Basic results on Grothendieck Duality

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November 2007

[1] Supported in part by NSA Grant # H98230-06-1-0010.

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Definition (Pseudofunctor: special case of 2-functor)

A *contravariant pseudofunctor* on a category **S** assigns to each $X \in \mathbf{S}$ a category $\mathbf{X}^{\#}$, to each map $f\colon X \to Y$ a functor $f^{\#}\colon \mathbf{Y}^{\#} \to \mathbf{X}^{\#}$ (with $\mathbf{1}^{\#} = \mathbf{1}$), and to each map-pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ a functorial isomorphism

$$d_{f,g}\colon f^{\#}g^{\#} \xrightarrow{\ \sim\ } (gf)^{\#}$$

satisfying $d_{\mathbf{1},g} = d_{g,\mathbf{1}} =$ identity, and such that for each triple of maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ the following commutes:

$$\begin{array}{ccc}
(hgf)^{\#} & \xleftarrow{\ d_{f,hg}\ } & f^{\#}(hg)^{\#} \\
{\scriptstyle d_{gf\,h}}\uparrow & & \uparrow{\scriptstyle d_{g,h}} \\
(gf)^{\#}h^{\#} & \xleftarrow[\ d_{f,g}\ ]{} & f^{\#}g^{\#}h^{\#}
\end{array}$$

Covariant pseudofunctor is similarly defined, with arrows reversed, i.e., it means contravariant functor on $\mathbf{S}^{\mathrm{op}}$.

Examples: Derived inverse-image (contravariant). Derived direct-image (covariant).

$$\mathbf{S} := \text{category of ringed spaces}$$

$$\mathbf{X}^{\#} := \mathbf{D}(X) \quad \big(\text{derived category of } \{\mathcal{O}_X\text{-modules}\}\big)$$

$$f^{\#} := \mathbf{L}f^* \qquad \text{resp.} \qquad f_{\#} := \mathbf{R}f_*$$

Relations between $\mathbf{L}f^*$ and $\mathbf{R}f_*$

1. For any ringed-space map $f\colon X \to Y$,

$\mathbf{L}f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X)$ is left-adjoint to $\mathbf{R}f_*$, i.e., for $E \in \mathbf{D}(Y)$, $F \in \mathbf{D}(X)$,

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{L}f^*E, F) \cong \mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathbf{R}f_*F).$$

2. For any commutative square of ringed-space maps

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

one has the functorial map $\theta = \theta_\sigma \colon \mathbf{L}u^*\mathbf{R}f_* \to \mathbf{R}g_*\mathbf{L}v^*$, adjoint to the natural composition

$$\mathbf{R}f_* \to \mathbf{R}f_*\mathbf{R}v_*\mathbf{L}v^* \xrightarrow{\ \sim\ } \mathbf{R}u_*\mathbf{R}g_*\mathbf{L}v^*.$$

If σ is a fiber square of concentrated (= quasi-compact, quasi-separated) schemes then, with $\mathbf{D}_{\mathrm{qc}}$ the full subcategory of $\mathbf{D}$ whose objects are the complexes with quasi-coherent homology,

θ_σ *is an isomorphism of functors on* $\mathbf{D}_{\mathrm{qc}}$ $\iff$ σ *is tor-independent*.

The adjoint pseudofunctors $\mathbf{R}f_*$ and $\mathbf{L}f^*$, and the derived sheaf-Hom and Tensor functors—also adjoint, i.e., for any ringed-space X there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X)}(E \underline{\otimes}_X F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(X)}\big(E, \mathbf{R}\mathcal{H}om_X(F, G)\big)$$

—are four of the six operations of Grothendieck. A fifth, right adjoint to $\mathbf{R}f_*$, is about to be introduced.

These operations, and their interrelations, generate an incredibly rich structure, around which e.g., Grothendieck Duality is built. For examples,

Sheafified adjointness of $\mathbf{L}f^*$ and $\mathbf{R}f_*$

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathbf{L}f^*E, F) \cong \mathbf{R}\mathcal{H}om_Y(E, \mathbf{R}f_*F) \quad \big(E \in \mathbf{D}(Y),\ F \in \mathbf{D}(X)\big)$$

Projection isomorphism for concentrated $f\colon X \to Y$

$$\mathbf{R}f_*(\mathbf{L}f^*E \underline{\otimes}_X F) \xrightarrow{\sim} E \underline{\otimes}_Y \mathbf{R}f_*F \qquad \big(E \in \mathbf{D}_{\mathrm{qc}}(Y),\ F \in \mathbf{D}_{\mathrm{qc}}(X)\big)$$

2. Global Duality Theorem

Grothendieck Duality begins with this theorem:

Let X be a concentrated scheme and $f\colon X \to Y$ a concentrated scheme-map. Then the Δ-functor $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}(Y)$ has a bounded-below right Δ-adjoint.

More elaborately,

For $f\colon X \to Y$ as before, there is a bounded-below Δ-functor $(f^\times, \text{identity})\colon \mathbf{D}(Y) \to \mathbf{D}_{\mathrm{qc}}(X)$ and a map of Δ-functors $\tau\colon \mathbf{R}f_*f^\times \to \mathbf{1}$ such that for all $F \in \mathbf{D}_{\mathrm{qc}}(X)$ and $G \in \mathbf{D}(Y)$, the natural composite Δ-functorial map (in the derived category of abelian groups)

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_X(F, f^\times G) &\longrightarrow \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*\mathbf{R}f_*F, f^\times G)\\ &\longrightarrow \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^\times G)\\ &\underset{\tau}{\longrightarrow} \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, G)\end{aligned}$$

is a Δ-functorial isomorphism.

(Needed for not-necessarily-noetherian situations.)

Definition

An $\mathcal{O}_X$-complex (X a scheme) is pseudo-coherent if its restriction to each affine open subscheme is **D**-isomorphic to a bounded-above complex of finite-rank locally free sheaves.

When X is **noetherian,** *pseudo-coherent* just means:

has coherent homology modules, vanishing in all large degrees.

Definition

A scheme-map $f\colon X \to Y$ is quasi-proper if $\mathbf{R}f_*$ takes pseudo-coherent $\mathcal{O}_X$-complexes to pseudo-coherent $\mathcal{O}_Y$-complexes.

When X is **noetherian,** and f finite type and separated, *quasi-proper* simply means *proper*.

For quasi-proper f we write $f^!$ in place of $f^\times$.

3. Tor-independent Base Change Theorem

Here is the other basic building block of the theory.

Theorem

Suppose there is given a tor-independent fiber square

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array} \qquad \begin{cases} f\ \text{(hence } g\text{) quasi-proper} \\ u \text{ of finite tor-dimension} \end{cases}$$

Then the functorial map adjoint to the natural composition

$$\mathbf{R}g_*\mathbf{L}v^*f^!G \xrightarrow[\text{above}]{\ \sim\ } \mathbf{L}u^*\mathbf{R}f_*f^!G \xrightarrow[\mathbf{L}u^*\tau]{} \mathbf{L}u^*G,$$

is an isomorphism

$$\beta_\sigma(G)\colon \mathbf{L}v^*f^!G \xrightarrow{\ \sim\ } g^!\mathbf{L}u^*G \qquad \big(G \in \mathbf{D}^+_{\mathrm{qc}}(Y)\big)$$

(where $G \in \mathbf{D}^+_{\mathrm{qc}}$ means $G \in \mathbf{D}_{\mathrm{qc}}$ and $H^n(G) = 0$ for all $n \ll 0$.

Corollary: Sheafified Duality

The Base-change Theorem for open immersions u is equivalent to the following Sheafified Duality Theorem.

Theorem

Let $f\colon X \to Y$ be quasi-proper. Then for any $F \in \mathbf{D}_{\mathrm{qc}}(X)$, $G \in \mathbf{D}^{+}_{\mathrm{qc}}(Y)$, the composite duality map

$$\begin{aligned} \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(F, f^{\times}G) &\longrightarrow \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathbf{L}f^*\mathbf{R}f_*F, f^{\times}G) \\ &\longrightarrow \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^{\times}G) \\ &\underset{\tau}{\longrightarrow} \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*F, G) \end{aligned}$$

is an isomorphism.

- Global Duality results from this by application of the functor $\mathbf{R}\Gamma(Y, -)$.
- (Neeman) Theorem fails without the boundedness restriction on G.

4. Twisted Inverse-image Pseudofunctor

On the category $\mathbf{S}_\mathsf{f}$ of finite-type separated maps of noetherian schemes, $\exists$ a $\mathbf{D}^+_{\mathsf{qc}}$-valued pseudofunctor $^!$ that is uniquely determined up to isomorphism by the following three properties:

(i) The pseudofunctor $^!$ restricts on the subcategory of proper maps to a right adjoint of the derived direct-image pseudofunctor.

(ii) The pseudofunctor $^!$ restricts on the subcategory of étale maps to the (derived or not) inverse-image pseudofunctor.

(iii) For any fiber square

$$\begin{array}{ccc} \bullet & \xrightarrow{\;v\;} & \bullet \\ g\big\downarrow & \sigma & \big\downarrow f \\ \bullet & \xrightarrow[\;u\;]{} & \bullet \end{array} \qquad (f, g \text{ proper; } u, v \text{ étale}),$$

the base-change map $\beta_\sigma \colon v^*f^! \to g^!u^*$, adjoint to the natural composition

$$\mathbf{R}g_*v^*f^! \xrightarrow[\text{above}]{\;\sim\;} u^*\mathbf{R}f_*f^! \longrightarrow u^*,$$

is the natural composite isomorphism

$$v^*f^! = v^!f^! \xrightarrow{\;\sim\;} (fv)^! = (ug)^! \xrightarrow{\;\sim\;} g^!u^! = g^!u^*.$$

Remarks

1. The proof uses Nagata's compactification theorem:
Every finite-type separated map of noetherian (or just concentrated) schemes factors as proper $\circ$ open immersion.

Use the base change theorem to paste the pseudofunctors on proper maps and on open immersions. The problem is to show that everything is independent of choice of compactification for the maps.

2. Without noetherian hypotheses, Nayak showed, without needing Nagata' theorem, that there is a $^!$ as above, but over the smallest subcategory of *arbitrary concentrated schemes* which contains all *flat* finitely-presented proper maps and all separated étale maps.

3. (Something concrete.) When f is both étale and proper—hence finite and flat—then for any such $^!$, the natural $f_*f^* = f_*f^! \to \mathbf{1}$ is nothing but the standard trace map.

For any scheme-map $f\colon X \to Y$ there is a natural pseudofunctorial (i.e., transitive w.r.t. composition in $\mathbf{S}_\mathsf{f}$) map

$$\psi^f_{E,F}\colon \mathbf{R}\mathcal{H}om_X(\mathbf{L}f^*E, f^!F) \to f^!\mathbf{R}\mathcal{H}om_Y(E,F)$$

agreeing with the obvious one when f is étale (so that $f^! = f^*$), and dual, when f is proper, to the natural composition

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathbf{L}f^*E, f^!F) \xrightarrow[\text{above}]{} \mathbf{R}\mathcal{H}om_Y(E, \mathbf{R}f_*f^!F) \to \mathbf{R}\mathcal{H}om_Y(E,F).$$

$\psi^f_{E,F}$ is an isomorphism if E is pseudo-coherent and $F \in \mathbf{D}^+_{\mathrm{qc}}(Y)$.

Interaction of twisted inverse image with $\underline{\underline{\otimes}}$

For any $\mathbf{S}_\mathsf{f}$-map $f\colon X \to Y$ there is a natural functorial map, defined via compactification and the "projection isomorphism" (details on request)

$$\chi^f_E\colon f^!\mathcal{O}_Y \underline{\underline{\otimes}} \mathbf{L}f^*E \to f^!E \qquad \big(E \in \mathbf{D}^+_{\mathrm{qc}}(Y)\big);$$

and, at least in the noetherian case,

f is perfect (i.e., has finite tor-dimension) $\Longrightarrow$ χ^f_E iso for all E;
and conversely when f is proper.

Consequently (and more generally in appearance):

When f is perfect there is a natural functorial isomorphism

$$\chi^f_{E,F}\colon f^!E \underline{\underline{\otimes}} \mathbf{L}f^*F \xrightarrow{\ \sim\ } f^!(E \underline{\underline{\otimes}} F) \qquad \big(E, F \in \mathbf{D}^+_{\mathrm{qc}}(Y)\big).$$

Thus for perfect f the study of $f^!$ is reduced, modulo properties of $\underline{\underline{\otimes}}$, to that of the relative dualizing complex $f^!\mathcal{O}_Y$.

5. Perfection

Definitions

Given a scheme-map $f\colon X \to Y$, we say an $\mathcal{O}_X$-complex E is *f-perfect* if E is *pseudo-coherent* and has *finite relative tor-dimension* (i.e., there are integers $m \le n$ such that the stalk E_x at each $x \in X$ is,
as an $\mathcal{O}_{Y,f(x)}$-complex, **D**-isomorphic to a flat complex vanishing in degrees outside $[m, n]$).

E is *perfect* if X is covered by open sets over which E is **D**-isomorphic to a bounded complex of finite-rank free $\mathcal{O}_X$-modules. ($\iff$ E is 1_X-perfect.)

Theorem

Equivalent for a finite-type separated $f\colon X \to Y$ with X, Y, noetherian:
(i) *The map f is perfect, i.e., the complex $\mathcal{O}_X$ is f-perfect.*
(ii) *The complex $f^!\mathcal{O}_Y$ is f-perfect.*

continuation of Theorem

(iii) $f^!\mathcal{O}_Y \in \overline{\mathbf{D}}^-_{\mathrm{qc}}(X)$, and for every $F \in \overline{\mathbf{D}}^+_{\mathrm{qc}}(Y)$, the $\mathbf{D}_{\mathrm{qc}}(X)$-map

$$\chi^f_{\mathcal{O}_Y,F}\, f^!\mathcal{O}_Y \otimes f^*F \xrightarrow{\;\sim\;} f^!F$$

is an isomorphism.

(iii)′ For every perfect $\mathcal{O}_Y$-complex E, $f^!E$ is f-perfect; and for all $E, F \in \mathbf{D}(Y)$ such that E and $E \otimes F$ are in $\overline{\mathbf{D}}^+_{\mathrm{qc}}(Y)$, the $\mathbf{D}_{\mathrm{qc}}(X)$-map

$$\chi^f_{E,F}\, f^!E \otimes f^*F \xrightarrow{\;\sim\;} f^!(E \otimes F).$$

is an isomorphism.

For a *proper* map $f\colon X \to Y$ of noetherian schemes:

f is perfect $\iff$
$\mathbf{R}f_*$ takes perfect $\mathcal{O}_X$-complexes to perfect $\mathcal{O}_Y$-complexes.

f is perfect $\iff$ *unrestricted tor-independent base change* holds:

For any tor-independent fiber square of noetherian schemes

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

and $G \in \mathbf{D}_{\mathrm{qc}}(Y)$ the above-defined base-change map is an isomorphism

$$\beta_\sigma(G)\colon \mathbf{L}v^*f^!G \xrightarrow{\ \sim\ } g^!\mathbf{L}u^*G.$$

(u need not have finite tor-dimension, and G need not be bounded below.)

If f is perfect and (F_α) is a small filtered direct system of flat quasi-coherent $\mathcal{O}_Y$-modules then for all $n \in \mathbb{Z}$ the natural map is an isomorphism

$$\varinjlim_{\alpha} H^n(f^! F_\alpha) \xrightarrow{\ \sim\ } H^n(f^! \varinjlim_{\alpha} F_\alpha).$$

6. Dualizing Complexes

Definition

A *dualizing complex* R on a noetherian scheme X is a complex with coherent homology that is $\mathbf{D}$-isomorphic to a bounded injective complex, and has the following equivalent properties:

(i) For every $F \in \mathbf{D}_{\mathrm{c}}(X)$, the map that is (derived) Hom-Tensor adjoint to the natural composition

$$F \otimes \mathbf{R}\mathcal{H}om(F, R) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(F, R) \otimes F \to R$$

is an isomorphism

$$F \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(F, R), R).$$

(ii) Condition (i) holds for $F = \mathcal{O}_X$, i.e., the map $\mathcal{O}_X \to \mathbf{R}\mathcal{H}om(R, R)$ which takes $1 \in \Gamma(X, \mathcal{O}_X)$ to the identity map of R is an isomorphism.

Remarks

Grothendieck's original strategy for proving duality for proper not-necessarily-projective maps of noetherian schemes, at least for bounded-below complexes with coherent homology, is based on pseudofunctorial properties of dualizing complexes.

The basic problem in this approach is the construction of a

coherent family of dualizing complexes,

or, roughly speaking,

a pseudofunctor on proper maps with properties like those of $(-)^!$, but taking values in dualizing complexes.

Though this approach gives less general results than stated before, it is not without interest—historical and otherwise; and indeed, for formal schemes, it yields results not otherwise obtainable (as of 2007), see Sastry's paper in **Contemporary Math. 375.**

This paper is preceded by one (by Sastry, Nayak and L.) in which the original construction in "Residues and Duality" is simplified, and generalized to Cousin complexes on formal schemes.

More Remarks

For connected X, dualizing $\mathcal{O}_X$-complexes, if they exist, are unique up to tensoring with a shifted invertible $\mathcal{O}_X$-module.

The existence of a dualizing complex places restrictions on X. For instance, X must be universally catenary and of finite Krull dimension. (Very recently, Neeman generalized the definition of dualizing complex to where it applies to bounded coherent complexes without the finite-dimensionality restriction on X.)

Any scheme of finite type over a regular (or even Gorenstein) scheme of finite Krull dimension has a dualizing complex.

The relation between dualizing complexes and the twisted inverse image pseudofunctor $(-)^!$ is rooted in the following, by now classical, Proposition.

Proposition

Let $f\colon X \to Y$ be a finite-type separated map of noetherian schemes, and let R be a dualizing $\mathcal{O}_Y$-complex.
Then with $R_f := f^!R$, and $\mathcal{D}_{R_f}(-) := \mathbf{R}\mathcal{H}om(-, R_f)$, it holds that

(i) R_f is a dualizing $\mathcal{O}_X$-complex.

(ii) There is a functorial isomorphism

$$f^!\mathcal{D}_R F \xrightarrow{\ \sim\ } \mathcal{D}_{R_f}\mathbf{L}f^*F \qquad \big(F \in \overline{\mathbf{D}}_{\mathrm{c}}^{-}(Y)\big)$$

or equivalently,

$$f^!E \xrightarrow{\ \sim\ } \mathcal{D}_{R_f}\mathbf{L}f^*\mathcal{D}_R E \qquad \big(E \in \overline{\mathbf{D}}_{\mathrm{c}}^{+}(Y)\big).$$

(i) $\Rightarrow$(ii) via the above iso $\psi^f_{E,F}\colon \mathbf{R}\mathcal{H}om_X(\mathbf{L}f^*E, f^!F) \xrightarrow{\ \sim\ } f^!\mathbf{R}\mathcal{H}om_Y(E,F)$.

This Proposition suggests how a coherent system of dualizing complexes, when such exists can give rise to a twisted inverse-image pseudofunctor.

Details in **Notes on Derived Functors and Grothendieck Duality,**

http://www.math.purdue.edu/˜lipman

7. Comments and Problems

Even leaving aside applications, there's lots more to Duality theory, especially enlivening concrete interpretations, e.g., via differentials and familiar maps like traces and residues.

There is also generalization to formal schemes, which unifies global and local duality; and analogous theories for étale cohomology, analytic spaces, etc., etc.

Mention one problem in the abstract vein:

Problem

Can the twisted inverse-image pseudofunctor and its basic properties (as above) be extended to the category of *essentially finite-type* separated maps of noetherian schemes?

Probably so, but not trivially.

Being essentially of finite type is a local condition, so it's not clear that useful global properties like *compactifiability* obtain.

More comments and problems

The idea is then to use Nayak's methods of pasting pseudofunctors, methods which don't require compactifiability, but do require (as do all of the preceding results!) verification of commutativity of complicated diagrams.

In fact these massively time-consuming verifications take up a major (and essential) part of the above-mentioned notes, suggesting that:

Problem

It would be very nice if someone came up with a "coherence" theorem, or at least an expert computer program, to make such tedium unnecessary.

A serious attempt to do this could lead to a much deeper understanding of how and why the formalism works, not to mention the potential interest of the artificial intelligence aspects.

We can also ask:

Problem

Can the twisted inverse-image pseudofunctor and its basic properties be extended to a nonnoetherian context, or to category of *essentially pseudo-finite-type* separated maps of noetherian *formal* schemes?

As already indicated, Nayak has done this to a considerable extent. Pursuing this further might give some insight into:

Final (for today) Problem

In some sense the twisted inverse image is too good to be true.

Why do these pastings of two quite different pseudofunctors, for proper resp. étale maps, and of canonical maps of these pseudofunctors—pastings which depend on complicated compatibilities, whether from the abstract or the concrete point of view—work out so well?

There may well be an as yet undiscovered deeper underlying structure which would explain it all.