Derived Hochschild Cohomology and Grothendieck Duality: local-global interplay

Joseph Lipman[1]

Purdue University
Department of Mathematics

lipman@math.purdue.edu
http://www.math.purdue.edu/~lipman

May 20, 2008

[1]Supported in part by NSA Grant # H98230-06-1-0010

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1. Dramatis Personae. For details, see

Notes on Derived Functors and Grothendieck Duality, to appear, SLN.

A *contravariant pseudofunctor* (pullback) over a category **S** assigns to each $X \in \mathbf{S}$ a category $\mathbf{X}^\#$, to each map $f\colon X \to Y$ a functor $f^\#\colon \mathbf{Y}^\# \to \mathbf{X}^\#$ ($\mathbf{1}^\# = \mathbf{1}$), and to each $X \xrightarrow{f} Y \xrightarrow{g} Z$ a 'transitivity' isomorphism

$$d_{f,\,g}\colon f^\# g^\# \xrightarrow{\ \sim\ } (gf)^\#$$

satisfying $d_{\mathbf{1},\,g} = d_{g,\mathbf{1}} =$ identity, and 'associative,' meaning that for each triple of maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ the following commutes:

$$\begin{array}{ccc}
(hgf)^\# & \xleftarrow{\ d_{f,\,hg}\ } & f^\#(hg)^\# \\
{\scriptstyle d_{gf,\,h}}\Big\uparrow & & \Big\uparrow{\scriptstyle f^\# d_{g,\,h}} \\
(gf)^\# h^\# & \xleftarrow[\ d_{f,\,g}\ ]{} & f^\# g^\# h^\#
\end{array}$$

Covariant pseudofunctor (pushforth) is similarly defined, with arrows reversed, i.e., it means contravariant functor over $\mathbf{S}^{\mathrm{op}}$.

Examples.

$$\mathbf{S} := \text{category of rings}$$

$$\mathbf{X}^{\#} := \{X\text{-modules}\}$$

$f_{\#} :=$ extension of scalars resp. $f^{\#} :=$ restriction of scalars

$$\mathbf{S} := \text{category of rings}$$

$\mathbf{X}^{\#} := \mathbf{D}(X)$ (derived category of $\{X$-modules$\}$)

$f_{\#} :=$ left-derived extension of scalars resp. $f^{\#} :=$ restriction of scalars

$$\mathbf{S} := \text{category of ringed spaces}$$

$\mathbf{X}^{\#} := \mathbf{D}(X)$ (derived category of $\{\mathcal{O}_X$-modules$\}$)

$f^{\#} := \mathbf{L}f^{*}$ (derived inverse-image) resp. $f_{\#} := \mathbf{R}f_{*}$ (derived direct-image)

Relations between $\mathbf{L}f^*$ and $\mathbf{R}f_*$

1. For any ringed-space map $f\colon X \to Y$,

$\mathbf{L}f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X)$ *is left-adjoint to* $\mathbf{R}f_*$, i.e., for $E \in \mathbf{D}(Y)$, $F \in \mathbf{D}(X)$,

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{L}f^*E, F) \cong \mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathbf{R}f_*F).$$

2. For any commutative square of ringed-space maps

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

one has the functorial map $\quad \theta = \theta_\sigma\colon \mathbf{L}u^*\mathbf{R}f_* \to \mathbf{R}g_*\mathbf{L}v^*$,
adjoint to the natural composition

$$\mathbf{R}f_* \to \mathbf{R}f_*\mathbf{R}v_*\mathbf{L}v^* \xrightarrow{\;\sim\;} \mathbf{R}u_*\mathbf{R}g_*\mathbf{L}v^*.$$

If σ is a fiber square of noetherian schemes, and $\mathbf{D}_{\mathrm{qc}}(X)$ the full subcategory of $\mathbf{D}(X)$ whose objects are the complexes having quasi-coherent homology,

θ_σ is an isomorphism of functors on $\mathbf{D}_{\mathrm{qc}}(X) \iff \sigma$ is tor-independent.

The adjoint pseudofunctors $\mathbf{R}f_*$ and $\mathbf{L}f^*$, and
the derived sheaf-Hom and Tensor functors
—also adjoint, i.e., for any ringed-space X there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X)}(E \underset{=}{\otimes}_X F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(X)}\big(E, \mathbf{R}\mathcal{H}om_X(F, G)\big)$$

—and the right adjoint to $\mathbf{R}f_*$, about to be introduced,
are five of the six operations of Grothendieck.

These operations, and their category-theoretic interrelations, generate an incredibly rich structure, around which e.g., Grothendieck Duality is built.

Projection isomorphism

$$E \underset{=}{\otimes}_Y \mathbf{R}f_*F \xrightarrow{\sim} \mathbf{R}f_*(\mathbf{L}f^*E \underset{=}{\otimes}_X F) \qquad \big(E \in \mathbf{D}_{\mathrm{qc}}(Y),\ F \in \mathbf{D}_{\mathrm{qc}}(X)\big)$$

Sheafified adjointness of $\mathbf{L}f^*$ and $\mathbf{R}f_*$

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathbf{L}f^*E, F) \cong \mathbf{R}\mathcal{H}om_Y(E, \mathbf{R}f_*F) \quad \big(E \in \mathbf{D}(Y),\ F \in \mathbf{D}(X)\big)$$

2. Global Duality

Grothendieck Duality begins with this theorem:

For any map $f\colon X \to Y$ of quasi-compact quasi-separated (e.g., noetherian) schemes, the functor $\mathbf{R}f_*\colon \mathbf{D}_{qc}(X) \to \mathbf{D}(Y)$ has a cohomologically bounded-below (homologically bounded-above) right adjoint.

More elaborately,

There is a cohomologically bounded-below functor $f^\times\colon \mathbf{D}(Y) \to \mathbf{D}_{qc}(X)$ and a map of functors $\tau\colon \mathbf{R}f_*f^\times \to \mathbf{1}$ such that for all $F \in \mathbf{D}_{qc}(X)$ and $G \in \mathbf{D}(Y)$, the composite functorial map, in $\mathbf{D}$(abelian groups),

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_X(F, f^\times G) &\longrightarrow \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*\mathbf{R}f_*F, f^\times G)\\ &\longrightarrow \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^\times G)\\ &\underset{\tau}{\longrightarrow} \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, G)\end{aligned}$$

is an isomorphism.

3. Tor-independent Base Change Theorem

Henceforth schemes are noetherian. Here is the other pillar of duality.

Theorem

Suppose there is given a tor-independent fiber square

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array} \qquad \begin{cases} f \text{ (hence } g\text{) proper} \\ u \text{ of finite flat dimension} \end{cases}$$

Then the functorial map adjoint to the natural composition

$$\mathbf{R}g_*\mathbf{L}v^*f^\times G \xrightarrow[\text{above}]{\sim} \mathbf{L}u^*\mathbf{R}f_*f^\times G \xrightarrow[\mathbf{L}u^*\tau]{} \mathbf{L}u^*G,$$

is an isomorphism

$$\beta_\sigma(G)\colon \mathbf{L}v^*f^\times G \xrightarrow{\sim} g^\times\mathbf{L}u^*G \qquad \big(G \in \mathbf{D}^+_{\mathrm{qc}}(Y)\big)$$

(where $G \in \mathbf{D}^+_{\mathrm{qc}}$ means $G \in \mathbf{D}_{\mathrm{qc}}$ and $H^n(G) = 0$ for all $n \ll 0$).

Corollary: Sheafified duality

The Base-change Theorem for open immersions u is equivalent to the following Sheafified Duality Theorem.

Theorem

Let $f\colon X \to Y$ be proper. Then for any $F \in \mathbf{D}_{\mathrm{qc}}(X)$, $G \in \mathbf{D}^{+}_{\mathrm{qc}}(Y)$, the composite duality map

$$\begin{aligned}\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(F, f^{\times}G) &\longrightarrow \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathbf{L}f^*\mathbf{R}f_*F, f^{\times}G)\\ &\longrightarrow \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^{\times}G)\\ &\underset{\tau}{\longrightarrow} \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*F, G)\end{aligned}$$

is an isomorphism.

- Global Duality results from this by application of the functor $\mathbf{R}\Gamma(Y, -)$.
- (Neeman) Theorem fails without the boundedness restriction on G.
- For $F = \mathcal{O}_X$, the Theorem says $\mathbf{R}f_*f^{\times}G \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*\mathcal{O}_X, G)$.

This fixes $f^{\times}$ when f is a *finite* map (so that $\mathbf{R}f_*$ can be replaced by f_*).

4. Twisted inverse-image pseudofunctor

One can use global duality and base change to extend the pseudofunctor $(-)^{\times}$ over (the category of) proper scheme-maps to a pseudofunctor over arbitrary separated maps by combining it with the a priori very different functor $\mathbf{L}u^{*}$, u an open immersion (or even étale).

This extended pseudofunctor is the basic object of study in the theory.

Theorem. *On the category* $\mathbf{S}_\mathsf{f}$ *of finite-type separated scheme-maps,* $\exists$ *a* $\mathbf{D}^+_{\mathsf{qc}}$*-valued pseudofunctor* $^!$ *that is uniquely determined up to isomorphism by the following three properties*:

(i) *The restriction of the pseudofunctor* $(-)^!$ *to the subcategory of proper maps is isomorphic to* $(-)^\times$, *i.e.,*
it is right adjoint to the derived direct-image pseudofunctor.

(ii) *The pseudofunctor* $^!$ *restricts on the subcategory of étale maps to the (derived or not) inverse-image pseudofunctor.*

(iii) *For any fiber square*

$$\begin{array}{ccc} \bullet & \xrightarrow{v} & \bullet \\ {\scriptstyle g}\downarrow & \sigma & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[u]{} & \bullet \end{array} \qquad (f, g \text{ proper; } u, v \text{ étale}),$$

the base-change map $\beta_\sigma\colon v^*f^! \to g^!u^*$, *adjoint to the natural composition*

$$\mathbf{R}g_*v^*f^! \underset{\text{above}}{\xrightarrow{\;\sim\;}} u^*\mathbf{R}f_*f^! \longrightarrow u^*,$$

is the natural composite isomorphism

$$v^*f^! = v^!f^! \xrightarrow{\;\sim\;} (fv)^! = (ug)^! \xrightarrow{\;\sim\;} g^!u^! = g^!u^*.$$

Example (cf. Serre duality)

$f\colon X \to Y$ smooth (so Ω_f locally free, of rank, say, n_f) $\implies$
$\exists$ functorial iso

$$f^{\#}E := \big(\Lambda^{n_f}\Omega_f\big)[n_f] \otimes_X f^*E \xrightarrow{\ \sim\ } f^!E.$$

The functor $^{\#}$ extends to a pseudofunctor on the category of smooth maps, via the natural isomorphism

$$\big(\Lambda^{n_f}\Omega_f\big)[n_f] \otimes_X f^*\big(\Lambda^{n_g}\Omega_g\big)[n_g] \xrightarrow{\ \sim\ } \big(\Lambda^{n_{gf}}\Omega_{gf}\big)[n_{gf}]$$

relative to a pair of smooth maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.
The above isomorphism $f^{\#} \xrightarrow{\ \sim\ } f^!$ is *pseudofunctorial,* i.e., compatible with the pair of natural isomorphisms $f^{\#}g^{\#} \xrightarrow{\ \sim\ } (gf)^{\#}$, $f^!g^! \xrightarrow{\ \sim\ } (gf)^!$.

Corollary

$$f^!\mathbf{D}_{\mathrm{c}}(Y) \subset \mathbf{D}_{\mathrm{c}}(X).$$

Since f factors locally as (smooth)$\circ$(closed immersion), one need only show this for f finite or smooth, where it results from preceding examples.

Remarks

1. The theorem's proof uses Nagata's compactification theorem:
Every finite-type separated map factors as proper $\circ$ (open immersion).
One uses the base change theorem to paste the proper-map pseudofunctor and the open-immersion pseudofunctor. The problem is to show that everything is independent of choice of compactification for the maps.

2. The theorem applies to affine schemes, that is, to finite-type ring homomorphisms. But one often wishes to look at *essentially*-finite-type ring homomorphisms, e.g., local homomorphisms of local rings. So one would like to extend the theorem to *essentially finite-type* $f\colon X \to Y$, i.e., f such that each $y \in Y$ has an affine neighborhood $V = \operatorname{Spec} A$ such that $f^{-1}V = \cup_i \operatorname{Spec} B_i$ with the B_i essentially-finite-type A-algebras. Unfortunately, the appropriate analog of compactification is not yet known, so the question remains open.
But fortunately, less suffices for algebraic applications:

Remarks, continued

If in the definition of essentially finite-type scheme-maps, all the maps $A \to B_i$ are localizations then we say that f is localizing.

The scheme-map f is essentially compactifiable if there exists a factorization $f = \bar{f}u$ with $\bar{f}$ proper and u localizing.

Example

Maps corresponding to essentially-finite-type ring homomorphisms are easily seen to be essentially compactifiable.

We don't know that a composition of two essentially compactifiable maps is essentially compactifiable. But Suresh Nayak has shown, via his theorem on pasting pseudofunctors, that the basic facts about the twisted inverse-image pseudofunctor hold over the least category $\mathbf{S}_f$ of noetherian schemes which contains all the essentially compactifiable maps.

For any $\mathbf{S}_f$-map $f\colon X \to Y$ there is a natural functorial map, defined via compactification and the above projection isomorphism,

$$\chi^f_E\colon f^!\mathcal{O}_Y \underline{\underline{\otimes}} \mathbf{L}f^*E =: f^\# E \to f^! E \qquad \big(E \in \mathbf{D}^+_{\mathrm{qc}}(Y)\big).$$

Given a compactification $X \xrightarrow{u} \overline{X} \xrightarrow{\bar f} Y$ ($f = \bar f u$, u an open immersion, $\bar f$ proper), one gets χ^f_E by applying u^* to the adjoint of the natural composition

$$\bar f_*(\bar f^! E \otimes \bar f^* F) \underset{\text{proj'n}}{\xrightarrow{\sim}} \bar f_* \bar f^! E \otimes F \longrightarrow E \otimes F,$$

Of course one has to show this is independent of the choice of compactification.

Theorem

f is perfect (i.e., has finite flat dimension) $\iff$
$f^!\mathcal{O}_Y$ has bounded homology and χ^f_E iso for all E.

Pseudofunctoriality of χ^f_E

For perfect maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, *and* $E \in \mathbf{D}^+_{\mathrm{qc}}(Z)$, *the following commutes*:

$$\begin{array}{ccccc}
f^\# g^\# E = f^!\mathcal{O}_Y \otimes f^*(g^!\mathcal{O}_Z \otimes g^*E) & \xrightarrow{\chi} & f^!(g^!\mathcal{O}_Z \otimes g^*E) & \xrightarrow{f^!\chi} & f^!g^!E \\
\big\downarrow & & & & \big\downarrow \\
(f^!\mathcal{O}_Y \otimes f^*g^!\mathcal{O}_Z) \otimes f^*g^*E & & & & \\
\big\downarrow{\scriptstyle \chi\otimes 1} & & & & \\
f^!g^!\mathcal{O}_Z \otimes f^*g^*E & & & & \\
\big\downarrow & & & & \big\downarrow \\
(gf)^\# E = (gf)^!\mathcal{O}_Z \otimes (gf)^*E & & \xrightarrow[\chi]{\hspace{6em}} & & (gf)^!E
\end{array}$$

Thus for perfect f the study of $f^!$ is reduced, modulo properties of $\underline{\underline{\otimes}}$, to that of the relative dualizing complex $f^!\mathcal{O}_Y$—a central player in this talk.

Example (Formally smooth $\mathbf{S}_f$-maps)

In this case, $f^!\mathcal{O}_Y \cong \Omega_f^{n_f}[n_f]$ (cf. above).

Example (Cohen-Macaulay and Gorenstein maps)

A flat, finite-type, $f\colon X \to Y$ is, by definition, *Cohen-Macaulay* (resp. *Gorenstein*) if its fibers are such.

Exercise 9.7 in Hartshorne's *Residues and Duality* states that *this holds $\iff$ locally on X, the relative dualizing complex $f^!\mathcal{O}_Y$ has a single nonzero homology sheaf, which is flat over $\mathcal{O}_Y$ (resp. invertible on X).*

A suitable generalization of the base-change theorem reduces this assertion to the corresponding one for the fibers, i.e., the case where Y is Spec of a field. In this case, the question being local, one is just saying something familiar about the canonical module of a local ring.

Other proofs: Lemma 1 in Compositio 38, 37-43; Thm. 3.5.1 in SLN 1750.

Recently Avramov and Iyengar gave yet another proof (modulo the commutative-algebra version of relative dualizing complex, see below).

5. Relatively perfect complexes

Definitions

Given a scheme-map $f\colon X \to Y$, we say an $\mathcal{O}_X$-complex E is *f-perfect* if $E \in \mathbf{D}_c(X)$ and has *finite relative flat dimension* (i.e., there are integers $m \le n$ such that the stalk E_x at each $x \in X$ is, as an $\mathcal{O}_{Y,f(x)}$-complex, $\mathbf{D}$-isomorphic to a flat complex vanishing in degrees outside $[m, n]$).

E is *perfect* if X is covered by open sets over which E is $\mathbf{D}$-isomorphic to a bounded complex of finite-rank free $\mathcal{O}_X$-modules. ($\iff$ E is 1_X-perfect.)

Example

f is perfect $\iff$ $\mathcal{O}_X$ is f-perfect $\iff$ $f^!\mathcal{O}_Y$ is f-perfect.

The first $\iff$ holds essentially by definition. An efficient (omitted) proof of the second uses the following recent result of Avramov and Iyengar:

On any scheme W, an $\mathcal{O}_W$-complex F is perfect $\iff$
F has bounded, coherent homology and $\mathbf{R}\mathcal{H}om_W(F, \mathcal{O}_W)$ is perfect.

Theorem (Illusie, SGA 6, p. 259, 4.2.9.)

If $E \in \mathbf{D}(X)$ is f-perfect, then so is $\mathcal{D}_f E := \mathbf{R}\mathcal{H}om_X(E, f^!\mathcal{O}_Y)$; and the canonical map is an isomorphism $E \xrightarrow{\sim} \mathcal{D}_f\mathcal{D}_f E$. So $\mathcal{D}_f$ is an involution of the full subcategory $\mathbf{P}(f) \subset \mathbf{D}(X)$ whose objects are the f-perfect complexes.

Corollary

If f is perfect then $\mathcal{D}_f$ is semidualizing: the natural map is an isomorphism

$$\mathcal{O}_X \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_X(\mathcal{D}_f, \mathcal{D}_f) = \mathcal{D}_f\mathcal{D}_f E.$$

So if Y is locally Gorenstein, then $\mathcal{D}_f$ is a dualizing complex.

Remark. The last assertion holds because $f^!$ preserves coherence of homology and finiteness of injective dimension.

Conversely, Avramov and Iyengar have shown (again, modulo the commutative-algebra version of relative dualizing complex) that *if $\mathcal{D}_f$ is a dualizing complex then Y is locally Gorenstein.*

6. Relative dualizing complexes in Commutative Algebra

Applying the above to maps of affine schemes, in view of the standard correspondence between modules over a ring and quasi-coherent sheaves on its Spec—an equivalence which extends to derived categories—we can associate to any essentially finite-type map $\sigma\colon K \to S$ a relative dualizing complex, as follows.

Express S as a homomorphic image of (or a module-finite algebra over) a localization P of a polynomial ring $K[x_1, \dots, x_n]$; and set

$$\Delta_\sigma := \mathbf{R}\mathrm{Hom}_P(S, \Lambda^n\Omega_{P/K}[n]) \cong \mathbf{R}\mathrm{Hom}_P(S, P).$$

Then show that Δ_σ *depends, up to isomorphism in* $\mathbf{D}(S)$, *only on* σ, (not on the choice of $P \to S$).

One reason for this independence is that the sheafification of Δ_σ is $f^!\mathcal{O}_Y$, where $f\colon X := \operatorname{Spec} S \to \operatorname{Spec} K =: Y$ corresponds to σ. But $f^!$ involves all sorts of global considerations, so this is not very aesthetic.

One could just argue directly, reducing to the case where S itself is a localized polynomial ring over K. (Exercise.)

More Remarks

There's another way, leading to some intriguing global questions about structures involving DGAs.

One can factor $\sigma\colon K \to S$ in the category of (positively graded, graded-commutative) K-DGAs as $K \to A \to S$ with A *flat* over K and $A \to S$ a quasi-isomorphism, i.e., the induced map $H_*(A) \to H_*(S) = S$ is an isomorphism.

Let $S \underline{\otimes}_K S$ be the DGA $A \otimes_K A$.

Let $\mathbf{P}(\sigma) \subset \mathbf{D}(X)$ be the full subcategory with objects the complexes with finitely generated homology and finite flat K-dimension.

Sheafification gives an equivalence from $\mathbf{P}(\sigma)$ to $\mathbf{P}(\operatorname{Spec}\sigma)$ ($\operatorname{Spec}\sigma$-perfect complexes).

Assume henceforth that $\sigma\colon K \to S$ has finite flat dimension, i.e., $\operatorname{Spec}\sigma$ is perfect.

The following theorem gives an *intrinsic characterization* of Δ_σ.

Theorem (Avramov, Iyengar)

There exists a complex $D^\sigma \in \mathbf{P}(\sigma)$ *satisfying*

$$\mathbf{R}\mathrm{Hom}_S(D^\sigma, D^\sigma) \cong S\,,$$

(i.e., D^σ *is semidualizing) and a canonical bifunctorial isomorphism*

$$\mathbf{R}\mathrm{Hom}_{S \underline{\otimes}_K S}(S, M \underline{\otimes}_K N) \cong \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(M, D^\sigma), N) \qquad \big(M \in \mathbf{P}(\sigma),\ N \in \mathbf{D}(S)\big).$$

Such a D^σ *is unique up to* $\mathbf{D}(S)$*-isomorphism.*

In fact, for every factorization $K \to P \to S$ *of* σ *with* S *finite over* P *and* P *essentially smooth of relative dimension* n *over* K,

$$D^\sigma \cong \mathbf{R}\mathrm{Hom}_P(S, (\Lambda^n_P \Omega_{P/K})[n]),$$

so that D^σ *is a relative dualizing complex, as defined above.*

Remarks

1. For any S-bimodule B, $\mathbf{R}\mathrm{Hom}_{S \underline{\underline{\otimes}}_K S}(S, B)$ is the *derived Hochschild cohomology* of the K-algebra S, with B-coefficients. When S is *flat* over K, $S \underline{\underline{\otimes}}_K S$ can be replaced by $S \otimes_K S$. In any case, at the homology level, the isomorphism in the theorem is a *reduction formula* expressing derived Hochschild cohomology with tensor-decomposable coefficients in terms of Exts.

2. The proof is mainly a complex diagram chase involving DG resolutions and various quasi-isomorphisms.

3. The special case where K is a regular finite-dimensional ring, and $M = N = D^\sigma$, was first done in 2004 by Yekutieli and Zhang. In this case, as mentioned above, D^σ is actually a dualizing complex.

4. Avramov and Iyengar came across the theorem in the course of investigating homological criteria for Gorensteinness of local homomorphisms. As indicated before, they use the theorem in showing that *Gorensteinness is characterized by (among other conditions) invertibility of D^σ, or even perfection (absolute, not relative!) of D^σ.*

7. Globalization: back to schemes

Since the derived-Hochschild approach to relative dualizing complexes is so different than the approach via global duality, one naturally asks whether the former has some interesting extension to the global context. So far only the case of *flat* scheme maps $f\colon X \to Y$ (where DGAs are not needed) can be dealt with.

For such f, let $\delta\colon X \to X \times_Y X$ be the diagonal. Then, as before,

$$\delta_*\delta^! G \cong \mathbf{R}\mathcal{H}om_{X\times_Y X}(\delta_*\mathcal{O}_X, G),$$

so that $\delta^! G$ *is the global version of derived Hochschild cohomology with coefficients in* G.

Theorem (Global reduction formulae)

Let π_1 *and* $\pi_2\colon X \times_Y X \to X$ *be the projections. There exist natural isomorphisms*

$$\delta^!(\pi_1^* M \underline{\underline{\otimes}}_{X'} \pi_2^* N) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om_X\big(\mathbf{R}\mathcal{H}om_X(M, f^!\mathcal{O}_Y), N\big);$$

$$\delta^!\mathbf{R}\mathcal{H}om_{X'}(\pi_1^* M, \pi_2^* N) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om_X(M \underline{\underline{\otimes}}_X f^!\mathcal{O}_Y, N).$$

Final Remarks and Questions

1. The proof is mainly a complex diagram chase involving various derived-category isomorphisms arising from previously-mentioned relations among the 'five operations.'

2. Does this produce the same map in the affine case as the Avramov-Iyengar procedure? Typically, such comparisons between abstract and concrete instances of the duality formalism are not very easy to carry through; and this one is no exception. In fact, we haven't worked it out.

3. What about possible globalizations for non-flat perfect maps?

Question

Is there some homotopical structure on schemes, involving DGAs, for which, in the affine situation, some category related to $S \underline{\otimes}_K S$ (defined only up to quasi-isomorphism) is closely related to the product of Spec S *with itself, over* Spec(K)*; and to which the duality formalism used in the proof of the preceding theorem extends??*