and (1.5.3) we have a triangle (with $\mathcal{H}_X = \mathbf{R}\mathcal{H}om^\bullet_X$):

$$\mathcal{H}_X(F/F', G) \to \mathcal{H}_X(F, G) \to \mathcal{H}_X(F', G) \to \mathcal{H}_X(F/F', G)[1]\,.$$

The complex $\mathcal{H}_X(F/F', G)$ vanishes in degree $\leq r-n$; and so from the exact homology sequence associated (as in (1.4.5)) to the triangle, we get isomorphisms

$$H^j\mathcal{H}_X(F, G) \xrightarrow{\;\sim\;} H^j\mathcal{H}_X(F', G) \qquad (j < r-n).$$

A simple induction on the number of degrees in which F' doesn't vanish (using [**H**, p. 70, (1)] to pass from n to $n+1$) yields $\mathcal{H}_X(F', G) \in \mathbf{D}_{\mathsf{qc}}(X)$, whence the assertion. Q.E.D.

There results a generalization of (4.2.2), with a similar proof (given (4.3.3.2) and (4.3.5)):

Corollary (4.3.6). *If* $f\colon X \to Y$ *is a quasi-proper concentrated scheme-map, with* X *concentrated, then for all pseudo-coherent* $F \in \mathbf{D}(X)$ *and all* $G \in \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Y)$, *the duality map* (4.2.1) *is an isomorphism*

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^{\bullet}_X(F, f^{\times}G) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^{\bullet}_Y(\mathbf{R}f_*F, G).$$

Here is a fact needed in the proof of Theorem (4.4.1), and elsewhere.

Lemma (4.3.7). *Let* $f\colon X \to Y$ *be a finitely-presentable scheme-map, and let* $\varphi\colon A_1 \to A_2$ *be a map in* $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}(X)$. *Suppose that for every pseudo-coherent* $F \in \mathbf{D}(X)$, *the resulting map*

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^{\bullet}_X(F, A_1) \to \mathbf{R}f_*\mathbf{R}\mathcal{H}om^{\bullet}_X(F, A_2) \tag{4.3.7.1}$$

is an isomorphism. Then φ *is an isomorphism.*

Proof. There are functorial isomorphisms (see (3.2.3.3), (2.5.10)(b)):

$$\mathbf{R}\Gamma_Y\mathbf{R}f_*\mathbf{R}\mathcal{H}om^{\bullet}_X \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_X\mathbf{R}\mathcal{H}om^{\bullet}_X \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_X.$$

Application of the functor $H^0\mathbf{R}\Gamma_Y$ to (4.3.7.1) gives then, via (2.4.2), an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X)}(F, A_1) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(X)}(F, A_2)\,. \tag{4.3.7.2}$$

Let $C \in \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(X)$ be the summit of a triangle with base φ. The exact homology sequence $(1.4.5)^H$ of this triangle shows, in view of (1.2.2), that φ is an isomorphism iff $H^n(C) = 0$ for all $n \in \mathbb{Z}$.

Let us suppose that $H^n(C) = 0$ for all $n < m$ while $H^m(C) \neq 0$, and derive a contradiction. The whole question being local on Y, we may assume that Y is affine. Since $H^m(C)$ is quasi-coherent, there exists then a finitely-presentable $\mathcal{O}_X$-module E together with a *non-zero* map $E \to H^m(C)$ [**GD**, p. 320, (6.9.12)].[50] By [**EGA**, IV, (8.9.1)], there exists a noetherian ring R, a map $Y \to \mathrm{Spec}(R)$, a finite-type map $X_0 \to \mathrm{Spec}(R)$, and a coherent $\mathcal{O}_{X_0}$-module E_0, such that, up to isomorphism, $X = X_0 \otimes_R Y$ and, with $w\colon X \to X_0$ the resulting map, $E = w^*E_0 = H^0(\mathbf{L}w^*E_0)$. It will be convenient to set $F := \mathbf{L}w^*E_0[-m]$, so that $\tau_{\geq m}F \cong E[-m]$ (see §1.10). Since X_0 is noetherian, therefore E_0 is pseudo-coherent, and hence, by (4.3.4), so is F.

Now by (1.4.2.1) there is an exact sequence (with $\mathrm{Hom} := \mathrm{Hom}_{\mathbf{D}(X)}$):

$$\begin{tikzcd}[column sep=small]
\mathrm{Hom}(F, A_1) \arrow[r, "\varphi"] & \mathrm{Hom}(F, A_2) \arrow[r] & \mathrm{Hom}(F, C) \arrow[r] & \mathrm{Hom}(F, A_1[1]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(F, A_2[1]) \arrow[d, equal] \\
 & & & \mathrm{Hom}(F[-1], A_1) \arrow[r, "\varphi"'] & \mathrm{Hom}(F[-1], A_2)
\end{tikzcd}$$

[50] Recall that finitely-presentable maps are quasi-compact and quasi-separated, by definition [**GD**, p. 305, (6.3.7)], so that X is quasi-compact and quasi-separated.

where, F and $F[-1]$ being pseudo-coherent, the maps labeled φ are isomorphisms, see (4.3.7.2). Thus,

$$\begin{aligned}
0 &= \operatorname{Hom}\big(F,\, C\big) \\
&\cong \operatorname{Hom}\big(\tau_{\ge m}F,\, C\big) && \text{see (1.10.1.2)} \\
&\cong \operatorname{Hom}\big(E[-m],\, C\big) \\
&\cong \operatorname{Hom}\big(E[-m],\, \tau_{\le m}C\big) && \text{see (1.10.1.1)} \\
&\cong \operatorname{Hom}\big(E[-m],\, (H^m(C))[-m]\big) && \text{see (1.2.3)} \\
&\ne 0,
\end{aligned}$$

a contradiction. Q.E.D.

Example (4.3.8). *Let $f\colon X \to Y$ be a proper map of schemes, and let $\mathcal{L}$ be an f-ample invertible sheaf* [**EGA**, II, p. 89, Déf. (4.6.1)]. *Then f is pseudo-coherent if and only if the $\mathcal{O}_Y$-complex $\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$ is pseudo-coherent for all $n \gg 0$.*

Proof. If f is pseudo-coherent then $\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$ is pseudo-coherent, by the Finiteness Theorem (4.3.3) (in fact—since f is projective locally on Y [**EGA**, II, p. 104, Thm. (5.5.3)]—by [**I**, p. 236, Thm. 2.2 and Cor. 1.12]).

We first illustrate the converse by treating the special case where f is finite and $f_*\mathcal{O}_X$ is a pseudo-coherent $\mathcal{O}_Y$-module. To check that f is pseudo-coherent, we may assume that Y—and hence X—is affine, so that for some $r > 0$, f factors as $f = pi$ with $p\colon \mathbb{A}^r_Y \to Y$ the (smooth) projection and $i\colon X \hookrightarrow \mathbb{A}^r_Y$ a closed immersion; and we need to show that $i_*\mathcal{O}_X$ is pseudo-coherent.

In algebraic terms, we have a finite ring-homomorphism $A \to B = A[t_1, \dots, t_r]$, such that the A-module B is resolvable by a complex $E_\bullet$ of finite-type free A-modules [**I**, p. 160, Prop. 1.1]. Let $T := (T_1, \dots, T_r)$ be a sequence of indeterminates, and let $\varphi\colon B[T] = B[T_1, \dots, T_r] \to B$ be the unique B-homomorphism such that $\varphi(T_k) = t_k$ $(1 \le k \le r)$. Then B is resolved as a $B[T]$-module by the Koszul complex $K_\bullet$ on $(T_1 - t_1, \dots, T_r - t_r)$. Since the $A[T]$-module $B[T]$ is resolved by $E_\bullet \otimes_A A[T]$, therefore the free $B[T]$-modules K_j can be resolved by finite-type free $A[T]$-modules, whence so can B, giving the desired pseudo-coherence of $i_*\mathcal{O}_X$.

Now let us treat (sketchily) the general case. Assuming, as we may, that Y is affine, we have for some $r > 0$, a factorization $f = pi$ where $p\colon \mathbb{P}^r_Y \to Y$ is the (smooth) projection and $i\colon X \hookrightarrow \mathbb{P}^r_Y$ is a closed immersion [**EGA**, II, p. 104, (5.5.4)(ii)]. With $\gamma\colon X \to X \times_Y \mathbb{P}^r_Y = \mathbb{P}^r_X$ the graph of i, there is a natural diagram

$$\begin{tikzcd}
X \arrow[r, "\gamma"] & \mathbb{P}^r_X \arrow[r, "F"] \arrow[d, "q"'] & \mathbb{P}^r_Y \arrow[d, "p"] \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and it needs to be shown that $i_*\mathcal{O}_X = \mathbf{R}F_*(\gamma_*\mathcal{O}_X)$ is pseudo-coherent. Note that since γ is a regular immersion [**Bt**, p. 429, Prop. 1.10], therefore $\gamma_*\mathcal{O}_X$ is pseudo-coherent. So it's enough to show that F is quasi-proper.

By [**EGA**, II, p. 91, (4.6.13)(iii)], $L := q^*\mathcal{L}$ is F-ample; and for $n \gg 0$, say $n \ge m$,

$$\mathbf{R}F_*(L^{\otimes -n}) = \mathbf{R}F_*\big(q^*(\mathcal{L}^{\otimes -n})\big) \underset{(3.9.5)}{\cong} p^*\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$$

is pseudo-coherent (4.3.4).

Imitating the proof of [**I**, p. 238, Thm. 2.2.2], we can then reduce the problem to showing that $\mathbf{R}F_*(E')$ is pseudo-coherent for any bounded $\mathcal{O}_X$-complex E' whose component in each degree is a finite direct sum of sheaves of the form $L^{\otimes -n}$; and this is easily done by induction on the number of nonzero components of E'. Q.E.D.

Exercises **(4.3.9).** (a) (Curve selection.) Let $\overline{Z}$ be a noetherian scheme, $Z \subset \overline{Z}$ a dense open subset, and $W := \overline{Z} \setminus Z$. Show that for each closed point $w \in W$ there is an integral one-dimensional subscheme $C \subset \overline{Z}$ such that w is an isolated point of $C \cap W$.

Hint. Use the *local nullstellensatz*: in any noetherian local ring A with $\dim A \geq 1$, the intersection of all those prime ideals p such that $\dim A/p = 1$ is the nilradical of A. (For this, note that the maximal ideal is contained in the union of all the height one primes, so that when $\dim A > 1$ there must be infinitely many height one primes; and deduce that if $q \subset A$ is a prime ideal with $\dim A/q > 1$ and $a \notin q$ then there exists a prime ideal $q' \neq m$ such that $q' \supsetneq q$ and $a \notin q'$.)

(b) Prove that if $f\colon X \to Y$ is a finite-type separated map of noetherian schemes such that $f_*(\mathcal{O}_X/\mathcal{I})$ is coherent for every coherent $\mathcal{O}_X$-ideal $\mathcal{I}$, then f is proper. In particular, if f is quasi-proper then f is proper.

Outline. If not, let $Z \subset X$ be a closed subscheme of Z minimal among those for which the restriction of f is not proper. Then Z is integral [**EGA**, II, p. 101, 5.4.5]. Let $\bar f\colon \overline{Z} \to Y$ be a *compactification* of $f|_Z$, see [**C′**], [**Lt**], [**Vj**], that is, $f = \bar f v$ with $\bar f$ proper and $v\colon Z \hookrightarrow \overline{Z}$ an open immersion. If $\dim Z > 1$ then by (a) there is a curve on Z for which the restriction of f is not proper, contradiction. So the problem is reduced to where X is integral, of dimension 1. Then if $\dim Y = 0$, and f is not proper, we may assume that $Y = \operatorname{Spec}(k)$, k a field, whence X is affine, and $f_*\mathcal{O}_X$ is not coherent. If $\dim(Y) = 1$ and $\bar f\colon \overline{X} \to Y$ is a compactification of f, then the map $\bar f$ is finite; and if $u\colon X \hookrightarrow \overline{X}$ is the inclusion, $u_*\mathcal{O}_X$ is coherent, whence, by [**EGA**, IV, p. 117, (5.10.10)(ii)], $X = \overline{X}$.

4.4. Sheafified Duality, Base Change

Unless otherwise indicated, *all schemes—and hence all scheme-maps—are assumed henceforth to be concentrated. All proper and quasi-proper maps are assumed to be finitely presentable.*

As in §4.3, a scheme-map $f\colon X \to Y$ is called quasi-proper if $\mathbf{R}f_*$ takes pseudo-coherent $\mathcal{O}_X$-complexes to pseudo-coherent $\mathcal{O}_Y$-complexes. For example, when Y is noetherian and f is of finite type and separated then f is quasi-proper iff it is proper, see (4.3.3.3). We will need the nontrivial fact that *quasi-properness of maps is preserved under tor-independent base change* [**LN**, Prop. 4.4].

The following abbreviations will be used, for a scheme-map h or a scheme Z:

$$h_* := \mathbf{R}h_*\,, \qquad h^* := \mathbf{L}h^*,$$

$$\mathcal{H}_Z := \mathbf{R}\mathcal{H}om^{\bullet}_Z\,, \qquad \mathbf{H}_Z := \mathbf{R}\mathrm{Hom}^{\bullet}_Z\,,$$

$$\otimes_Z := \underline{\underline{\otimes}}_Z\,, \qquad \boldsymbol{\Gamma}_Z(-) := \mathbf{R}\Gamma(Z,-).$$

Recall the characterizations of *independent fiber square* (3.10.3), of *finite tor-dimension map* (2.7.6), and of the "dualizing pair" $(f^\times, \tau)$ in (4.1.1). We write $f^!$ for $f^\times$ when f is quasi-proper.

Recall also the natural map (3.5.4.1) = (3.5.4.4) (see (3.5.2)(d)) associated to any ringed-space map $f\colon X \to Y$,

$$\nu\colon f_*\mathcal{H}_X(F,H) \to \mathcal{H}_Y(f_*F, f_*H) \qquad \big(F, H \in \mathbf{D}(X)\big). \tag{4.4.0}$$

The composition (3.2.3.2) $\circ$ (3.2.1.0) in (4.2.1) is an instance of this map. (See the line immediately following (3.5.4.2).)

Theorem (4.4.1). *Suppose one has an independent fiber square*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

with f (hence g) quasi-proper and u of finite tor-dimension.

Then for any $F' \in \mathbf{D}_{\mathsf{qc}}(X')$ and $G \in \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Y)$, the composition

$$\begin{aligned} g_*\mathcal{H}_{X'}(F', v^*f^!G) &\xrightarrow{\ \nu\ } \mathcal{H}_{Y'}(g_*F', g_*v^*f^!G) \\ &\underset{(3.10.3)}{\xrightarrow{\ \sim\ }} \mathcal{H}_{Y'}(g_*F', u^*f_*f^!G) \underset{\tau}{\longrightarrow} \mathcal{H}_{Y'}(g_*F', u^*G) \end{aligned}$$

is an isomorphism.

If u and v are identity maps then so is the map labeled (3.10.3), and the resulting composition (with $F := F'$)

$$\delta(F,G)\colon f_*\mathcal{H}_X(F, f^!G) \xrightarrow{\nu} \mathcal{H}_Y(f_*F, f_*f^!G) \xrightarrow{\tau} \mathcal{H}_Y(f_*F, G)$$

is just the duality map (4.2.1), whence the following generalization of (4.3.6):

Corollary (4.4.2) (Duality). *Let $f\colon X \to Y$ be quasi-proper. Then for any $F \in \mathbf{D}_{\mathsf{qc}}(X)$ and $G \in \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Y)$, the duality map $\delta(F,G)$ is an isomorphism.*

Moreover:

Corollary (4.4.3) (Base Change). *In* (4.4.1), *the functorial map adjoint to the composition*

$$g_*v^*f^!G \underset{(3.10.3)}{\xrightarrow{\ \sim\ }} u^*f_*f^!G \underset{u^*\tau}{\longrightarrow} u^*G,$$

is an isomorphism

$$\beta(G) = \beta_\sigma(G)\colon v^*f^!G \xrightarrow{\ \sim\ } g^!u^*G \qquad \big(G \in \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Y)\big).$$

To deduce (4.4.3) from (4.4.1), let $F' \in \mathbf{D}_{\mathsf{qc}}(X')$ and consider the next diagram, whose commutativity follows from the definition of $\beta = \beta(G)$:

(4.4.3.1)
$$\begin{tikzcd}
g_*\mathcal{H}_{X'}(F',\, v^*f^!G) \arrow[r, "\beta"] \arrow[d, "\nu"'] & g_*\mathcal{H}_{X'}(F',\, g^!u^*G) \arrow[d, "\nu"] \\
\mathcal{H}_{Y'}(g_*F',\, g_*v^*f^!G) \arrow[r, "\beta"] \arrow[d, "(3.10.3)"', "\simeq"] & \mathcal{H}_{Y'}(g_*F',\, g_*g^!u^*G) \arrow[d, "\tau"] \\
\mathcal{H}_{Y'}(g_*F',\, u^*f_*f^!G) \arrow[r, "\tau"'] & \mathcal{H}_{Y'}(g_*F',\, u^*G)
\end{tikzcd}$$

By (4.4.1), $\tau \circ (3.10.3) \circ \nu$ is an isomorphism; and by (4.4.2) (a special case of (4.4.1)), the right column is an isomorphism too. (Note that by (2.7.5)(d) and (3.9.1), $u^*G \in \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Y')$.) It follows that the top row is an isomorphism, and applying the functor $\mathrm{H}^0\mathbf{\Gamma}_{Y'}$ we get as in (4.3.7.2) an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X')}(F', v^*f^!G) \xrightarrow{\text{via } \beta} \mathrm{Hom}_{\mathbf{D}(X')}(F', g^!u^*G);$$

and since this holds for any $F' \in \mathbf{D}_{\mathsf{qc}}(X')$, in particular for $F' = v^*f^!G$ and $F' = g^!u^*G$, it follows that β itself is an isomorphism. Q.E.D.

Remarks **(4.4.4).** (a) Conversely, the commutativity of (4.4.3.1) shows that (4.4.2) and (4.4.3) together imply (4.4.1).

(b) An example of Neeman [**N**, p. 233, 6.5], with f the unique map $\mathrm{Spec}(\mathbb{Z}[T]/(T^2)) \to \mathrm{Spec}(\mathbb{Z})$ (T an indeterminate), shows that (4.4.2) and (4.4.3) can fail when G is not bounded below.

(c) In (4.4.1), $\mathrm{tordim}\, v \le \mathrm{tordim}\, u < \infty$.

To see this, let $x' \in X'$, $x = v(x')$, $y' = g(x')$, $y = u(y') = f(x)$, $A = \mathcal{O}_{Y,y}$, $A' = \mathcal{O}_{Y',y'}$, $B = \mathcal{O}_{X,x}$, and $B' = \mathcal{O}_{X',x'}$. By (2.7.6.4), the A-module A' has a flat resolution $P_\bullet$ of length $d := \mathrm{tordim}\, u < \infty$; and so by (i) in (3.10.2), $P_\bullet \otimes_A B$ is a flat resolution of the B-module $B^* = A' \otimes_A B$. Since B' is a localization of B^*, it holds for any B-module M that

$$\mathrm{Tor}_j^B(B', M) = B' \otimes_{B^*} \mathrm{Tor}_j^B(B^*, M) = 0 \qquad (j > d);$$

and it follows then from (2.7.6.4) that $\mathrm{tordim}\, v \le d$.

(d) By definition, β is the unique functorial map making the following diagram commute:

$$\begin{tikzcd}
g_*v^*f^! \arrow[r, "g_*\beta"] \arrow[d, "(3.10.3)"', "\simeq"] & g_*g^!u^* \arrow[d, "\tau_g"] \\
u^*f_*f^! \arrow[r, "u^*\tau_f"'] & u^*
\end{tikzcd}$$

This diagram generalizes [**H**, p. 207, TRA 4.]

4.5. Proof of Duality and Base Change: outline

In describing the organization of the proof of (4.4.1), we will attach symbols to labels of the form (4.4.x) to refer to special cases of (4.4.x):

$(4.4.1)^*_{\mathrm{pc}} :=$ (4.4.1) with $F' = v^*F$, where $F \in \mathbf{D}(X)$ is pseudo-coherent.

$(4.4.2)_{\mathrm{pc}} :=$ Corollary (4.3.6) $:= (4.4.1)^*_{\mathrm{pc}}$ with $u = v =$ identity.

$(4.4.3)^{\mathrm{o}} := (4.4.3)$ with the map u an open immersion.

$(4.4.3)^{\mathrm{af}} := (4.4.3)$ with the map u affine.

Having already proved $(4.4.2)_{\mathrm{pc}}$, our strategy is to prove the chain of implications

$$(4.4.2)_{\mathrm{pc}} \Leftrightarrow (4.4.1)^*_{\mathrm{pc}} \Rightarrow \big((4.4.3)^{\mathrm{o}}+(4.4.3)^{\mathrm{af}}\big) \Rightarrow (4.4.3) \Rightarrow (4.4.3)^{\mathrm{o}} \Leftrightarrow (4.4.2).$$

By (4.4.4)(a), then, (4.4.1) results.

Remark **(4.5.1).** For arbitrary finitely-presentable f, the assertions (4.4.1)–(4.4.3) are meaningful—though not necessarily true—with $(f^{\times}, g^{\times})$ in place of $(f^!, g^!)$. As will be apparent from the following proofs, the equivalence $(4.4.1) \Leftrightarrow (4.4.2) + (4.4.3)$ holds in this generality, as do the preceding implications except for $(4.4.2)_{\mathrm{pc}} \Rightarrow (4.4.1)^*_{\mathrm{pc}}$.

4.6. Steps in the proof

I. Proof of $(4.4.2)_{\mathrm{pc}}$

This has already been done (Corollary (4.3.6)).

II. $(4.4.2)_{\mathrm{pc}} \Leftrightarrow (4.4.1)^*_{\mathrm{pc}}$

The implication $\Leftarrow$ is trivial.

The implication $\Rightarrow$ follows at once from:

Lemma (4.6.4). *With the assumptions of* $(4.4.1)^*_{\mathrm{pc}}$, *and* δ *the duality map in* (4.4.2), *there is a natural commutative* $\mathbf{D}(Y')$*-diagram*

$$\begin{tikzcd}
u^*f_*\mathcal{H}_X(F,\, f^!G) \arrow[r, "u^*\delta"] \arrow[d, "\simeq"'] & u^*\mathcal{H}_Y(f_*F,\, G) \arrow[d, "\simeq"] \\
g_*\mathcal{H}_{X'}(v^*F,\, v^*f^!G) \arrow[r, "(4.4.1)^*_{\mathrm{pc}}"'] & \mathcal{H}_{Y'}(g_*v^*F,\, u^*G)
\end{tikzcd}$$

in which the vertical arrows are isomorphisms.

Commutativity in (4.6.4) is derived from the following relation—to be proved below—among the canonical maps ν, θ (3.7.2), and ρ (3.5.4.5):

Lemma (4.6.5). *For any commutative diagram of ringed-space maps*

$$
\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}
\tag{4.6.5.1}
$$

and $F \in \mathbf{D}_{\mathsf{qc}}(X)$, $H \in \mathbf{D}(X)$, *the following diagram commutes:*

$$
\begin{tikzcd}
u^*f_*\mathcal{H}_X(F,\,H) \arrow[rr, "\nu"] \arrow[d, "\theta"'] & & u^*\mathcal{H}_Y(f_*F,\,f_*H) \arrow[d, "\rho"] \\
g_*v^*\mathcal{H}_X(F,\,H) \arrow[d, "\rho"'] & & \mathcal{H}_{Y'}(u^*f_*F,\,u^*f_*H) \arrow[d, "{(1,\theta)}"] \\
g_*\mathcal{H}_{X'}(v^*F,\,v^*H) \arrow[r, "\nu"'] & \mathcal{H}_{Y'}(g_*v^*F,\,g_*v^*H) \arrow[r, "{(\theta,1)}"'] & \mathcal{H}_{Y'}(u^*f_*F,\,g_*v^*H)
\end{tikzcd}
$$

Indeed, if (4.6.5.1) is an independent fiber square of scheme-maps, so that by (3.10.3), $\theta(F)\colon u^*f_*F \to g_*v^*F$ is an isomorphism, and if $G \in \mathbf{D}(Y)$, $H := f^\times G$, so that there is a natural map $f_*H \to G$ (see (4.1.1)), then we get (a generalization of) commutativity in (4.6.4) by gluing the $\mathbf{D}(X')$-diagram in (4.6.5) and the following natural commutative diagram along the common column:

$$
\begin{tikzcd}
u^*\mathcal{H}_Y(f_*F,\,f_*H) \arrow[rr] \arrow[d, "\rho"'] & & u^*\mathcal{H}_Y(f_*F,\,G) \arrow[d, "\rho"] \\
\mathcal{H}_{Y'}(u^*f_*F,\,u^*f_*H) \arrow[r, equal] \arrow[d, "{(1,\theta)}"'] & \mathcal{H}_{Y'}(u^*f_*F,\,u^*f_*H) \arrow[r] \arrow[d, "{(\theta^{-1},1)}"', "\simeq"] & \mathcal{H}_{Y'}(u^*f_*F,\,u^*G) \arrow[d, "{(\theta^{-1},1)}", "\simeq"'] \\
\mathcal{H}_{Y'}(u^*f_*F,\,g_*v^*H) \arrow[r, "\sim", "{(\theta^{-1},\theta^{-1})}"'] & \mathcal{H}_{Y'}(g_*v^*F,\,u^*f_*H) \arrow[r] & \mathcal{H}_{Y'}(g_*v^*F,\,u^*G)
\end{tikzcd}
$$

Here is where we need f to be quasi-proper: since F is, by assumption, pseudo-coherent, therefore f_*F is pseudo-coherent. In view of (4.4.4)(c), the following Proposition gives then the isomorphism assertion in (4.6.4).

Proposition (4.6.6). *Let* $u\colon Y' \to Y$ *be any scheme-map of finite tor-dimension, and let* $H \in \overline{\mathbf{D}}^{+}(Y)$. *Then there is an integer* e *such that for all* $m \in \mathbb{Z}$ *and all* m*-pseudo-coherent* $C \in \mathbf{D}(Y)$, *the map*

$$\rho_u\colon u^*\mathcal{H}_Y(C,\,H) \to \mathcal{H}_{Y'}(u^*C,\,u^*H)$$

induces homology isomorphisms in all degrees $\le e-m$*. In particular, if* C *is pseudo-coherent then* ρ_u *is an isomorphism.*

PROOF. The question is local on Y, because if $i\colon U \to Y$ is an open immersion, $U' := U \times_Y Y'$, and $w\colon U' \to U$, $j\colon U' \to Y'$ are the projections (so that j is an open immersion), then $j^*\rho_u \cong \rho_w$—more precisely, the following natural diagram commutes for any $F, G \in \mathbf{D}(Y)$:

$$\begin{tikzcd}
j^*u^*\mathcal{H}_Y(F,G) \arrow[r, "j^*\rho_u"] \arrow[d, "\simeq"'] & j^*\mathcal{H}_{Y'}(u^*f, u^*G) \arrow[d, "\rho_j", "\simeq"'] \\
w^*i^*\mathcal{H}_Y(F,G) \arrow[d, "w^*\rho_i"', "\simeq"] & \mathcal{H}_{U'}(j^*u^*f, j^*u^*G) \arrow[d, "\simeq"'] \\
w^*\mathcal{H}_U(i^*F, i^*G) \arrow[r, "\rho_w"'] & \mathcal{H}_{U'}(w^*i^*F, w^*i^*G)
\end{tikzcd}$$

Here ρ_i and ρ_j are isomorphisms by the last assertion in (4.6.7) (whose proof does not depend on (4.6.6)); and commutativity follows from (3.7.1.1).

So by [**I**, p. 98, 2.3] we may assume there is a $\mathbf{D}(Y)$-map $E \to C$ with E *strictly perfect* (i.e., E is a bounded complex of finite-rank locally free $\mathcal{O}_Y$-modules), such that the induced map is an isomorphism $\tau_{\ge m+1}E \xrightarrow{\sim} \tau_{\ge m+1}C$. The contravariant Δ-functors

$$\Phi_1(C) := u^*\mathcal{H}_Y(C,\, H), \qquad \Phi_2(C) := \mathcal{H}_{Y'}(u^*C,\, u^*H)$$

are both bounded below (1.11.1), and so arguing as in the proof of (4.3.3.2) we find that there is an integer e such that for $i = 1, 2$, the natural maps

$$\tau_{\le e-m}\Phi_i(E) \leftarrow \tau_{\le e-m}\Phi_i(\tau_{m+1}E) \xrightarrow{\sim} \tau_{\le e-m}\Phi_i(\tau_{m+1}C) \to \tau_{\le e-m}\Phi_i(C)$$

are isomorphisms.

Thus it will be more than enough to prove:

Proposition (4.6.7). *Let* $u\colon Y' \to Y$ *be a scheme-map, let* E *be a bounded-above complex of finite-rank locally free* $\mathcal{O}_Y$*-modules, and let* $H \in \overline{\mathbf{D}}^+(Y)$*. If* E *is strictly perfect or if* u *has finite tor-dimension then the map*

$$\rho\colon u^*\mathcal{H}_Y(E,\, H) \to \mathcal{H}_{Y'}(u^*E,\, u^*H)$$

is an isomorphism.

The same holds for any $E, H \in \mathbf{D}(Y)$ *if* u *is an open immersion.*

Except for the proofs of (4.6.5) and (4.6.7), which are postponed to the end of this section 4.6, the proof of (4.6.4)—and hence of the the implication $(4.4.2)_{\mathrm{pc}} \Rightarrow (4.4.1)^*_{\mathrm{pc}}$—is now complete.

III. $(4.4.1)^*_{\mathbf{pc}} \Rightarrow \big((4.4.3)^{\mathbf{o}} + (4.4.3)^{\mathbf{af}}\big)$

Let $\beta = \beta(G)$ be as in (4.4.3). When u, hence v, is an open immersion or affine, then v is isofaithful ((3.10.2.1)(d) or (3.10.2.2)), so that for β to be an isomorphism it suffices that $v_*\beta$ be an isomorphism.

Let $F \in \mathbf{D}(X)$ be pseudo-coherent. From (4.4.3.1) with $F' = v^*F$ and with $^!$ replaced by $^\times$, one derives the following commutative diagram:

$$\begin{tikzcd}
f_*\mathcal{H}_X(F,\, v_*v^*f^\times G) \arrow[r, "\text{via } v_*\beta"] \arrow[d, "(3.2.3.2)^{-1}"', "\simeq"] & f_*\mathcal{H}_X(F,\, v_*g^\times u^*G) \arrow[d, "\simeq"', "(3.2.3.2)^{-1}"] \\
f_*v_*\mathcal{H}_{X'}(v^*F,\, v^*f^\times G) \arrow[r, "\text{via } \beta"] \arrow[d, "\simeq"] & f_*v_*\mathcal{H}_{X'}(v^*F,\, g^\times u^*G) \arrow[d, "\simeq"'] \\
u_*g_*\mathcal{H}_{X'}(v^*F,\, v^*f^\times G) \arrow[r, "\text{via } \beta"'] \arrow[d, equal] & u_*g_*\mathcal{H}_{X'}(v^*F,\, g^\times u^*G) \arrow[d, "\simeq"', "u_*\delta"] \\
u_*g_*\mathcal{H}_{X'}(v^*F,\, v^*f^\times G) \arrow[r, "\sim", "u_*(4.4.1)^*_{\mathrm{pc}}"'] & u_*\mathcal{H}_{Y'}(g_*v^*F,\, u^*G)
\end{tikzcd}$$

The bottom row is an isomorphism by assumption, as is the right column, by the special case $(4.4.2)_{\mathrm{pc}}$ of $(4.4.1)^*_{\mathrm{pc}}$. Thus the top row is an isomorphism, and hence, by (4.3.7), so is $v_*\beta$.

IV. $\big((4.4.3)^{\mathbf{o}} + (4.4.3)^{\mathbf{af}}\big) \Rightarrow (4.4.3)$

The essence of what follows is contained in the four lines preceding "CASE 1" on p. 401 of [**V**].

Denote the independent square in (4.4.1) by σ, and the corresponding functorial map $v^*f^\times \to g^\times u^*$ by β_σ (cf. (4.4.3), without assuming f and g to be quasi-proper). Let us first record the following elementary *transitivity* properties of β_σ.

Proposition (4.6.8). *For any commutative diagram*

$$\begin{tikzcd}
X'' \arrow[r, "v_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma_1"] & X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y'' \arrow[r, "u_1"'] & Y' \arrow[r, "u"'] & Y
\end{tikzcd}
\qquad \textit{or} \qquad
\begin{tikzcd}
Z' \arrow[r, "w"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma_1"] & Z \arrow[d, "f_1"] \\
X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

where both σ and σ_1 are independent squares—whence so is the composed square $\sigma_0 := \sigma\sigma_1$ see (3.10.3.2)*—the following resulting diagrams of func-*

torial maps commute:

$$\begin{tikzcd}
(vv_1)^*f^\times \arrow[rr, "\beta_{\sigma_0}"] \arrow[d, "\simeq"'] & & h^\times(uu_1)^* \arrow[d, "\simeq"] \\
v_1^*v^*f^\times \arrow[r, "v_1^*\beta_\sigma"'] & v_1^*g^\times u^* \arrow[r, "\beta_{\sigma_1}"'] & h^\times u_1^*u^*
\end{tikzcd}$$

$$\begin{tikzcd}
w^*(ff_1)^\times \arrow[rr, "\beta_{\sigma_0}"] \arrow[d, "\simeq"'] & & (gg_1)^\times u^* \arrow[d, "\simeq"] \\
w^*f_1^\times f^\times \arrow[r, "\beta_{\sigma_1}"'] & g_1^\times v^*f^\times \arrow[r, "g_1^\times\beta_\sigma"'] & g_1^\times g^\times u^*
\end{tikzcd}$$

PROOF. (Sketch.) Using the definition of β, one reduces mechanically to proving the transitivity properties for θ in (3.7.2), (ii) and (iii). Q.E.D.

Assuming (4.4.3)°, we first reduce (4.4.3) to the case where Y is affine. Let $(\mu_i\colon Y_i \to Y)_{i\in I}$ be an open covering of Y with each Y_i affine. Consider the diagrams, with σ as in (4.4.1),

$$\begin{tikzcd}
X_i' \arrow[r, "v_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\sigma_i"] & X_i \arrow[r, "\nu_i"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\tau_i"] & X \arrow[d, "f"] \\
Y_i' \arrow[r, "u_i"'] & Y_i \arrow[r, "\mu_i"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X_i' \arrow[r, "\nu_i'"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\tau_i'"] & X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y_i' \arrow[r, "\mu_i'"'] & Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

where $Y_i' := Y' \times_Y Y_i$, u_i and μ_i' are the projections, and all the squares are fiber squares. The composed squares $\tau_i\sigma_i$ and $\sigma\tau_i'$ are identical. The squares τ_i and τ_i' are independent because μ_i and μ_i' are open immersions; and by (4.4.3)°, β_{τ_i} and $\beta_{\tau_i'}$ are isomorphisms.

Furthermore, since f is quasi-proper therefore so are the maps f_i. The map u_i, which agrees over Y_i with u, has finite tor-dimension. By (3.10.3.4), the square $\sigma_i \cong \mu_i^*\sigma$ is independent. Thus if (4.4.3) holds whenever Y is affine, then β_{σ_i} is an isomorphism, and (4.6.8) shows that so are $\beta_{\sigma\tau_i'}$ $(= \beta_{\tau_i\sigma_i})$ and $\nu_i'^*\beta_\sigma$. Since $(\nu_i'\colon X_i' \to X')_{i\in I}$ is an open covering of X', and since isomorphism can be checked locally (see (1.2.2)), it follows that β_σ is an isomorphism, whence the asserted reduction.

Next, again assuming $(4.4.3)^{\mathrm{o}}$, we reduce (4.4.3) with affine Y to where Y' too is affine. That will complete the proof, since when both Y and Y' are affine then so is u, and $(4.4.3)^{\mathrm{af}}$ applies.

Let $(\nu_j\colon Y'_j \to Y')_{j\in J}$ be an open covering of Y' with each Y'_j affine. Consider the diagram, with affine Y and σ as in (4.4.1),

$$\begin{tikzcd}
X'_j \arrow[r, "v_j"] \arrow[d, "g_j"'] \arrow[dr, phantom, "\sigma_j"] & X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y'_j \arrow[r, "\nu_j"'] & Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

where σ_j is a fiber square, hence independent. By $(4.4.3)^{\mathrm{o}}$, β_{σ_j} is an isomorphism. If (4.4.3) holds for independent squares whose bottom corners are affine, then $\beta_{\sigma\sigma_j}$ is an isomorphism; and so by (4.6.8), $v_j^*\beta_\sigma$ is also an isomorphism. As before, then, β_σ is an isomorphism, and we have the desired reduction. Q.E.D.

V. (4.4.3) ⇒ (4.4.3)$^{\mathrm{o}}$ ⇔ (4.4.2)

The first implication is trivial. The implication (4.4.2) ⇒ $(4.4.3)^{\mathrm{o}}$ is contained in what we have already done, but it's more direct than that, as we'll see. Incidentally, the following argument does not need f to be quasi-proper.

Let us first deduce (4.4.2) from $(4.4.3)^{\mathrm{o}}$. As in (4.6.4), via (4.6.5), there is for any $F \in \mathbf{D}(X)$, $G \in \mathbf{D}(Y)$ a commutative diagram

$$\begin{tikzcd}
u^*f_*\mathcal{H}_X(F,\, f^\times G) \arrow[r, "u^*\delta"] \arrow[d] & u^*\mathcal{H}_Y(f_*F,\, G) \arrow[d] \\
g_*\mathcal{H}_{X'}(v^*F,\, v^*f^\times G) \arrow[r, "(4.4.1)"'] & \mathcal{H}_{Y'}(g_*v^*F,\, u^*G)
\end{tikzcd} \tag{4.6.9}$$

When u (hence v) is an open immersion, then the vertical arrows in this diagram are *isomorphisms*. Indeed, these arrows are combinations of ρ and θ, ρ being an isomorphism by (4.6.7), and $\theta(L)\colon u^*f_*L \to g_*v^*L$ being an isomorphism for *any* $L \in \mathbf{D}(X)$, as follows easily from (2.4.5.2) after L is replaced by a q-injective resolution. Furthermore, the functor $\mathbf{\Gamma}_{Y'} := \mathbf{R}\Gamma(Y', -)$ transforms the bottom row of (4.6.9) into an isomorphism. This follows from commutativity of the next diagram, obtained via Exercise (3.2.5)(f) by application of $\mathbf{\Gamma}_{Y'}$ to the commutative diagram (4.4.3.1), and where, under the present assumption of $(4.4.3)^{\mathrm{o}}$, β is an isomorphism:

$$
(4.6.10)\qquad
\begin{tikzcd}
 & \mathbf{H}_{X'}(F', v^*f^\times G) \arrow[dl, "\mathbf{\Gamma}_{Y'}(4.4.1)"'] \arrow[dr, "\text{via } \beta"] & \\
\mathbf{H}_{Y'}(g_*F', u^*G) & & \mathbf{H}_{X'}(F', g^\times u^*G) \arrow[ll, "\sim", "(4.1.1)"']
\end{tikzcd}
$$

We conclude that $\mathbf{\Gamma}_{Y'}u^*\delta$ is an isomorphism whenever $u\colon Y' \to Y$ is an open immersion; and then (4.4.2) results from:

Lemma (4.6.11). *Let $\phi\colon G_1 \to G_2$ be a map in $\mathbf{D}(Y)$. Then ϕ is an isomorphism iff for every open immersion $u\colon Y' \hookrightarrow Y$ with Y' affine, the map*

$$\mathbf{\Gamma}_{Y'}u^*(\phi)\colon \mathbf{\Gamma}_{Y'}u^*(G_1) \to \mathbf{\Gamma}_{Y'}u^*(G_2)$$

is an isomorphism.

Proof. Write $\Gamma_{Y'}$ for the sheaf-functor $\Gamma(Y', -)$. We may assume that G_1 and G_2 are q-injective and that ϕ is actually a map of complexes, see (2.3.8)(v), so that $\mathbf{\Gamma}_{Y'}u^*(\phi)$ is the map $\Gamma_{Y'}(\phi)\colon \Gamma_{Y'}(G_1) \to \Gamma_{Y'}(G_2)$. If $\mathbf{\Gamma}_{Y'}u^*(\phi)$ is an isomorphism, then the homology maps

$$\mathrm{H}^p\Gamma_{Y'}(\phi)\colon \mathrm{H}^p\Gamma_{Y'}(G_1) \to \mathrm{H}^p\Gamma_{Y'}(G_2) \qquad (p \in \mathbb{Z})$$

are all isomorphisms; and since $H^p(G_i)$ is the sheaf associated to the presheaf $Y' \mapsto \mathrm{H}^p\Gamma_{Y'}(G_i)$ $(i = 1, 2)$, it follows for every $p \in \mathbb{Z}$ that the map $H^p(\phi)\colon H^p(G_1) \to H^p(G_2)$ is an isomorphism, so that by (1.2.2), ϕ is an isomorphism. The converse is obvious. Q.E.D.

Conversely, if (4.4.2) holds, then the top row—and hence the bottom row—in (4.6.9) is an isomorphism. We deduce from (4.6.10) that

$$\mathbf{H}_{X'}(F', v^*f^\times G) \xrightarrow{\text{via } \beta} \mathbf{H}_{X'}(F', g^\times u^*G)$$

is an isomorphism for all F', whence (taking homology, see (2.4.2)) that

$$\mathrm{Hom}_{\mathbf{D}(X')}(F', v^*f^\times G) \xrightarrow{\text{via } \beta} \mathrm{Hom}_{\mathbf{D}(X')}(F', g^\times u^*G)$$

is an isomorphism for all F', so that β itself is an isomorphism. Q.E.D.

It remains to prove (4.6.5) and (4.6.7).

Proof of (4.6.5). One verifies, using the definitions of ν, of θ (via (3.7.2)(a)) and of ρ, and the line following (3.5.4.2), that in the big diagram on the following page—with natural maps, and in which α denotes the map (3.5.4.2) = (3.5.4.3) (of which the isomorphism (3.2.3.2) is an instance, see (3.2.4)(i))—the outer border is adjoint to the diagram in (4.6.5). Therefore it will suffice to show that all the subdiagrams in the big diagram commute.

For the unnumbered subdiagrams commutativity is clear. Commutativity of ① follows from the definition of ρ; of ② from the definition of θ via (3.7.2)(a); of ③ from (3.7.1.1) (with β replaced by α, etc.); and of ④ from the definition of θ via (3.7.2)(c). Q.E.D.

Proof of (4.6.7). For this proof, we drop the abbreviations introduced at the beginning of §4.4. Thus u_* and u^* will now denote the usual sheaf-functors, and $\mathbf{R}u_*$, $\mathbf{L}u^*$ their respective derived functors. Similarly, $\mathcal{H}$ will denote the functor $\mathcal{H}om^\bullet$ of complexes, and $\mathbf{R}\mathcal{H}om^\bullet$ its derived functor.

We need to understand ρ more concretely, and to that end we will establish commutativity of the following diagram of natural maps, for any complexes E, H of $\mathcal{O}_Y$-modules:

(4.6.7.1)
$$\begin{tikzcd}
\mathbf{L}u^*\mathcal{H}_Y(E,\,H) \arrow[r, "b"] \arrow[d, "a"'] & u^*\mathcal{H}_Y(E,\,H) \arrow[d, "\rho_0"] \\
\mathbf{L}u^*\mathbf{R}\mathcal{H}_Y(E,\,H) \arrow[ddd, "\rho"'] & \mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[d, "c"] \\
 & \mathbf{R}\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[d, "d"] \\
 & \\
\mathbf{R}\mathcal{H}_{Y'}(\mathbf{L}u^*E,\,\mathbf{L}u^*H) \arrow[r, "e"'] & \mathbf{R}\mathcal{H}_{Y'}(\mathbf{L}u^*E,\,u^*H)
\end{tikzcd}$$

Here ρ_0 is adjoint to the natural composite map of complexes

$$\xi\colon \mathcal{H}_Y(E,\,H) \to \mathcal{H}_Y(E,\,u_*u^*H) \underset{(3.1.6)}{\xrightarrow{\ \sim\ }} u_*\mathcal{H}_{Y'}(u^*E,\,u^*H).$$

This ξ is such that for any open $U \subset Y$, $\Gamma(U,\xi)$ is the map

$$\prod_{i\in\mathbb{Z}} \mathrm{Hom}_U(E^i,\,H^{i+n}) \to \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{f^{-1}U}(u^*E^i,\,u^*H^{i+n})$$

arising from the functoriality of u^*.

$$
\begin{tikzcd}
f_*\mathcal{H}_X(E,\,H) \arrow[r] \arrow[dd] & f_*\mathcal{H}_X(f^*f_*E,\,H) \arrow[r, equal] \arrow[dd] & f_*\mathcal{H}_X(f^*f_*E,\,H) \arrow[r, "\alpha"] \arrow[d] & \mathcal{H}_Y(f_*E,\,f_*H) \arrow[r, equal] \arrow[d] & \mathcal{H}_Y(f_*E,\,f_*H) \arrow[ddd] \\
 & {} \arrow[r, phantom, "\textcircled{1}"] & f_*\mathcal{H}_X(f^*f_*E,\,v_*v^*H) \arrow[r, "\alpha"'] & \mathcal{H}_Y(f_*E,\,f_*v_*v^*H) \arrow[dd, "\simeq"] \arrow[r, phantom, "\textcircled{2}"] & {} \\
f_*v_*v^*\mathcal{H}_X(E,\,H) \arrow[r] \arrow[d, "\simeq"'] & f_*v_*v^*\mathcal{H}_X(f^*f_*E,\,H) \arrow[r, "\rho"] \arrow[d, "\simeq"'] & f_*v_*\mathcal{H}_{X'}(v^*f^*f_*E,\,v^*H) \arrow[u, "\alpha", "\simeq"'] \arrow[d, "\simeq"] & & \\
u_*g_*v^*\mathcal{H}_X(E,\,H) \arrow[r] \arrow[d, "\rho"'] & u_*g_*v^*\mathcal{H}_X(f^*f_*E,\,H) \arrow[r, "\rho"'] & u_*g_*\mathcal{H}_{X'}(v^*f^*f_*E,\,v^*H) \arrow[d, equal] \arrow[r, phantom, "\textcircled{3}"] & \mathcal{H}_Y(f_*E,\,u_*g_*v^*H) \arrow[dd, "\alpha^{-1}"', "\simeq"] & \mathcal{H}_Y(f_*E,\,u_*u^*f_*H) \arrow[l, "{(1,\theta)}"'] \arrow[dd, "\simeq"', "\alpha^{-1}"] \\
u_*g_*\mathcal{H}_{X'}(v^*E,\,v^*H) \arrow[rr] \arrow[d] & {} \arrow[d, phantom, "\textcircled{4}"] & u_*g_*\mathcal{H}_{X'}(v^*f^*f_*E,\,v^*H) \arrow[d, "\simeq"] & & \\
u_*g_*\mathcal{H}_{X'}(g^*g_*v^*E,\,v^*H) \arrow[rr, "{(\theta,1)}"'] \arrow[d, equal] & {} & u_*g_*\mathcal{H}_{X'}(g^*u^*f_*E,\,v^*H) \arrow[r] & u_*\mathcal{H}_{Y'}(u^*f_*E,\,g_*v^*H) & u_*\mathcal{H}_{Y'}(u^*f_*E,\,u^*f_*H) \arrow[l, "{(1,\theta)}"] \\
u_*g_*\mathcal{H}_{X'}(g^*g_*v^*E,\,v^*H) \arrow[rrr] & & & u_*\mathcal{H}_{Y'}(g_*v^*E,\,g_*v^*H) \arrow[u, "{(\theta,1)}"'] &
\end{tikzcd}
$$

Commutativity of (4.6.7.1) is equivalent to commutativity of the following "adjoint" diagram:[51]

$$
\begin{tikzcd}
\mathcal{H}_Y(E,\,H) \arrow[r] \arrow[d] & \mathbf{R}u_*u^*\mathcal{H}_Y(E,\,H) \arrow[d, "\mathbf{R}u_*(\rho_0)"] \\
\mathbf{R}\mathcal{H}_Y(E,\,H) \arrow[d] & \mathbf{R}u_*\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[d] \\
\mathbf{R}\mathcal{H}_Y(E,\,\mathbf{R}u_*\mathbf{L}u^*H) \arrow[d, "(3.2.3.2)^{-1}"'] & \mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[d] \\
\mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(\mathbf{L}u^*E,\,\mathbf{L}u^*H) \arrow[r] & \mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(\mathbf{L}u^*E,\,u^*H)
\end{tikzcd}
$$

But in this diagram the two maps obtained by going around from the top left to the bottom right clockwise and counterclockwise respectively, are both equal to the natural composition

$$
\begin{aligned}
\mathcal{H}_Y(E,\,H) \longrightarrow \mathcal{H}_Y(E,\,u_*u^*H) &\xrightarrow{(3.1.5)^{-1}} u_*\mathcal{H}_{Y'}(u^*E,\,u^*H)\\
&\longrightarrow \mathbf{R}u_*\mathcal{H}_{Y'}(u^*E,\,u^*H) \longrightarrow \mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(u^*E,\,u^*H)\\
&\longrightarrow \mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(\mathbf{L}u^*E,\,u^*H),
\end{aligned}
$$

as shown by the commutativity of the following two diagrams. (In the first, the top three horizontal arrows come from the natural functorial composition $\mathbf{1} \to u_*u^* \to \mathbf{R}u_*u^*$; and the right column is $\mathbf{R}u_*(\rho_0)$.)

$$
\begin{tikzcd}
\mathcal{H}_Y(E,\,H) \arrow[r] \arrow[d] & \mathbf{R}u_*u^*\mathcal{H}_Y(E,\,H) \arrow[d] \\
\mathcal{H}_Y(E,\,u_*u^*H) \arrow[r] \arrow[d] & \mathbf{R}u_*u^*\mathcal{H}_Y(E,\,u_*u^*H) \arrow[d] \\
u_*\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "①"] & \mathbf{R}u_*u^*u_*\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[d] \\
u_*\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[r] & \mathbf{R}u_*\mathcal{H}_{Y'}(u^*E,\,u^*H)
\end{tikzcd}
$$

[51] Recall that by (3.2.4)(i), the map (3.2.3.2) is an instance of the map (3.5.4.3).

$$\begin{tikzcd}
\mathcal{H}_Y(E,\,H) \arrow[r] \arrow[d] & \mathcal{H}_Y(E,\,u_*u^*H) \arrow[r, "\sim"] \arrow[d] & u_*\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[d] \\
\mathbf{R}\mathcal{H}_Y(E,\,H) \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & \mathbf{R}\mathcal{H}_Y(E,\,u_*u^*H) \arrow[d] \arrow[dr, phantom, "③"] & \mathbf{R}u_*\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[dd] \\
\mathbf{R}\mathcal{H}_Y(E,\,\mathbf{R}u_*\mathbf{L}u^*H) \arrow[r] \arrow[d] & \mathbf{R}\mathcal{H}_Y(E,\,\mathbf{R}u_*u^*H) \arrow[d] & \\
\mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(\mathbf{L}u^*E,\,\mathbf{L}u^*H) \arrow[r] & \mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(\mathbf{L}u^*E,\,u^*H) & \mathbf{R}u_*\mathbf{R}\mathcal{H}_{Y'}(u^*E,\,u^*H) \arrow[l]
\end{tikzcd}$$

Commutativity of subdiagram ① follows from the natural functorial composition $u_* \to u_*u^*u_* \to u_*$ being the identity. Commutativity of ② follows from that of (3.2.1.3). Commutativity of ③ follows from that of the diagram immediately following (3.2.3.2).

Thus (4.6.7.1) does indeed commute.

Proceeding now with the proof of (4.6.7), suppose that E is a bounded-above complex of finite-rank locally free $\mathcal{O}_Y$-modules, and that $H \in \overline{\mathbf{D}}^{+}(Y)$. To show that ρ is an isomorphism, we may assume that H is a complex of u^*-acyclic $\mathcal{O}_Y$-modules, bounded below if u has finite tor-dimension, see (2.7.5)(vi). Then in (4.6.7.1), d and e are isomorphisms; and $\mathcal{H}_Y(E,H)$ is also a complex of u^*-acyclic $\mathcal{O}_Y$-modules (the question being local on Y), so that b too is an isomorphism, see (2.7.5)(a). That ρ_0 is an isomorphism follows from the fact that (exercise) its stalk at $y' \in Y'$ is—with $y := u(y')$, $R' := \mathcal{O}_{Y',y'}$ and $R := \mathcal{O}_{Y,y}$—the natural map

$$R' \otimes_R \operatorname{Hom}_R(E_y, H_y) \to \operatorname{Hom}_{R'}(R' \otimes_R E_y, R' \otimes_R H_y).$$

It remains to be shown that a and c are isomorphisms. For a, it suffices that if $H \to I$ is a quasi-isomorphism with I injective and bounded-below, then the resulting map $\mathcal{H}_Y(E,\,H) \to \mathcal{H}_Y(E,\,I)$ be an isomorphism. Since $\mathcal{H}_Y$ is a Δ-functor, and by the footnote under (1.5.1), it is equivalent to show that if C is the summit of a triangle whose base is $H \to I$ (so that C is exact), then $\mathcal{H}_Y(E,\,C)$ is exact. For any $n \in \mathbb{Z}$, to show that $H^n\mathcal{H}_Y(E,\,C) = 0$ we may assume that $E \neq 0$, let $m_0 = m_0(E)$ be the least integer such that $E^m = 0$ for all $m > m_0$, and argue by induction on m_0, as follows.

If $m_0 \ll 0$, then $\mathcal{H}_Y(E,\,C)$ vanishes in degree n, so the assertion is obvious. Proceeding inductively, set $i = m_0(E)$, and let $E_{<i}$ be the complex which agrees with E in all degrees $< i$, and vanishes in all degrees $\geq i$, so that we have a natural semi-split exact sequence

$$0 \to E^i[-i] \to E \to E_{<i} \to 0\,,$$

and a corresponding triangle, cf. (1.4.3.3). There results an exact homology sequence, see $(1.4.5)^H$:

$$H^n\mathcal{H}_Y(E_{<i}\,, C) \to H^n\mathcal{H}_Y(E, C) \to H^{n+i}\mathcal{H}_Y(E^i, C)$$

in which the first term vanishes by the inductive hypothesis, and the last term vanishes because E^i is locally free of finite rank and C is exact. Hence $H^n\mathcal{H}_Y(E, C)$ also vanishes, as desired. Thus a is indeed an isomorphism. Similarly c is an isomorphism. Hence, finally, so is ρ.

For the last assertion in (4.6.7), suppose u is an open immersion. It is left as an exercise to show that now ρ_0 is just the obvious restriction map. To show that ρ is an isomorphism we may assume that H—and hence u^*H—is q-injective, see (2.4.5.2). Clearly, then, all the maps in (4.6.7.1) other than ρ are isomorphisms, whence so is ρ. Q.E.D.

4.7. Quasi-perfect maps

Again, all schemes are assumed to be concentrated.

In this section, for a scheme-map $f\colon X \to Y$ the functor $f^\times$ will be as in (4.1.1), *but* restricted to $\mathbf{D}_{\mathsf{qc}}(Y)$; in other words, $f^\times$ is always to be regarded as a functor from $\mathbf{D}_{\mathsf{qc}}(Y)$ to $\mathbf{D}_{\mathsf{qc}}(X)$.

Quasi-perfect maps are scheme-maps $f\colon X \to Y$ characterized by any one of several nice properties preserved by tor-independent base change (see (4.7.3.1)). Among those properties are the following, the first two by (4.7.1), and the next two by (4.7.4) and (4.7.6)(d):

• $f^\times$ commutes with small direct sum in $\mathbf{D}_{\mathsf{qc}}$ (i.e., direct sum of any family indexed by a small set, see §4.1).

• For all $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ the natural map is an isomorphism

$$\chi_F\colon f^\times\mathcal{O}_Y \underset{=}{\otimes} \mathsf{L}f^*F \xrightarrow{\ \sim\ } f^\times F.$$

• $f^\times$ is a bounded functor, and it satisfies *universal* tor-independent base change, that is, for *any* independent square as in (4.4.1), and any $G \in \mathbf{D}_{\mathsf{qc}}(Y)$—not necessarily in $\overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$—the base-change map $\beta(G)$ in (4.4.3) is an isomorphism.

• $f^\times$ is a bounded functor, and these two conditions hold:

(i) For all $F \in \mathbf{D}_{\mathsf{qc}}(X)$ the duality map (4.2.1) is an isomorphism

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om^\bullet(F, f^\times\mathcal{O}_Y) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om^\bullet_Y(\mathsf{R}f_*F, \mathcal{O}_Y).$$

(ii) If (F_α) is a small directed system of flat quasi-coherent $\mathcal{O}_Y$-modules then for any $n \in \mathbb{Z}$ the natural map is an isomorphism

$$\varinjlim_{\alpha} H^n(f^\times F_\alpha) \xrightarrow{\ \sim\ } H^n(f^\times \varinjlim_{\alpha} F_\alpha).$$

It follows that quasi-perfection of f implies the following; and in fact when Y is separated the converse is true, see (4.7.4):

• $f^\times$ is a bounded functor, and the above natural map χ_F is an isomorphism whenever F is a flat quasi-coherent $\mathcal{O}_Y$-module.

Further, though we won't prove it here, the main result Theorem 1.2 in [**LN**] is the equivalence of the following conditions:

(i) *f is quasi-perfect.*

(ii) *f is quasi-proper* (4.3.3.1) *and has finite tor-dimension.*

(iii) *f is quasi-proper and the functor $f^\times$ is bounded.*

We call a scheme-map f *perfect* if f is pseudo-coherent and of finite tor-dimension. (For pseudo-coherent f, being of finite tor-dimension is equivalent to boundedness of $f^\times$, see [**LN**, Thm. 1.2]).

For example, since finite-type maps of noetherian schemes are always pseudo-coherent, the foregoing and (4.3.9) show that a separated such map is quasi-perfect if and only if it is proper and perfect.

Perfect maps of noetherian schemes will be treated in §4.9.

Before proceeding, we review a few basic facts about *perfect complexes.* A complex in $E \in \mathbf{D}(X)$ (X a scheme) is said to be perfect if it is locally-$\mathbf{D}$-isomorphic to a strictly perfect complex, i.e., a bounded complex of finite-rank free $\mathcal{O}_X$-modules. More precisely, E is said to have *perfect amplitude in* $[a, b]$ ($a \le b \in \mathbb{Z}$) if locally on X, E is $\mathbf{D}$-isomorphic to a strictly perfect complex vanishing in all degrees which are $< a$ or $> b$. Thus E is perfect iff it has perfect amplitude in some interval $[a, b]$. By [**I**, p. 134, 5.8], this condition is equivalent to E being pseudo-coherent and also having flat amplitude in $[a, b]$ (i.e., being globally $\mathbf{D}$-isomorphic to a flat complex vanishing in all degrees $< a$ and $> b$). So E is perfect iff it is pseudo-coherent and of finite tor-dimension (that is, $\mathbf{D}$-isomorphic to a bounded flat complex, see (3.9.4.2)(b)).

Proposition (4.7.1) (Neeman). *For any scheme-map $f\colon X \to Y$, the following conditions, with $f^\times$ as in* (4.1.1), *are equivalent:*

(i) *$f^\times$ respects direct sums* (*see* (3.8.1)) *in $\mathbf{D}_{\mathsf{qc}}$, i.e., for any small $\mathbf{D}_{\mathsf{qc}}(Y)$-family (F_α) the natural map is an isomorphism*

$$\bigoplus_\alpha f^\times F_\alpha \xrightarrow{\sim} f^\times(\bigoplus_\alpha F_\alpha).$$

(ii) *The functor $\mathsf{R}f_*$ takes perfect complexes to perfect complexes.*

(iii) *The functor $f^\times$ has a right adjoint.*

(iv) *For all $F \in \mathbf{D}_{\mathsf{qc}}(Y)$, the map adjoint to*

$$\mathsf{R}f_*(f^\times\mathcal{O}_Y \underset{=}{\otimes} \mathsf{L}f^*F) \underset{(3.9.4)}{\xrightarrow{\sim}} \mathsf{R}f_*f^\times\mathcal{O}_Y \underset{=}{\otimes} F \underset{\text{via } \tau}{\longrightarrow} F$$

is an isomorphism

$$f^\times\mathcal{O}_Y \underset{=}{\otimes} \mathsf{L}f^*F \xrightarrow{\sim} f^\times F.$$

Proof. (i) $\Leftrightarrow$ (ii): [**N**, p. 215, Prop. 2.5 and Cor. 2.3; and p. 224, Thm. 5.1 (where every $s \in S$ is implicitly assumed to be compact)].

(i) $\Rightarrow$ (iii): [**N**, p. 215, Prop. 2.5; p. 207, lines 12–13; and p. 223, Thm. 4.1].

(iii) $\Rightarrow$ (i): simple.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (i): For the first $\Rightarrow$ see [**N**, p. 226, Thm. 5.4]. The second implication follows from (3.8.2).

Strictly speaking, the referenced results in [**N**] are proved for *separated* schemes; but in view of [**BB**, p. 9, Thm. 3.1.1] one readily verifies that the proofs are valid for any concentrated scheme. Q.E.D.

Definition (4.7.2). *A map* $f\colon X \to Y$ *is quasi-perfect* if it satisfies the conditions in (4.7.1).

Remark. The fact, mentioned above, that quasi-perfect maps are quasi-proper results from (4.7.1)(ii) and [**LN**, Cor. 4.3.2], which says that f is quasi-proper if and (clearly) only if $\mathbf{R}f_*$ takes perfect complexes to pseudo-coherent complexes.

Examples (4.7.3). (a) *Any quasi-proper scheme-map* f *of finite tor-dimension—so by* (4.3.3.2), *any proper perfect map, in particular, any flat finitely-presentable proper map—is quasi-perfect.*

Indeed $\mathbf{R}f_*$ preserves both pseudo-coherence of complexes and—by [**I**, p. 250, 3.7.2] (a consequence of (3.9.4) above)—finite tor-dimensionality of complexes; so (4.7.1)(ii) holds.

(b) Let $f\colon X \to Y$ be a scheme-map with X *divisorial,* i.e., X has an ample family $(\mathcal{L}_i)_{i\in I}$ of invertible $\mathcal{O}_X$-modules [**I**, p. 171, Défn. 2.2.5].

Then [**N**, p. 211, Example 1.11 and p. 224, Theorem 5.1] imply that f *is quasi-perfect* $\Leftrightarrow$ *for each* $i \in I$, *there is an integer* n_i *such that the* $\mathcal{O}_Y$*-complex* $\mathbf{R}f_*(\mathcal{L}_i^{\otimes -n})$ *is perfect for all* $n \ge n_i$.

(c) (Cf. (4.3.8).) Let f be quasi-projective and let $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. Then:

f *is quasi-perfect* $\Leftrightarrow$ *the* $\mathcal{O}_Y$*-complex* $\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$ *is perfect for all* $n \gg 0$
$\Rightarrow$ f *is perfect.*

Indeed, condition (4.7.1)(ii), together with the compatibility of $\mathbf{R}f_*$ and open base change, implies that quasi-perfection is a property of f which is local on Y, and the same holds for perfection of $\mathbf{R}f_*(\mathcal{L}^{\otimes -n})$; so for the $\Leftrightarrow$ we may assume Y affine, and apply (b). The $\Rightarrow$ is given by (4.7.3.3) below.

(d) For a *finite* map $f\colon X \to Y$ the following are equivalent:

(i) f *is quasi-perfect.*

(ii) f *is perfect.*

(iii) *The complex* $f_*\mathcal{O}_X \cong \mathbf{R}f_*\mathcal{O}_X$ *is perfect.*

Indeed, the implication (i) $\Rightarrow$ (iii) is given by (4.7.1)(ii). If (iii) holds then f has finite tor-dimension (see (2.7.6.4)), and as in the first part of the proof of (4.3.8), f is pseudo-coherent; thus f is perfect. The implication (ii) $\Rightarrow$ (i) is given by (a).

Proposition (4.7.3.1). *For any independent square of scheme-maps,*

$$\begin{CD} X' @>v>> X \\ @VgVV @VVfV \\ Y' @>>u> Y \end{CD}$$

(i) *if f is quasi-perfect then so is g; and*

(ii) *if the* (*bounded-below*) *functor $f^\times\colon \mathbf{D}_{\mathsf{qc}}(Y) \to \mathbf{D}_{\mathsf{qc}}(X)$ is bounded above, then so is $g^\times\colon \mathbf{D}_{\mathsf{qc}}(Y') \to \mathbf{D}_{\mathsf{qc}}(X')$.*

Hence, if $(Y_i)_{i\in I}$ is an open cover of Y then

(iii) *f is quasi-perfect $\Leftrightarrow$ for all i, the same is true of the induced map $f^{-1}Y_i \to Y_i$; and*

(iv) *if f is quasi-proper then $f^\times$ is bounded above $\Leftrightarrow$ for all i, the same is true of the induced map $f^{-1}Y_i \to Y_i$.*

Proof. To begin with, (iii) follows easily from (i) and (4.7.1)(ii); and (iv) follows from (ii) and (4.4.3).

In the rest of this proof, quasi-perfection is characterized by (4.7.1)(i).

Suppose first that Y' is separated. We induct on $q = q(Y')$, the least number of affine open subschemes needed to cover Y'.

If $q = 1$ then the map u is affine, whence so is v [**GD**, p. 358, (9.1.16), (v) and (iii)]; so to prove (i) (resp. (ii)) it suffices, by (3.10.2.2), to show that for any small $\mathbf{D}_{\mathsf{qc}}(Y')$-family (F_α) the natural map is an isomorphism

$$\bigoplus_\alpha \mathbf{R}v_*g^\times F_\alpha \overset{(3.9.3.3)}{\cong} \mathbf{R}v_*\big(\bigoplus_\alpha g^\times F_\alpha\big) \xrightarrow{\ \sim\ } \mathbf{R}v_*g^\times\big(\bigoplus_\alpha F_\alpha\big)$$

(resp.—since every $G \in \mathbf{D}_{\mathsf{qc}}(X')$ is isomorphic to a quasi-coherent, hence v_*-acyclic, $\mathcal{O}_{X'}$-complex G', see (2.7.5)(a), so that

$$H^n(\mathbf{R}v_*G) \cong H^n(v_*G') \cong v_*H^n(G') = 0 \implies H^n(G) \cong H^n(G') = 0$$

—that $\mathbf{R}v_*g^\times\colon \mathbf{D}_{\mathsf{qc}}(Y') \to \mathbf{D}_{\mathsf{qc}}(X)$ is bounded). Since $\mathbf{R}u_*$ is bounded (see (3.9.2.3)), the second of these facts results from the natural isomorphism $\mathbf{R}v_*g^\times \xrightarrow{\sim} f^\times\mathbf{R}u_*$ of (3.10.4). The first results from the (easily-checked) commutativity of

$$\begin{CD}
\bigoplus_\alpha \mathbf{R}v_*g^\times F_\alpha @>\sim>(3.9.3.3)> \mathbf{R}v_*\big(\bigoplus_\alpha g^\times F_\alpha\big) @>>> \mathbf{R}v_*g^\times\big(\bigoplus_\alpha F_\alpha\big) \\
@V{\simeq}V{(3.10.4)}V @. @V{(3.10.4)}V{\simeq}V \\
\bigoplus_\alpha f^\times\mathbf{R}u_*F_\alpha @>\sim>> f^\times\big(\bigoplus_\alpha \mathbf{R}u_*F_\alpha\big) @>\sim>(3.9.3.3)> f^\times\mathbf{R}u_*\big(\bigoplus_\alpha F_\alpha\big)
\end{CD}$$

Suppose $q > 1$, so $Y' = Y_1' \cup Y_2'$ with Y_i' open in Y', $q(Y_1') = q-1$, and $q(Y_2') = 1$. Set $Y_{12}' := Y_1' \cap Y_2'$, so that $q(Y_{12}') \le q-1$. (Y' being separated, the intersection of affine subschemes of Y' is affine). We have the commutative diagram of immersions

$$\begin{tikzcd} Y'_{12} \arrow[r, "w_1"] \arrow[d, "w_2"'] & Y'_1 \arrow[d, "u_1"] \\ Y'_2 \arrow[r, "u_2"'] & Y' \end{tikzcd}$$

With $u_{12} := u_1 w_1 = u_2 w_2$ there is, for any $F \in \mathbf{D}(Y')$, a natural triangle

$$F \to \mathbf{R}u_{1*}u_1^*F \oplus \mathbf{R}u_{2*}u_2^*F \to \mathbf{R}u_{12*}u_{12}^*F \to F[1] \tag{4.7.3.2}$$

obtained by applying the standard exact sequence—holding for any injective (or even flasque) $\mathcal{O}_{Y'}$-module G—

$$0 \to G \to u_{1*}u_1^*G \oplus u_{2*}u_2^*G \to u_{12*}u_{12}^*G \to 0$$

to an injective q-injective resolution of F (see paragraph around (1.4.4.2)).

The inductive hypothesis applied to the natural composite independent square (see (3.10.3.2)), with $i = 1, 2, 12$,

$$\begin{tikzcd} X'_i \arrow[r, "v_i"] \arrow[d, "g_i"'] & X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y'_i \arrow[r, "u_i"'] & Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

gives that $g_i^\times$ is bounded. Since $\mathbf{R}v_{i*}$ is bounded (3.9.2.3), therefore so is

$$g^\times \mathbf{R}u_{i*}u_i^* \underset{(3.10.4)}{\cong} \mathbf{R}v_{i*}g_i^\times u_i^*.$$

Hence, application of the Δ-functor $g^\times$ to the triangle (4.7.3.2) shows that $g^\times$ is bounded above, proving (ii).

As for (i), in view of $(\Delta 3)^*$ of §1.4 it similarly suffices to show (left as an exercise) that the following natural diagram—whose columns are triangles (see (3.8.3)), and where the two middle arrows are isomorphisms by (3.9.3.3), by the inductive hypothesis, and by (3.8.2)(ii) (for the trivial case of an open immersion)—commutes:

$$\begin{tikzcd} \bigoplus_\alpha g^\times F_\alpha \arrow[r] \arrow[d] & g^\times\big(\bigoplus_\alpha F_\alpha\big) \arrow[d] \\ \bigoplus_\alpha\big(\mathbf{R}v_{1*}g_1^\times u_1^*F_\alpha \oplus \mathbf{R}v_{2*}g_2^\times u_2^*F_\alpha\big) \arrow[r, "\sim"] \arrow[d] & \big(\mathbf{R}v_{1*}g_1^\times u_1^* \bigoplus_\alpha F_\alpha\big) \oplus \big(\mathbf{R}v_{2*}g_2^\times u_2^* \bigoplus_\alpha F_\alpha\big) \arrow[d] \\ \bigoplus_\alpha \mathbf{R}v_{12*}g_{12}^\times u_{12}^*F_\alpha \arrow[r, "\sim"] \arrow[d] & \mathbf{R}v_{12*}g_{12}^\times u_{12}^* \bigoplus_\alpha F_\alpha \arrow[d] \\ \bigoplus_\alpha g^\times F_\alpha[1] \arrow[r] & g^\times\big(\bigoplus_\alpha F_\alpha\big)[1] \end{tikzcd}$$

Having thus settled the separated case, we can proceed similarly for arbitrary concentrated Y', with $q(Y')$ the least number of separated open subschemes needed to cover Y'. Q.E.D.

Proposition (4.7.3.3). *Let $f\colon X \to Y$ be a* locally embeddable *scheme-map, i.e., every $y \in Y$ has an open neighborhood V over which the induced map $f^{-1}V \to V$ factors as $f^{-1}V \xrightarrow{i} Z \xrightarrow{p} V$ where i is a closed immersion and p is smooth.* (*For instance, any quasi-projective f satisfies this condition* [**EGA**, II, (5.3.3)].) *If f is quasi-perfect then f is perfect.*

Proof. (i) By (4.7.3.1)(iii), quasi-perfection is local over Y, and the same clearly holds for perfection; so we may as well assume that $X = f^{-1}V$. Then by [**I**, p. 252, Prop. 4.4] it suffices to show that the complex $i_*\mathcal{O}_X$ is perfect, or, more generally, that the map i is quasi-perfect. But i factors as $X \xrightarrow{\gamma} X \times_Y Z \xrightarrow{g} Z$ where γ is the graph of i and g is the projection. The map γ is a local complete intersection [**EGA**, IV, (17.12.3)], so the complex $\gamma_*\mathcal{O}_X$ is perfect, and by Example (4.7.3)(d) (or otherwise) γ is quasi-perfect. Also, g arises from f by flat base change, so by (4.7.3.1)(i), g is quasi-perfect. Hence $i = g\gamma$ is quasi-perfect, as desired. Q.E.D.

Remark. Using the analog of (4.7.3.1)(i) with "quasi-proper" in place of "quasi-perfect" [**LN**, Prop. 4.4], one shows similarly for locally embeddable f that f *quasi-proper* $\Rightarrow$ f *pseudo-coherent.* The converse holds when f is also proper, see (4.3.3.2). Thus, e.g., a projective map is quasi-proper if and only if it is pseudo-coherent.

Exercises **(4.7.3.4).** For a scheme-map $f\colon X \to Y$ and for $E, F \in \mathbf{D}_{\mathsf{qc}}(Y)$, let

$$\chi_{E,F}\colon f^\times E \underline{\underline{\otimes}} \mathbf{L}f^*F \longrightarrow f^\times(E \underline{\underline{\otimes}} F).$$

be the map adjoint to

$$\mathbf{R}f_*(f^\times E \underline{\underline{\otimes}} \mathbf{L}f^*F) \underset{(3.9.4)}{\xrightarrow{\sim}} \mathbf{R}f_*f^\times E \underline{\underline{\otimes}} F \xrightarrow[\text{via } \tau]{} E \underline{\underline{\otimes}} F.$$

In particular, $\chi_{\mathcal{O}_Y,F}$ is the map in (4.7.1)(iv).

(a) Show that for any $E, F, G \in \mathbf{D}_{\mathsf{qc}}(Y)$, the following diagram commutes.

$$\begin{tikzcd}
f^\times E \underline{\underline{\otimes}} (\mathbf{L}f^*F \underline{\underline{\otimes}} \mathbf{L}f^*G) \arrow[r, "\sim", "\text{via (3.2.4)}"'] \arrow[d, "\simeq"'] & f^\times E \underline{\underline{\otimes}} \mathbf{L}f^*(F \underline{\underline{\otimes}} G) \arrow[r, "\chi_{E,F\underline{\underline{\otimes}}G}"'] & f^\times(E \underline{\underline{\otimes}} (F \underline{\underline{\otimes}} G)) \arrow[d, "\simeq"] \\
(f^\times E \underline{\underline{\otimes}} \mathbf{L}f^*F) \underline{\underline{\otimes}} \mathbf{L}f^*G \arrow[r, "\chi_{E,F}\underline{\underline{\otimes}}1"'] & f^\times(E \underline{\underline{\otimes}} F) \underline{\underline{\otimes}} \mathbf{L}f^*G \arrow[r, "\chi_{E\underline{\underline{\otimes}}F,G}"'] & f^\times((E \underline{\underline{\otimes}} F) \underline{\underline{\otimes}} G)
\end{tikzcd}$$

Taking $E = \mathcal{O}_Y$, deduce that *f is quasi-perfect if and only if $\chi_{F,G}$ is an isomorphism for all F and G.* (For this one needs that for any f the map defined in (4.7.1)(iv) is an isomorphism

$$(\#) \qquad f^\times\mathcal{O}_Y \underline{\underline{\otimes}} \mathbf{L}f^*\mathcal{O}_Y \xrightarrow{\sim} f^\times\mathcal{O}_Y,$$

since, e.g., it factors naturally as $f^\times\mathcal{O}_Y \underline{\underline{\otimes}} \mathbf{L}f^*\mathcal{O}_Y \xrightarrow{\sim} f^\times\mathcal{O}_Y \underline{\underline{\otimes}} \mathcal{O}_X \xrightarrow{\sim} f^\times\mathcal{O}_Y$. In fact (#) obtains with any perfect complex in place of $\mathcal{O}_Y$: see [**N**, pp. 227–228 and p. 213]. Cf. also (4.7.5) below.)

Hint. Using 3.4.7(iv), show that the adjoint of the preceding diagram commutes.

(b) Show that, with $\mathbf{1}$ the identity map of Y, the map

$$\chi_{E,F}\colon E \underset{=}{\otimes} F = \mathbf{1}^{\times}E \otimes \mathbf{1}^{*}F \to E \underset{=}{\otimes} F$$

is the identity map

(c) (Compatibility of χ and base change.) In this exercise, v^* is an abbreviation for $\mathbf{L}v^*$, and u^*, f^* and g^* are analogously understood. Also, $\otimes$ stands for $\underset{=}{\otimes}$.

For any independent square

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

show that the following diagram, in which β comes from (4.4.3), and the unlabeled isomorphisms are the natural ones, commutes:

$$\begin{tikzcd}
v^*f^{\times}E \otimes v^*f^*F \arrow[r, "\beta(E)\otimes 1"] \arrow[d, "\simeq"] & g^{\times}u^*E \otimes v^*f^*F \arrow[d, "\simeq"'] \\
v^*(f^{\times}E \otimes f^*F) \arrow[dd, "v^*\chi_{E,F}"'] & g^{\times}u^*E \otimes g^*u^*F \arrow[d, "\chi_{u^*E,u^*F}"] \\
& g^{\times}(u^*E \otimes u^*F) \arrow[d, "\simeq"] \\
v^*f^{\times}(E \otimes F) \arrow[r, "\beta(E\otimes F)"'] & g^{\times}u^*(E \otimes F)
\end{tikzcd}$$

Hint. It suffices to check commutativity of the following natural diagram, whose outer border is adjoint to that of the one in question.

$$\begin{tikzcd}
g_*(v^*f^{\times}E \otimes v^*f^*F) \arrow[rr, "\beta(E)"] \arrow[d, equal] & & g_*(g^{\times}u^*E \otimes v^*f^*F) \arrow[d, "\simeq"] \\
g_*(v^*f^{\times}E \otimes v^*f^*F) \arrow[r, "\sim"] \arrow[dddd, "\simeq"'] & g_*(v^*f^{\times}E \otimes g^*u^*F) \arrow[r, "\beta(E)"] \arrow[d, "p"'] & g_*(g^{\times}u^*E \otimes g^*u^*F) \arrow[d, "p"] \\
& g_*v^*f^{\times}E \otimes u^*F \arrow[r, "\beta(E)"] \arrow[d] & g_*g^{\times}u^*E \otimes u^*F \arrow[d] \\
\arrow[r, phantom, "\text{cf. (3.7.3)}"] & u^*f_*f^{\times}E \otimes u^*F \arrow[r] \arrow[d] & u^*E \otimes u^*F \arrow[d, "\simeq"] \\
& u^*(f_*f^{\times}E \otimes F) \arrow[r] \arrow[d, "p"'] & u^*(E \otimes F) \arrow[dd, equal] \\
g_*v^*(f^{\times}E \otimes f^*F) \arrow[r] \arrow[d] & u^*f_*(f^{\times}E \otimes f^*F) \arrow[d] & \\
g_*v^*f^{\times}(E \otimes F) \arrow[r, "\sim"] & u^*f_*f^{\times}(E \otimes F) \arrow[r] & u^*(E \otimes F)
\end{tikzcd}$$

(d) (Transitivity of χ). If $g\colon Y \to Z$ is a second scheme-map then the following natural diagram is commutative:

$$\begin{tikzcd}
f^{\times}g^{\times}E \underset{=}{\otimes} \mathbf{L}f^{*}\mathbf{L}g^{*}F \arrow[r] \arrow[d, "\simeq"'] & f^{\times}(g^{\times}E \underset{=}{\otimes} \mathbf{L}g^{*}F) \arrow[r] & f^{\times}\big(g^{\times}(E \underset{=}{\otimes} F)\big) \arrow[d, equal] \\
(gf)^{\times}E \underset{=}{\otimes} \mathbf{L}(gf)^{*}F \arrow[r] & (gf)^{\times}(E \underset{=}{\otimes} F) \arrow[r, "\sim"] & f^{\times}g^{\times}(E \underset{=}{\otimes} F)
\end{tikzcd}$$

Hint. Using (3.7.1), show that the adjoint diagram commutes.

(e) Show that $\chi_{E,F}$ corresponds via $(2.6.1)'$ to the composite map

$$\begin{aligned}
f^{\times}E \xrightarrow[\text{natural}]{} f^{\times}\mathbf{R}\mathcal{H}om^{\bullet}(F, E \underset{=}{\otimes} F) &\xrightarrow[(4.2.3)(c)]{\sim} f^{\times}[F, E \underset{=}{\otimes} F]_{Y} \\
&\xrightarrow[(4.2.3)(e)]{\sim} [\mathbf{L}f^{*}F, f^{\times}(E \underset{=}{\otimes} F)]_{X} \\
&\xrightarrow[\text{natural}]{} \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{L}f^{*}F, f^{\times}(E \underset{=}{\otimes} F)).
\end{aligned}$$

(f) With notation as in (4.2.3)(e), and $E, F, G \in \mathbf{D}_{\mathsf{qc}}(Y)$, establish a natural commutative functorial diagram

$$\begin{tikzcd}
f^{\times}F \underset{=}{\otimes} \mathbf{L}f^{*}[E,G]_{Y} \arrow[r, "\chi"] \arrow[d] & f^{\times}\big(F \underset{=}{\otimes} [E,G]_{Y}\big) \arrow[r] & f^{\times}[E, F \underset{=}{\otimes} G]_{Y} \\
f^{\times}F \underset{=}{\otimes} [\mathbf{L}f^{*}E, \mathbf{L}f^{*}G]_{X} \arrow[r] & [\mathbf{L}f^{*}E, f^{\times}F \underset{=}{\otimes} \mathbf{L}f^{*}G]_{X} \arrow[r, "\text{via } \chi"'] & [\mathbf{L}f^{*}E, f^{\times}(F \underset{=}{\otimes} G)]_{X} \arrow[u, "\simeq"']
\end{tikzcd}$$

We adopt again the notations introduced at the beginning of §4.4.

Apropos of the next theorem, recall from the beginning of §4.7 that f quasi-perfect $\Longrightarrow f^{\times}$ bounded.

Theorem (4.7.4). *Let*

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

be an independent square of scheme-maps, with f quasi-perfect. Then for all $E \in \mathbf{D}_{\mathsf{qc}}(Y)$ the base-change map of (4.4.3)—with $^{\times}$ in place of $^{!}$—is an isomorphism

$$\beta(E)\colon v^{*}f^{\times}E \xrightarrow{\sim} g^{\times}u^{*}E.$$

The same holds, with no assumption on f, whenever u is finite and perfect.

Conversely, the following conditions on a scheme-map $f\colon X\to Y$ *are equivalent; and if* Y *is separated and* $f^{\times}$ *bounded above, they imply that* f *is quasi-perfect*:

(i) *For any flat affine universally bicontinuous map* $u\colon Y'\to Y$, (*i.e., for any* $Y''\to Y$ *the resulting projection* $Y'\times_Y Y''\to Y''$ *is a homeomorphism onto its image* [**GD**, p. 249, Défn. (3.8.1)]) *the base-change map associated to the independent fiber square*

$$\begin{array}{ccc} Y'\times_Y X = X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

is an isomorphism $\beta(\mathcal{O}_Y)\colon v^*f^{\times}\mathcal{O}_Y \xrightarrow{\,\sim\,} g^{\times}u^*\mathcal{O}_Y$.

(ii) *The map in* (4.7.1)(iv) *is an isomorphism*

$$\chi_F\colon f^{\times}\mathcal{O}_Y \underset{=}{\otimes} \mathbf{L}f^*F \xrightarrow{\,\sim\,} f^{\times}F$$

whenever F *is a flat quasi-coherent* $\mathcal{O}_Y$*-module.*

PROOF. For the first assertion, using (4.7.3.1)(i) we reduce as in IV of §4.6 to where u, hence v, is an open immersion or affine, so that v is isofaithful ((3.10.2.1)(d) or (3.10.2.2)), and for β to be an isomorphism it suffices that $v_*\beta$ be an isomorphism.

For this purpose it will clearly suffice that the following diagram—in which $\mathcal{O}' := \mathcal{O}_{Y'}$, ϕ is the isomorphism in (3.10.4), θ' is as in (3.10.2) (see (3.10.3)), $\chi := \chi_{E,u_*\mathcal{O}'}$ is as in (4.7.3.4)(a), q is the natural composite isomorphism

$$f^{\times}E\otimes v_*g^*\mathcal{O}' \underset{(3.9.4)}{\xrightarrow{\,\sim\,}} v_*(v^*f^{\times}E\otimes g^*\mathcal{O}') \xrightarrow{\,\sim\,} v_*v^*f^{\times}E$$

and r is the natural composite isomorphism

$$E\otimes u_*\mathcal{O}' \underset{(3.9.4)}{\xrightarrow{\,\sim\,}} u_*(u^*E\otimes \mathcal{O}') \xrightarrow{\,\sim\,} u_*u^*E,$$

—is commutative:

$$\begin{array}{ccccc} f^{\times}E\otimes v_*g^*\mathcal{O}' & \underset{q}{\xrightarrow{\,\sim\,}} & v_*v^*f^{\times}E & \xrightarrow[v_*\beta(E)]{} & v_*g^{\times}u^*E \\ {\scriptstyle 1\otimes\theta'}\big\uparrow{\scriptstyle\simeq} & & & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\phi} \\ f^{\times}E\otimes f^*u_*\mathcal{O}' & \underset{\chi}{\xrightarrow{\,\sim\,}} & f^{\times}(E\otimes u_*\mathcal{O}') & \underset{f^{\times}r}{\xrightarrow{\,\sim\,}} & f^{\times}u_*u^*E \end{array} \tag{4.7.4.1}$$

Since χ is an isomorphism whenever $u_*\mathcal{O}'$ is perfect (see the end of exercise (4.7.3.4)(a)), and since finite maps are isofaithful (3.10.2.2), commutativity of (4.7.4.1) also implies the theorem's assertion about finite perfect u.

Now, commutativity of (4.7.4.1) results from commutativity of the following diagram (4.7.4.1)*, where q' is the composite isomorphism

$$f_*f^\times E\otimes u_*\mathcal{O}' \underset{(3.9.4)}{\xrightarrow{\ \sim\ }} u_*(u^*f_*f^\times E\otimes \mathcal{O}') \xrightarrow{\ \sim\ } u_*u^*f_*f^\times E$$

and t and t' are the natural maps, a diagram whose outer border, with the isomorphism (3.4.9) replaced by its inverse, is adjoint to (4.7.4.1):

$$(4.7.4.1)^*\qquad \begin{tikzcd}
f_*(f^\times E\otimes v_*g^*\mathcal{O}') \arrow[r, "f_*q"] & f_*v_*v^*f^\times E \arrow[r, "f_*v_*\beta"] \arrow[d, equal] & f_*v_*g^\times u^*E \arrow[d, equal] \\
f_*(f^\times E\otimes f^*u_*\mathcal{O}') \arrow[u, "f_*(1\otimes\theta')", "\simeq"'] \arrow[r, phantom, "①"] & u_*g_*v^*f^\times E \arrow[r, "u_*g_*\beta"] \arrow[dr, phantom, "②"] & u_*g_*g^\times u^*E \arrow[d, "u_*t'"] \\
f_*f^\times E\otimes u_*\mathcal{O}' \arrow[u, "(3.9.4)", "\simeq"'] \arrow[r, "\sim", "q'"'] & u_*u^*f_*f^\times E \arrow[u, "u_*\theta", "\simeq"'] \arrow[r, "u^*u_*t"'] & u_*u^*E
\end{tikzcd}$$

Subdiagram ② commutes by the very definition of β.

Expand subdiagram ① as follows, with an arbitrary $F\in\mathbf{D}(X)$ in place of $f^\times E$, with unlabeled maps being the natural ones, and with p denoting projection maps from (3.4.6) or (3.9.4):

$$\begin{tikzcd}
f_*(F\otimes v_*g^*\mathcal{O}') \arrow[r] & f_*(v_*v^*F\otimes v_*g^*\mathcal{O}') \arrow[r, "(3.4.2.1)"] & f_*v_*(v^*F\otimes g^*\mathcal{O}') \arrow[r, "\sim"] \arrow[d, equal] & f_*v_*v^*F \arrow[d, equal] \\
f_*(F\otimes f^*u_*\mathcal{O}') \arrow[u, "\theta'"] \arrow[r] & f_*(v_*v^*F\otimes f^*u_*\mathcal{O}') \arrow[u, "\theta'"] \arrow[dr, phantom, "③"] & u_*g_*(v^*F\otimes g^*\mathcal{O}') \arrow[r, "\sim"] \arrow[dr, phantom, "④"] & u_*g_*v^*F \arrow[dd, equal] \\
f_*F\otimes u_*\mathcal{O}' \arrow[u, "p"] \arrow[r] \arrow[dd, equal] \arrow[ddr, phantom, "⑤"] & f_*v_*v^*F\otimes u_*\mathcal{O}' \arrow[u, "p"] \arrow[d, equal] & & \\
& u_*g_*v^*F\otimes u_*\mathcal{O}' \arrow[r, "(3.4.2.1)"] & u_*(g_*v^*F\otimes \mathcal{O}') \arrow[r, "\sim"] \arrow[uu, "u_*p"] & u_*g_*v^*F \\
f_*F\otimes u_*\mathcal{O}' \arrow[r] & u_*u^*f_*F\otimes u_*\mathcal{O}' \arrow[u, "\theta"] \arrow[r, "(3.4.2.1)"'] & u_*(u^*f_*F\otimes \mathcal{O}') \arrow[u, "\theta"] \arrow[r, "\sim"] & u_*u^*f_*F \arrow[u, "\theta"]
\end{tikzcd}$$

Commutativity of the unlabeled subdiagrams is clear. That of ⑤ follows from the definition (3.7.2)(a) of θ; and that of ④ follows from (3.4.7)(iii). Subdiagram ③ expands as follows:

$$\begin{tikzcd}
f_*(v_*v^*F\otimes v_*g^*\mathcal{O}') \arrow[r, equal] & f_*(v_*v^*F\otimes v_*g^*\mathcal{O}') \arrow[r, "(3.4.2.1)"] & f_*v_*(v^*F\otimes g^*\mathcal{O}') \arrow[dd, equal] \\
& f_*v_*v^*F\otimes f_*v_*g^*\mathcal{O}' \arrow[u, "(3.4.2.1)"] \arrow[d, equal] \arrow[r, phantom, "⑦"] & \\
f_*(v_*v^*F\otimes f^*u_*\mathcal{O}') \arrow[uu, "\theta'"] \arrow[ur, phantom, "⑥"] & u_*g_*v^*F\otimes u_*g_*g^*\mathcal{O}' \arrow[r, "(3.4.2.1)"] \arrow[dr, phantom, "⑧"] & u_*g_*(v^*F\otimes g^*\mathcal{O}') \\
f_*v_*v^*F\otimes u_*\mathcal{O}' \arrow[u, "p"] \arrow[r, equal] & u_*g_*v^*F\otimes u_*\mathcal{O}' \arrow[u] \arrow[r, "(3.4.2.1)"'] & u_*(g_*v^*F\otimes \mathcal{O}') \arrow[u, "u_*p"]
\end{tikzcd}$$

For commutativity of subdiagram ⑧, replace p by its definition (3.4.6), and apply commutativity of (3.6.7.2). Commutativity of ⑦ also follows from that of (3.6.7.2). Finally, subdiagram ⑥ expands as follows:

$$
\begin{tikzcd}
f_*(v_*v^*F \otimes f^*u_*\mathcal{O}') \arrow[rr, "\theta'"] & & f_*(v_*v^*F \otimes v_*g^*\mathcal{O}') \\
f_*v_*v^*F \otimes f_*f^*u_*\mathcal{O}' \arrow[u, "(3.4.2.1)"] \arrow[rr, "\theta'"] & & f_*v_*v^*F \otimes f_*v_*g^*\mathcal{O}' \arrow[u, "(3.4.2.1)"'] \arrow[d, equal] \\
f_*v_*v^*F \otimes u_*\mathcal{O}' \arrow[u] \arrow[r] \arrow[d, equal] & f_*v_*v^*F \otimes u_*g_*g^*\mathcal{O}' \arrow[u, phantom, "⑨"] \arrow[r] & u_*g_*v^*F \otimes u_*g_*g^*\mathcal{O}' \\
f_*v_*v^*F \otimes u_*\mathcal{O}' \arrow[rr, equal] & & u_*g_*v^*F \otimes u_*\mathcal{O}' \arrow[u]
\end{tikzcd}
$$

Commutativity of ⑨ is an easy consequence of the definition (3.7.2)(a) of θ'; and that of the other two subdiagrams is clear.

It is thus established that $(4.7.4.1)^*$ commutes.

We show next that (i) $\Leftrightarrow$ (ii).

Assume (i). Let F be a flat quasi-coherent $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be the $\mathcal{O}_Y$-algebra $\mathcal{O}_Y \oplus F$ with $F^2 = 0$ (i.e., the symmetric algebra on F, modulo everything of degree ≥ 2), and let $u\colon Y' \to Y$ be an affine scheme-map such that $u_*\mathcal{O}_{Y'} = \mathcal{F}$ (see [**GD**, p. 355, (9.1.4) and p. 370, (9.4.4)]). This u is a flat affine universally bicontinuous map. With $E = \mathcal{O}_Y$, all the maps in the commutative diagram (4.7.4.1) other than $\chi = \chi_{\mathcal{O}_Y} \oplus \chi_F$ are isomorphisms, and so χ must be an isomorphism too. But $\chi_{\mathcal{O}_Y}$ is an isomorphism (exercise), so χ_F is an isomorphism, i.e., (ii) holds.

Conversely, if u is any flat affine map and (ii) holds for the flat quasi-coherent $\mathcal{O}_Y$-module $F = u_*\mathcal{O}_{Y'}$ then (4.7.4.1) with $E = \mathcal{O}_Y$ shows that $v_*\beta(\mathcal{O}_Y)$ is an isomorphism, whence, v being affine, so is $\beta(\mathcal{O}_Y)$, see (3.10.2.2).

Finally, assuming (ii) and that Y is separated and $f^\times$ bounded-above, let us deduce that the map $\chi_E\colon f^\times\mathcal{O}_Y \underset{=}{\otimes} \mathbf{L}f^*E \to f^\times E$ is an isomorphism for all $E \in \mathbf{D}_{\mathsf{qc}}(Y)$, so that f is quasi-perfect (see (4.7.1)(iv)).

Since Y is separated, we can replace E by a $\mathbf{D}$-isomorphic q-flat quasi-coherent complex, which is a $\varinjlim$ of bounded-above flat complexes, see [**AJL**, p. 10, (1.1)] and its proof. Since the functors $f^\times\mathcal{O}_Y \underset{=}{\otimes} \mathbf{L}f^*(-)$ and $f^\times(-)$ are both bounded-above, we may assume that E is bounded-below: for each $n \in \mathbb{Z}$, if E' is obtained by replacing all sufficiently-negative-degree components of E by (0) then χ_E and $\chi_{E'}$ induce identical homology maps in degree n, and (1.2.2) can be applied. Similarly, since $f^\times$ is bounded below, and $\mathbf{L}f^*E = f^*E$ when E is a $\varinjlim$ of bounded-above flat complexes, we can reduce further to where E is bounded, flat, and quasi-coherent. Now an induction on the number of nonvanishing components of E (using the triangle [**H**, p. 70, (1)]) gives the desired conclusion. Q.E.D.

For more along these lines see exercise 4.7.6(f) below.

Proposition (4.7.5). *If $f\colon X \to Y$ is quasi-proper and $F \in \mathbf{D}_{\mathsf{qc}}(Y)$ has finite tor-dimension then for all $E \in \mathbf{D}_{\mathsf{qc}}(Y)$ the map $\chi_{E,F}$ of* (4.7.3.4) *is an isomorphism*

$$f^{\times}E \underset{=}{\otimes} \mathsf{L}f^{*}F \xrightarrow{\ \sim\ } f^{\times}(E \underset{=}{\otimes} F).$$

Proof. If $U \hookrightarrow Y$ is an open immersion, then by [**LN**, Prop. 4.4], the projection $X \times_Y U \to U$ is quasi-proper. Together with (4.4.3) and (4.7.3.4)(c), this implies that the assertion in (4.7.5) is local on Y, so we may assume that Y is affine.

We can then replace F by a **D**-isomorphic bounded-above quasi-coherent complex—see (3.9.6)(a)—which by [**H**, p. 42, 4.6 1)] (dualized) may be assumed flat. Since F has finite tor-dimension, an application of [**I**, p. 131, 5.1.1] to a suitable **D**-isomorphic truncation of F allows one to assume further that F is bounded. Then an induction on the number of nonvanishing components of F (using the triangle [**H**, p. 70, (1)]) reduces the problem to where F is a single flat quasi-coherent $\mathcal{O}_Y$-module.

As in the proof of (4.7.4) ((i) $\Leftrightarrow$ (ii)), let $u\colon Y' \to Y$ be an affine scheme-map such that $u_*\mathcal{O}_{Y'} = \mathcal{O}_Y \oplus F$. The map u is flat, so u and f are two sides of an independent square, and by (4.4.3) the corresponding base-change map $\beta(E)$ in the commutative diagram (4.7.4.1) is an isomorphism. One concludes as before that $\chi_{E,F}$ is an isomorphism. Q.E.D.

Exercises **(4.7.6).** (a). Let $f\colon X \to Y$ be a quasi-perfect scheme-map. Assume that X is *divisorial*—i.e., X has an ample family of invertible $\mathcal{O}_X$-modules—so that by [**I**, p. 173, 2.2.8 b)] every pseudo-coherent $\mathcal{O}_X$-complex is **D**-isomorphic to a bounded above complex of finite-rank locally free $\mathcal{O}_X$-modules. Show that an $\mathcal{O}_X$-complex F is pseudo-coherent iff for every $n \in \mathbb{Z}$ there is a triangle $P \to F \to R \to P[1]$ with P perfect and $R \in (\mathbf{D}_{\mathsf{qc}})_{<\mathbf{n}}$; and using (3.9.2.3) above, deduce that f is quasi-proper.

(A similar result without the divisoriality assumption is [**LN**, Thm. 4.1].)

(b). Let $f\colon X \to Y$ be a quasi-proper scheme-map. Let $r \in \mathbb{Z}$ and let $(G_\alpha)_{\alpha\in A}$ be a family of complexes in $\mathbf{D}_{\mathsf{qc}}(X)_{\geq \mathbf{r}}$, i.e., for every α, $H^m(G_\alpha) = 0$ whenever $m < r$. Show that the natural map is an isomorphism[52]

$$\underset{\alpha}{\oplus} f^{\times}G_\alpha \xrightarrow{\ \sim\ } f^{\times}(\underset{\alpha}{\oplus} G_\alpha)\,.$$

Hint. Write f_* for $\mathsf{R}f_*$, $\mathcal{H}_X$ for $\mathsf{R}\mathcal{H}om^{\bullet}_X$, etc. The triangulated category $\mathbf{D}_{\mathsf{qc}}(X) \equiv \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X))$ is generated by perfect complexes (see [**N**, pp. 215–216], or [**LN**, Thm. 4.2]), so a $\mathbf{D}_{\mathsf{qc}}$-map $\varphi\colon A_1 \to A_2$ is an isomorphism iff the induced map $\mathrm{Hom}(E, A_1) \to \mathrm{Hom}(E, A_2)$ is an isomorphism for all perfect $E \in \mathbf{D}(X)$. In the following natural diagram, easily seen to commute,

$$\begin{tikzcd}
f_*\mathcal{H}_X(E, \oplus f^{\times}G_\alpha) \arrow[r] & f_*\mathcal{H}_X(E,\, f^{\times}(\oplus G_\alpha)) \arrow[r, "\sim", "(4.3.6)"'] & \mathcal{H}_Y(f_*E,\, \oplus G_\alpha) \\
f_*(\oplus\mathcal{H}_X(E,\, f^{\times}G_\alpha)) \arrow[u] & \oplus f_*\mathcal{H}_X(E,\, f^{\times}G_\alpha) \arrow[l, "\sim"', "(3.9.3.1)"] \arrow[u] \arrow[r, "\sim", "(4.3.6)"'] & \oplus\mathcal{H}_Y(f_*E,\, G_\alpha) \arrow[u]
\end{tikzcd}$$

the left and right vertical arrows are isomorphisms whenever E is pseudo-coherent.

[52] Cf. [**V′**, p. 396, Lemma 1], where the necessary *uniform* lower bound on the G_α is omitted.

(The question being local on X, one can, as in the proof of (4.3.5), replace E by a bounded finite-rank free complex E' and then, using the triangle [**H**, p. 70, (1)], proceed by induction on the number of degrees in which E' doesn't vanish.) Finally, apply the functor $\mathrm{H}^0\mathbf{R}\Gamma(Y,-)$.

(c) Deduce from (b) that *a quasi-proper scheme-map f with $f^\times$ bounded above is quasi-perfect.* (This is part of [**LN**, Thm. 1.2.])

(d) Let $f\colon X\to Y$ be a scheme-map. Show that if f is quasi-perfect then the following two conditions hold, and that the converse is true when $f^\times$ is bounded. (Apropos, recall again from the beginning of this section that f quasi-perfect $\implies f^\times$ bounded.)

(i) If $u\colon Y'\to Y$ is an open immersion, and $v\colon f^{-1}U\to X$, $g\colon f^{-1}U\to U$ are the obvious induced maps, then the base-change map is an isomorphism

$$\beta(\mathcal{O}_Y)\colon v^*f^\times\mathcal{O}_Y \xrightarrow{\ \sim\ } g^\times u^*\mathcal{O}_Y.$$

Equivalently (see subsection V in §4.6), for all $F\in\mathbf{D}_{\mathsf{qc}}(X)$ the duality map $\delta(F,\mathcal{O}_Y)$ defined as in (4.4.2) is an isomorphism

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F,\, f^\times\mathcal{O}_Y) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F,\mathcal{O}_Y)$$

(ii) If (F_α) is a small filtered direct system of flat quasi-coherent $\mathcal{O}_Y$-modules then for all $n\in\mathbb{Z}$ the natural map is an isomorphism

$$\varinjlim_\alpha H^n(f^\times F_\alpha) \xrightarrow{\ \sim\ } H^n(f^\times \varinjlim_\alpha F_\alpha).$$

Hint. Use (4.7.3.4)(c) and Lazard's theorem that over a commutative ring A any flat module is a $\varinjlim$ of finite-rank free A-modules [**GD**, p. 163, (6.6.24)] to show that (i) and (ii) imply condition (ii) in (4.7.4).

(e) (i) (Neeman). Using, e.g., (i) in (d) (with $F=\mathcal{O}_X$), show that if $f\colon X\to Y$ is quasi-perfect then the $\mathcal{O}_Y$-complex $\mathbf{R}f_*f^\times\mathcal{O}_Y$ is perfect; and deduce that for any perfect $\mathcal{O}_Y$-complex E, $\mathbf{R}f_*f^\times E$ is perfect.

(ii) (cf. [**I**, p. 257, 4.8]). Let $f\colon X\to Y$ be a concentrated quasi-proper map of quasi-compact schemes. Then for any f-perfect $\mathcal{O}_X$-complex E, $\mathbf{R}f_*E$ is a perfect $\mathcal{O}_Y$-complex.

(f) Let $U\xrightarrow{u}X\xrightarrow{f}Y$ be scheme-maps, with f quasi-proper, and let $E\in\mathbf{D}_{\mathsf{qc}}(Y)$. Show that the following are equivalent.

(i) The functor $\mathbf{L}u^*f^\times(E\underset{=}{\otimes}F)$ $(F\in\mathbf{D}_{\mathsf{qc}}(Y))$ is bounded above.

(ii) $\mathbf{L}u^*f^\times E\in\overline{\mathbf{D}}^-(X))$, and the map (see exercise (4.7.3.4) above)

$$\mathbf{L}u^*\chi_{E,F}\colon \mathbf{L}u^*f^\times E\underset{=}{\otimes}\mathbf{L}(fu)^*F\to\mathbf{L}u^*f^\times(E\underset{=}{\otimes}F),$$

is an isomorphism for all $F\in\mathbf{D}_{\mathsf{qc}}(Y)$.

(iii) $\mathbf{L}u^*f^\times E\in\overline{\mathbf{D}}^-(X))$, and the functor $\mathbf{L}u^*f^\times(E\underset{=}{\otimes}F)$ $(F\in\mathbf{D}_{\mathsf{qc}}(Y))$ respects direct sums (cf. (4.7.1)(i)).

Moreover, if u has finite tor-dimension, then the following are equivalent.

(i)$'$ The functor $\mathbf{L}u^*f^\times(E\underset{=}{\otimes}F)$ $(F\in\mathbf{D}_{\mathsf{qc}}(Y))$ is bounded.

(ii)$'$ The complex $\mathbf{L}u^*f^\times E$ has finite flat fu-amplitude (2.7.6), and $\mathbf{L}u^*\chi_{E,F}$ is an isomorphism for all $F\in\mathbf{D}_{\mathsf{qc}}(Y)$.

(iii)$'$ $\mathbf{L}u^*f^\times E$ has finite flat fu-amplitude, and the functor $\mathbf{L}u^*f^\times(E\underset{=}{\otimes}F)$ $(F\in\mathbf{D}_{\mathsf{qc}}(Y))$ respects direct sums.

Hint. Given (i), one sees as in exercise (c) above that the functor $\mathbf{L}u^*f^\times(E\underset{=}{\otimes}F)$ respects direct sums; and then arguing as in [**N**, p. 226, Thm. 5.4], one see that $\mathbf{L}u^*\chi_{E,F}$ in (ii) is an isomorphism. It follows then from [**I**, p. 242, 3.3(iv)], and the fact that if $V\subset Y$ is open then any quasi-coherent $\mathcal{O}_V$-module M is the restriction of a quasi-coherent $\mathcal{O}_Y$-module, that if (i)$'$ holds then $\mathbf{L}u^*f^\times E$ has finite flat fu-amplitude.

4.8. Two fundamental theorems

Up to now we have dealt with the pseudofunctor ${}^\times$ (see (4.1.1)) for quite general maps—it cost nothing to do so. But for non-proper maps this pseudofunctor may still be of limited interest (see [**De′**, p. 416, line 3]).

As indicated in the Introduction to these notes, Grothendieck Duality is fundamentally concerned with a $\overline{\mathbf{D}}^+_{\mathsf{qc}}$-valued pseudofunctor ${}^!$ over the category of say, separated finite-type maps of noetherian schemes, agreeing with ${}^\times$ on proper maps, but, unlike ${}^\times$ (see (4.2.3)(d)), agreeing with the usual pseudofunctor * on open immersions (more generally, on separated étale maps see [**EGA**, IV, §§17.3, 17.6]), and compatible in a suitable sense with flat base change. The existence and uniqueness, up to isomorphism, of this remarkable pseudofunctor is given by Theorem (4.8.1), and its behavior vis-à-vis flat base change is described in Theorem (4.8.3).

The proof of (4.8.1) presented here is based on a formal method of Deligne for pasting pseudofunctors (see Proposition (4.8.4)), and on the compactification theorem of Nagata, that any finite-type separable map of noetherian schemes factors as an open immersion followed by a proper map (see [**Lt**], [**C′**], [**Vj**]). The proof of (4.8.3) is based on a formal pasting procedure for *base-change setups* (see (4.8.2), (4.8.5)).

There are other pasting techniques, due to Nayak [**Nk**], to establish the two basic theorems, (4.8.1) and (4.8.3).[53] As mentioned in the Introduction, Nayak's methods avoid using Nagata's theorem, and so apply in contexts where Nagata's theorem may not hold. For example, the results in [**Nk**, §7.1] are generalizations of (4.8.1) and (4.8.3) to the case of noetherian formal schemes (except for "thickening" as in (4.8.11) below, which allows flat base-change isomorphisms for *admissible squares* (4.8.3.0) rather than just fiber squares, see Exercise (4.8.12)(d).)

All commutative squares will be considered to be *oriented,* as in §3.10.

The first main result defines (up to isomorphism) the *twisted inverse image pseudofunctor.*

Theorem (4.8.1). *On the category* $\mathbf{S_f}$ *of finite-type separated maps of noetherian schemes, there is a* $\overline{\mathbf{D}}^+_{\mathsf{qc}}$*-valued pseudofunctor* ${}^!$ *that is uniquely determined up to isomorphism by the following three properties*:

(i) *The pseudofunctor* ${}^!$ *restricts on the subcategory of proper maps to a right adjoint of the derived direct-image pseudofunctor, see* (3.6.7)(d)*.*

(ii) *The pseudofunctor* ${}^!$ *restricts on the subcategory of étale maps to the usual inverse-image pseudofunctor* **.*

[53] [**Nk**, §7.5] discusses the relation between Nayak's methods and Deligne's. On the other hand, in [**Nk′**] Nayak extends Nagata compactification—and hence Theorems (4.8.1) and (4.8.3)—to separated maps which are *essentially* of finite type.

(iii) *For any fiber square in* $\mathbf{S_f}$:

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "u"'] & \bullet \end{tikzcd} \qquad (f, g \text{ proper; } u, v \text{ étale}),$$

the base-change map β_σ *of* (4.4.3) *is the natural composite isomorphism*

$$v^*f^! = v^!f^! \xrightarrow{\sim} (fv)^! = (ug)^! \xrightarrow{\sim} g^!u^! = g^!u^*.$$

Remark **(4.8.1.1).** It follows that when f is both étale and proper (hence by [**EGA**, III, 4.4.11], finite), then the natural map $f_*f^* = f_*f^! \to \mathbf{1}$ is precisely—not just up to isomorphism—the standard trace map, see Exercise (4.8.12)(b)(vii).

For subsequent considerations, involving base-change isomorphisms and their properties, the following definition will be convenient to have.

Definition (4.8.2). A *base-change setup* $\mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ consists of the following data (a)–(d), subject to conditions (1)–(3):

(a) Subcategories $\mathbf{P}$ and $\mathbf{F}$ of a category $\mathbf{S}$, each containing every object of $\mathbf{S}$.

(b) Contravariant pseudofunctors ${}^!$ on $\mathbf{P}$ and * on $\mathbf{F}$ such that for all objects $X \in \mathbf{S}$, the categories $\mathbf{X}^!$ and $\mathbf{X}^*$ coincide (see §3.6.5).

(c) A class $\square$ of (oriented) commutative $\mathbf{S}$-squares, the *distinguished squares,* each member of which has the form

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "u"'] & \bullet \end{tikzcd} \qquad (f, g \in \mathbf{P};\ u, v \in \mathbf{F})$$

(where u precedes f in the orientation of σ, see §3.10).

(d) For each distinguished σ as in (c), an isomorphism of functors

$$\beta_\sigma \colon v^*f^! \xrightarrow{\sim} g^!u^*.$$

(1) If two commutative $\mathbf{S}$-squares

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "u"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "v_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma_1"] & \bullet \arrow[d, "f_1"] \\ \bullet \arrow[r, "u_1"'] & \bullet \end{tikzcd}$$

are *isomorphic,* i.e., there exists a commutative cube with front and rear faces σ and σ_1 respectively, and i, i_1, j, j_1 isomorphisms:

$$\begin{tikzcd}
& \bullet \arrow[rr, "v_1"] \arrow[dl, "j_1"'] \arrow[dd, "g_1" near start] & & \bullet \arrow[dl, "j"'] \arrow[dd, "f_1"] \\
\bullet \arrow[rr, "v" near end] \arrow[dd, "g"'] & & \bullet \arrow[dd, "f" near start] & \\
& \bullet \arrow[rr, "u_1" near start] \arrow[dl, "i_1"'] & & \bullet \arrow[dl, "i"] \\
\bullet \arrow[rr, "u"'] & & \bullet &
\end{tikzcd}$$

then σ is distinguished $\Leftrightarrow$ σ_1 is distinguished.

(2) For every $\mathbf{P}$-map f, the square

$$\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}$$

is distinguished, and $\beta_\sigma \colon f^! \to f^!$ is the identity map.

(2)′ For every $\mathbf{F}$-map u, the square

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "1"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "1"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}$$

is distinguished, and $\beta_\sigma \colon u^* \to u^*$ is the identity map.

(3) (Horizontal and vertical transitivity.) If the square $\sigma_0 = \sigma_2 \circ \sigma_1$ (with g resp. v deleted)

$$\begin{tikzcd}
\bullet \arrow[r, "v_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma_1"] & \bullet \arrow[r, "v_2"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma_2"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u_1"'] & \bullet \arrow[r, "u_2"'] & \bullet
\end{tikzcd}
\qquad \textit{resp.} \qquad
\begin{tikzcd}
\bullet \arrow[r, "w"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma_1"] & \bullet \arrow[d, "f_1"] \\
\bullet \arrow[r, "v"] \arrow[d, "g_2"'] \arrow[dr, phantom, "\sigma_2"] & \bullet \arrow[d, "f_2"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}$$

as well as its constituents σ_2 and σ_1 are all distinguished, then the corresponding natural diagram of functorial maps commutes:

$$
\begin{tikzcd}
(v_2v_1)^*f^! \arrow[rr, "\beta_{\sigma_0}"] \arrow[d, "\simeq"'] & & h^!(u_2u_1)^* \arrow[d, "\simeq"] \\
v_1^*v_2^*f^! \arrow[r, "v_1^*\beta_{\sigma_2}"'] & v_1^*g^!u_2^* \arrow[r, "\beta_{\sigma_1}"'] & h^!u_1^*u_2^*
\end{tikzcd}
$$

resp.

$$
\begin{tikzcd}
(g_2g_1)^!u^* \arrow[d, "\simeq"'] & & w^*(f_2f_1)^! \arrow[ll, "\beta_{\sigma_0}"'] \arrow[d, "\simeq"] \\
g_1^!g_2^!u^* & g_1^!v^*f_2^! \arrow[l, "g_1^!\beta_{\sigma_2}"] & w^*f_1^!f_2^! \arrow[l, "\beta_{\sigma_1}"]
\end{tikzcd}
$$

REMARKS **(4.8.2.1).** (a) Let u and v be $\mathbf{S}$-isomorphisms. If f and g are $\mathbf{S}$-maps such that $fv = ug$ is in $\mathbf{P}$, then the squares

$$
\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "ug"'] \arrow[dr, phantom, "\tilde\sigma"] & \bullet \arrow[d, "fv"] \\
\bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}
$$

are isomorphic, so that by (1) and (2), σ is distinguished—which entails that u and v are in $\mathbf{F}$ and that f and g are in $\mathbf{P}$. In particular,

$$
\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "v"'] & \bullet \arrow[d, "v^{-1}"] \\
\bullet \arrow[r, "v^{-1}"'] & \bullet
\end{tikzcd}
$$

is distinguished, so that every $\mathbf{S}$-isomorphism lies in $\mathbf{P} \cap \mathbf{F}$ (whence $fv \in \mathbf{P} \iff f \in \mathbf{P}$, and $ug \in \mathbf{P} \iff g \in \mathbf{P}$).

Similarly, if f and g are $\mathbf{S}$-isomorphisms, and u and v are any $\mathbf{F}$-maps such that $fv = ug$, then σ is distinguished.

(b) That the isomorphism β_σ in (2) is idempotent, hence the identity, actually follows from (3), with $u_i = v_i = 1$ (resp. $f_i = g_i = 1$).

(c) To each base-change setup $\mathcal{B} = \mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ is associated a *dual setup* $\mathcal{B}^{\mathsf{op}} := \mathcal{B}\big(\mathbf{S}, \mathbf{F}, \mathbf{P}, {}^*, {}^!, (\beta_{\sigma'} := \beta_\sigma^{-1})_{\sigma'\in\square'}\big)$, where σ' is the transpose of σ (i.e., σ with its orientation reversed, or, visually, the reflection of σ in its upper-left to lower-right diagonal), and $\square'$ consists of all transposes of squares in $\square$.

Example **(4.8.2.2).** Let $\mathbf{S}$ be a category, take $\mathbf{P}=\mathbf{F}=\mathbf{S}$, let $^{!}={}^{*}$ be a contravariant pseudofunctor on $\mathbf{S}$, let all commutative squares in $\mathbf{S}$ be distinguished, and for any such square σ, let

$$\beta_\sigma\colon v^*f^* \xrightarrow{\ \sim\ } (fv)^* = (ug)^* \xrightarrow{\ \sim\ } g^*u^*$$

be the isomorphism naturally associated with the pseudofunctor *.

Then (4.8.2)(1) holds trivially, and (2), (2)′, (3) follow readily from the definition of "pseudofunctor."

We will denote such a base-change setup by $\mathcal{B}(\mathbf{S}, {}^{*})$.

Example **(4.8.2.3).** Let $\mathbf{S}$ be a subcategory of the category of quasi-compact separated schemes, $\mathbf{P}\subset\mathbf{S}$ the subcategory of *quasi-proper* maps, and $\mathbf{F}\subset\mathbf{S}$ the subcategory of *finite-tor-dimension* maps. On $\mathbf{P}$ there is the $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}$-valued pseudofunctor $^{\times}$ (see (4.1.2)); and on $\mathbf{F}$ there is the $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}$-valued pseudofunctor * with $u^*:=\mathsf{L}u^*$ for any $\mathbf{F}$-map u. Let $\square$ be the class of *independent fiber squares* of the form specified in 4.8.2(c). For $\sigma\in\square$, let $\beta_\sigma\colon v^*f^\times\to g^\times u^*$ be the corresponding base-change isomorphism from (4.4.3).

Conditions (1), (2) and (2)′ in (4.8.2) are then easily verified; and as in (4.6.8), (3) follows formally from (3.7.2), (ii) and (iii). So we have a base-change setup $\mathcal{B}\big(\mathbf{S},\mathbf{P},\mathbf{F},{}^{\times},{}^{*},(\beta_\sigma)_{\sigma\in\square}\big)$.

Example **(4.8.2.4).** As a special case, we have the base-change setup $\mathcal{B}\big(\mathbf{S_f},\mathbf{P},\mathbf{E},{}^{\times},{}^{*},(\beta_\sigma)_{\sigma\in\square}\big)$ with $\mathbf{S_f}$ as in (4.8.1), $\mathbf{P}\subset\mathbf{S_f}$ the subcategory of proper maps, $\mathbf{E}\subset\mathbf{S_f}$ the subcategory of étale maps, and $^{\times}$, *, $\square$, β_σ as in the preceding example (4.8.2.3) (with $\mathbf{F}$ replaced by $\mathbf{E}$).

To prove (4.8.1), we will need to show that there is a unique way to enlarge the preceding setup to a setup $\mathcal{B}\big(\mathbf{S_f},\mathbf{P},\mathbf{E},{}^{\times},{}^{*},(\beta'_\sigma)_{\sigma\in\square'}\big)$ where $\square'$ consists of *all* commutative $\mathbf{S_f}$-squares

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

with f,g proper and u,v étale.

This, and more, will be done in (4.8.11). Meanwhile, we'll refer to this *unique* enlarged setup as Example (4.8.2.4)′.

Notation-Definition (4.8.3.0). A category $\mathbf{S}$ having been given, for $\mathbf{S}$-maps v,f,g,u with $fv=ug$, $\sigma_{v,f,g,u}$ is the commutative square

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "g"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "u"'] & \bullet \end{tikzcd}$$

In the category of schemes, such a $\sigma_{v,f,g,u}$:

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

is an *admissible square* if u is flat, f is finitely presentable, and in the associated diagram

$$\begin{tikzcd} X' \arrow[r, "i"] & X \times_Y Y' \arrow[r, "q_1"] \arrow[d, "q_2"'] & X \arrow[d, "f"] \\ & Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

where q_1, q_2 are the projections, $q_1 i = v$ and $q_2 i = g$, the map i is *étale.* (Note that then $g = q_2 i$ is finitely presentable, and $v = q_1 i$ is flat, so that $\mathsf{L}v^* = v^*$.)

Theorem (4.8.3). *Let* $\mathbf{S}$ *be the category of separated maps of noetherian schemes, let* $\mathbf{S_f} \subset \mathbf{S}$ *and* $^!$ *be as in* (4.8.1), *let* $\mathbf{F} \subset \mathbf{S}$ *be the subcategory of flat maps, and let* * *be the usual* $\overline{\mathbf{D}}^+_{\mathsf{qc}}$*-valued inverse-image pseudofunctor on* $\mathbf{F}$*. Then there is a unique base-change setup* $\mathcal{B}\big(\mathbf{S}, \mathbf{S_f}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ *with* $\square$ *the class of admissible* $\mathbf{S}$*-squares, such that the following conditions hold for any admissible* $\mathbf{S}$*-square* $\sigma = \sigma_{v,f,g,u}$:

(i) *If* σ *is a fiber square with* f *proper then* β_σ *is the base-change isomorphism in* (4.4.3).

(ii) *If* f*—and hence* g*—is étale, so that* $f^! = f^*$ *and* $g^! = g^*$*, then* β_σ *is the natural isomorphism* $v^*f^* \xrightarrow{\sim} g^*u^*$.

(iii) *If* u*—and hence* v*—is étale, so that* $u^* = u^!$ *and* $v^* = v^!$*, then* β_σ *is the natural isomorphism* $v^!f^! \xrightarrow{\sim} g^!u^!$.

Remarks **(4.8.3.1).** (a) Since étale maps are unramified [**EGA**, IV, (17.6.2)], therefore by [**EGA**, IV, (17.3.3)(iii) and (17.3.4)], every commutative $\mathbf{S_f}$-square $\sigma_{v,f,g,u}$ with u and v flat and such that either f and g or u and v are étale is admissible.

(b) *Uniqueness* in (4.8.3) is implied by (i), (ii) and vertical transitivity as in (4.8.2)(3), because if $\sigma_{v,f,g,u}$ is admissible, then, by Nagata's theorem, $f = f_2 f_1$ with f_2 proper and f_1 an open immersion, whence σ decomposes as in the second diagram in (4.8.2)(3), with σ_1 having v, w flat and f_1, g_1 étale, and with σ_2 an admissible fiber square.

(c) As for *existence,* the preceding suggests defining β_σ via a choice of such factorizations, one for each f, then showing that the definition does not depend on the choice, and that (i)–(iii) in (4.8.3) are satisfied.

This purely formal procedure is straightforward in principle but, as will emerge, lengthy in practice.

In view of Nagata's compactification theorem, it is readily verified that the *existence* of the pseudofunctor $^{\mathbf{!}}$ in Theorem (4.8.1) results from the next Proposition (4.8.4) on the pasting of pseudofunctors, as applied to the base-change setup $(4.8.2.4)'$.

Proposition 4.8.4 ([**De**, p. 318, Prop. 3.3.4])**.** *Let there be given a base-change setup* $\mathcal{B} = \mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{E}, {}^{\times}, {}^{*}, (\beta_\sigma)_{\sigma\in\square}\big)$ *such that:*

(a) *the fiber product in* $\mathbf{P}$ *of any two* $\mathbf{P}$*-maps with the same target exists, and is a fiber product in* $\mathbf{S}$ *of the same two maps;*

(b) *every map* $f \in \mathbf{S}$ *has a "compactification," i.e., a factorization* $f = \bar f i$ *with* $\bar f \in \mathbf{P}$ *and* $i \in \mathbf{E}$; *and*

(c) $\square$ *consists of all of the commutative* $\mathbf{S}$*-squares* $\sigma_{v,f,g,u}$ *for which* $f, g \in \mathbf{P}$ *and* $u, v \in \mathbf{E}$.

Then there exists a contravariant pseudofunctor $^{\mathbf{!}}$ *on* $\mathbf{S}$, *uniquely determined up to isomorphism by the properties that* $X^{\mathbf{!}} = X^{\times} = X^{*}$ *for all* $X \in \mathbf{S}$ *and that there exist isomorphisms of pseudofunctors* (*see* (3.6.6)) $\alpha_{\mathbf{P}}\colon {}^{\mathbf{!}}|_{\mathbf{P}} \xrightarrow{\sim} {}^{\times}$ *and* $\alpha_{\mathbf{E}}\colon {}^{\mathbf{!}}|_{\mathbf{E}} \xrightarrow{\sim} {}^{*}$ *such that for any* $\sigma = \sigma_{v,f,g,u} \in \square$, β_σ *is the natural composition* (*with first and last isomorphisms coming from* $\alpha_{\mathbf{P}}$ *and* $\alpha_{\mathbf{E}}$):

$$v^*f^\times \xrightarrow{\sim} v^!f^! \xrightarrow{\sim} (fv)^! = (ug)^! \xrightarrow{\sim} g^!u^! \xrightarrow{\sim} g^\times u^*.$$

In other words, $\mathcal{B}(\mathbf{S}, {}^{\mathbf{!}})$ (*see* (4.8.2.2)) *extends* $\mathcal{B}$, *via* $\alpha_{\mathbf{P}}$ *and* $\alpha_{\mathbf{E}}$.

In fact there is a $^{\mathbf{!}}$ *such that, furthermore,* ${}^{\mathbf{!}}|_{\mathbf{E}} = {}^{*}$ *and* $\alpha_{\mathbf{E}}$ *is the identity isomorphism.*

Remark. *Uniqueness* (up to isomorphism) in (4.8.1) also results from (4.8.4), as follows. Let $\mathbf{P} \subset \mathbf{S_f}$, $\mathbf{E} \subset \mathbf{S_f}$ and $\square'$ be as in (4.8.2.4). If the pseudofunctor $^{\mathbf{!}}$ satisfies the conditions in (4.8.1) then there is a natural pseudofunctorial isomorphism $\alpha_{\mathbf{P}}\colon {}^{\mathbf{!}}|_{\mathbf{P}} \xrightarrow{\sim} {}^{\times}|_{\mathbf{P}}$ (since both ${}^{\mathbf{!}}|_{\mathbf{P}}$ and ${}^{\times}|_{\mathbf{P}}$ have the same pseudofunctorial left adjoint). For any $\sigma_{v,f,g,u} \in \square'$ let β''_σ be the natural composite isomorphism

$$v^*f^\times \underset{v^*\alpha_{\mathbf{P}}^{-1}}{\xrightarrow{\sim}} v^*f^! = v^!f^! \xrightarrow{\sim} g^!u^! = g^!u^* \underset{\alpha_{\mathbf{P}}}{\xrightarrow{\sim}} g^\times u^*.$$

This gives a setup $\mathcal{B}'' = \mathcal{B}\big(\mathbf{S_f}, \mathbf{P}, \mathbf{E}, {}^{\times}, {}^{*}, (\beta''_\sigma)_{\sigma\in\square'}\big)$. (Check directly, or see Exercise (4.8.12)(a).) When σ is a fiber square then, one checks, β''_σ is the base change map of (4.4.3). Thus $\mathcal{B}''$ is the unique enlargement $(4.8.2.4)'$ of the setup (4.8.2.4), so that the uniqueness assertion in (4.8.4) gives the uniqueness in (4.8.1).

Proof of (4.8.4). (Outline: more details are in [**De**, pp. 304–318].[54])

If the pseudofunctor $^!$ exists then to each compactification $f=\bar f i$ there is naturally associated an isomorphism $f^! \xrightarrow{\,\sim\,} i^*\bar f^{\times}$; and for a composite **S**-map f_1f_2 and compactifications $f_1=\bar f_1 i_1,\ f_2=\bar f_2 i_2,\ i_1\bar f_2=\bar g j$, with $\sigma:=\sigma_{j,\bar g,\bar f_2,i_1}$, the canonical isomorphism $f_2^!f_1^! \xrightarrow{\,\sim\,} (f_1f_2)^!$ factors naturally as

$$
\begin{aligned}
(4.8.4.1)\qquad (\bar f_2 i_2)^!(\bar f_1 i_1)^! &\xrightarrow{\,\sim\,} i_2^*\bar f_2^{\times} i_1^*\bar f_1^{\times} \xrightarrow[\beta_\sigma^{-1}]{\,\sim\,} i_2^* j^* \bar g^{\times}\bar f_1^{\times}\\
&\xrightarrow{\,\sim\,} (ji_2)^*(\bar f_1\bar g)^{\times} \xrightarrow{\,\sim\,} (\bar f_1\bar g j i_2)^! = (\bar f_1 i_1 \bar f_2 i_2)^!.
\end{aligned}
$$

If $^{!!}$ is another pseudofunctor with the same property as $^!$ then for each compactification $f=\bar f i$ we have a natural composite functorial isomorphism

$$
(4.8.4.2)\qquad f^! = (\bar f i)^! \xrightarrow{\,\sim\,} i^!\bar f^! \xrightarrow{\,\sim\,} i^*\bar f^{\times} \xrightarrow{\,\sim\,} i^{!!}\bar f^{!!} \xrightarrow{\,\sim\,} (\bar f i)^{!!} = f^{!!}.
$$

One must show that (4.8.4.2) depends only on the **S**-map $f\colon X\to Y$, not on any particular compactification. Then it is a simple exercise to check via (4.8.4.1) that these isomorphisms, for variable f, constitute an isomorphism of pseudofunctors, giving *uniqueness* of $^!$ (up to a pseudofunctorial isomorphism—itself unique if we require compatibility with $\alpha_{\mathbf P}$ and $\alpha_{\mathbf E}$).

For comparing (4.8.4.2) relative to various compactifications of f,

$$
(i_s,\bar f_s) := \big(X \xrightarrow{i_s} X_s \xrightarrow{\bar f_s} Y\big),
$$

let $[(i_1,\bar f_1),(i_2,\bar f_2)]$ be the natural composite isomorphism

$$
i_2^*\bar f_2^{\times} \xrightarrow{\,\sim\,} i_2^!\bar f_2^! \xrightarrow{\,\sim\,} f^! \xrightarrow{\,\sim\,} i_1^!\bar f_1^! \xrightarrow{\,\sim\,} i_1^*\bar f_1^{\times}.
$$

Noting that the compactifications of f are the objects of a category $\mathcal C$ in which a morphism $(i_1,\bar f_1)\to(i_2,\bar f_2)$ is a **P**-map $g\colon X_1\to X_2$ such that $gi_1=i_2$ and $\bar f_2 g=\bar f_1$, one shows the following identity, transitivity and normalization properties (sketch the diagrams!):

(i) $[(i_1,\bar f_1),(i_1,\bar f_1)] = \text{identity}$.

(ii) $[(i_1,\bar f_1),(i_2,\bar f_2)]\circ[(i_2,\bar f_2),(i_3,\bar f_3)] = [(i_1,\bar f_1),(i_3,\bar f_3)]$.

(iii) For any $g\colon(i_1,\bar f_1)\to(i_2,\bar f_2)$, and $\sigma:=\sigma_{i_1,g,1,i_2}$, the isomorphism $[(i_2,\bar f_2),(i_1,\bar f_1)]$ factors naturally as $i_1^*\bar f_1^{\times} \xrightarrow{\,\sim\,} i_1^*g^{\times}\bar f_2^{\times} \xrightarrow[\beta_\sigma]{\,\sim\,} i_2^*\bar f_2^{\times}$.

---

[54] where there are a few minor misprints (for example, (3.2.4.∗) should be (3.2.5.∗)), and omissions of symbols.

Making use of condition (4.8.4)(a), Deligne shows in [**De**, p. 308, 3.2.6(ii)] that the opposite category $\mathcal{C}^{\mathsf{op}}$ is *filtered* (see [**M**, p. 211]).[55] It follows that the independence verification for (4.8.4.2) need only be done for a pair of compactifications of which one maps to the other. This is now a straightforward exercise, using isomorphisms of the form $[(i_1, \bar f_1), (i_2, \bar f_2)]$.

To prove *existence* of $^!$ Deligne constructs, for each map f, a family of functorial isomorphisms $[(i_1, \bar f_1), (i_2, \bar f_2)]\colon i_2^* \bar f_2^\times \xrightarrow{\ \sim\ } i_1^* \bar f_1^\times$, indexed by pairs of compactifications of f, and satisfying (i)–(iii) [**De**, p. 313, 3.3.2.1]. (There is a pretty obvious such isomorphism when $(i_1, \bar f_1)$ maps to $(i_2, \bar f_2)$; and the rest follows from the fact that $\mathcal{C}^{\mathsf{op}}$ is filtered.) He then makes an arbitrary choice of a compactification $f = \bar f i$, and sets $f^! := i^* \bar f^\times$. Thus for any compactification $f = \bar f_\bullet i_\bullet$ one has an isomorphism

$$[(i_\bullet, \bar f_\bullet), (i, \bar f)]\colon f^! = i^* \bar f^\times \xrightarrow{\ \sim\ } i_\bullet^* \bar f_\bullet^\times. \tag{4.8.4.3}$$

For $f \in \mathbf{E}$, taking $\bar f_\bullet = 1$, $i_\bullet = f$, one gets $f^! \xrightarrow{\ \sim\ } f^*$, giving $\alpha_{\mathbf{E}}$ at the functorial—but not yet the pseudofunctorial—level. Analogous remarks lead to $\alpha_{\mathbf{P}}$.

Substituting isomorphisms as in (4.8.4.3) at each of the three appropriate places in (4.8.4.1), one gets a *definition* of $d_{f_1, f_2}\colon f_2^! f_1^! \xrightarrow{\ \sim\ } (f_1 f_2)^!$, provided it is first shown that the result of this substitution does not depend on the choice of $\bar g$ and j. As before, since $\mathcal{C}^{\mathsf{op}}$ is filtered it suffices to show that (4.8.4.1) (as here modified) is unaltered by the substitution for $(j, \bar g)$ of a compactification $(j_1, \bar g_1)$ of $i_1 \bar f_2$ such that there exists a $\mathbf{P}$-map $\bar h$ with $j = j_1 \bar h$ and $\bar g \bar h = \bar g_1$. This is done in [**De**, pp. 314–316].

Finally, a brief check [**De**, p. 317, 3.3.2.4] ensures that this d endows $^!$, $\alpha_{\mathbf{P}}$ and $\alpha_{\mathbf{E}}$ with all the desired pseudofunctorial properties. The last assertion in (4.8.4) simply reflects the possibility in the above definition of $^!$ of making the obvious choice $\bar f = 1$, $i = f$ whenever $f \in \mathbf{E}$. Q.E.D.

The proof of (4.8.3) will be based on the following pasting result for base-change setups.[56]

Proposition (4.8.5). *With notation and assumptions as in* (4.8.4), *let* $\overline{\mathbf{S}}$ *be a category containing* $\mathbf{S}$ *as a subcategory. Let*

$$\mathcal{B}' := \mathcal{B}\big(\overline{\mathbf{S}}, \mathbf{E}, \mathbf{F}, {}^*, {}^\#, (\beta'_\sigma)_{\sigma \in \square'}\big), \qquad \mathcal{B}'' := \mathcal{B}\big(\overline{\mathbf{S}}, \mathbf{P}, \mathbf{F}, {}^\times, {}^\#, (\beta''_\sigma)_{\sigma \in \square''}\big),$$

be base-change setups with $\square'$ (*resp.* $\square''$) *the class of* $\overline{\mathbf{S}}$*-fiber squares* $\sigma_{v,f,g,u}$ *such that* $f, g \in \mathbf{E}$ (*resp.* $\mathbf{P}$) *and* $u, v \in \mathbf{F}$. *Assume that for any* $f \in \mathbf{E}$ (*resp.* $\mathbf{P}$) *and* $u \in \mathbf{F}$, *such a* $\sigma_{v,f,g,u}$ *exists.*

[55] In that proof take K to be the inverse image of the diagonal under the map $(r, s)\colon \overline{Y}_1 \to \overline{Y}_2 \times_X \overline{Y}_2$.

[56] This result should be compared with [**Nk**, p. 205, Thm. 2.3.2].

Then there is at most one base-change setup

$$\overline{\mathcal{B}} := \mathcal{B}\big(\overline{\mathbf{S}}, \mathbf{S}, \mathbf{F}, {}^{!}, {}^{\#}, (\bar{\beta}_\sigma)_{\sigma\in\overline{\square}}\big)$$

which extends—in the obvious sense, via $\alpha_{\mathbf{P}}$ and $\alpha_{\mathbf{E}}$—both $\mathcal{B}'$ and $\mathcal{B}''$, and with $\overline{\square}$ the class of $\overline{\mathbf{S}}$-fiber squares $\sigma_{v,f,g,u}$ such that $f, g \in \mathbf{S}$ and $u, v \in \mathbf{F}$. Such a $\overline{\mathcal{B}}$ exists if and only if, for any $\overline{\mathbf{S}}$-cube with $i, i_1, j, j_1 \in \mathbf{E}$, $f, f_1, g, g_1 \in \mathbf{P}$, and $u, u_1, v, v_1 \in \mathbf{F}$, and in which all the faces are distinguished (for the appropriate one of $\mathcal{B}$, $\mathcal{B}'$, or $\mathcal{B}''$):

$$\begin{tikzcd}
& \bullet \arrow[rr, "v_1"] \arrow[dl, "j_1"'] \arrow[dd, "g_1"] & & \bullet \arrow[dl, "j"'] \arrow[dd, "f_1"] \\
\bullet \arrow[rr, "v"] \arrow[dd, "g"'] & & \bullet \arrow[dd, "f"] & \\
& \bullet \arrow[rr, "u_1"] \arrow[dl, "i_1"'] & & \bullet \arrow[dl, "i"'] \\
\bullet \arrow[rr, "u"'] & & \bullet &
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
v_1^{\#} j^* f^{\times} \arrow[r, "\beta'"] \arrow[d, "\beta"'] & j_1^* v^{\#} f^{\times} \arrow[r, "\beta''"] & j_1^* g^{\times} u^{\#} \arrow[d, "\beta"] \\
v_1^{\#} f_1^{\times} i^* \arrow[r, "\beta''"'] & g_1^{\times} u_1^{\#} i^* \arrow[r, "\beta'"'] & g_1^{\times} i_1^* u^{\#}
\end{tikzcd} \tag{4.8.5.1}$$

Remark **(4.8.5.2).** The existence part of Theorem (4.8.3), weakened by substituting for $\square$ the class of *fiber squares* $\sigma_{v,f,g,u}$ with u, v flat and f, g finitely presentable, and by leaving aside conditions (4.8.3)(iii), results from an application of (4.8.5) to the following base-change setups $\mathcal{B}'$ and $\mathcal{B}''$.

For $\mathcal{B}'$, let $\overline{\mathbf{S}}$ be the category of separated maps of noetherian schemes; $\mathbf{F}$ the subcategory of flat maps, with ${}^{\#} = {}^{*}$, the usual $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}$-valued inverse-image pseudofunctor; $\mathbf{E} \subset \mathbf{F}$ the subcategory of étale maps, with the same inverse-image pseudofunctor *; $\square'$ the class of all $\overline{\mathbf{S}}$-fiber squares $\sigma_{v,f,g,u}$ with f, g étale and u, v flat; and $\beta_\sigma \colon v^*f^* \xrightarrow{\ \sim\ } g^*u^*$ the natural isomorphism. (This is just a "subsetup" of $\mathcal{B}(\overline{\mathbf{S}}, \mathsf{L}{-}^{*})$, see (4.8.2.2).)

For $\mathcal{B}''$, let $\overline{\mathbf{S}}$ and $(\mathbf{F}, {}^{\#})$ be the same as for $\mathcal{B}'$; let $\mathbf{P}$ be the subcategory of proper maps, with the $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}$-valued pseudofunctor ${}^{\times}$ (see (4.1.2)); $\square''$ the class of those $\overline{\mathbf{S}}$-fiber squares $\sigma_{v,f,g,u}$ with f and g proper, u and v flat; and β_σ $(\sigma \in \square')$ the base-change isomorphism from (4.4.3).

In this situation, commutativity of (4.8.5.1) is easily checked, via "horizontal transitivity" in Example (4.8.2.3).

In (4.8.6)–(4.8.11), the resulting base-change setup $\overline{\mathcal{B}}$ will be extended to where $\overline{\Box}$ consists of all admissible $\overline{\mathbf{S}}$-squares.

Proof of (4.8.5). Fiber products being unique up to isomorphism, it follows from (4.8.2.1)(a) and the assumption in (4.8.5) that *any* $\overline{\mathbf{S}}$-fiber square $\sigma_{v,f,g,u}$ with $f \in \mathbf{E}$ (resp. $\mathbf{P}$) and $u \in \mathbf{F}$ is in $\Box'$ (resp. $\Box''$). It is then straightforward to see via (4.8.4)(b) that any $\sigma \in \overline{\Box}$ is a vertical composite $\sigma_2 \circ \sigma_1$ with $\sigma_1 \in \Box'$ and $\sigma_2 \in \Box''$:

$$\sigma \;=\; \begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "j"'] \arrow[dr, phantom, "\sigma_1"] & \bullet \arrow[d, "i"] \\ \bullet \arrow[r, "w"] \arrow[d, "\bar g"'] \arrow[dr, phantom, "\sigma_2"] & \bullet \arrow[d, "\bar f"] \\ \bullet \arrow[r, "u"'] & \bullet, \end{tikzcd} \tag{4.8.5.3}$$

and to check that if $\overline{\mathcal{B}}$ exists then $\bar\beta_\sigma$ has to be the natural composition

$$\begin{aligned} v^{\#}(\bar f i)^! \xrightarrow{\ \sim\ } v^{\#} i^! \bar f^! \xrightarrow[\alpha_{\mathbf{P}}]{\ \sim\ } v^{\#} i^! \bar f^{\times} \xrightarrow[\alpha_{\mathbf{E}}]{\ \sim\ } v^{\#} i^* \bar f^{\times} \xrightarrow[\beta']{\ \sim\ } j^* w^{\#} \bar f^{\times} \\ \xrightarrow[\beta'']{\ \sim\ } j^* \bar g^{\times} u^{\#} \xrightarrow[\alpha_{\mathbf{E}}^{-1}]{\ \sim\ } j^! \bar g^{\times} u^{\#} \xrightarrow[\alpha_{\mathbf{P}}^{-1}]{\ \sim\ } j^! \bar g^! u^{\#} \xrightarrow{\ \sim\ } (\bar g j)^! u^{\#}, \end{aligned}$$

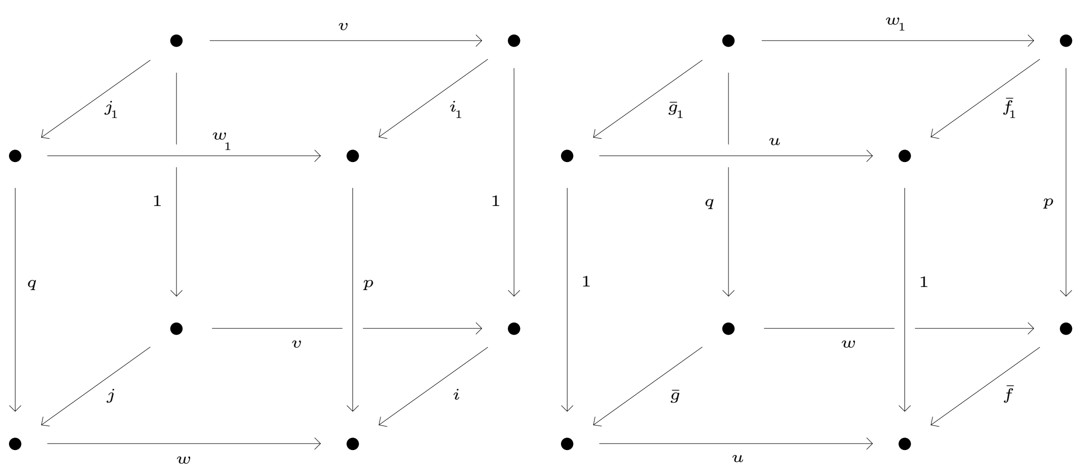
whence the uniqueness of $\overline{\mathcal{B}}$ (if it exists). Expanding the two instances of β in (4.8.5.1) according to the description of β_σ in (4.8.4), one finds then that (4.8.5.1) commutes. (The commutativity amounts to two ways of expanding $\bar\beta\colon v_1^{\#}(fj)^! = v_1^{\#}(if_1)^! \xrightarrow{\ \sim\ } (gj_1)^! u^{\#} = (i_1 g_1)^! u^{\#}$ according to vertical transitivity (4.8.2)(3).)

To prove the existence of $\overline{\mathcal{B}}$, we first show that the above expression for $\bar\beta_\sigma$ depends only on σ.

For this purpose, consider the category $\widetilde{\mathbf{S}}$ whose objects are $\mathbf{F}$-maps, the morphisms from an $\mathbf{F}$-map $v\colon X' \to X$ to an $\mathbf{F}$-map $u\colon Y' \to Y$ being the fibre squares $\sigma_{v,f,g,u} \in \overline{\Box}$, with the obvious definition of composition. Define the subcategory $\widetilde{\mathbf{E}} \subset \widetilde{\mathbf{S}}$ (resp. $\widetilde{\mathbf{P}} \subset \widetilde{\mathbf{S}}$) to be the one having the same objects as $\widetilde{\mathbf{S}}$, but with morphisms $\sigma_{v,f,g,u} \in \overline{\Box}$ such that $f, g \in \mathbf{E}$ (resp. $\mathbf{P}$). The above decomposition $\sigma = \sigma_2 \circ \sigma_1$ signifies that every $\widetilde{\mathbf{S}}$-morphism has an $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{P}})$-*compactification,* i.e., it factors as an $\widetilde{\mathbf{E}}$-morphism followed by a $\widetilde{\mathbf{P}}$-morphism.

It is left as an exercise to deduce from (4.8.4)(a) its analogue for $\widetilde{\mathbf{P}} \subset \widetilde{\mathbf{S}}$.

It follows then, as in the proof of (4.8.4), that it will be enough to show that two different compactifications of $\sigma \in \overline{\square}$ give the same $\bar{\beta}_\sigma$ when one of them maps to the other, via $\widetilde{\mathbf{P}}$—cf. the definition of morphisms of compactifications which appears in the proof of (4.8.4). Let the target compactification be given by factorizations $f = \bar{f}i$, $g = \bar{g}j$ (see (4.8.5.3)); let the source compactification be given similarly by factorizations $f = \bar{f}_1 i_1$, $g = \bar{g}_1 j_1$. Then the map of compactifications is given by $\mathbf{P}$-maps p and q fitting into commutative cubes (with a common face), whose horizontal arrows are $\mathbf{F}$-maps:



The first cube entails, via (4.8.5.1), a commutative diagram

$$
\begin{tikzcd}
v^{\#}i_1^{*}p^{\times} \arrow[r, "\beta'"] \arrow[d, "\beta"'] & j_1^{*}w_1^{\#}p^{\times} \arrow[r, "\beta''"] & j_1^{*}q^{\times}w^{\#} \arrow[d, "\beta"] \\
v^{\#}i^{*} \arrow[r, equal] & v^{\#}i^{*} \arrow[r, "\beta'"'] & j^{*}w^{\#}
\end{tikzcd}
\tag{4.8.5.4}
$$

Vertical transitivity (4.8.2)(3) for the setup $\mathcal{B}\big(\overline{\mathbf{S}}, \mathbf{P}, \mathbf{F}, {}^{\times}, {}^{\#}, (\beta''_\sigma)_{\sigma\in\square''}\big)$, applied to the composite diagram consisting of the rear and bottom faces of the second cube, yields a commutative diagram

$$
\begin{tikzcd}
\bar{g}_1^{\times}u^{\#} \arrow[rr] & & w_1^{\#}\bar{f}_1^{\times} \\
q^{\times}\bar{g}^{\times}u^{\#} \arrow[u] & q^{\times}w^{\#}\bar{f}^{\times} \arrow[l] & w_1^{\#}p^{\times}\bar{f}^{\times} \arrow[l] \arrow[u]
\end{tikzcd}
\tag{4.8.5.5}
$$

Now, by the definition of $\bar{\beta}_\sigma$ with respect to a given compactification, the present problem is to show commutativity of the outer border of the following diagram, in which the maps are the obvious isomorphisms. (Recall that $i \circ 1 = i = pi_1$, $\bar{f}_1 = \bar{f}p$, $j \circ 1 = j = qj_1$, $wq = pw_1$ and $\bar{g}_1 = \bar{g}q$.)

$$
\begin{tikzcd}
v^{\#}f^{!} \arrow[rrr, equal] \arrow[d] & & & v^{\#}f^{!} \arrow[d] \\
v^{\#}i^{!}\bar{f}^{!} \arrow[rr] \arrow[d] \arrow[drr, phantom, "①"] & & v^{\#}i_1^{!}p^{!}\bar{f}^{!} \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & v^{\#}i_1^{!}\bar{f}_1^{!} \arrow[d] \\
v^{\#}i^{*}\bar{f}^{\times} \arrow[rr] \arrow[d] \arrow[drr, phantom, "③"] & & v^{\#}i_1^{*}p^{\times}\bar{f}^{\times} \arrow[r] \arrow[d] & v^{\#}i_1^{*}\bar{f}_1^{\times} \arrow[d] \\
j^{*}w^{\#}\bar{f}^{\times} \arrow[r] \arrow[d] & j_1^{*}q^{\times}w^{\#}\bar{f}^{\times} \arrow[r] \arrow[d] & j_1^{*}w_1^{\#}p^{\times}\bar{f}^{\times} \arrow[r] \arrow[d, phantom, "④"] & j_1^{*}w_1^{\#}\bar{f}_1^{\times} \arrow[d] \\
j^{*}\bar{g}^{\times}u^{\#} \arrow[r] \arrow[d] & j_1^{*}q^{\times}\bar{g}^{\times}u^{\#} \arrow[rr] \arrow[d] \arrow[drr, phantom, "⑤"] & & j_1^{*}\bar{g}_1^{\times}u^{\#} \arrow[d] \\
j^{!}\bar{g}^{!}u^{\#} \arrow[r] \arrow[d] & j_1^{!}q^{!}\bar{g}^{!}u^{\#} \arrow[rr] & & j_1^{!}\bar{g}_1^{!}u^{\#} \arrow[d] \\
g^{!}u^{\#} \arrow[rrr, equal] & & & g^{!}u^{\#}
\end{tikzcd}
$$

Subdiagram ① commutes by (4.8.4) (for $v := i_1$, $f := p$, $u := i$ and $g := 1$), ③ by (4.8.5.4), and ④ by (4.8.5.5). Subdiagrams ② and ⑤ commute because the isomorphism $\alpha_{\mathbf{P}}$ is pseudofunctorial. Commutativity of the remaining subdiagrams is clear. Thus the entire diagram does commute, and so $\bar{\beta}_\sigma$ depends only on σ.

It remains to check conditions (1)–(3) in (4.8.2), of which only "vertical transitivity for $\bar{\beta}_\sigma$" is not straightforward enough to be left to the reader.

So we need to consider a commutative diagram, with $\bar{f}_t, \bar{g}_t \in \mathbf{P}$ and $i_t, j_t \in \ \mathbf{E}$ $(t = 1, 2)$, $w, x, y, z, u \in \mathbf{F}$, and in which all the squares are fiber squares:

$$\begin{tikzcd}
\bullet \arrow[r, "w"] \arrow[d, "j_1"'] & \bullet \arrow[d, "i_1"] \\
X' \arrow[r, "x"] \arrow[d, "g_1"'] & X \arrow[d, "f_1"] \\
\bullet \arrow[r, "y"] \arrow[d, "j_2"'] & \bullet \arrow[d, "i_2"] \\
Z' \arrow[r, "z"] \arrow[d, "g_2"'] & Z \arrow[d, "f_2"] \\
\bullet \arrow[r, "u"] & \bullet
\end{tikzcd}$$

Let $i_2 f_1 = f i$ with $f \colon Y \to Z \in \mathbf{P}$ and $i \colon X \to Y \in \mathbf{E}$.

Let $g \colon Z' \times_Z Y \to Z'$ and $v \colon Z' \times_Z Y \to Y$ be the projections, so that $g \in \mathbf{P}$ and $v \in \mathbf{F}$.

Then there is a unique $\mathbf{E}$-map $j \colon X' \to Z' \times_Z Y$ such that $gj = j_2 g_1$ and $vj = ix$. One sees then that in the cube

$$\begin{tikzcd}
& X' \arrow[rr, "x"] \arrow[dl, "j"'] \arrow[dd, "g_1"] & & X \arrow[dl, "i"] \arrow[dd, "f_1"] \\
Z' \times_Z Y \arrow[rr, "v"] \arrow[dd, "g"'] & & Y \arrow[dd, "f"] & \\
& \bullet \arrow[rr, "y"] \arrow[dl, "j_2"'] & & \bullet \arrow[dl, "i_2"] \\
Z' \arrow[rr, "z"] & & Z &
\end{tikzcd}$$

the top and bottom faces are $\mathcal{B}'$-distinguished, the front and back faces are $\mathcal{B}''$-distinguished, and the other two faces are $\mathcal{B}$-distinguished.

Now vertical transitivity amounts to commutativity of the diagram

$$\begin{tikzcd}
w^{\#}(ii_1)^*(f_2f)^{\times} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\textcircled{1}"] & w^{\#}i_1^*i^*f^{\times}f_2^{\times} \arrow[r] \arrow[d] & w^{\#}i_1^*f_1^{\times}i_2^*f_2^{\times} \arrow[d] \\
 & j_1^*x^{\#}i^*f^{\times}f_2^{\times} \arrow[r] \arrow[d] \arrow[ddr, phantom, "\textcircled{3}"] & j_1^*x^{\#}f_1^{\times}i_2^*f_2^{\times} \arrow[d] \\
(jj_1)^*v^{\#}(f_2f)^{\times} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\textcircled{2}"] & j_1^*j^*v^{\#}f^{\times}f_2^{\times} \arrow[d] & j_1^*g_1^{\times}y^{\#}i_2^*f_2^{\times} \arrow[d] \\
 & j_1^*j^*g^{\times}z^{\#}f_2^{\times} \arrow[r] \arrow[d] & j_1^*g_1^{\times}j_2^*z^{\#}f_2^{\times} \arrow[d] \\
(jj_1)^*(g_2g)^{\times}u^{\#} \arrow[r] & j_1^*j^*g^{\times}g_2^{\times}u^{\#} \arrow[r] & j_1^*g_1^{\times}j_2^*g_2^{\times}u^{\#}
\end{tikzcd}$$

Subsquares ① and ② commute by vertical transitivity for $\mathcal{B}''$. Commutativity of ③ is the instance of (4.8.5.1) corresponding to the preceding cube. Commutativity of the remaining two subsquares is obvious.

This completes the proof of Proposition (4.8.5). Q.E.D.

As previously noted, to finish the proof of (4.8.1) we need to enlarge the setup (4.8.2.4) to (4.8.2.4)$'$. Similarly, to finish the proof of (4.8.3) we need to show that there exists a unique enlargement $\widetilde{\mathcal{B}}$ of the setup $\overline{\mathcal{B}}$ at the end of (4.8.5.2) such that all admissible **S**-squares are $\widetilde{\mathcal{B}}$-distinguished. In addition, we need to check that (4.8.3)(ii) and (iii) hold for this $\widetilde{\mathcal{B}}$.

All this will be done in (4.8.11), after the supporting formal details are developed in (4.8.6)–(4.8.10).

Definition (4.8.6). For a base-change setup $\mathcal{B}\big(\mathbf{S},\mathbf{P},\mathbf{F},{}^{!},{}^{*},(\beta_\sigma)_{\sigma\in\square}\big)$ a subcategory $\mathbf{E}\subset\mathbf{S}$ is *special* if for any maps $i\colon X\to Y$ in $\mathbf{E}$, $g\colon X'\to X$ in $\mathbf{P}$, and $v\colon X'\to X$ in $\mathbf{F}$, the squares

$$\begin{tikzcd}
X' \arrow[r, "1"] \arrow[d, "g"'] & X' \arrow[d, "ig"] \\
X \arrow[r, "i"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "1"'] & X \arrow[d, "i"] \\
X' \arrow[r, "iv"'] & Y
\end{tikzcd}$$

are distinguished.

Remarks **(4.8.6.1).** (a) If $\mathbf{E}$ is special then $\mathbf{E}\subset\mathbf{P}\cap\mathbf{F}$.

(b) If $\mathbf{E}$ is special for $\mathcal{B}$, then $\mathbf{E}$ is also special for the dual of $\mathcal{B}$ (see (4.8.2.1)(c)).

Example **(4.8.6.2).** For (4.8.2.4), or for $\mathcal{B}'$, $\mathcal{B}''$ or $\overline{\mathcal{B}}$ in (4.8.5.2), the category $\mathbf{E}$ whose maps are all the open-and-closed immersions of noetherian schemes is special. Indeed, since i is a monomorphism, both squares in (4.8.6) are fiber squares.

After fixing a special subcategory $\mathbf{E}$, we will call its maps special. For any special map $i\colon X \to Y$,

$$\beta_i\colon i^! \xrightarrow{\ \sim\ } i^* \tag{4.8.7.0}$$

is defined to be the isomorphism β_τ associated to the distinguished square

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\tau"] & X \arrow[d, "i"] \\ X \arrow[r, "i"'] & Y \end{tikzcd}$$

Proposition (4.8.7). *Let $\mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ be a base-change setup and $\mathbf{E}$ a special subcategory. Then the restrictions of the pseudofunctors ${}^!$ and * to $\mathbf{E}$ are naturally isomorphic.*

Proof. The family of isomorphisms β_i ($i \in \mathbf{E}$) of (4.8.7.0) is pseudofunctorial (see (3.6.6)): if $i\colon X \to Y$ and $j\colon Y \to Z$ are in $\mathbf{E}$, apply (3) and (2) of (4.8.2) to

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[dd, "1"'] & X \arrow[r, "1"] \arrow[d, "i"'] & X \arrow[d, "i"] \\ & Y \arrow[r, "1"] \arrow[d, "1"'] & Y \arrow[d, "j"] \\ X \arrow[r, "i"'] & Y \arrow[r, "j"'] & Z \end{tikzcd}$$

to see that the left and right halves of the following diagram commute:

$$\begin{tikzcd} (ji)^! \arrow[r, equal] \arrow[d, "\simeq"'] & (ji)^! \arrow[r, "\beta_{ji}"] \arrow[d, "\simeq"'] & (ji)^* \arrow[d, "\simeq"] \\ i^!j^! \arrow[r, "i^!\beta_j"'] & i^!j^* \arrow[r, "\beta_i"'] & i^*j^* \end{tikzcd}$$

Q.E.D.

Proposition (4.8.8). *Let $\mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ be a base-change setup,* $\mathbf{E}$ *a special subcategory, and* β_i $(i \in \mathbf{E})$ *as in* (4.8.7.0). *Then:*

(i) *For each distinguished square*

$$\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}$$

with f and g in $\mathbf{E}$, *the following diagram commutes:*

$$\begin{tikzcd}
v^*f^! \arrow[rr, "\beta_\sigma"] \arrow[d, "v^*\beta_f"'] & & g^!u^* \arrow[d, "\beta_g"] \\
v^*f^* \arrow[r, "\sim"] & (fv)^* = (ug)^* & g^*u^* \arrow[l, "\sim"']
\end{tikzcd}$$

(ii) *For each distinguished square*

$$\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}$$

with u and v in $\mathbf{E}$, *the following diagram commutes:*

$$\begin{tikzcd}
v^*f^! \arrow[rr, "\beta_\sigma"] & & g^!u^* \\
v^!f^! \arrow[u, "\beta_v"] & (fv)^! = (ug)^! \arrow[l, "\sim"'] \arrow[r, "\sim"] & g^!u^! \arrow[u, "g^!\beta_u"']
\end{tikzcd}$$

Proof. Definition (4.8.6) shows that the following composite square ρ is distinguished, as are its constituents:

$$\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "1"'] & \bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "g"'] & \bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}$$

so horizontal transitivity (4.8.2)(3) gives a commutative diagram

$$\begin{tikzcd}
v^*f^! \arrow[r, "\beta_\sigma"] \arrow[d, "\beta_\rho"'] & g^!u^* \arrow[d, "\beta_g"] \\
(ug)^* & g^*u^* \arrow[l, "\sim"']
\end{tikzcd} \tag{4.8.8.1}$$

Also, the following decomposition of ρ

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "1"'] & \bullet \arrow[r, "1"] \arrow[d, "1"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "v"'] & \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

yields—via $(2)'$ and (3) of (4.8.2)—the commutative diagram

(4.8.8.2)
$$\begin{tikzcd} v^*f^! \arrow[rr, equal] \arrow[d, "v^*\beta_f"'] & & v^*f^! \arrow[d, "\beta_\rho"] \\ v^*f^* \arrow[r, "\sim"] & (fv)^* \arrow[r, equal] & (ug)^* \end{tikzcd}$$

Pasting (4.8.8.1) and (4.8.8.2) along their common edge, we get (i).

Assertion (ii) is just (i) for the dual setup (see (4.8.2.1)(c)).

Q.E.D.

(4.8.9) We will now see how to enlarge certain base-change setups.

Consider a category $\mathbf{S}$ in which for any maps $X \to Y$ and $Y' \to Y$ a fiber product $X \times_Y Y'$ exists. A square $\sigma_{v,f,g,u}$ in $\mathbf{S}$:

(4.8.9.1)
$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

is, as usual, called a *fiber square* if the corresponding map $X' \to X \times_Y Y'$ is an isomorphism.

Let $\mathcal{B} := \mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ be a base-change setup, and $\mathbf{E}$ a special subcategory, see (4.8.6).

We make the following assumptions, in addition to those in (4.8.2).

(4) In the following $\mathbf{S}$-diagrams, suppose that $u_1 \in \mathbf{F}$ (resp. $f_1 \in \mathbf{P}$).

$$\begin{tikzcd} \bullet \arrow[r, "v_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma_1"] & \bullet \arrow[r, "v_2"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma_2"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "u_1"'] & \bullet \arrow[r, "u_2"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "w"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma_1"] & \bullet \arrow[d, "f_1"] \\ \bullet \arrow[r, "v"] \arrow[d, "g_2"'] \arrow[dr, phantom, "\sigma_2"] & \bullet \arrow[d, "f_2"] \\ \bullet \arrow[r, "u"'] & \bullet \end{tikzcd}$$

In either diagram, if σ_2 is a fiber square and the composed square $\sigma_2\sigma_1$ is in $\square$, then $\sigma_1 \in \square$.

(5) For any fiber square (4.8.9.1) in $\square$, if u (resp. f) is special (i.e., lies in $\mathbf{E}$) then so is v (resp. g).

(6) If the square (4.8.9.1) is in $\square$ then so is any fiber square with the same u and f,

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

and furthermore, the resulting map $X' \to X''$ is special.

Example **(4.8.9.2)** Conditions (4)–(6) are easily seen to be satisfied in any of the situations in Example (4.8.6.2), where all distinguished squares are fiber squares.

Remark **(4.8.9.3)** Let $\mu\colon X' \to X''$ be an isomorphism and consider the following fiber squares, the first of which is, by (4.8.2)(2), distinguished:

$$\begin{tikzcd} X'' \arrow[r, "1"] \arrow[d, "1"'] & X'' \arrow[d, "1"] \\ X'' \arrow[r, "1"'] & X'' \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "\mu"] \arrow[d, "\mu"'] & X'' \arrow[d, "1"] \\ X'' \arrow[r, "1"'] & X'' \end{tikzcd}$$

From (6) it follows that μ is special. Thus *every isomorphism is special.*

Proposition (4.8.10). *Under the preceding assumptions, there is a unique base-change setup* $\mathcal{B}' = \mathcal{B}'_{\mathbf{E}} = \mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta'_\sigma)_{\sigma\in\square'}\big)$ *such that:*

(i) *A commutative square*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

is in $\square'$ if and only if there is a fiber square in $\square$

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

such that the resulting map $X' \to X''$ is special.

So by (4.8.9)(6) *and* (4.8.9.3), $\square \subseteq \square'$; *and by* (4.8.2)(1), *every fiber square in $\square'$ is in $\square$.*

(ii) *For every $\sigma \in \square \subseteq \square'$ it holds that $\beta_\sigma = \beta'_\sigma$.*

Proof. For uniqueness, suppose that $\mathcal{B}'$ satisfies (i) (which determines $\square'$) and (ii). We note first that if $i\colon X \to Y$ is a special map, then by (i), the square τ' in the following diagram is in $\square'$, as are the squares τ (by (4.8.6)) and $\tau'\tau$ (by (4.8.2)(2)$'$):

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\tau"] & X \arrow[r, "i"] \arrow[d, "i"] \arrow[dr, phantom, "\tau'"] & Y \arrow[d, "1"] \\
X \arrow[r, "i"'] & Y \arrow[r, "1"'] & Y
\end{tikzcd}$$

It follows then from (4.8.2)(3) and (4.8.2)(2)$'$ that

$$\beta'_{\tau'} = (\beta'_{\tau})^{-1} \overset{\text{(ii)}}{=} (\beta_{\tau})^{-1} \overset{(4.8.7.0)}{=} (\beta_i)^{-1}.$$

Now, any $\sigma \in \square'$:

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

can, according to (i), be decomposed as

$$\begin{tikzcd}
X' \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\sigma_1"] & X'' \arrow[r, "w"] \arrow[d, "1"] & X \arrow[dd, "f"] \\
X'' \arrow[r, "1"'] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma_2"] & X'' \arrow[d, "h"] \arrow[r, phantom, "\sigma_3"] & {} \\
Y' \arrow[r, "1"'] & Y' \arrow[r, "u"'] & Y
\end{tikzcd} \tag{4.8.10.1}$$

with $\sigma_3 \in \square$ a fiber square (so that $h \in \mathbf{P}$), and i special. The fiber square σ_2 is in $\square$, by (4.8.2)(2); and by (i), σ_1 and $\sigma_2\sigma_1 \in \square'$. We saw above that $\beta'_{\sigma_1} = (\beta_i)^{-1}$; and the maps β'_{σ_k} $(k = 2, 3)$ are determined by (ii). Hence $\beta'_{\sigma_2\sigma_1}$ is determined, and then so is β'_{σ} (see (4.8.2)(3)). Thus $\mathcal{B}'$ is unique.

For the existence, let $\square'$ be the class of all squares

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

satisfying (i), that is, decomposing as in (4.8.10.1)—where $i \in \mathbf{P} \cap \mathbf{F}$ (see (4.8.6.1)), $h \in \mathbf{P}$ and $w \in \mathbf{F}$, so that $f, g \in \mathbf{P}$ and $u, v \in \mathbf{F}$, as required of distinguished squares.

To such a decomposition we associate the natural composite map

$$v^*f^! \xrightarrow{\sim} i^*w^*f^! \xrightarrow[i^*\beta_{\sigma_3}]{} i^*h^!u^* \xrightarrow[\beta_i^{-1}]{\sim} i^!h^!u^* \xrightarrow{\sim} g^!u^*. \tag{4.8.10.2}$$

We will define β'_σ for $\mathcal{B}'$ to be (4.8.10.2), but first we need to show it independent of the chosen decomposition.

Suppose then that we have another decomposition with (X'', i, h, w) replaced by (X''_1, i_1, h_1, w_1), i.e., there is an isomorphism $\mu\colon X'' \xrightarrow{\sim} X''_1$ such that

$$i_1 = \mu i, \qquad h_1 = h\mu^{-1}, \qquad w_1 = w\mu^{-1}.$$

For the special map μ (see (4.8.9.3)), we have the isomorphism β_μ of (4.8.7.0). We have also the isomorphism β_ρ associated to the square

$$\begin{tikzcd}
X' \arrow[r, "1"] \arrow[d, "i"'] \arrow[dr, phantom, "\rho"] & X' \arrow[d, "i_1"] \\
X'' \arrow[r, "\mu"'] & X''_1
\end{tikzcd}$$

which is in $\square$ by (4.8.2.1)(a).

We want to show that the following diagram of natural maps (with outside columns as in (4.8.10.2)) commutes:

$$\begin{tikzcd}
v^*f^! \arrow[rrr, equal] \arrow[d] & & & v^*f^! \arrow[d] \\
i^*w^*f^! \arrow[r] \arrow[dd] & i^*\mu^*w_1^*f^! \arrow[rr] \arrow[d] & & i_1^*w_1^*f^! \arrow[d] \\
\textcircled{1} & i^*\mu^*h_1^!u^* \arrow[r, equal] \arrow[d, "i^*\beta_\mu^{-1}"] & i^*\mu^*h_1^!u^* \arrow[r] \arrow[dd, "\beta_i^{-1}"'] & i_1^*h_1^!u^* \arrow[dd, "\beta_{i_1}^{-1}"] \\
i^*h^!u^* \arrow[r] \arrow[d, "\beta_i^{-1}"'] & i^*\mu^!h_1^!u^* \arrow[d, "\beta_i^{-1}"] & \arrow[r, phantom, "\textcircled{2}"] & {} \\
i^!h^!u^* \arrow[r] \arrow[d] & i^!\mu^!h_1^!u^* \arrow[r, "i^!\beta_\mu"] \arrow[d] & i^!\mu^*h_1^!u^* \arrow[r, "\beta_\rho^{-1}"] \arrow[d, phantom, "\textcircled{3}"] & i_1^!h_1^!u^* \arrow[d] \\
g^!u^* \arrow[r] & i_1^!h_1^!u^* \arrow[rr] & {} & g^!u^*
\end{tikzcd}$$

Commutativity of ② (resp. ③) follows from (4.8.8)(i) (resp. (4.8.8)(ii)) applied to ρ.

Commutativity of ① follows from (4.8.2)(3) and (4.8.8)(ii), applied respectively to the following fiber squares $\sigma_3 = \sigma_3'\sigma'$ and σ' (σ' being distinguished, by (4.8.2.1)(a)):

$$\begin{tikzcd}
X'' \arrow[r, "\mu"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma'"] & X''_1 \arrow[r, "w_1"] \arrow[d, "h_1"] \arrow[dr, phantom, "\sigma'_3"] & X \arrow[d, "f"] \\
Y' \arrow[r, "1"'] & Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

Commutativity of the remaining subdiagrams is clear.

So we can indeed define β'_σ as indicated above.

Condition (i) in (4.8.10) is then obvious.

As for (ii), referring to a decomposition (4.8.10.1) of $\sigma \in \square$ (where $wi = v$ and $hi = g$), note that by (4.8.9)(4) the square $\sigma_2\sigma_1$ is in $\square$, so by (4.8.2)(3) the diagram

$$\begin{tikzcd}
v^*f^! \arrow[rr, "\beta_\sigma"] \arrow[d, "\simeq"'] & & g^!u^* \arrow[d, equal] \\
i^*w^*f^! \arrow[r, "i^*\beta_{\sigma_3}"'] & i^*h^!u^* \arrow[r, "\beta_{\sigma_2\sigma_1}"'] & g^!1^*u^*
\end{tikzcd}$$

commutes. Also, (4.8.8)(ii) applied to $\sigma_2\sigma_1$ shows that $\beta_{\sigma_2\sigma_1}$ factors as

$$i^*h^!u^* \xrightarrow[\beta_i^{-1}]{} i^!h^!u^* \xrightarrow{\;\sim\;} g^!u^*.$$

Hence the composite map (4.8.10.2) is equal to β_σ, proving (ii).

Having thus defined $\mathcal{B}'$, we are left with proving (1)–(3) in (4.8.2).

For (1), assume, with notation as in (4.8.2), that $\sigma_1 \in \square'$. Consider a commutative decomposition of σ

$$\begin{tikzcd}
\bullet \arrow[r, "j_1^{-1}"] \arrow[d, "kj_1^{-1}"'] & \bullet \arrow[r, "v_1"] \arrow[d, "k"'] & \bullet \arrow[r, "j"] \arrow[d, "1"] & \bullet \arrow[d, "1"] \\
\bullet \arrow[r, equal] \arrow[d, "i_1h"'] \arrow[dr, phantom, "\sigma'"] & \bullet \arrow[r, "w_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\tau"] & \bullet \arrow[r, "j"] \arrow[d, "f_1"] \arrow[dr, phantom, "\sigma''"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "i_1^{-1}"'] & \bullet \arrow[r, "u_1"'] & \bullet \arrow[r, "i"'] & \bullet
\end{tikzcd}$$

in which the middle third of the diagram is a decomposition of σ_1 with $\tau \in \square$ a fiber square and k special, and $v_1 := w_1k$; and the right third exists by assumption, σ'' being a fiber square because i and j are isomorphisms. (Note: $i_1hkj_1^{-1} = i_1g_1j_1^{-1} = g$.) The composed fiber square $\sigma''\tau\sigma'$, being isomorphic to τ, is in $\square$; and thus, since kj_1^{-1} is special (see (4.8.6.1)(a)), therefore $\sigma \in \square'$, proving (1).

Conditions (2) and (2)′ for $\mathcal{B}'$ follow from the same for $\mathcal{B}$, because of (4.8.10)(ii).

As for (3), consider a composite diagram $\sigma_0 = \sigma_2\sigma_1$:

$$\begin{tikzcd}
\bullet \arrow[r, "v_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma_1"] & \bullet \arrow[r, "v_2"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma_2"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u_1"'] & \bullet \arrow[r, "u_2"'] & \bullet
\end{tikzcd}$$

with σ_2, σ_1 and σ_0 in $\square'$. Using all the assumptions in (4.8.9), we find that this decomposes further as

(4.8.10.3)
$$\begin{tikzcd}
\bullet \arrow[d, "j"'] \arrow[dr, "v_1"] & & \\
\bullet \arrow[r, "p"'] \arrow[d, "h_2"'] \arrow[dr, phantom, "\sigma''"] & \bullet \arrow[d, "k"'] \arrow[dr, "v_2"] & \\
\bullet \arrow[r, "q"'] \arrow[d, "h_1"'] \arrow[dr, phantom, "\sigma'"] & \bullet \arrow[r, "w"'] \arrow[d, "g_1"'] \arrow[dr, phantom, "\tau"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u_1"'] & \bullet \arrow[r, "u_2"'] & \bullet
\end{tikzcd}$$

where σ'', σ' and τ are fiber squares in $\square$; the maps g_1, w, h_1, q, h_2, p are the natural projections; the maps j and k are special—whence so are h_2 and $h_2 j$ (see (4.8.9)(5)); the triangles commute; $g_1 k = g$ and $h_1 h_2 j = h$.

What (3) asserts is, first, that the following natural diagram commutes:

(4.8.10.4)
$$\begin{tikzcd}
(v_2v_1)^*f^! \arrow[rr, "\beta_{\sigma_0}"] \arrow[d] & & h^!(u_2u_1)^* \arrow[d] \\
v_1^*v_2^*f^! \arrow[d, "\simeq"'] & v_1^*g^!u_2^* \arrow[r, "\beta_{\sigma_1}"'] & h^!u_1^*u_2^* \\
j^*p^*v_2^*f^! \arrow[r, "\text{via } \beta_{\sigma_2}"'] & j^*p^*(g_1k)^!u_2^* \arrow[u, "\simeq"'] &
\end{tikzcd}$$

Expanding β_{σ_2}, β_{σ_1}, and $\beta_{\sigma_2\sigma_1}$, as in (4.8.10.2), one sees that for this it is enough to show commutativity of the outer border of the natural diagram on the following page, or just to show that each of its twelve undecomposed subdiagrams commutes.

But for the eight unlabeled subdiagrams, commutativity holds by elementary (pseudo)functorial considerations; for subdiagram ①, one can use (4.8.7); for ② and ④, (4.8.2)(3); and for ③, (4.8.8)(i).

This completes the proof of the "horizontal" part of (3).

The proof of the "vertical" part of (3) is similar. Alternatively, one can just dualize everything in sight, as indicated in (4.8.2.1)(c). The conditions in (4.8.6) defining a special subcategory are self-dual, so that if $\mathbf{E}$ is special for a setup $\mathcal{B}$, then $\mathbf{E}$ is also special for the dual setup $\mathcal{B}^{\mathsf{op}}$. Likewise, conditions (4)–(6) in (4.8.9) hold for $\mathcal{B}$ iff they hold for $\mathcal{B}^{\mathsf{op}}$. Then, one checks, vertical transitivity for $(\mathcal{B}^{\mathsf{op}})'$ (constructed as above) is identical with the just-proved horizontal transitivity for $\mathcal{B}'$.

This completes the proof of Proposition (4.8.10). Q.E.D.

Corollary (4.8.10.5). *With notation and assumptions as in* (4.8.10), *let* $\mathbf{E}'$ *be a subcategory of* $\mathbf{S}$ *such that for every map* $i\colon X \to Y \in \mathbf{E}'$ *the diagonal map* $\delta_i\colon X \to X \times_Y X$ *is in* $\mathbf{E}$. *Assume further that for any fiber square* $\sigma_{v,f,g,u}$ *in* $\mathbf{S}$, *if* u (*resp.* f) *is in* $\mathbf{E}'$ *then so is* v (*resp.* g). *Then*:

(i) $\mathbf{E}'$ *is* $\mathcal{B}'$*-special; and conditions* (4)-(6) *in* (4.8.9) *hold for* $(\mathcal{B}', \mathbf{E}')$. *Thus it is meaningful to set* $\mathcal{B}'' := (\mathcal{B}')'_{\mathbf{E}'}$.

(ii) *If a fiber square* $\sigma = \sigma_{v,f,g,u}$ *with* $u \in \mathbf{E}'$ *is in* $\square$, *then any commutative* $\sigma_{v',f,g',u}$ *with* $v' \in \mathbf{E}'$ *and* $g' \in \mathbf{P}$ *is* $\mathcal{B}''$*-distinguished.*

Proof. (i) The second diagram in (4.8.6)—call it σ—expands as

$$\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "1"'] & \bullet \arrow[r, "1"] \arrow[d, "1"'] & \bullet \arrow[d, "i"] \\
\bullet \arrow[r, "v"'] & \bullet \arrow[r, "i"'] & \bullet
\end{tikzcd}$$

which when $i \in \mathbf{E}'$ can be further expanded in the form (4.8.10.3), with $j = 1$ and $k \in \mathbf{E}$, whence (since $\sigma'' \in \square$) $h_2 \in \mathbf{E}$, whence by (4.8.10)(i), $\sigma \in \square'$. In a similar way, or by dualizing (see (4.8.6.1(b)), one finds that the first diagram in (4.8.6) is in $\square'$.

For (4.8.9)(4), decompose the horizontal $\sigma_2\sigma_1$ of that condition as

$$\begin{tikzcd}
\bullet \arrow[d, "j"'] \arrow[dr, "v_1"] & & \\
\bullet \arrow[r, "q"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\sigma'"] & \bullet \arrow[r, "v_2"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma_2"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u_1"'] & \bullet \arrow[r, "u_2"'] & \bullet
\end{tikzcd}$$

with $j \in \mathbf{E}$, $qj = v_1$, $h_1 j = h$, and σ_2, σ' fiber squares such that the fiber square $\sigma_2\sigma'$ is in $\square$.

$$
\begin{tikzcd}
(v_2v_1)^*f^! \arrow[r] \arrow[d] & (h_2j)^*(wq)^*f^! \arrow[r] \arrow[d] & (h_2j)^*h_1^!(u_2u_1)^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\textcircled{1}"] & (h_2j)^!h_1^!(u_2u_1)^* \arrow[r] \arrow[d] & (h_1h_2j)^!(u_2u_1)^* \arrow[d] \\
v_1^*v_2^*f^! \arrow[d] & j^*h_2^*(wq)^*f^! \arrow[r] \arrow[d] \arrow[dr, phantom, "\textcircled{2}"] & j^*h_2^*h_1^!(u_2u_1)^* \arrow[r] \arrow[d] & j^!h_2^!h_1^!(u_2u_1)^* \arrow[d] & (h_1h_2j)^!u_1^*u_2^* \\
j^*p^*v_2^*f^! \arrow[r] \arrow[dd] & j^*h_2^*q^*w^*f^! \arrow[d] & j^*h_2^*h_1^!u_1^*u_2^* \arrow[r] & j^!h_2^!h_1^!u_1^*u_2^* \arrow[r, equal] & j^!h_2^!h_1^!u_1^*u_2^* \arrow[u] \\
 & j^*h_2^*q^*g_1^!u_2^* \arrow[r] \arrow[d] \arrow[ur] \arrow[dr, phantom, "\textcircled{3}"] & j^*h_2^!q^*g_1^!u_2^* \arrow[r] & j^!h_2^!q^*g_1^!u_2^* \arrow[u] \arrow[dr, phantom, "\textcircled{4}"] & \\
j^*p^*k^*w^*f^! \arrow[r] \arrow[uur] & j^*p^*k^*g_1^!u_2^* \arrow[r] & j^*p^*k^!g_1^!u_2^* \arrow[r] \arrow[u] \arrow[d] & j^!p^*k^!g_1^!u_2^* \arrow[u] \arrow[d] & \\
 & & j^*p^*(g_1k)^!u_2^* \arrow[r] & j^!p^*(g_1k)^!u_2^* \arrow[r] & j^!(h_1h_2)^!u_1^*u_2^* \arrow[uuu]
\end{tikzcd}
$$

It follows from (4.8.9)(4) for $\mathcal{B}$ that $\sigma' \in \square$, whence $\sigma_1 \in \square'$, proving the horizontal part of (4.8.9)(4) for $\mathcal{B}'$. The vertical part is similar (or dual).

Since any fiber square in $\square'$ is in $\square$, (4.8.9)(5) is essentially the "further" assumption on $\mathbf{E}'$.

Finally, (4.8.9)(6) for $\mathcal{B}'$ follows from (4.8.10)(i), (4.8.9)(6) for $\mathcal{B}$, and (4.8.9.3).

(ii) Consider a decomposition of $\sigma_{v',f,g',u}$

$$\begin{tikzcd} Z \arrow[d, "j"'] \arrow[dr, "v'"] & \\ X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

with $v' = vj$. We need only show that $j \in \mathbf{E}'$.

With Γ_j the graph map of j and $\pi_2 \colon Z \times_X X' \to X'$ the projection, the map j factors as

$$Z \xrightarrow{\Gamma_j} Z \times_X X' \xrightarrow{\pi_2} X'.$$

The fiber square

$$\begin{tikzcd} Z \times_X X' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X' \arrow[d, "v"] \\ Z \arrow[r, "v'"'] & X \end{tikzcd}$$

shows that $\pi_2 \in \mathbf{E}'$; and the fiber square

$$\begin{tikzcd} Z \arrow[r, "\Gamma_j"] \arrow[d, "j"'] & Z \times_X X' \arrow[d, "j \times 1_{X'}"] \\ X' \arrow[r, "\delta_v"'] & X' \times_X X' \end{tikzcd}$$

shows that $\Gamma_j \in \mathbf{E}'$, whence the conclusion. Q.E.D.

(4.8.11). Let us now complete the proof of (4.8.1) and (4.8.3) by doing what was indicated just before Definition (4.8.6).

For either the setup $\mathcal{B}$ of (4.8.2.4) or the larger setup $\overline{\mathcal{B}}$ of (4.8.5.2), the category $\mathbf{E}$ of open-and-closed immersions is special, see (4.8.6.2).

The diagonal of a separated étale map is an open-and-closed immersion [**EGA IV**, (17.4.2)(b)]; and maps which are étale (resp. separated, resp. proper) remain so after arbitrary base change [**EGA IV**, (17.3.3)(iii)].

Therefore the category $\mathbf{E}'$ of separated étale maps (resp. proper étale maps) satisfies the hypotheses of (4.8.10.5) with respect to $(\overline{\mathcal{B}}, \mathbf{E})$ (resp. $(\mathcal{B}, \mathbf{E})$). Keeping in mind the uniqueness part of (4.8.10), one see that the resulting base-change setup $\widetilde{\mathcal{B}} := \overline{\mathcal{B}}''$ is the sought-after unique enlargement of $\overline{\mathcal{B}}$, and that $\mathcal{B}''$ is the unique enlargement $(4.8.2.4)'$ of $\mathcal{B}$.

It remains to show that conditions (4.8.3)(ii) and (iii) hold for $\widetilde{\mathcal{B}}$.

Using the definition (4.8.10.2) of β_σ, one readily reduces the question to where σ is a fiber square. In that case, (ii) follows from the description of $\mathcal{B}'$ in (4.8.5.2).

As for (iii), let $f = \bar{f}i$ be a compactification, and apply vertical transitivity (4.8.2)(3), to reduce to where *either* $f = i$ is an open immersion, a case covered by (ii), *or* $f = \bar{f}$ is proper, a case covered by (4.8.1)(iii). Q.E.D.

Exercises **(4.8.12).** (a) Let $\mathcal{B}(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square})$ be a base-change setup, and let there be given pseudofunctorial isomorphisms ${}^! \xrightarrow{\sim} {}^\times$, ${}^* \xrightarrow{\sim} {}^\#$. For any $\sigma_{v,f,g,u} \in \square$ let $\bar{\beta}_\sigma$ be the natural composite isomorphism

$$v^\# f^\times \xrightarrow{\sim} v^* f^! \underset{\beta_\sigma}{\xrightarrow{\sim}} g^! u^* \xrightarrow{\sim} g^\times u^\#.$$

Show that $\mathcal{B}(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^\times, {}^\#, (\bar{\beta}_\sigma)_{\sigma\in\square})$ is a base-change setup.

(b) (generalizing (4.1.9)(c)). Notation is as in (4.8.2.4). For a finite étale scheme-map $f\colon X \to Y$, the natural map is an isomorphism $f_* \xrightarrow{\sim} \mathbf{R}f_*$ of functors from $\mathbf{D}_{\mathsf{qc}}(X)$ to $\mathbf{D}_{\mathsf{qc}}(Y)$, see proof of (3.10.2.2). Define the functorial "trace" map

$$f_*f^*E \underset{(3.9.4)}{\cong} f_*\mathcal{O}_X \otimes E \to \mathcal{O}_Y \otimes E \cong E \qquad (E \in \mathbf{D}_{\mathsf{qc}}(Y))$$

to be $\mathrm{tr}_f \otimes 1$ where tr_f is the natural composition

$$f_*\mathcal{O}_X \longrightarrow \mathcal{H}om^\bullet(f_*\mathcal{O}_X, f_*\mathcal{O}_X) \cong \mathcal{H}om^\bullet(f_*\mathcal{O}_X, \mathcal{O}_Y) \otimes f_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y,$$

given locally by the usual linear-algebra trace map. (Note that, f being flat and finitely presented, $f_*f^*\mathcal{O}_Y$ is a locally free $\mathcal{O}_Y$-module.) There corresponds a functorial map $t_f\colon f^* \to f^\times$.

(i) Show that on finite étale maps, the map $t_{(-)}\colon (-)^* \to (-)^\times$ is *pseudofunctorial,* see (3.6.6). (Reduction to the affine case may help.) Also, $t_{\text{identity}} = \text{identity}$.

(ii) (Compatibility of trace with base change.) Given a fiber square $\sigma = \sigma_{v,f,g,u}$ with f and g finite étale, u and v flat, show that the following diagram commutes:

$$\begin{tikzcd} u^*f_*f^* \arrow[r, "u^*\mathrm{tr}_f"] \arrow[d, "(3.7.2)"', "\simeq"] & u^* \\ g_*v^*f^* \arrow[r, "\sim", "\text{natural}"'] & g_*g^*u^* \arrow[u, "\mathrm{tr}_g"'] \end{tikzcd}$$

(iii) For σ as in (ii), show that the following diagram commutes:

$$\begin{tikzcd} v^*f^* \arrow[r, "\text{natural}"] \arrow[d, "v^*t_f"'] & g^*u^* \arrow[d, "t_g"] \\ v^*f^\times \arrow[r, "\beta_\sigma"'] & g^\times u^* \end{tikzcd}$$

(Commutativity of the adjoint diagram is a consequence of (ii).)

(iv) For any finite étale f show, using, e.g., (i), (iii), and (4.8.10.2), that with $\beta_f\colon f^\times \xrightarrow{\,\sim\,} f^*$ (see (4.8.7.0)) as in the base-change setup $(4.8.2.4)'$, $\beta_f t_f$ *is the identity* (whence t_f *is an isomorphism*—which can also be proved more directly).

(v) Deduce from (iv) that when $^!$ is constructed as in the proof of (4.8.1), via application of (4.8.4) to $(4.8.2.4)'$, then the canonical map $f_*f^* = f_*f^! \to \mathbf{1}$ (arising from right-adjointness of $f^!$ to f_*) is just the trace map.

(vi) For any finite étale $f\colon X \to Y$, and $E,\ F \in \mathbf{D}_{\mathsf{qc}}(X)$, show, using (v), or otherwise, that the map $\chi_{E,F}$ of (4.7.3.4) is just the isomorphism $f^*E \otimes f^*F \xrightarrow{\,\sim\,} f^*(E\otimes F)$ of (3.2.4).

(vii) Suppose that on the category $\mathbf{E}$ of finite étale maps of noetherian schemes there is associated to each $f\colon X \to Y$ a functorial map $\tau_f\colon f_*f^* \to \mathbf{1}$ in such a way that the pairs (f^*, τ_f) $(f \in \mathbf{E})$ form a pseudofunctorial right adjoint to the $\mathbf{D}_{\mathsf{qc}}$-valued direct image pseudofunctor, and such that furthermore, the diagram in (ii) above still commutes when tr_f is replaced by τ_f. Prove that $\tau_f = \mathrm{tr}_f$ for all f.

Deduce that (v) holds for *any* $f^!$ satisfying (4.8.1).

Hint. Show that $\tau_f = \mathrm{tr}_f \circ \theta_f$ for some automorphism θ_f of the functor f^*, i.e., $\theta_f =$ multiplication by e_f for some unit $e_f \in \mathrm{H}^0(X, \mathcal{O}_X)$. Then check that pseudofunctoriality implies, for any composition $X \xrightarrow{f} Y \xrightarrow{g} Z$, that $e_{gf} = e_g(g^*e_f)$; and check that for any σ as in (ii), $e_g = v^*e_f$. Then deduce from (iii), *mutatis mutandis,* that for any open-and closed immersion δ, $e_\delta = 1$; and finally, from the diagram

$$\begin{tikzcd} X \arrow[r, "\delta"] & X\times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

($\delta :=$ diagonal), that $e_f = 1$ for all f.

(c) Show that a horizontal or vertical composite of admissible squares is admissible.

(d) Adapt the arguments in §4.11 to extend [**Nk**, p. 268, Thm. 7.3.2]—which avoids noetherian hypotheses—to where $\mathfrak{s}$ can be any admissible square $\sigma_{v,f,g,u}$ with f and g composites of finitely-presentable proper flat maps and étale maps. (Recall that finitely-presentable flat maps are pseudo-coherent (4.3.1).)

4.9. Perfect maps of noetherian schemes

In this section *all schemes are assumed noetherian and all scheme-maps finite-type and separated.* The abbreviations introduced at the beginning of §4.4 will be used throughout.

We will associate to any such scheme-map $f\colon X \to Y$ a canonical bifunctorial map, with $f^!$ as in (4.8.1), and both E and $E\otimes F$ in $\overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$,

$$\chi^f_{E,F}\colon f^!E\otimes f^*F \to f^!(E\otimes F),$$

agreeing with the map $\chi_{E,F}$ in (4.7.3.4) when f is proper, and with the inverse of the isomorphism in (3.2.4) when f is étale.

Any functorial relation involving $(-)^!$ ought to be examined with regard to pseudofunctoriality and base change (cf., e.g., (4.2.3)(h)–(j)). For χ, this is done in Corollary (4.9.5) and Exercise (4.9.3)(c).

The main result, Theorem (4.9.4), inspired by [**V**′, p. 396, Lemma 1 and Corollary 2], gives several criteria for f to be *perfect* (i.e., since f is pseudo-coherent, to have *finite tor-dimension*). Included there is the implication f *perfect* $\implies \chi^{f}_{E,F}$ *an isomorphism.*

In [**Nk**′,Theorem 5.9] Nayak extends these results to separated maps that are only *essentially* of finite type.

(4.9.1). For scheme-maps $X \xrightarrow{u} \overline{X} \xrightarrow{\overline{f}} Y$, u an open immersion, $\overline{f}$ proper, we define the bifunctorial map

$$\chi^{\overline{f}}_{E,F}\colon \overline{f}^{\,!}E\otimes \overline{f}^{\,*}F \longrightarrow \overline{f}^{\,!}(E\otimes F) \qquad \big(E,F\in \mathbf{D}_{\mathsf{qc}}(Y)\big)$$

to be the map adjoint to the natural composite map

$$\overline{f}_*(\overline{f}^{\,!}E\otimes \overline{f}^{\,*}F) \underset{(3.9.4)}{\xrightarrow{\ \sim\ }} \overline{f}_*\overline{f}^{\,!}E\otimes F \longrightarrow E\otimes F,$$

and we define the bifunctorial map

$$\chi^{\overline{f},u}_{E,F}\colon u^*\overline{f}^{\,!}E\otimes f^*F \longrightarrow u^*\overline{f}^{\,!}(E\otimes F) \qquad \big(E,F\in \mathbf{D}_{\mathsf{qc}}(Y)\big)$$

to be the natural composite map

$$u^*\overline{f}^{\,!}E\otimes f^*F \xrightarrow{\ \sim\ } u^*\overline{f}^{\,!}E\otimes u^*\overline{f}^{\,*}F \xrightarrow{\ \sim\ } u^*(\overline{f}^{\,!}E\otimes \overline{f}^{\,*}F) \xrightarrow{u^*\chi^{\overline{f}}_{E,F}} u^*\overline{f}^{\,!}(E\otimes F).$$

When E and $E\otimes F$ are in $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Y)$, setting $f:=\overline{f}u$ we can write $f^!$ for $u^*\overline{f}^{\,!}$. In that case, we'll see below, in (4.9.2.2), that $\chi^{\overline{f},u}_{E,F}$ depends only on f, not on the factorization $f=\overline{f}u$, so we can denote the map $\chi^{\overline{f},u}_{E,F}$ by

$$\chi^{f}_{E,F}\colon f^!E\otimes f^*F \to f^!(E\otimes F). \tag{4.9.1.1}$$

In this connection, recall that by Nagata's compactification theorem, *any* (finite-type separated) scheme-map f factors as $f=\overline{f}u$.

Lemma (4.9.2). *Let there be given a commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \arrow[rr, "g"] \arrow[dr, "v"'] & & Z \\
& \overline{X} \arrow[ur, "\overline{f}"'] \arrow[dr, "w"'] & & \overline{Y} \arrow[ur, "\bar{g}"'] & \\
& & \overline{\overline{X}} \arrow[ur, "\bar{h}"'] & &
\end{tikzcd}$$

with u, v and w open immersions, $\overline{f}$, $\bar{g}$ and $\bar{h}$ proper.

Then for all $E,F\in\mathbf{D}(Z)$ such that E and $E\otimes F$ are in $\overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Z)$, the following natural diagram commutes.

$$
\begin{tikzcd}
(gf)^!E\otimes(gf)^*F \arrow[rr, "\chi^{\bar g\bar h,\,wu}_{E,F}"] \arrow[d, "\simeq"'] & & (gf)^!(E\otimes F) \arrow[d, "\simeq"] \\
f^!g^!E\otimes f^*g^*F \arrow[r, "\chi^{\bar f,u}_{g^!E,\,g^*F}"'] & u^*\bar f^!(g^!E\otimes g^*F) \arrow[r, "u^*\bar f^!\chi^{\bar g,v}_{E,F}"'] & f^!g^!(E\otimes F)
\end{tikzcd}
$$

Proof (Sketch). Set $\bar E:=\bar g^!E,\ \bar F:=\bar g^*F$ (so that $v^*\bar E\cong g^!E$ and $v^*\bar F\cong g^*F$). Let β be the natural composite functorial isomorphism

$$w^*\bar h^! \xrightarrow{\ \sim\ } (\bar h w)^! = (v\bar f)^! \xrightarrow{\ \sim\ } \bar f^! v^*. \tag{4.9.2.1}$$

Straightforward—if a bit tedious—considerations, using the definitions of the maps involved (see, e.g., (4.8.4)), translate Lemma (4.9.2) into commutativity of the natural diagram

$$
\begin{tikzcd}
(wu)^*\big((\bar g\bar h)^!E\otimes(\bar g\bar h)^*F\big) \arrow[rr, "(wu)^*\chi^{\bar g\bar h}_{E,F}"] \arrow[d, "\simeq"'] & \textcircled{1} & (wu)^*(\bar g\bar h)^!(E\otimes F) \arrow[d, "\simeq"] \\
u^*w^*\bar h^!\bar E\otimes u^*w^*\bar h^*\bar F \arrow[r, "u^*w^*\chi^{\bar h}_{\bar E,\bar F}"] \arrow[d, "\text{via }\beta"', "\simeq"] \arrow[dr, phantom, "\textcircled{2}"] & u^*w^*\bar h^!(\bar E\otimes\bar F) \arrow[r, "u^*w^*\bar h^!\chi^{\bar g}_{E,F}"] \arrow[d, "\text{via }\beta"', "\simeq"] \arrow[dr, phantom, "\textcircled{3}"] & u^*w^*\bar h^!\bar g^!(E\otimes F) \arrow[d, "\simeq"', "\text{via }\beta"] \\
u^*\bar f^!v^*\bar E\otimes u^*\bar f^*v^*\bar F \arrow[r, "u^*\chi^{\bar f}_{v^*\bar E,\,v^*\bar F}"'] & u^*\bar f^!v^*(\bar E\otimes\bar F) \arrow[r, "u^*\bar f^!v^*\chi^{\bar g}_{E,F}"'] & u^*\bar f^!v^*\bar g^!(E\otimes F),
\end{tikzcd}
$$

in which, commutativity of subdiagram ③ is obvious.

Commutativity of subdiagram ① follows from "transitivity" of χ with respect to proper maps (Exercise (4.7.3.4)(d)).

As for the remaining subdiagram ②, decomposing $\sigma_{w,\bar h,\bar f,v}$ as

$$
\begin{tikzcd}
\overline{X} \arrow[r, "i"] & Y\times_{\overline{Y}}\overline{\overline{X}} \arrow[r, "w_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\sigma"] & \overline{\overline{X}} \arrow[d, "\bar h"] \\
& Y \arrow[r, "v"'] & \overline{Y}
\end{tikzcd}
$$

with $w_1 i = w,\ f_1 i = \bar f$, and σ an independent fiber square (since v is flat), we see from (4.8.10.2) that β factors naturally as

$$w^*\bar h^! \xrightarrow{\ \sim\ } i^*w_1^*\bar h^! \xrightarrow[\beta_\sigma]{} i^*f_1^!v^* \xrightarrow[\beta_i^{-1}]{} i^!f_1^!v^* \xrightarrow{\ \sim\ } \bar f^! v^*.$$

Here i is an open and closed immersion, so that by (4.8.4), $i^! = i^*$ and the map β_i (see (4.8.7.0)) is the identity. Indeed, since if_1 and f_1 are both proper, therefore so is i [**EGA**, II, (5.4.3)(i)]; and since iw_1 and w_1 are both open immersions, therefore so is i (cf. (4.8.3.1)(a)).

It is left now to the reader to expand β as above and then to verify, with the aid of (4.7.3.4)(c) and (d), and of Exercise (4.8.12)(b)(vi) for open-and-closed immersions, that ② does commute. Q.E.D.

Corollary (4.9.2.2). *If a map* $f\colon X \to Z$ *factors in two ways as*

$$X \xrightarrow{u} Y \xrightarrow{\bar f} Z, \qquad X \xrightarrow{v} \overline{Y} \xrightarrow{\bar g} Z$$

($\bar f$ *and* $\bar g$ *proper,* u *and* v *open immersions) then for all* E, F *as in* (4.9.2), *it holds that* $\chi^{\bar f,u}_{E,F} = \chi^{\bar g,v}_{E,F}$.

Proof. The given data determine uniquely a map $\bar w\colon X \to Y \times_Z \overline{Y}$, whose schematic image we denote by $\overline{\overline{X}}$, see [**GD**, p. 324, (6.10.1) and p. 325, (6.10.5)]. The map $\bar w$ factors as $X \to X \times_Z X \to Y \times_Z \overline{Y}$, where the first map is the diagonal, a closed immersion, and the second is an open immersion. So $\bar w$ is an immersion, and hence induces an open immersion $w\colon X \to \overline{\overline{X}}$. Furthermore, the projections to Y and $\overline{Y}$ induce proper maps $h\colon \overline{\overline{X}} \to Y$ and $\bar h\colon \overline{\overline{X}} \to \overline{Y}$. It suffices then for (i) to prove the Corollary for each of the pairs of factorizations $f = \bar g v = (\bar g \bar h)w$ and $f = \bar f u = (\bar f h)w$.

For the first pair, one need only look at the case $u = f = \bar f = 1$ of Lemma (4.9.2). The second pair, being of the same form as the first, is handled similarly. Q.E.D.

Corollary (4.9.2.3). *For any étale* $g\colon Y \to Z$ *and* E, F *as in* (4.9.2), *the map* $\chi^g_{E,F}$ (4.9.1.1) *is the isomorphism* $f^*E \otimes f^*F \xrightarrow{\sim} f^*(E \otimes F)$ *coming from* (3.2.4).

Proof (Sketch). The idea is to redo everything in this section 4.9, up to this point, with "étale" in place of "open immersion." The first difficulty which arises is that in the last paragraph of the proof of Lemma (4.9.2), the map i is now finite étale, making it necessary to know (4.9.2.3) for finite étale f, a fact given by Exercise (4.8.12)(b)(vi). The only other nontrivial modification is in the proof of (4.9.2.2), where the map $X \times_Z X \to Y \times_Z \overline{Y}$ should now be factored as $X \times_Z X \hookrightarrow W \to Y \times_Z \overline{Y}$ with the first map an open immersion and the second proper, and then $\overline{\overline{X}}$ should be defined to be the schematic image of $X \to X \times_Z X \hookrightarrow W \dots$ Q.E.D.

Exercises (4.9.3). (a) In Ex. (4.7.3.4)(e) replace $f^\times$ by $\bar f^!$ and apply the functor u^* to get a natural map $u^*\bar f^! E \to \mathcal{H}_X(f^*F, u^*\bar f^!(E \otimes F))$. Then show that this map corresponds via (2.6.1)′ to $\chi^{\bar f,u}_{E,F}$.

(b) Let $f = \bar f u$ be as in (4.9.1). Show, for E, $F \in \mathbf{D}_{\mathsf{qc}}(Y)$, that the composite map

$$u^*\bar f^!\mathcal{H}_Y(E,F) \otimes u^*\bar f^*E \xrightarrow{\chi^{\bar f,u}} u^*\bar f^!(\mathcal{H}_Y(E,F) \otimes E) \xrightarrow{\text{natural}} u^*\bar f^! F$$

depends only on f, not on its factorization.

Deduce the existence, for any $E \in \mathbf{D}^-_{\mathsf{c}}(Y)$ and $F \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$, of a canonical isomorphism

$$\bar{f}^!\mathcal{H}_Y(E,F) \xrightarrow{\ \sim\ } \mathcal{H}_X(f^*E, f^!F),$$

inverse to $u^*\zeta$ where ζ comes from (4.2.3)(e) applied to $\bar{f}$. (This can also be done without recourse to χ.)

(c) (Compatibility of χ with base change.) After replacing $(-)^\times$ by $(-)^!$, do exercise (4.7.3.4)(c), assuming that the square is an *admissible* square, and interpreting β as in (4.8.3). Do something similar with the map ϕ of (3.10.4) in place of β.

(d) Proceeding as in (a), work out exercises (4.7.3.4)(a), (d), and (f), with $(-)^\times$ replaced by $(-)^!$. This will likely involve verifications of compatibility with restriction to open subschemes for a number of functorial maps. Do similarly for (4.2.3)(h)–(j).

(e) Show that if $f\colon X \to Y$ is ètale then the map in (b) is the same as the map coming from (3.5.4.5).

(f) Explain the formal tensor-hom symmetry in the pair of natural isomorphisms

$$f^*E \otimes f^!F \xrightarrow{\ \sim\ } f^!(E \otimes F) \qquad (E, F \in \mathbf{D}_{\mathsf{qc}(Y)}),$$
$$\mathcal{H}_X(f^*E, f^!F) \xrightarrow{\ \sim\ } f^!\mathcal{H}_Y(E,F) \qquad (E \in \overline{\mathbf{D}}^-_{\mathsf{c}}(Y),\, F \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)).$$

Another such pair, coming from (3.9.4) and (3.2.3.2), is

$$E \otimes f_*F \xrightarrow{\ \sim\ } f_*(f^*E \otimes F) \qquad (E, F \in \mathbf{D}_{\mathsf{qc}}(Y)),$$
$$\mathcal{H}_Y(E, f_*F) \xrightarrow{\ \sim\ } f_*\mathcal{H}_X(f^*E, F) \qquad (E, F \in \mathbf{D}(Y)).$$

(I don't have an answer.)

With respect to a scheme-map $f\colon X \to Y$, an $\mathcal{O}_X$-complex E is *f-perfect* if E has coherent homology and finite flat f-amplitude. As noted in (2.7.6), f is perfect (i.e., of finite tor-dimension) $\iff$ $\mathcal{O}_X$ is f-perfect.

When f is perfect, the natural map, taking $1 \in \mathrm{H}^0(X, \mathcal{O}_X)$ to the identity map of the *relative dualizing complex* $f^!\mathcal{O}_Y$ is an isomorphism

$$\xi\colon \mathcal{O}_X \xrightarrow{\ \sim\ } \mathcal{H}_X(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y).$$

In fact, the functor $\mathcal{H}_X(-, f^!\mathcal{O}_Y)$ induces an antiequivalence of the full subcategory of f-perfect complexes in $\mathbf{D}(X)$ to itself [**I**, p. 259, 4.9.2].

Theorem (4.9.4). *For any finite-type separated map $f\colon X \to Y$ of noetherian schemes, the following conditions are equivalent.*

(i) *The map f is perfect, i.e., the complex $\mathcal{O}_X$ is f-perfect.*

(ii) *The complex $f^!\mathcal{O}_Y$ is f-perfect.*

(iii) *$f^!\mathcal{O}_Y \in \overline{\mathbf{D}}^-_{\mathsf{qc}}(X)$, and for every $F \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$, the $\mathbf{D}_{\mathsf{qc}}(X)$-map*

$$\chi^f_{\mathcal{O}_Y, F}\colon f^!\mathcal{O}_Y \otimes f^*F \longrightarrow f^!F$$

is an isomorphism.

(iii)′ *For every perfect $\mathcal{O}_Y$-complex E, $f^!E$ is f-perfect; and for all $E, F \in \mathbf{D}(Y)$ such that E and $E \otimes F$ are in $\overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$, the $\mathbf{D}_{\mathsf{qc}}(X)$-map*

$$\chi^f_{E,F}\colon f^!E \otimes f^*F \longrightarrow f^!(E \otimes F).$$

is an isomorphism.

(iv) *The functor $f^!\colon \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y) \to \overline{\mathbf{D}}^+_{\mathsf{qc}}(X)$ is bounded.*

Proof. (i)$\Leftrightarrow$(ii). The question is local on X, so we may assume that f factors as $X \xrightarrow{i} Z \xrightarrow{p} Y$ where Z is an affine open subscheme of $Y \otimes_{\mathbb{Z}} \mathbb{Z}[T_1, \dots, T_n]$ (with independent indeterminates T_i), i is a closed immersion, and p is the obvious map.

By (4.4.2) (with $F = \mathcal{O}_X$), we have a functorial isomorphism

$$(4.9.4.1) \qquad i_*i^!G \xrightarrow{\sim} \mathcal{H}_Z(i_*\mathcal{O}_X, G) \qquad \big(G \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Z)\big).$$

Also, with Ω^n_p the invertible $\mathcal{O}_Z$-module of relative Kähler n-forms, there is a natural isomorphism

$$(4.9.4.2) \qquad p^!E \cong \Omega^n_p[n] \otimes p^*E \qquad (E \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)),$$

see [$\mathbf{V}'$, p. 397, Thm. 3].[57]

Now, by [**I**, p. 250, 4.1, and p. 252, 4.4], (i) holds if and only if the $\mathcal{O}_Z$-complex $i_*\mathcal{O}_X$ is perfect; and (ii) holds if and only if the $\mathcal{O}_Z$-complex

$$i_*f^!\mathcal{O}_Y \cong i_*i^!p^!\mathcal{O}_Y \cong \mathcal{H}_Z(i_*\mathcal{O}_X, p^!\mathcal{O}_Y) \cong \mathcal{H}_Z(i_*\mathcal{O}_X, \Omega^n_p[n])$$

is perfect. Hence the equivalence of (i) and (ii) results from the following fact, in the case $F = i_*\mathcal{O}_X$.

Lemma (4.9.4.3). *On any noetherian scheme W, an $\mathcal{O}_W$-complex F is perfect $\iff$ $F \in \overline{\mathbf{D}}^{\mathsf{b}}_{\mathsf{c}}(W)$ and $\mathcal{H}_W(F, \mathcal{O}_W)$ is perfect.*

Proof. The implication $\Rightarrow$ results from [**I**, p. 148, 7.1].

For the converse, the question being local, we may assume that W is affine, say $W = \mathrm{Spec}(R)$, that F is a bounded-above complex of finite-rank locally free $\mathcal{O}_W$-modules (see 4.3.2), and that $\mathcal{H}_W(F, \mathcal{O}_W)$ is $\mathbf{D}(W)$-isomorphic to a strictly perfect $\mathcal{O}_W$-complex.

Then $N := \Gamma(W, F)$ is a bounded-above complex of finite-rank projective R-modules, and with $^\sim$ the usual sheafification functor, $F \cong N^\sim$.

Let $R \to I^\bullet$ be an R-injective resolution of R. By [**H**, p. 130, 7.14], the resulting map $\mathcal{O}_W = R^\sim \to I^{\bullet\sim}$ is an injective resolution of $\mathcal{O}_W$. So $\mathcal{H}om_W(N^\sim, I^{\bullet\sim}) \cong \mathcal{H}_W(F, \mathcal{O}_W)$ is $\mathbf{D}(W)$-isomorphic—and hence, by (3.9.6)(a), $\mathbf{D}(\mathcal{A}^{\mathsf{qc}}_W)$-isomorphic—to a strictly perfect $\mathcal{O}_W$-complex. Since $\Gamma(W, -)$ is exact on $\mathcal{A}^{\mathsf{qc}}_W$, it follows that

$$\mathbf{R}\mathrm{Hom}_R(N, R) \cong \mathrm{Hom}_R(N, I^\bullet) \cong \Gamma\big(W, \mathcal{H}om_W(N^\sim, I^{\bullet\sim})\big)$$

is a perfect R-complex. So by [**AIL**, Prop. 4.1(ii)], N is perfect, whence so is $F \cong N^\sim$. Q.E.D.

[57] The proof in *loc. cit.* can be imitated, without the assumption of finite Krull dimension, and with E in place of $\mathcal{O}_Y$; but instead of Corollary 2 one should use [**H**, p. 180, Cor. 7.3], noting that the graph map denoted by Δ is a local complete intersection map of codimension n [**EGA**, IV, (17.12.3)]. It might appear simpler to use [$\mathbf{V}'$, p. 396, Lemma 1], whose proof, however, seems to need an isomorphism of the form (4.9.4.2) when Z is $\mathbf{P}^1_Y$. For this, see [**H**, p. 161, 5.1] (duality for $\mathbf{P}^n_Y$), except that the proof given there applies only to $F \in \overline{\mathbf{D}}^-_{\mathsf{qc}}(Y)$. That suffices, nevertheless, by (4.3.7) applied to the map $\phi\colon \Omega^1_p \to p^!\mathcal{O}_Y$ corresponding by duality to the canonical isomorphism $R^1p_*(\Omega^1_p) \xrightarrow{\sim} \mathcal{O}_Y$ [**H**, p. 155, 4.3].

(i) ⇒ (iii). One may assume f factors as above: $X \xrightarrow{i} Z \xrightarrow{p} Y$.

By (4.9.4.2), for $f^!\mathcal{O}_Y = i^! p^! \mathcal{O}_Y$ to be in $\overline{\mathbf{D}}^-_{\mathsf{qc}}(X)$ it suffices that the functor $i^!$ be bounded on $\overline{\mathbf{D}}^+_{\mathsf{qc}}(Z)$, which it is, by (4.9.4.1), because $i_*\mathcal{O}_X$ is perfect. (For this boundedness, as in the proof of [**I**, p. 148, 7.1], after replacing $i_*\mathcal{O}_X$ by an arbitrary perfect $\mathcal{O}_X$-complex E and localizing, one may assume that E is a bounded complex of finite-rank free $\mathcal{O}_Z$-modules, and proceed by "dévissage," i.e., induction on the number of nonzero components of E, to reduce to noting that $\mathcal{H}_Z(E, G)$ is a bounded functor of $G \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Z)$ when E is a finite-rank free $\mathcal{O}_Z$-module.)

Next, by (4.9.2), with $(f, g, u, \bar{f})$ replaced by $(i, p, 1, i)$, it suffices to show that $\chi^i_{p^!\mathcal{O}_Y,\, p^*F}$ and $\chi^p_{\mathcal{O}_Y,\, F}$ are isomorphisms.

By (4.9.3)(c), the question of whether $\chi^i_{p^!\mathcal{O}_Y,\, p^*F}$ is an isomorphism is local on Y, so we may assume Y affine, in which case every quasi-coherent $\mathcal{O}_Y$-module is a homomorphic image of a free one. Since p is flat and, by (4.9.4.2), the complex $p^!\mathcal{O}_Y$ is perfect, therefore $p^!\mathcal{O}_Y \otimes p^*F$ is a bounded functor of F; and again by (4.9.4.2), so is $p^!F$. Hence, by (1.11.3.1), one need only note that by (4.7.5) applied to a compactification of p, $\chi^i_{p^!\mathcal{O}_Y,\, p^*F}$ is an isomorphism whenever F is a *free* $\mathcal{O}_Y$-module.

That $\chi^i_{p^!\mathcal{O}_Y,\, G}$ is an isomorphism for any $G \in \mathbf{D}_{\mathsf{qc}}(Z)$ can be checked *after* application of the functor i_*. The source and target of $i_*\chi^i_{p^!\mathcal{O}_Y,\, G}$ are

$$i_*(i^! p^! \mathcal{O}_Y \otimes i^*G) \underset{(3.9.4)}{\cong} i_* i^! p^! \mathcal{O}_Y \otimes G \underset{(4.9.4.1)}{\cong} \mathcal{H}_Z(i_*\mathcal{O}_X, p^!\mathcal{O}_Y) \otimes G,$$

$$i_* i^!(p^!\mathcal{O}_Y \otimes G) \underset{(4.9.4.1)}{\cong} \mathcal{H}_Z(i_*\mathcal{O}_X, p^!\mathcal{O}_Y \otimes G).$$

Since $i_*\mathcal{O}_X$ is perfect, and, by (4.9.4.2), so is $p^!\mathcal{O}_Y$, therefore both the source and target are bounded functors of G, commuting with direct sums (see (3.8.2)). As before, one reduces to where Z is affine and G is a free $\mathcal{O}_Z$-module, in which case commutativity with direct sums gives a reduction to the trivial case $G = \mathcal{O}_Z$.

(Alternatively, it is a nontrivial exercise to show that (4.9.4.2) with $p^!\mathcal{O}_Y$ in place of $\Omega^n_p[n]$ is in fact $\chi^p_{\mathcal{O}_Y,\, E}$. One also shows, with $E := i_*\mathcal{O}_X$, $F := p^!\mathcal{O}_Y$, that $i_*\chi^i_{F,G}$ is isomorphic to the map

$$\zeta(E)\colon \mathcal{H}_Z(E, F) \otimes G \to \mathcal{H}_Z(E, F \otimes G)$$

associated by $(2.6.1)^*$ to the natural map $\mathcal{H}_Z(E, F) \otimes G \otimes E \to F \otimes G$, and then finds via dévissage to the trivial case $E = \mathcal{O}_Z$ that $\zeta(E)$ is an isomorphism for all perfect E. What is involved here is a *concrete* local interpretation of χ^f.)

(iii) ⇔ (iii)′ ⇒ (ii). The implications (iii)′ ⇒ (ii) and (iii)′ ⇒ (iii) are trivial

Assume, conversely, that (iii) holds.

To be shown first is that for a perfect $\mathcal{O}_Y$-complex E, $f^!E$ is f-perfect. Since $f^!$ commutes with open base change (4.8.3), one can replace Y by any open subset. Thus one may assume that E is a bounded complex of finite-rank free $\mathcal{O}_Y$-modules, and then proceed by dévissage to reduce to the case $E = \mathcal{O}_Y$, treated as follows.

Let $\mu\colon V \hookrightarrow Y$ be the inclusion of an open subscheme, $\nu\colon f^{-1}V \hookrightarrow X$ the inclusion, $g\colon f^{-1}V \to V$ the map induced by f, and M an $\mathcal{O}_V$-module. We have then the obvious isomorphisms

$$\nu^*f^!\mathcal{O}_Y \otimes g^*M \cong \nu^*(f^!\mathcal{O}_Y \otimes f^*\mu_*M) \underset{\text{(iii)}}{\cong} \nu^*f^!\mu_*M.$$

Since μ_*M is a bounded complex (3.9.2), and since $f^!$ is bounded below and, by (iii), bounded above, therefore there is an interval $[m,n]$ not depending on M such that

$$H^i(\nu^*f^!\mathcal{O}_Y \otimes g^*M) = 0 \quad \text{for all } i \notin [m,n].$$

So by [**I**, p. 242, 3.3(iv)], $f^!\mathcal{O}_Y$ has finite flat f-amplitude. Also, (4.9.4.1) and (4.9.4.2) imply that $f^!\mathcal{O}_Y \in \mathbf{D}_{\mathsf{c}}(X)$. Thus $f^!\mathcal{O}_Y$ is f-perfect.

For the isomorphism in (iii)$'$, apply (4.7.3.4)(a) with $E = \mathcal{O}_Y$ to a compactification of f.

(i)$\Rightarrow$(iv). Theorem (4.1) gives that $f^!$ is bounded below. If (i) holds then by definition, the (derived) functor f^* is bounded above; and as shown above, (iii) holds, whence $f^!$ is bounded above. Thus $f^!$ is bounded.

(iv)$\Rightarrow$(i). With notation as in the proof of (i) $\Leftrightarrow$ (ii), we will show that if $f^!$ is bounded then so is $i^!$. By [**LN**, Thm. 1.2] (or (4.9.6(e) below), this implies that i is perfect, whence so is $f = pi$.

Factor i as $X \xrightarrow{\gamma} X\times_Y Z \xrightarrow{g} Z$ where γ is the graph of i and g is the projection. The map γ, a local complete intersection [**EGA**, IV, (17.12.3)], is perfect, and so, as we've just seen, $\gamma^!$ is bounded.

Also, g arises from f by flat base change, so, as in (4.7.3.1)(ii) with $^\times$ replaced by $^!$, $g^!$ is bounded: to imitate the proof of (4.7.3.1)(ii) one just needs to associate a functorial isomorphism $v_*g^! \xrightarrow{\sim} f^!u_*$ to each composite fiber square

$$\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "s"'] & \bullet \arrow[d, "t"] \\
\bullet \arrow[r, "\bar v"] \arrow[d, "\bar g"'] & \bullet \arrow[d, "\bar f"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}$$

with u, $\bar v$ and v flat, $\bar f$ and $\bar g$ proper, t and s open immersions, $f = \bar f t$ and $g = \bar g s$. One such isomorphism is the natural composition

$$v_*g^! \xrightarrow{\sim} v_*s^*\bar g^! \xrightarrow{\sim} t^*\bar v_*\bar g^! \underset{(3.10.4)}{\xrightarrow{\sim}} t^*\bar f^! u_* \xrightarrow{\sim} f^!u_*.$$

Thus $i^! \cong \gamma^! g^!$ is bounded. Q.E.D.

Corollary (4.9.5). *On the category of perfect maps there is a pseudofunctor $(-)^\#$ which associates to each such map $f: X \to Y$ the functor $f^\#: \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y) \to \overline{\mathbf{D}}^+_{\mathsf{qc}}(X)$ given objectwise by*

$$f^\# F := f^!\mathcal{O}_Y \otimes F \qquad \big(F \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)\big).$$

For a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of perfect maps, the resulting functorial isomorphism $f^\# g^\# G \xrightarrow{\sim} (gf)^\# G$ $(G \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Z))$ is the left column of the following diagram of natural isomorphisms, whose commutativity results from (4.7.3.4)(a) and (d), as treated in (4.9.3)(d), or from (4.9.2) with $E := \mathcal{O}_X$ and $F := G$.

$$\begin{tikzcd}
f^\# g^\# G \arrow[r, phantom, "="] & f^!\mathcal{O}_Y \otimes f^*(g^!\mathcal{O}_Z \otimes g^*G) \arrow[r, "\chi"] \arrow[d] & f^!(g^!\mathcal{O}_Z \otimes g^*G) \arrow[r, "f^!\chi"] & f^!g^!G \arrow[ddd] \\
& (f^!\mathcal{O}_Y \otimes f^*g^!\mathcal{O}_Z) \otimes f^*g^*G \arrow[d, "\chi\otimes 1"'] & & \\
& f^!g^!\mathcal{O}_Z \otimes f^*g^*G \arrow[d] & & \\
(gf)^\# G \arrow[r, phantom, "="] & (gf)^!\mathcal{O}_Z \otimes (gf)^*G \arrow[rr, "\chi"'] & & (gf)^!G
\end{tikzcd}$$

Exercises (4.9.6). (a) Show that $\chi^f_{E,F}$ is an isomorphism whenever $F \in \mathbf{D}_{\mathsf{qc}}(X)$ has finite tor-dimension. (Cf. (4.7.5).)

(b) Noting Ex. (3.5.3)(g), establish a natural commutative diagram

$$\begin{tikzcd}
f^!F \otimes \mathcal{H}_X(f^*E, f^*G) \arrow[r] & \mathcal{H}_X(f^*E, f^!F \otimes f^*G) \arrow[d] & \mathcal{H}_X(f^*E, f^!F) \otimes f^*G \arrow[l] \arrow[d] \\
f^!F \otimes f^*\mathcal{H}_Y(E,G) \arrow[u] \arrow[d] & \mathcal{H}_X(f^*E, f^!(F \otimes G)) \arrow[d] & f^!\mathcal{H}_Y(E,F) \otimes f^*G \arrow[d] \\
f^!(F \otimes \mathcal{H}_Y(E,G)) \arrow[r] & f^!\mathcal{H}_Y(E, F \otimes G) & f^!(\mathcal{H}_Y(E,F) \otimes G) \arrow[l]
\end{tikzcd}$$

(c) (Neeman, van den Bergh). Show, for any perfect $f: X \to Y$ and $E \in \overline{\mathbf{D}}^+_{\mathsf{qc}}(Y)$, that the map $f^*E \to \mathcal{H}_X(f^!\mathcal{O}_Y, f^!E)$ induced via (2.6.1)$'$ by $\chi^f_{\mathcal{O}_Y,E}$ is an isomorphism.

Hint. Factor f locally as pi—see proof of (4.9.4), and apply i_*.

(d) Let X be a noetherian scheme, $E \in \mathbf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. Show that the functor $\mathcal{H}_X(E, -)$ from $\overline{\mathbf{D}}^+_{\mathsf{qc}}(X)$ to itself is bounded if and only if E is perfect.

Hint. Reduce to where $X = \mathrm{Spec}(A)$, and where E is the sheafificaton $\mathrm{E}^\sim$ of a bounded A-complex E of finitely generated A-modules. Use the fact that the sheafification of an A-injective module is $\mathcal{O}_X$-injective [**RD**, p. 130, 7.14], to show that for any $\mathrm{F} \in \overline{\mathbf{D}}^{+}(A)$, $\mathcal{H}_X(E,\mathrm{F}^{\sim}) = \mathbf{R}\mathrm{Hom}_A(\mathrm{E},\mathrm{F})^{\sim}$, and hence to reduce further to the corresponding statement for A-modules.

(e) Using (d) and (4.4.2) with $F = \mathcal{O}_X$, show that a finite map $f\colon X \to Y$ of noetherian schemes is perfect if and only if the functor $f^!\colon \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(Y) \to \overline{\mathbf{D}}^{+}_{\mathsf{qc}}(X)$ is bounded.

4.10. Appendix: Dualizing complexes

Grothendieck's original strategy for proving duality—at least the version in Corollary (4.2.2)—for proper not-necessarily-projective maps, is based on pseudofunctorial properties of dualizing complexes. In this section, we sketch the idea. The principal result, Thm. (4.10.4), makes clear how the basic problem—not treated here—in this approach is the construction of a "coherent" family of dualizing complexes (in other words, a "Dualizing Complex," see below). What emerges is less than Thm. (4.8.1). But for formal schemes, this kind of approach yields results not otherwise obtainable (as of early 2008), see the remarks following Thm. (4.10.4).

Throughout this section, without further mention we restrict to schemes which are noetherian and to scheme-maps that are separated, of finite type. Also, we continue to use the notations introduced at the beginning of §4.4.

Let $\mathcal{A}^{\mathsf{c}}(X) \subset \mathcal{A}(X)$ be the full subcategory whose objects are the coherent $\mathcal{O}_X$-modules; it is a plump subcategory [**GD**, 113, 5.3.5]. Additional notation will be as in §(1.9.1), with $\# = \mathsf{c}$.

For example, $\overline{\mathbf{D}}^{+}_{\mathsf{c}}(X)$ is the Δ-subcategory of $\mathbf{D}(X)$ whose objects are the complexes whose homology modules vanish in all sufficiently negative degrees, and are coherent in all degrees.

A *dualizing complex* R on a noetherian scheme X is a complex in $\mathbf{D}_{\mathsf{c}}(X)$ which is $\mathbf{D}$-isomorphic to a bounded injective complex, and has the following equivalent properties [**H**, p. 258, 2.1]:

(i) For every $F \in \mathbf{D}_{\mathsf{c}}(X)$, the map corresponding via $(2.6.1)'$ to the natural composition

$$F \otimes \mathbf{R}\mathcal{H}om(F,R) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om(F,R) \otimes F \to R$$

is an isomorphism (called by some other authors the *Grothendieck Duality isomorphism*):

$$F \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(F,R),R).$$

(ii) Condition (i) holds for $F = \mathcal{O}_X$, i.e., the map $\mathcal{O}_X \to \mathbf{R}\mathcal{H}om(R,R)$ which takes $1 \in \Gamma(X,\mathcal{O}_X)$ to the identity map of R is an isomorphism.

For connected X, dualizing $\mathcal{O}_X$-complexes, if they exist, are unique up to tensoring with a complex of the form $L[n]$ where L is an invertible $\mathcal{O}_X$-module and $n \in \mathbb{Z}$ [**H**, p. 266, 3.1].

The associated dualizing functor

$$\mathcal{D}_R := \mathbf{R}\mathcal{H}om_X(-,R)$$

satisfies $\mathcal{D}_R \circ \mathcal{D}_R \cong \mathbf{1}$, and it induces antiequivalences from $\mathbf{D}_{\mathsf{c}}(X)$ to itself, and between $\overline{\mathbf{D}}^{+}_{\mathsf{c}}(X)$ and $\overline{\mathbf{D}}^{-}_{\mathsf{c}}(X)$ (in either direction).

The existence of a dualizing complex places restrictions on X—for instance, X must then be universally catenary and of finite Krull dimension [**H**, p. 300]. Sufficient conditions for the existence are given in [**H**, p. 299]. For example, any scheme of finite type over a regular (or even Gorenstein) scheme of finite Krull dimension has a dualizing complex.[58]

Henceforth we restrict schemes to those which, in addition to being noetherian, have dualizing complexes.

The relation between dualizing complexes and the pseudofunctor $^!$ of Thm. (4.8.1) is rooted in the following Proposition, see [**H**, Chapter V, §8], [**V**′, p. 396, Corollary 3], or [**N**″, Theorems 3.12 and 3.14].

Proposition 4.10.1. *Let $f\colon X \to Y$ be a scheme-map, and let R be a dualizing $\mathcal{O}_Y$-complex. Then with $R_f := f^!R$,*

(i) *R_f is a dualizing $\mathcal{O}_X$-complex.*

(ii) *There is a functorial isomorphism*

$$f^!\mathcal{D}_R F \xrightarrow{\ \sim\ } \mathcal{D}_{R_f}\mathbf{L}f^*F \qquad \big(F \in \overline{\mathbf{D}}^{-}_{\mathsf{c}}(Y)\big)$$

or equivalently,

$$f^!E \xrightarrow{\ \sim\ } \mathcal{D}_{R_f}\mathbf{L}f^*\mathcal{D}_R E \qquad \big(E \in \overline{\mathbf{D}}^{+}_{\mathsf{c}}(Y)\big).$$

PROOF. First, it follows from the construction of the functor $f^\times$ (see just before (4.1.8)) that it preserves finite injective dimension. So when f is proper, $f^! = f^\times$ preserves finite injective dimension. The same is clearly true for $f^! = f^*$ when f is an open immersion, and hence—via compactification—for any f.

The question of whether $f^!R \in \mathbf{D}_{\mathsf{c}}(X)$ is local; hence an affirmative answer is provided by (4.9.4.1) and (4.9.4.2).

It remains to show that the natural map $\psi_f\colon \mathcal{O}_X \to \mathcal{D}_{R_f}\mathcal{D}_{R_f}\mathcal{O}_X$ is an isomorphism. Again, the question is local, so we reduce to the two cases (a) f is smooth, (b) f is a closed immersion.

(a) For smooth f, (4.9.4.2) and (4.6.7) provide natural isomorphisms

$$\mathbf{R}\mathcal{H}om_X(R_f, R_f) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om_X(p^*R, p^*R) \xrightarrow{\ \sim\ } p^*\mathbf{R}\mathcal{H}om_Y(R, R).$$

One verifies then that ψ_f is isomorphic, via the preceding isomorphisms, to p^* applied to the isomorphism $\mathcal{O}_Y \xrightarrow{\ \sim\ } \mathcal{D}_R\mathcal{D}_R\mathcal{O}_Y$.

(b) It suffices that $f_*\psi_f$ be an isomorphism, which it is, by (4.9.4.1) (with $i = f$), since $f_*\mathcal{O}_X \in \overline{\mathbf{D}}^{\mathsf{b}}_{\mathsf{c}}(Y)$ and therefore the canonical map $f_*\mathcal{O}_X \to \mathcal{D}_R\mathcal{D}_R f_*\mathcal{O}_X$ is an isomorphism.

Assertion (ii) follows immediately from Ex. (4.2.3)(e), as $\mathcal{D}_R$ and $\mathcal{D}_{R_f}$ are antiequivalences. Q.E.D.

[58] In [**N**″], Neeman studies a notion of dualizing complex which applies to infinite-dimensional schemes. Suresh Nayak observed, via [**C**, p. 121, Lemma 3.1.5], that Neeman's dualizing complexes are the same as pointwise dualizing complexes with bounded cohomology, cf. [**C**, p. 127, Lemma 3.2.1].

Definition (4.10.2). A *Dualizing Complex* on a scheme Y is a map which associates to each $f\colon X\to Y$ a dualizing complex R_f on X, to each open immersion $u\colon U\to X$ a $\mathbf{D}(X)$-isomorphism $\gamma_{f,u}\colon u^*R_f \xrightarrow{\ \sim\ } R_{fu}$, and to each proper map $g\colon X'\to X$ a $\mathbf{D}(X)$-map $\tau_{f,g}\colon g_*R_{fg}\to R_f$, subject to the following conditions on each such f, u and g:

(a) If $v\colon V\to U$ is an open immersion, then the following diagram commutes:

$$\begin{tikzcd}
v^*u^*R_f \arrow[r, "\sim", "(3.6.4)^*"'] \arrow[d, "v^*\gamma_{f,u}"'] & (uv)^*R_f \arrow[d, "\gamma_{f,uv}"] \\
v^*R_{fu} \arrow[r, "\gamma_{fu,v}"'] & R_{fuv}
\end{tikzcd}$$

(b) The pair $(R_{fg},\tau_{f,g})$ represents the functor

$$\mathrm{Hom}_{\mathbf{D}(X)}(g_*E,R_f)\colon \overline{\mathbf{D}}{}^{+}_{\mathsf{c}}(X')\to\overline{\mathbf{D}}{}^{+}_{\mathsf{c}}(X),$$

that is, the natural composite map

$$\mathrm{Hom}_{\mathbf{D}(X')}(E,R_{fg})\longrightarrow \mathrm{Hom}_{\mathbf{D}(X)}(g_*E,g_*R_{fg})\xrightarrow[\text{via }\tau]{}\mathrm{Hom}_{\mathbf{D}(X)}(g_*E,R_f)$$

is an isomorphism. Further, if $h\colon X''\to X'$ is proper then the following diagram commutes:

$$\begin{tikzcd}
g_*h_*R_{fgh} \arrow[r, "\sim", "(3.6.4)_*"'] \arrow[d, "g_*\tau_{fg,h}"'] & (gh)_*R_{fgh} \arrow[d, "\tau_{f,gh}"] \\
g_*R_{fg} \arrow[r, "\tau_{f,g}"'] & R_f
\end{tikzcd}$$

(c) For any fiber square

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "h"'] & Z \arrow[d, "g"] \\
U \arrow[r, "u"'] & X
\end{tikzcd}$$

with g (hence h) proper and u (hence v) an open immersion, the following natural diagram commutes:

$$\begin{tikzcd}
u^*g_*R_{fg} \arrow[rr, "\sim"] \arrow[d, "u^*\tau_{f,g}"'] & & h_*v^*R_{fg} \arrow[d, "\simeq"', "h_*\gamma_{fg,v}"] \\
u^*R_f \arrow[r, "\sim", "\gamma_{f,u}"'] & R_{fu} & h_*R_{fuh}=h_*R_{fgv} \arrow[l, "\tau_{fu,h}"']
\end{tikzcd}$$

Remarks. In (4.10.2)(a) take $U = V = X$ and let u and v be identity maps, to get $\gamma_{f,u} \circ \gamma_{f,u} = \gamma_{f,u}$, whence the isomorphism $\gamma_{f,u}$ is the identity map $\mathbf{1}$ of R_f. Similarly, when g is the identity map of X, one deduces from (b) that $\tau_{f,g} \circ \tau_{f,g} = \tau_{f,g}$; but $(R_f, \tau_{f,g})$ and $(R_f, \mathbf{1})$ both represent the same functor, whence $\tau_{f,g}$ is an isomorphism, so $\tau_{f,g} = \mathbf{1}$. Also, when $Z = U = V$ and $g = u$ is an open and closed immersion, (c) shows that $\gamma_{f,g} \circ g^*\tau_{f,g}$ is the canonical isomorphism $g^*g_*R_{fg} \xrightarrow{\sim} R_{fg}$.

Examples **(4.10.2.1).** (A) If R is a dualizing $\mathcal{O}_Y$-complex and $^!$ is as in (4.8.1), one can associate to each map $f\colon X \to Y$ the dualizing $\mathcal{O}_X$-complex $R_f := f^!R$, to each open immersion $u\colon U \to X$ the natural composition

$$\gamma_{f,u}\colon u^*R_f = u^!f^!R_f \xrightarrow{\sim} (fu)^!R = R_{fu},$$

and to each proper map $g\colon X' \to X$ the map $\tau = \tau_{f,g}\colon g_*(fg)^!R \to f^!R$ resulting from (4.1.1). Condition (a) is then clear, (b) follows from (4.1.2), and (c) from (4.4.4)(d).

(B) Let $\mathcal{R} = (R, \gamma, \tau)$ be a Dualizing Complex on Y. Then for any map $e\colon Y' \to Y$ we have a Dualizing Complex $\mathcal{R} \times_Y Y' := (R', \gamma', \tau')$ on Y', where for all $f\colon X \to Y'$ we set $R'_f := R_{ef}$, $\gamma'_{f,u} := \gamma'_{ef,u}$ and $\tau'_{f,g} := \tau_{ef,g}$.

That $\mathcal{R} \times_Y Y'$ satisfies conditions (a), (b) and (c) is simple to check.

(C) Let $\mathcal{R} = (R, \gamma, \tau)$ be a Dualizing Complex on Y. Then for any invertible $\mathcal{O}_Y$-module $\mathcal{L}$ and any locally constant function $n\colon Y \to \mathbb{Z}$, we have a Dualizing Complex

$$\mathcal{R} \otimes \mathcal{L}[n] = (R \otimes \mathcal{L}[n], \gamma \otimes \mathcal{L}[n], \tau \otimes \mathcal{L}[n])$$

on Y, where for all $f\colon X \to Y$,

- $(R \otimes \mathcal{L}[n])_f := R_f \otimes f^*\mathcal{L}[n]$ (easily seen to be a dualizing $\mathcal{O}_X$-complex),
- $(\gamma \otimes \mathcal{L}[n])_{f,u}$ is the natural composition
$$u^*\big(R_f \otimes f^*\mathcal{L}[n]\big) \xrightarrow{\sim} u^*R_f \otimes u^*f^*\mathcal{L}[n] \xrightarrow{\sim} R_{fu} \otimes (fu)^*\mathcal{L}[n],$$
- $\big(\tau \otimes \mathcal{L}[n]\big)_{f,g}$ is the natural composition
$$g_*\big(R_{fg} \otimes (fg)^*\mathcal{L}[n]\big) \xrightarrow{\sim} g_*\big(R_{fg} \otimes g^*f^*\mathcal{L}[n]\big) \underset{(3.9.4)}{\xrightarrow{\sim}} g_*R_{fg} \otimes f^*\mathcal{L}[n] \longrightarrow R_f \otimes f^*\mathcal{L}[n].$$

Here, condition (a) is given by the (readily verified) commutativity of the natural diagram

$$\begin{tikzcd}
v^*u^*(R_f \otimes f^*\mathcal{L}[n]) \arrow[rr] \arrow[d] & & (uv)^*(R_f \otimes f^*\mathcal{L}[n]) \arrow[d] \\
v^*(u^*R_f \otimes u^*f^*\mathcal{L}[n]) \arrow[r] \arrow[d] & v^*u^*R_f \otimes v^*u^*f^*\mathcal{L}[n] \arrow[r] \arrow[d] & (uv)^*R_f \otimes (uv)^*f^*\mathcal{L}[n] \arrow[d] \\
v^*(R_{fu} \otimes (fu)^*\mathcal{L}[n]) \arrow[r] & v^*R_{fu} \otimes v^*(fu)^*\mathcal{L}[n] \arrow[r] & R_{fuv} \otimes (fuv)^*\mathcal{L}[n]
\end{tikzcd}$$

Fix a $\mathbf{D}(X)$-isomorphism $\alpha\colon \mathcal{L}[n]\otimes\mathcal{L}^{-1}[-n] \xrightarrow{\;\sim\;} \mathcal{O}_Y$. The first part of condition (b) results from commutativity of the natural diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}(X)}\big(E, R_{fg}\otimes(fg)^*\mathcal{L}[n]\big) \arrow[r, "\sim"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathbf{D}(X)}\big(E\otimes(fg)^*\mathcal{L}^{-1}[-n], R_{fg}\big) \arrow[d, "\simeq"'] \\
\mathrm{Hom}_{\mathbf{D}(X)}\big(E, R_{fg}\otimes g^*f^*\mathcal{L}[n]\big) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{D}(X)}\big(E\otimes g^*f^*\mathcal{L}^{-1}[-n], R_{fg}\big) \arrow[d] \\
\mathrm{Hom}_{\mathbf{D}(Y)}\big(g_*E, g_*(R_{fg}\otimes g^*f^*\mathcal{L}[n])\big) \arrow[d, "\text{via (3.9.4)}"', "\simeq"] & \mathrm{Hom}_{\mathbf{D}(Y)}\big(g_*(E\otimes g^*f^*\mathcal{L}^{-1}[-n]), g_*R_{fg}\big) \arrow[d, "\simeq"', "\text{via (3.9.4)}"] \\
\mathrm{Hom}_{\mathbf{D}(Y)}\big(g_*E, g_*R_{fg}\otimes f^*\mathcal{L}[n]\big) \arrow[d] & \mathrm{Hom}_{\mathbf{D}(Y)}\big(g_*E\otimes f^*\mathcal{L}^{-1}[-n], g_*R_{fg}\big) \arrow[l, "\sim"'] \arrow[d] \\
\mathrm{Hom}_{\mathbf{D}(Y)}\big(g_*E, R_f\otimes f^*\mathcal{L}[n]\big) & \mathrm{Hom}_{\mathbf{D}(Y)}\big(g_*E\otimes f^*\mathcal{L}^{-1}[-n], R_f\big) \arrow[l, "\sim"']
\end{tikzcd}$$

where, with $\mathcal{L}_n := \mathcal{L}[n]$ and $\mathcal{L}^{-1}_{-n} := \mathcal{L}^{-1}[-n]$, the first row takes a map $\eta\colon E\to R_{fg}\otimes(fg)^*\mathcal{L}_n$ to the natural composition

$$\begin{aligned}
E\otimes(fg)^*\mathcal{L}^{-1}_{-n} &\xrightarrow{\text{via }\eta} \big(R_{fg}\otimes(fg)^*\mathcal{L}_n\big)\otimes(fg)^*\mathcal{L}^{-1}_{-n} \\
&\xrightarrow{\;\sim\;} R_{fg}\otimes(fg)^*\big(\mathcal{L}_n\otimes\mathcal{L}^{-1}_{-n}\big) \xrightarrow{\text{via }\alpha} R_{fg}\otimes(fg)^*\mathcal{O}_Y \xrightarrow{\;\sim\;} R_{fg}
\end{aligned}$$

and the second row takes $\eta'\colon E\to R_{fg}\otimes g^*f^*\mathcal{L}_n$ to the natural composition

$$\begin{aligned}
E\otimes g^*f^*\mathcal{L}^{-1}_{-n} &\xrightarrow{\text{via }\eta'} \big(R_{fg}\otimes g^*f^*\mathcal{L}_n\big)\otimes g^*f^*\mathcal{L}^{-1}_{-n} \\
&\xrightarrow{\;\sim\;} R_{fg}\otimes g^*f^*\big(\mathcal{L}_n\otimes\mathcal{L}^{-1}_{-n}\big) \xrightarrow{\text{via }\alpha} R_{fg}\otimes g^*f^*\mathcal{O}_Y \xrightarrow{\;\sim\;} R_{fg}.
\end{aligned}$$

The arrows in the last two rows are defined in a similar manner.

Commutativity of the bottom subrectangle is obvious. Checking commutativity of the other two subdiagrams is left as an exercise. (For the middle one, a variant of diagram (3.4.7)(iv) may prove useful.)

The second part of condition (b) follows from (3.7.1). (Details left as an exercise.)

Condition (c) is given by commutativity of the following natural diagram, where $\mathcal{L}[n]$ has been abbreviated to $\mathcal{L}$:

$$\begin{tikzcd}
u^*(R_f\otimes f^*\mathcal{L}) \arrow[d] & u^*(g_*R_{fg}\otimes f^*\mathcal{L}) \arrow[l] \arrow[d] & u^*g_*(R_{fg}\otimes g^*f^*\mathcal{L}) \arrow[l] \arrow[r] & h_*v^*(R_{fg}\otimes g^*f^*\mathcal{L}) \arrow[dd] \\
u^*R_f\otimes u^*f^*\mathcal{L} \arrow[dddd] & u^*g_*R_{fg}\otimes u^*f^*\mathcal{L} \arrow[l] \arrow[d] & ① & \\
 & h_*v^*R_{fg}\otimes u^*f^*\mathcal{L} \arrow[ddd] & h_*(v^*R_{fg}\otimes h^*u^*f^*\mathcal{L}) \arrow[l] \arrow[ddd] & h_*(v^*R_{fg}\otimes v^*g^*f^*\mathcal{L}) \arrow[l] \arrow[d] \\
 & & & h_*(R_{fgv}\otimes(fgv)^*\mathcal{L}) \arrow[dd, equal] \\
 & & & \\
R_{fu}\otimes(fu)^*\mathcal{L} & h_*R_{fuh}\otimes(fu)^*\mathcal{L} \arrow[l] & h_*(R_{fuh}\otimes h^*(fu)^*\mathcal{L}) \arrow[l] & h_*(R_{fuh}\otimes(fuh)^*\mathcal{L}) \arrow[l]
\end{tikzcd}$$

Commutativity of subdiagram ① is given by (3.7.3). Commutativity of the other subdiagrams is easy to check.

A *morphism of Dualizing Complexes on* Y, $\psi\colon (R,\gamma,\tau) \xrightarrow{\sim} (R',\gamma',\tau')$ is a map associating to each scheme-map $f\colon X \to Y$ a $\mathbf{D}(X)$-map $\psi_f\colon R_f \xrightarrow{\sim} R'_f$, such that for each open immersion $u\colon U \to X$ (resp. each proper map $g\colon X' \to X$) the following diagrams commute:

(4.10.2.2)
$$\begin{tikzcd} u^*R_f \arrow[r, "\gamma_{f,u}"] \arrow[d, "u^*\psi_f"'] & R_{fu} \arrow[d, "\psi_{fu}"] \\ u^*R'_f \arrow[r, "\gamma'_{f,u}"'] & R'_{fu} \end{tikzcd} \qquad \begin{tikzcd} g_*R_{fg} \arrow[r, "\tau_{f,g}"] \arrow[d, "g_*\psi_{fg}"'] & R_f \arrow[d, "\psi_f"] \\ g_*R'_{fg} \arrow[r, "\tau'_{f,g}"'] & R'_f \end{tikzcd}$$

In the next Proposition, 1 denotes the identity map of Y.

Proposition (4.10.3). *Let (R,γ,τ) and (R',γ',τ') be Dualizing Complexes on Y, and let $\psi_0\colon R_1 \to R'_1$ be a $\mathbf{D}(Y)$-map. Then there exists a unique morphism $\psi\colon (R,\gamma,\tau) \xrightarrow{\sim} (R',\gamma',\tau')$ with $\psi_1 = \psi_0$.*

Corollary (4.10.3.1) (Uniqueness of Dualizing Complexes). *If $\mathcal{R}$ and $\mathcal{R}'$ are Dualizing Complexes on Y then there exists an invertible $\mathcal{O}_Y$-module $\mathcal{L}$, unique up to isomorphism, and a unique locally constant function $n\colon Y \to \mathbb{Z}$ such that $\mathcal{R}' \cong \mathcal{R} \otimes \mathcal{L}[n]$. Moreover, if ψ and χ are two isomorphisms from $\mathcal{R}'$ to $\mathcal{R} \otimes \mathcal{L}[n]$ then $\psi^{-1}\chi$ is multiplication by a unit in* $\mathrm{H}^0(Y,\mathcal{O}_Y)$.

Proof of (4.10.3.1). One reduces easily to where Y is connected. In view of (4.10.3), the first assertion follows then from the corresponding assertion for dualizing $\mathcal{O}_Y$-complexes [**H**, p. 266, Thm. 3.1]. The second assertion results from the sequence of natural ring isomorphisms and anti-isomorphisms—with R a dualizing $\mathcal{O}_Y$-complex and $\mathcal{D}_R(-) := \mathbf{R}\mathcal{H}om_X(-,R)$:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(Y)}(R,R) &\cong \mathrm{Hom}_{\mathbf{D}(Y)}\big(\mathcal{D}_R(R),\mathcal{D}_R(R)\big) \cong \mathrm{Hom}_{\mathbf{D}(Y)}(\mathcal{O}_Y,\mathcal{O}_Y) \\ &\cong \mathrm{H}^0\mathbf{R}\Gamma\mathbf{R}\mathcal{H}om(\mathcal{O}_Y,\mathcal{O}_Y) \cong \mathrm{H}^0\mathbf{R}\Gamma(\mathcal{O}_Y) \cong \mathrm{H}^0(Y,\mathcal{O}_Y). \end{aligned}$$

Proof of (4.10.3). For any proper map $g: X \to Y$, since $(R'_g, \tau'_{1,g})$ represents the functor $\mathrm{Hom}_{\mathbf{D}(Y)}(g_*E, R'_1)$ (see (4.10.2)(b)), there exists a unique $\mathbf{D}(X)$-map $\psi_g\colon R_g \to R'_g$ making the following diagram commute:

$$\begin{tikzcd} g_*R_g \arrow[r, "g_*\psi_g"] \arrow[d, "\tau_{1,g}"'] & g_*R'_g \arrow[d, "\tau'_{1,g}"] \\ R_1 \arrow[r, "\psi_0"'] & R'_1 \end{tikzcd}$$

A general map $f\colon X \to Y$ factors as $X \xrightarrow{u} Z \xrightarrow{g} Y$ with g proper and u an open immersion. Let $\psi_{g,u}\colon R_f \to R'_f$ be the unique $\mathbf{D}(X)$-map making the following diagram commute:

$$\begin{tikzcd}
R_f \arrow[r, "\psi_{g,u}"] & R'_f \\
u^*R_g \arrow[u, "\gamma_{g,u}", "\simeq"'] \arrow[r, "u^*\psi_g"'] & u^*R'_g \arrow[u, "\simeq", "\gamma'_{g,u}"']
\end{tikzcd}$$

Let us show that $\psi_{g,u}$ depends only on f, allowing us to write ψ_f instead of $\psi_{g,u}$. So let $X \xrightarrow{\tilde u} \widetilde Z \xrightarrow{\tilde g} Y$ also be a factorization of f ($\tilde u$ an open immersion, $\tilde g$ proper). There results a natural diagram

$$\begin{tikzcd}
 & Z \arrow[dr, "g"] & \\
X \arrow[ur, "u"] \arrow[r, "\bar u"] \arrow[dr, "\tilde u"'] & \overline{Z} \arrow[u] \arrow[d] \arrow[r, "\bar g"] & Y \\
 & \widetilde Z \arrow[ur, "\tilde g"'] &
\end{tikzcd}$$

with $\overline{Z}$ the scheme-theoretic image [**GD**, p. 324, §6.10] of the composite immersion $X \xrightarrow{\text{diag}} X \times_Y X \xrightarrow{(u,\tilde u)} Z \times_Y \widetilde Z$, and $\bar u\colon X \to \bar Z$ the resulting open immersion; and where the vertical maps, induced by the canonical projections, are proper.

We need only show that $\psi_{u,g} = \psi_{\bar u,\bar g} = \psi_{\tilde u,\tilde g}$; so it's enough to treat the case $(\tilde u, \tilde g) = (\bar u, \bar g)$, that is, we may assume that there is a proper map $p\colon \widetilde Z \to Z$ such that $gp = \tilde g$ and $p\tilde u = u$, and that furthermore $\tilde u(X)$ is a dense open subset of $\widetilde Z$:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\tilde u"'] \arrow[dr, phantom, "①"] & X \arrow[r, equal] \arrow[d, "u"'] & X \arrow[d, "\tilde g"] \\
\widetilde Z \arrow[r, "p"'] & Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

Here subdiagram ① is a fiber square, since the map $\tilde u_0\colon X \to p^{-1}(uX)$ induced by $\tilde u$ is both an open immersion (clearly) and a closed immersion (because $\tilde u_0$ has a left inverse, essentially $p|_{p^{-1}(uX)}$), so that $\tilde uX$ is open, closed and dense in $p^{-1}(uX)$, hence equal to $p^{-1}(uX)$. Consequently, there is a natural functorial isomorphism $\theta\colon u^*p_* \xrightarrow{\sim} \tilde u^*$.

It will be enough to show that the following diagram—whose top and bottom rows compose to $\psi_{g,u}$ and $\psi_{\tilde g,\tilde u}$ respectively—commutes:

$$
\begin{tikzcd}
R_{gu} \arrow[r, "\sim", "\gamma_{g,u}^{-1}"'] \arrow[dd, equal] & u^*R_g \arrow[r, "u^*\psi_g"] \arrow[r, phantom, shift right=6, "③"] & u^*R'_g \arrow[r, "\sim", "\gamma'_{g,u}"'] & R'_{gu} \arrow[dd, equal] \\
② & u^*p_*R_{gp} \arrow[u, "u^*\tau_{g,p}"] \arrow[r, "u^*p_*\psi_{gp}"'] \arrow[d, "\theta"', "\simeq"] \arrow[r, phantom, shift right=6, "④"] & u^*p_*R'_{gp} \arrow[u, "u^*\tau'_{g,p}"'] \arrow[d, "\simeq"', "\theta"] & ⑤ \\
R_{\tilde g\tilde u} = R_{gp\tilde u} \arrow[r, "\sim", "\gamma_{gp,\tilde u}^{-1}"'] & \tilde u^*R_{gp} \arrow[r, "\tilde u^*\psi_{gp}"'] & \tilde u^*R'_{gp} \arrow[r, "\sim", "\gamma'_{gp,\tilde u}"'] & R'_{gp\tilde u} = R'_{\tilde g\tilde u}
\end{tikzcd}
$$

Commutativity of subdiagram ④ is clear. Subdiagrams ② and ⑤ commute by condition (c) in (4.10.2), applied to the above fiber square ①. Finally, the first part of (4.10.2)(b) guarantees the existence of a map $\hat\psi_{gp}\colon R_{gp} \to R'_{gp}$ such that the following diagram commutes:

$$
\begin{tikzcd}
R_g \arrow[r, "\psi_g"] & R'_g \\
p_*R_{gp} \arrow[u, "\tau_{g,p}"] \arrow[r, "p_*\hat\psi_{gp}"'] & p_*R'_{gp}\,; \arrow[u, "\tau'_{g,p}"']
\end{tikzcd}
$$

and in view of the of the commutative diagram in (4.10.2)(b), and of the definition of ψ_f for proper f, application of the functor g_* to the preceding diagram shows that $\hat\psi_{gp} = \psi_{gp}$, whence ③ commutes.

We have now defined ψ_f for all f. The commutativity in (4.10.2.2) shows that no other family (ψ_f) can satisfy (4.10.3). It remains to be proved that with the present (ψ_f), commutativity does hold for the two diagrams in (4.10.2.2).

For the first of those diagrams, the problem is to show, given a sequence $U \xrightarrow{u} X \xrightarrow{v} Z \xrightarrow{g} Y$ with u and v open immersions and g proper, that the following natural diagram commutes:

$$
\begin{tikzcd}
u^*R_{gv} \arrow[r, "\sim"] \arrow[d] & u^*v^*R_g \arrow[r] \arrow[d] & u^*v^*R'_g \arrow[r, "\sim"] \arrow[d] & u^*R'_{gv} \arrow[d] \\
R_{gvu} \arrow[r] & (vu)^*R_g \arrow[r] & (vu)^*R'_g \arrow[r] & R'_{gvu}\,;
\end{tikzcd}
$$

but this is an immediate consequence of (4.10.2)(a).

For the second diagram in (4.10.2.2), suppose there is given a sequence

$X' \xrightarrow{g} X \xrightarrow{v} Z \xrightarrow{h} Y$ with u an open immersion and g, h both proper. As above, there are maps $X' \xrightarrow{w} W \xrightarrow{\bar g} Z$ such that w maps X' isomorphically onto a dense open subscheme of W, $\bar g$ is proper, and $\bar g w = vg$:

$$\begin{tikzcd}
X' \arrow[r, "w"] \arrow[d, "g"'] & W \arrow[d, "\bar g"] & \\
X \arrow[r, "v"'] & Z \arrow[r, "h"'] & Y
\end{tikzcd}$$

The proper map g factors naturally as $X' \to \bar g^{-1}(vX) \to X$, whence $w(X')$ is open, closed and dense in—hence equal to—$\bar g^{-1}(vX)$, and so there is a natural isomorphism $\theta\colon v^*\bar g_* \xrightarrow{\ \sim\ } g_*w^*$.

The problem is to show commutativity of the natural diagram

$$\begin{tikzcd}
g_*R_{hvg} \arrow[rr] \arrow[d, equal] & & R_{hv} \arrow[dd, "\simeq"] \\
g_*R_{h\bar g w} \arrow[d, "\simeq"'] & ⑥ & \\
g_*w^*R_{h\bar g} \arrow[r, "\theta^{-1}"] \arrow[d, "g_*w^*\psi_{h\bar g}"'] & v^*\bar g_*R_{h\bar g} \arrow[r] \arrow[d, "v^*\bar g_*\psi_{h\bar g}"'] \arrow[dr, phantom, "⑦"] & v^*R_h \arrow[d, "v^*\psi_h"] \\
g_*w^*R'_{h\bar g} \arrow[r, "\theta^{-1}"'] \arrow[d, "\simeq"'] & v^*\bar g_*R'_{h\bar g} \arrow[r] & v^*R'_h \arrow[dd, "\simeq"] \\
g_*R'_{h\bar g w} \arrow[d, equal] & ⑧ & \\
g_*R'_{hvg} \arrow[rr] & & R'_h
\end{tikzcd}$$

Commutativity of subdiagrams ⑥ and ⑧ is given by (4.10.2)(c). The argument that subdiagram ⑦ commutes is similar to that used above for ③. Commutativity of the remaining subdiagram is obvious. Q.E.D.

Here is the main result of this section.[59]

Theorem (4.10.4). *Let* **S** *be a category of noetherian schemes such that if* $Y \in \mathbf{S}$ *and* $f\colon X \to Y$ *is a separated finite-type map then* $X \in \mathbf{S}$*. Suppose every scheme in* **S** *has a Dualizing Complex.*

[59] Cf. [**H**, p. 383, Cor. 3.4], and its proof.

Then there exists on $\mathbf{S}$ *a* $\overline{\mathbf{D}}_{\mathsf{c}}^{+}$*-valued pseudofunctor* $^{!}$ *which is uniquely determined up to isomorphism by the properties that it restricts to the inverse-image pseudofunctor* * *on the subcategory of open immersions, that for a proper* $f \in \mathbf{S}$*, the functor* $f^{!}$ *is right-adjoint to* $f_{*}\colon \overline{\mathbf{D}}_{\mathsf{c}}^{+}(X) \to \overline{\mathbf{D}}_{\mathsf{c}}^{+}(Y)$ *(see* (3.9.2.6)(c)*), and that for any fiber square* σ *in* $\mathbf{S}$

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "j"'] & Y \end{tikzcd}$$

with j *an open immersion and* p *proper, the base-change map* β_{σ} *of* (4.4.3) *is the natural composite isomorphism*

$$j'^{*}p^{!} = j'^{!}p^{!} \xrightarrow{\sim} (pj')^{!} = (jp')^{!} \xrightarrow{\sim} p'^{!}j^{!} = p'^{!}j^{*}.$$

With this $^{!}$*, each Dualizing Complex* $(\bar{R}, \bar{\gamma}, \bar{\tau})$ *on* Y *is isomorphic to the one in* (4.10.2.1)(A) *with* $R := \bar{R}_{(\text{identity of } Y)}$.

Remarks. This says less than Theorem (4.8.1): the restriction to $\mathbf{S}$ of the pseudofunctor in that Theorem satisfies this one. The point is, however, that Theorem (4.10.4) captures Grothendieck's strategy for constructing a duality pseudofunctor by means of Dualizing Complexes. Indeed, showing the existence of Dualizing Complexes is a major theme of the second half of [**H**]. (See also the discussion and clarification of this material in [**C**, §§3.1–3.4].)[60]

Let us add a few words, in passing, about noetherian *formal schemes.* Applying his results about pasting pseudofunctors to the duality theory in [**AJL′**], Nayak gets the existence of a duality pseudofunctor for composites of any number of pseudo-proper maps and open immersions [**Nk**, §7.1]. (As of 2008, one doesn't know whether or not any pseudo-finite separated map of formal schemes is such a composite.) On the other hand, using an analog of Theorem (4.10.4), Sastry constructs a duality pseudofunctor on the category of all formal schemes admitting a dualizing complex (suitably defined for formal schemes), with "essentially pseudo-finite type" maps; and he shows that this pseudofunctor agrees with Nayak's whenever both are defined [**S′**, §9].

[60] Recently, Yekutieli and Zhang have exploited the notion of "rigid dualizing complex," introduced by Van den Bergh in the context of noncommutative algebra, to give an elegant new approach to the existence question, at least for finite tor-dimension maps of schemes of finite type over a regular scheme. See [**YZ**] for a preliminary account.

Sastry's approach has some resemblance to the one in [**H**], but there are a number of new techniques involved in the construction of Dualizing Complexes. In short, Chapter 6 of [**H**] is localized, generalized, and extended to the context of formal schemes in [**LNS**]; and then, among other things, the main results of Chapter 7 of [**H**], are extended to this context in [**S′**].

Thus, at the present time (2008), the theory of Dualizing Complexes for formal schemes gives rise in certain situations to the *only* way to construct dualizing pseudofunctors.

PROOF OF (4.10.4) (Outline only). For each $Y \in \mathbf{S}$ choose a Dualizing Complex $\mathcal{R}^Y = (R^Y, \gamma^Y, \tau^Y)$. For any $\mathbf{S}$-map $f\colon X \to Y$ let $\mathcal{D}_f^Y$ be the functor from $\mathbf{D_c}(X)$ to $\mathbf{D_c}(X)$ given by

$$\mathcal{D}_f^Y(E) := \mathcal{H}_X(E, R_f^Y) \qquad (\mathcal{H}_X := \mathbf{R}\mathcal{H}om^\bullet_X).$$

We set $R_Y := R^Y_{1_Y}$ and $\mathcal{D}^Y := \mathcal{D}^Y_{1_Y}$ where 1_Y is the identity map of Y.

For any $\mathbf{S}$-map $f\colon X \to Y$, the functor $f^!\colon \mathbf{D_c}(Y) \to \mathbf{D_c}(X)$ is defined to be

$$f^! := \mathcal{D}_f^Y f^* \mathcal{D}^Y.$$

This functor has the following properties.

(1) If f is an open immersion, there is a natural functorial isomorphism $f^! \xrightarrow{\sim} f^*$, namely, the natural composition, with $E \in \mathbf{D_c}(Y)$,

$$\begin{aligned} f^!E = \mathcal{H}_X\big(f^*\mathcal{D}^Y E, R_f^Y\big) &\xrightarrow[\text{via (4.6.7)}]{\sim} \mathcal{H}_X\big(\mathcal{H}_X(f^*E, f^*R_Y), R_f^Y\big) \\ &\xrightarrow[\text{via } \gamma^{-1}_{1_Y,u}]{\sim} \mathcal{H}_X\big(\mathcal{H}_X(f^*E, R_f^Y), R_f^Y\big) \xrightarrow{\sim} f^*E \end{aligned}$$

(the last isomorphism resulting from R_f^Y being a dualizing $\mathcal{O}_X$-complex).

(2) If f is proper then $f^!$ is right-adjoint to $f_*\colon \mathbf{D_c}(X) \to \mathbf{D_c}(Y)$. Indeed, for $E \in \mathbf{D_c}(X)$, $F \in \mathbf{D_c}(Y)$ we have, in view of (4.10.2)(b), natural functorial isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(X)}\big(E, f^!F\big) &\xrightarrow[(2.6.1)']{\sim} \mathrm{Hom}_{\mathbf{D}(X)}\big(E \otimes f^*\mathcal{H}_Y(F, R_Y), R_f^Y\big) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}\big(f_*(E \otimes f^*\mathcal{H}_Y(F, R_Y)), R_Y\big) \\ &\xrightarrow[(3.9.4)]{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}\big(f_*E \otimes \mathcal{H}_Y(F, R_Y), R_Y\big) \\ &\xrightarrow[(2.6.1)']{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}\big(f_*E, \mathcal{H}_Y(\mathcal{H}_Y(F, R_Y), R_Y)\big) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}\big(f_*E, R_Y\big). \end{aligned}$$

(3) There is a natural isomorphism $f^!R_Y \xrightarrow{\sim} R_f^Y$. This follows easily from the natural isomorphism $\mathcal{D}^Y R_Y \xrightarrow{\sim} \mathcal{O}_Y$.

(4) The functor ${}^!$ extends to a pseudofunctor.

For the proof, we need:

Lemma (4.10.4.1). *For any sequence* $V \xrightarrow{h} W \xrightarrow{g} X \xrightarrow{f} Y$ *in* $\mathbf{S}$ *there is a natural isomorphism*

$$\phi_{f,g,h}\colon \mathcal{D}^X_{gh}h^*\mathcal{D}^X_g \xrightarrow{\sim} \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg},$$

such that

$$\phi_{f,g,h}\circ\phi_{g,1_W,h} = \phi_{fg,1_W,h}\colon \mathcal{D}^W_h h^*\mathcal{D}^W \xrightarrow{\sim} \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}.$$

Proof. By (4.10.3.1) there is an invertible $\mathcal{O}_X$-module $\mathcal{L}_f$, a locally constant, integer-valued function n_f, and an isomorphism of Dualizing Complexes

$$\alpha\colon \mathcal{R}^X \xrightarrow{\sim} (\mathcal{R}^Y \times_Y X)\otimes\mathcal{L}_f[n_f],$$

see (4.10.2.1), (B) and (C). Set $\mathcal{I} := \mathcal{L}_f[n_f]$ and $\mathcal{I}^{-1} := \mathcal{H}_X(\mathcal{I},\mathcal{O}_X)$, so that there is a canonical isomorphism $\mathcal{I}\otimes\mathcal{I}^{-1} \xrightarrow{\sim} \mathcal{O}_X$. Also, for any map $e\colon Z\to X$ and $F,G\in\mathbf{D}(Z)$, the map coming from (3.5.3)(g) is an *isomorphism* $\mathcal{H}_Z(F,G)\otimes e^*\mathcal{I}^{-1} \xrightarrow{\sim} \mathcal{H}_Z(F\otimes e^*\mathcal{I},G)$. (The question being local, the proof reduces easily to the simple case $\mathcal{I}=\mathcal{O}_X$.)

There results, for any $E\in\mathbf{D}_{\mathsf{c}}(W)$, a composite isomorphism

$$\begin{aligned}\varphi_{\alpha,\mathcal{L}}\colon \mathcal{D}^X_{gh}h^*\mathcal{D}^X_g E &\xrightarrow{\sim} \mathcal{D}^Y_{fgh}(h^*\mathcal{D}^X_g E)\otimes(gh)^*\mathcal{I}\\ &\xrightarrow{\sim} \mathcal{D}^Y_{fgh}(h^*\mathcal{D}^Y_{fg}E\otimes(gh)^*\mathcal{I})\otimes(gh)^*\mathcal{I}\\ &\xrightarrow{\sim} \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}E\otimes(gh)^*\mathcal{I}^{-1}\otimes(gh)^*\mathcal{I}\\ &\xrightarrow{\sim} \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}E\otimes(gh)^*(\mathcal{I}^{-1}\otimes\mathcal{I})\\ &\xrightarrow{\sim} \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}E.\end{aligned}$$

It is easily checked that $\varphi_{\alpha,\mathcal{L}}$ is independent of the choice of α and of $\mathcal{L}$, i.e., if μ is a unit in $\mathrm{H}^0(X,\mathcal{O}_X)$, and if $\mathcal{L}'\cong\mathcal{L}$, then $\varphi_{\alpha,\mathcal{L}} = \varphi_{\mu\alpha,\mathcal{L}'}$. So we can set $\phi_{f,g,h} = \varphi_{\alpha,\mathcal{L}}$.

The final assertion is left to the very patient reader. (A direct approach seems to involve a formidable diagram—although the analogous statement (3.3.13) in [**C**, p. 135] is said there to be "easy to check.") Q.E.D.

Next, with f, g, h as in (4.10.4.1), we define the functorial isomorphism $d_{g,f}\colon g^!f^! \xrightarrow{\sim} (fg)^!$ to be the natural composition

$$\begin{aligned}g^!f^! = \mathcal{D}^X_g g^*\mathcal{D}^X\mathcal{D}^Y_f f^*\mathcal{D}^Y &\xrightarrow[\phi_{f,1_Y,g}]{\sim} \mathcal{D}^Y_{fg}g^*\mathcal{D}^Y_f\,\mathcal{D}^Y_f f^*\mathcal{D}^Y\\ &\xrightarrow{\sim} \mathcal{D}^Y_{fg}g^*f^*\mathcal{D}^Y \xrightarrow{\sim} \mathcal{D}^Y_{fg}(fg)^*\mathcal{D}^Y = (fg)^!\end{aligned}$$

Pseudofunctoriality requires the following diagram to commute:[61]

$$\begin{tikzcd} (fgh)^! & h^!(fg)^! \arrow[l, "d_{h,fg}"'] \\ (gh)^!f^! \arrow[u, "d_{gh,f}"] & h^!g^!f^! \arrow[l, "d_{g,h}"] \arrow[u, "h^!d_{f,g}"'] \end{tikzcd}$$

[61] Strictly speaking, we need also to "normalize" $^!$, i.e., to replace $(1_Y)^!$ by the identity functor of $\mathbf{D}_{\mathsf{c}}(Y)$ for every $Y\in\mathbf{S}$.

Expanding this diagram according to the definition of $d_{g,f}$, one finds quickly that the problem is to show commutativity of the following diagram of natural isomorphisms:

$$
\begin{tikzcd}
\mathcal{D}^Y_{fgh}(gh)^*\mathcal{D}^Y_f \arrow[r] & \mathcal{D}^Y_{fgh}h^*g^*\mathcal{D}^Y_f & \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}\mathcal{D}^Y_{fg}g^*\mathcal{D}^Y_f \arrow[l] \\
\mathcal{D}^X_{gh}(gh)^*\mathcal{D}^X \arrow[u, "\phi_{f,1_X,gh}"] & \mathcal{D}^X_{gh}h^*g^*\mathcal{D}^X \arrow[l] & \mathcal{D}^W_h h^*\mathcal{D}^W\mathcal{D}^Y_{fg}g^*\mathcal{D}^Y_f \arrow[u, "\phi_{fg,1_W,h}"'] \\
 & \mathcal{D}^X_{gh}h^*\mathcal{D}^X_g\mathcal{D}^X_g g^*\mathcal{D}^X \arrow[u] & \mathcal{D}^W_h h^*\mathcal{D}^W\mathcal{D}^X_g g^*\mathcal{D}^X \arrow[l, "\phi_{g,1_W,h}"] \arrow[u, "\mathcal{D}^W_h h^*\mathcal{D}^W(\phi_{f,1_X,g})"]
\end{tikzcd}
$$

Using the equality in (4.10.4.1), one transforms the question to commutativity of

$$
\begin{tikzcd}
\mathcal{D}^Y_{fgh}(gh)^*\mathcal{D}^Y_f \arrow[r] & \mathcal{D}^Y_{fgh}h^*g^*\mathcal{D}^Y_f & \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}\mathcal{D}^Y_{fg}g^*\mathcal{D}^Y_f \arrow[l] \\
\mathcal{D}^X_{gh}(gh)^*\mathcal{D}^X \arrow[u, "\phi_{f,1_X,gh}"] & \mathcal{D}^X_{gh}h^*g^*\mathcal{D}^X \arrow[l] & \\
 & \mathcal{D}^X_{gh}h^*\mathcal{D}^X_g\mathcal{D}^X_g g^*\mathcal{D}^X \arrow[u] & \mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}\mathcal{D}^X_g g^*\mathcal{D}^X \arrow[l, "\phi_{f,g,h}"] \arrow[uu, "\mathcal{D}^Y_{fgh}h^*\mathcal{D}^Y_{fg}(\phi_{f,1_X,g})"]
\end{tikzcd}
$$

Checking this commutativity is left to the few (if any) extremely patient readers who might be willing to do it. Again, the complete expansion according to definitions is intimidating—but the analogous associativity statement is said in [**C**, p. 136] to be "straightforward to check."

Pseudofunctoriality being thus established, one must now verify that the isomorphism in (1) above is *pseudofunctorial;* that on proper maps, $_*$ and $^!$ are adjoint as *pseudofunctors* (see (2) and (3.6.7(d)); that the isomorphism in (3) extends to an isomorphism of Dualizing Complexes; and that β_σ is as described in Theorem (4.10.4). And finally, the *uniqueness* (up to isomorphism) of the pseudofuctor $^!$ can be verified as at the beginning of the proof of (4.8.4).

Each of these verifications amounts, upon expansion according to definitions, to checking commutativity of a rather unpleasant diagram.

For the purposes of these Notes, Thm. (4.10.4) is not one of the "main results" referred to in Section (0.3) of the Introduction; so I leave it at that.

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