## $\mathbf{x^3 + y^3 + z^3 \neq 0}$

**Theorem 1** (Fermat—with first known proof by Euler)**.** *No integers $x, y, z$ with $xyz \neq 0$ satisfy $x^3 + y^3 + z^3 = 0$.*

*Proof.* We may assume that $x$, $y$, and $z$ are pairwise coprime.

If $xyz$ is not divisible by 3, then the equation has no solution even in $\mathbb{Z}/(9)$, where every nonzero cube is $\pm 1$. Suppose then, without loss of generality, that $3|z$.

We will work in the UFD $R := \mathbb{Z}[\zeta]$ with $\zeta = (-1 + i\sqrt{3}\,)/2$, a root of the polynomial $X^2 + X + 1$ (so that $\zeta^3 = 1$). The complex conjugate $\bar{\zeta}$ is

$$\bar{\zeta} = \zeta^2 = -(1 + \zeta).$$

The *norm* of $a + b\zeta \in R$ $(a, b \in \mathbb{Z})$ is

$$N(a + b\zeta) := (a + b\zeta)(a + b\bar{\zeta}) = a^2 - ab + b^2 \geq 0.$$

One has

$$N(r_1 r_2) = N(r_1)N(r_2) \qquad (r_1,\ r_2 \in R). \tag{1.1}$$

It follows that the units in $R$ are those $a + b\zeta$ whose norm is 1, easily seen to be the six elements $\pm 1$, $\pm\zeta$ and $\pm\zeta^2 = \mp(1 + \zeta)$.

The norm of

$$\pi := 1 - \zeta$$

is

$$N(\pi) = (1 - \zeta)(1 - \zeta^2) = 3.$$

Since 3 is prime in $\mathbb{Z}$, (1.1) implies then that $1 - \zeta$ is prime in $R$.

For $a$, $b \in \mathbb{Z}$, $a + b\zeta \equiv a + b \pmod{\pi}$, so the natural map $\mathbb{Z}/(3) \to R/(\pi)$ is a surjection of fields, hence an isomorphism.

Now it is more than enough to show that *the equation*

$$x^3 + y^3 = u(\pi^k z)^3 \tag{1.2}$$

*has no solution in $R$ with $u$ a unit, $k \geq 1$, $x$, $y$, and $z$ pairwise coprime and $xyz$ not divisible by $\pi$.*

We will show that:

(i) if there is a solution, then $k \geq 2$; and

(ii) if there is a solution with $k \geq 2$, then there is also a solution when $k$ is replaced by $k - 1$.

Together, these two statements clearly imply that there is no solution.

So let $x$, $y$, $z$, $u$, and $k \geq 2$ be as in (1.2). We can write

$$x = a_0 + a_1\pi, \quad y = b_0 + b_1\pi,$$

where $a_i$ and $b_i$ are in $\mathbb{Z}$ and 3 does not divide $a_0 b_0$. (Recall that $\pi$ does not divide $xy$). Then for any $j \geq 0$,

$$\zeta^j y = (1 - \pi)^j (b_0 + b_1\pi) \equiv b_0 + (b_1 - jb_0)\pi \pmod{\pi^2}.$$

Note that $\pi^2 = -3\zeta$. Pick $j$ so that $a_1 + b_1 - jb_0 \equiv 0 \pmod 3$. Then, after replacing $y$ by $\zeta^j y$ in a solution of (1.2) we may assume that

$$x + y \equiv a_0 + b_0 \pmod 3.$$

It follows that for any $i \geq 0$,

$$x + \zeta^i y = x + y - (1 - \zeta^i)y \equiv x + y \equiv a_0 + b_0 \pmod \pi.$$

Then

$$0 \equiv x^3 + y^3 = (x + y)(x + \zeta y)(x + \zeta^2 y) \equiv (a_0 + b_0)^3 \pmod \pi,$$

whence 3 divides the rational integer $a_0 + b_0$. Thus $\pi^2$ divides $x + y$ and $\pi^4$ divides $x^3 + y^3$, making $k \geq 2$, as asserted in (i).

Since $(x, y) = 1$, therefore for $0 \leq i < j < 3$, $(x + \zeta^i y)/\pi$ and $(x + \zeta^j y)/\pi$ are relatively prime: any common factor would divide their difference

$$\frac{\zeta^i - \zeta^j}{\pi} \cdot y = \zeta^i \frac{1 - \zeta^{j-i}}{1 - \zeta} \cdot y = (\text{unit}) \cdot y,$$

and also would divide $x + \zeta^j y$, hence would divide $x$ as well as $y$.

Consequently,

$$\begin{aligned} x + \zeta y &= \pi e_1 t_1^3 \\ x + \zeta^2 y &= \pi e_2 t_2^3 \\ x + y &= \pi e_0 \pi^{3\ell} t_0^3 \end{aligned}$$

where the $e_i$ are units, the $t_i$ are pairwise coprime and not divisible by $\pi$, and $\ell = k - 1$. Since

$$(x + \zeta y) + \zeta(x + \zeta^2)y = (1 + \zeta)(x + y)$$

therefore

$$e_1 t_1^3 + \zeta e_2 t_2^3 = e_0 (1 + \zeta) \pi^{3\ell} t_0^3,$$

so that, since $\ell \geq 1$,

$$t_1^3 + \zeta \left( \frac{e_2}{e_1} \right) t_2^3 = \frac{e_0}{e_1} (1 + \zeta) \pi^{3\ell} t_0^3 \equiv 0 \pmod 3. \tag{1.3}$$

But

$$t_i^3 = (\text{say})\ (a_i + b_i \zeta)^3 \equiv a_i^3 + b_i^3 \not\equiv 0 \pmod 3,$$

where the inequality holds because $\pi$ doesn't divide $t_i$. So there is a rational integer $n$ such that $n(a_2^3 + b_2^3) \equiv 1 \pmod 3$; and after multiplying (1.3) by $n$ we see that the unit $\zeta e_2/e_1$ is a rational integer mod 3. Checking this condition for each of the six units shows that $\zeta e_2/e_1 = \pm 1$, and so (1.3) establishes (ii). □