Grothendieck operations and coherence in categories

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Introduction

I'm going to talk eventually about a problem in automated reasoning, growing out of Grothendieck Duality theory, a problem which I've been raising informally from time to time over the past twenty years, but haven't thought about intensively enough. I hope someday someone will be moved to do better.

To motivate the problem, indeed, to state it properly, I need to describe the formalism of Grothendieck duality. This involves the very rich formalism of relations among Grothendieck's six operations. (Actually I'll only talk about five of them.)

To make it more agreeable this formalism will be discussed in a simple commutative-algebra context—where it may appear to be a collection of trivialities. However, from time to time it will be noted that the very same formalism plays a central role in the much more complicated context of Grothendieck duality.

Outline

1. Extension and restriction of scalars.
2. Tensor and Hom: closed categories.
3. Compatibilities among the four operations.
4. Let the games begin: commutativities growing from the axiomatic soil.
5. A fifth operation.
6. Twisted inverse image: the basic pseudofunctor, more commutativities.
7. Illustration: the fundamental class of a flat map.
8. Coherence: mastering commutative diagrams in closed categories.
9. Summing up: how to deal with diagrams built out of Grothendieck operations?

1. Extension and restriction of scalars

Details in **Notes on Derived Functors and Grothendieck Duality,** SLN #1960.

Available (as are the slides for this lecture) at

< http://www.math.purdue.edu/~lipman >.

$\mathcal{R} :=$ category of commutative rings

For $f\colon R \to S$ in $\mathcal{R}$, set $\mathbf{R} := \{R\text{-modules}\}$, $\mathbf{S} := \{S\text{-modules}\}$,

$f^*\colon \mathbf{R} \to \mathbf{S} :=$ extension of scalars: for $M \in \mathbf{R}$, $f^*M := M \otimes_R S \in \mathbf{S}$ (covariant functor),

$f_*\colon \mathbf{S} \to \mathbf{R} :=$ restriction of scalars: for $N \in \mathbf{S}$, $f_*N := N \in \mathbf{R}$ (contravariant functor).

Adjointness of f^* and f_*

$$\mathrm{Hom}_S(f^*E, F) = \mathrm{Hom}_R(E, f_*F) \qquad (E \in \mathbf{R},\ F \in \mathbf{S}).$$

Pseudofunctoriality (behavior vis-à-vis composition of maps)

For an identity map $\mathbf{1}\colon R \to R$ we have

$$\mathbf{1}^* = \text{identity of } \mathbf{R}.$$

For $R \xrightarrow{f} S \xrightarrow{g} T$ in $\mathcal{R}$, $\exists$ a natural transitivity isomorphism of functors

$$d_{g,f}\colon (gf)^* \xrightarrow{\sim} g^*f^*$$

satisfying $d_{\mathbf{1},f} = d_{g,\mathbf{1}} =$ identity, and associative (sort of) in that for each triple of maps $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ the following commutes:

$$\begin{array}{ccc}
(hgf)^* & \xrightarrow{d_{hg,f}} & (hg)^*f^* \\
{\scriptstyle d_{hg,f}}\downarrow & & \downarrow{\scriptstyle d_{h,g}} \\
h^*(gf)^* & \xrightarrow[h^*d_{g,f}]{} & h^*g^*f^*
\end{array}$$

And analogously, with arrows reversed, for restriction of scalars.

Compatibility of pseudofunctoriality and adjointness

As with any adjunction, there is a functorial unit map $\epsilon\colon 1 \to f_*f^*$ adjoint to the identity map $f^* \to f^*$. One checks(!) that for any R-module M and $m \in M$, $\epsilon(M)\colon M \to M \otimes_R S$ takes m to $m \otimes 1$.

For $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ in $\mathcal{R}$, the following natural diagram of functors, involving three different unit maps and two transitivity isomorphisms, commutes:

$$
\begin{array}{ccccc}
1 & \longrightarrow & f_*f^* & \longrightarrow & f_*(g_*g^*f^*) \\
\big\downarrow & & & & \big\| \\
(gf)_*(gf)^* & \xrightarrow{\ \sim\ } & f_*g_*(gf)^* & \xrightarrow{\ \sim\ } & f_*g_*g^*f^*
\end{array}
$$

Equivalently (categorically), the “dual” diagram commutes:

$$
\begin{array}{ccccc}
1 & \longleftarrow & g^*g_* & \longleftarrow & g^*(f^*f_*g_*) \\
\big\uparrow & & & & \big\| \\
(gf)^*(gf)_* & \xleftarrow{\ \sim\ } & g^*f^*(gf)_* & \xleftarrow{\ \sim\ } & g^*f^*f_*g_*
\end{array}
$$

2. $\otimes$ and Hom: closed categories

$\otimes$ and Hom, over a ring R, are instances of two more Grothendieck operations. Their axiomatic properties are summarized in the notion of (symmetric monoidal) closed category.

Definition (Eilenberg-Kelly, 1965)

A *symmetric monoidal category*

$$\overline{\mathbf{R}} = (\mathbf{R}, \otimes, R, \alpha, \lambda, \rho, \gamma)$$

consists of a category $\mathbf{R}$, a "product" functor $\otimes\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, an object R of $\mathbf{R}$, and functorial isomorphisms (A, B, C in $\mathbf{R}$):

$$\alpha\colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C) \qquad \textit{(associativity)}$$

$$\lambda\colon R \otimes A \xrightarrow{\sim} A \qquad \rho\colon A \otimes R \xrightarrow{\sim} A \qquad \textit{(left and right units)}$$

$$\gamma\colon A \otimes B \xrightarrow{\sim} B \otimes A \qquad \textit{(symmetry)}$$

such that $\gamma^2 = 1$ and the following diagrams commute:

$$
\begin{CD}
(A\otimes R)\otimes B @>\alpha>> A\otimes(R\otimes B)\\
@V{\rho\otimes 1}VV @VV{1\otimes\lambda}V\\
A\otimes B @= A\otimes B
\end{CD}
\qquad\qquad
\begin{CD}
R\otimes A @>\gamma>> A\otimes R\\
@V{\lambda}VV @VV{\rho}V\\
A @= A
\end{CD}
$$

$$
\begin{CD}
((A\otimes B)\otimes C)\otimes D @>\alpha>> (A\otimes B)\otimes(C\otimes D) @>\alpha>> A\otimes(B\otimes(C\otimes D))\\
@V{\alpha\otimes 1}VV @. @VV{1\otimes\alpha}V\\
(A\otimes(B\otimes C))\otimes D @>>\alpha> @. A\otimes((B\otimes C)\otimes D)
\end{CD}
$$

$$
\begin{CD}
(A\otimes B)\otimes C @>\alpha>> A\otimes(B\otimes C) @>\gamma>> (B\otimes C)\otimes A\\
@V{\gamma\otimes 1}VV @. @VV{\alpha}V\\
(B\otimes A)\otimes C @>\alpha>> B\otimes(A\otimes C) @>1\otimes\gamma>> B\otimes(C\otimes A)
\end{CD}
$$

A *closed category* is a symmetric monoidal category $\overline{\mathbf{R}}$ as above, together with a functor, called internal hom:

$$[-,-]\colon \mathbf{R}^{\mathrm{op}} \times \mathbf{R} \to \mathbf{R}$$

(where $\mathbf{R}^{\mathrm{op}}$ is the dual category of $\mathbf{R}$) and a functorial isomorphism

$$\pi\colon \mathrm{Hom}_{\mathbf{R}}(A \otimes B,\ C) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{R}}(A,\ [B, C])\,.$$

The isomorphism π just expresses "tensor-hom" adjunction.

Example (Monoids as monoidal categories)

As a category having only identity maps, a monoid G together with its multiplication is a monoidal category, symmetric if G is commutative. If G is a group ("closed under inverses"), then the operation $[y, z] := zy^{-1}$ makes G into a closed category:

$$\mathrm{Hom}_G(xy, z) \neq \phi \iff xy = z \iff x = zy^{-1} \iff \mathrm{Hom}_G(x, [y, z]) \neq \phi.$$

Remark: Adjunction via unit and counit maps.

For later purposes, it is better to reformulate the preceding adjunction π in computer-friendly terms, via the unit and counit maps of π: with $\Phi(A) := A \otimes B$ and $\Psi(C) := [B, C]$ these are the functorial maps

$$\epsilon\colon A \to \Psi\Phi(A) = [B, A \otimes B], \qquad \epsilon := \pi(\text{identity map of } A \otimes B);$$

$$\eta\colon \Phi\Psi(C) = [B, C] \otimes B \to C, \qquad \eta := \pi^{-1}\big(\text{identity map of } [B, C]\big).$$

Indeed, it is standard that adjunctions between *any* two given functors

$$\bullet \underset{\Psi}{\overset{\Phi}{\leftrightarrows}} \bullet$$

correspond one-one to the existence of maps $\epsilon\colon \mathbf{1} \to \Psi\Phi$ and $\eta\colon \Psi\Phi \to \mathbf{1}$ such that both of the following compositions are identity maps:

$$\Phi \xrightarrow{\text{via } \epsilon} \Phi\Psi\Phi \xrightarrow{\text{via } \eta} \Phi, \qquad \Psi \xrightarrow{\text{via } \epsilon} \Psi\Phi\Psi \xrightarrow{\text{via } \eta} \Psi.$$

Deductions

From the axioms of monoidal categories, even ignoring the symmetry isomorphism γ, one can deduce that the following diagrams commute:

$$
\begin{array}{ccc}
(R \otimes A) \otimes B & \xrightarrow{\alpha} & R \otimes (A \otimes B) \\
{\scriptstyle \lambda \otimes 1}\downarrow & & \downarrow{\scriptstyle \lambda} \\
A \otimes B & = & A \otimes B
\end{array}
\qquad
\begin{array}{ccc}
A \otimes (B \otimes R) & \xrightarrow{\alpha} & (A \otimes B) \otimes R \\
{\scriptstyle 1 \otimes \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
A \otimes B & = & A \otimes B
\end{array}
$$

(see MacLane's **Categories for the working mathematician,** first exercise in Chap. 7.)

This is done by a clever direct argument, or by use of a coherence theorem (**ibid.,** Chap.7, §2) which says, roughly, that "all diagrams built up from the axioms must commute."

This is a primitive example of a basic question—to be explained later—underlying the present talk.

3. Compatibilities among the four operations

For $f\colon R \to S$ in $\mathcal{R}$, there are natural **R**-maps

$$R \to f_*S, \qquad f_*A \otimes_R f_*B \xrightarrow{\mu} f_*(A \otimes_S B) \qquad (A, B \in \mathbf{S})$$

(read this with f_* deleted), and natural **S**-maps

$$S \leftarrow f^*R, \qquad f^*E \otimes_S f^*F \xleftarrow{\nu} f^*(E \otimes_R F) \qquad (E, F \in \mathbf{R}).$$

ν is just the standard iso $(E \otimes_R S) \otimes_S (F \otimes_R S) \xleftarrow{\sim} (E \otimes_R F) \otimes_R S$.

These maps are related via f^*- f_* adjointness. For example,
ν is adjoint to the natural composite map

$$E \otimes_R F \to f_*f^*E \otimes_R f_*f^*F \xrightarrow{\mu} f_*(f^*E \otimes_S f^*F).$$

Note. As this categorical characterization isn't the usual definition of ν, there is something to check!

Ditto (without explicit mention) for subsequent examples.

Assume, axiomatically, that

$$\nu\colon f^*(E \otimes_R F) \to f^*E \otimes_S f^*F \text{ is an isomorphism.}$$

This holds over rings, and also in most other cases of interest.

One checks commutativity of the following natural diagrams, commutativities which serve as axioms for the interaction of f_* and $\otimes$. *Interactions involving f^* and $[\ ,\ ]$ result via adjunction.*

$$\begin{array}{ccc} f_*S \otimes f_*A & \xrightarrow{\ \mu\ } & f_*(S \otimes A) \\ \big\uparrow & & \big\downarrow{\scriptstyle f_*\lambda} \\ R \otimes f_*A & \xrightarrow[\lambda]{} & f_*A \end{array} \qquad\qquad \begin{array}{ccc} f_*A \otimes f_*B & \xrightarrow{\ \mu\ } & f_*(A \otimes B) \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle f_*\gamma} \\ f_*B \otimes f_*A & \xrightarrow[\mu]{} & f_*(B \otimes A) \end{array}$$

$$\begin{array}{ccccc} (f_*A \otimes f_*B) \otimes f_*C & \xrightarrow{\ \mu\otimes 1\ } & f_*(A \otimes B) \otimes f_*C & \xrightarrow{\ \mu\ } & f_*((A \otimes B) \otimes C) \\ {\scriptstyle\alpha}\big\downarrow & & & & \big\downarrow{\scriptstyle f_*(\alpha)} \\ f_*A \otimes (f_*B \otimes f_*C) & \xrightarrow[1\otimes\mu]{} & f_*A \otimes f_*(B \otimes C) & \xrightarrow[\mu]{} & f_*(A \otimes (B \otimes C)) \end{array}$$

Monoidal functors

Abstractly speaking, the preceding commutativities signify that $f_*\colon \mathbf{S} \to \mathbf{R}$ is compatible with the monoidal structures on its source and target; or, as we say, that f_* *is a monoidal functor.*

Example

If G and H are abelian groups, viewed as closed categories, then a functor $\phi\colon G \to H$ is monoidal $\iff$ ϕ is a group homomorphism.

Compatibility between pseudofunctoriality and monoidality

For $R \xrightarrow{f} S \xrightarrow{g} T$ in $\mathcal{R}$ and $A, B \in \mathbf{T}$, the following natural diagrams commute:

$$\begin{array}{ccc}
R & \longrightarrow & (gf)_*T \\
\downarrow & & \downarrow \\
f_*S & \longrightarrow & f_*g_*T
\end{array}$$

$$\begin{array}{ccccc}
(gf)_*A \otimes (gf)_*B & & \xrightarrow{\quad \mu_{gf} \quad} & & (gf)_*(A \otimes B) \\
\simeq \downarrow & & & & \downarrow \simeq \\
f_*g_*A \otimes f_*g_*B & \xrightarrow[f_*\mu_g]{} & f_*(g_*A \otimes g_*B) & \xrightarrow[\mu_f]{} & f_*g_*(A \otimes B)
\end{array}$$

4. Let the games begin.

The category-theoretic structure we have described up to now is the formalism of adjoint monoidal closed-category-valued pseudofunctors.

Examples

• There is a functorial map $f_*[A, B] \longrightarrow [f_*A, f_*B]$ corresponding under π to the composed map

$$f_*[A, B] \otimes f_*A \xrightarrow{\mu} f_*\big([A, B] \otimes A\big) \xrightarrow{f_*\eta_{AB}} f_*B$$

where η_{AB} is the counit map of the $\otimes - [\ ,\ ]$ adjunction.

• For fixed A the functorial isomorphism $\nu\colon f^*(C \otimes A) \xrightarrow{\ \sim\ } f^*C \otimes f^*A$ induces a *conjugate* "internal adjunction" isomorphism on right adjoints, namely $[A, f_*B] \underset{\xi}{\xleftarrow{\ \sim\ }} f_*[f^*A, B]$

• There is a functorial map $f^*[A, B] \longrightarrow [f^*A, f^*B]$ which is adjoint to the composition

$$[A, B] \longrightarrow [A, f_*f^*B] \xrightarrow{\xi^{-1}} f_*[f^*A, f^*B]\,.$$

Of course the above all turn out to be standard maps in the ring context; but again, this formalism applies to other contexts...

The original paper of Eilenberg and Kelly contains a crowd of maps and commutative diagrams, all coming out of the axioms. *Many* such diagrams force themselves on you when, for example, you delve into Grothendieck duality theory.

Here's an example, involving all four operations.

Example (Exercise)

Establish (from axioms) a natural commutative diagram

$$\begin{array}{ccccc} f^*\big(f_*[f^*F, G]\otimes F\big) & \longrightarrow & f^*\big([F, f_*G]\otimes F\big) & \longrightarrow & f^*f_*G \\ \big\downarrow & & & & \big\downarrow \\ f^*f_*[f^*F, G]\otimes f^*F & \longrightarrow & [f^*F, G]\otimes f^*F & \longrightarrow & G \end{array}$$

Interpret this in the context of rings.

5. A fifth operation

For $f : R \to S$ in $\mathbf{R}$, and $E \in \mathbf{S}$, $F \in \mathbf{R}$, there is a canonical isomorphism

$$\mathrm{Hom}_R(E, F) \xrightarrow{\sim} \mathrm{Hom}_S(E, \mathrm{Hom}_R(S, F)).$$

Thus:

The functor $f^{\times} : \mathbf{R} \to \mathbf{S}$ taking F to $\mathrm{Hom}_R(S, F)$ is right-adjoint to f_*.

In Grothendieck duality theory, the existence of a right adjoint for f_* is a fundamental (nontrivial) theorem.

In any case, we can add the right adjoint $f^{\times}$ to the preceding formalism.

Base Change

For any commutative fiber-sum (pushout) square

$$\begin{array}{ccc} S & \xrightarrow{v} & S' \cong S \otimes_R R' \\ f\uparrow & \sigma & \uparrow g \\ R & \xrightarrow[u]{} & R' \end{array}$$

with u *flat* and f *finite and finitely presented*, one has canonical isos

$$S' \otimes_S \mathrm{Hom}_R(S, F) \xrightarrow{\sim} R' \otimes_R \mathrm{Hom}_R(S, F) \xrightarrow{\sim} \mathrm{Hom}_R(S, R' \otimes_R F)$$
$$\xrightarrow{\sim} \mathrm{Hom}_{R'}(S', R' \otimes_R F).$$

This composition can be described formally as the functorial map

$$\beta_\sigma \colon v^* f^\times F \to g^\times u^* F$$

adjoint to the composition $g_* v^* f^\times F \xrightarrow[\theta_\sigma^{-1}]{\sim} u^* f_* f^\times F \longrightarrow u^* F$.

The theorem that for f a proper map of noetherian schemes, and u flat, this base-change map is an isomorphism, is a pillar of Grothendieck duality.

6. Twisted inverse image

Grothendieck duality theory is concerned basically with the twisted inverse-image pseudofunctor built by pasting together the pseudofunctors $f^{\times}$ over proper maps and f^{*} over étale maps, via the preceding base-change isomorphism. The pasting is possible because any finite-type separated map of noetherian schemes factors as (proper)∘(open immersion) (Nagata's compactification theorem).

Suresh Nayak showed recently that the process extends to *essentially* finite-type separable scheme-maps.

For simplicity, we state the defining theorem only for quasi-finite maps of noetherian rings—essentially finite-type maps with finite fibers. A formally similar statement holds in Grothendieck duality, with "proper" in place of "finite."

A quasi-finite map whose ring-theoretic fibers are (finite) products of separable field extensions is called étale.

Basic Duality Theorem (for quasi-finite ring-maps)

Theorem

On the category of quasi-finite maps of noetherian rings, there is a pseudofunctor $^!$ *that is uniquely determined up to isomorphism by the following three properties:*

(i) (Duality.) *The pseudofunctor* $^!$ *restricts on the subcategory of finite maps to a right adjoint of* $(-)_*$.

(ii) *The pseudofunctor* $^!$ *restricts on étale maps to* $(-)^*$.

(iii) *For any commutative fiber-sum (pushout) square*

$$\begin{array}{ccc} S & \xrightarrow{v} & S' \cong S \otimes_R R' \\ {\scriptstyle f}\big\uparrow & \sigma & \big\uparrow{\scriptstyle g} \\ R & \xrightarrow[u]{} & R' \end{array}$$

with u *(hence* v*) étale and* f *(hence* g*) finite, the base-change map* β_σ *is*

$$v^*f^! = v^!f^! \xrightarrow{\sim} (vf)^! = (gu)^! \xrightarrow{\sim} g^!u^! = g^!u^*.$$

Construction of $f^!$

In the quasi-finite context, the necessary compactification is given by Zariski's main theorem:

Any quasi-finite map $f\colon R \to S$ factors as $R \xrightarrow{p} T \xrightarrow{e} S$ with p finite and e étale.

Choose such a factorization, and set

$$f^! F := e^* p^\times F = S \otimes_T \mathrm{Hom}_R(T, F) \qquad (f \in \mathbf{R}).$$

The problem is to show $f^!$ independent of choice of factorization. (This means showing that various diagrams commute.)

Now explore various compatibilities of $f^!$ with f_*, f^*, $\otimes$ and $[\ ,\ ]$. ($\implies$ more commutative diagrams.)

7. The fundamental class of a flat map.

Let $f\colon R \to S$ be a flat quasi-finite map The fundamental class of f is a canonical map $S \to f^!R$, defined as a composition of a dozen or so maps coming from the preceding formalism. (Details irrelevant here.)

This map has a simple interpretation when f is finite: the Duality property of $(-)^!$ makes it correspond to an R-linear map $S \to R$—and that map turns out to be the usual trace map.

We'd like to reduce the general case to the concrete finite one via a factorization $R \xrightarrow{p} T \xrightarrow{e} S$ as above, but this is problematic because T need not be flat over R. A quasi-finite map *is* finite locally (i.e., after completion), and there we still have the relation to the trace map. But how do you show *ab ovo* that there's a single global map with this nice local behavior?

Digression: Hochschild homology and Grothendieck duality

This all becomes more interesting in the context of Grothendieck duality, where the fundamental class (defined via the same abstract formalism) is a canonical derived-category map **c** from the Hochschild complex of a flat scheme-map $f\colon X \to Y$ (i.e., $\mathbf{L}\delta^*\delta_*\mathcal{O}_X$, $\delta\colon X \to X \times_Y X$ being the diagonal) to the relative dualizing complex $f^!\mathcal{O}_Y$.

More concretely, apply cohomology H^n ($n :=$ fiber dimension of f) to get

$$\mathrm{c}_f\colon \Omega^n_f \to H^n(f^!\mathcal{O}_Y) =: \text{dualizing sheaf of } f.$$

(In the previous discussion, n was 0.) Via local duality, c corresponds to residues (the higher-dimensional generalization of traces).

For *smooth* f, c is an isomorphism–whence Serre duality for smooth maps. More generally, c is a key to the role played by differential forms in Duality theory. However, it hasn't yet been shown that this c—via Hochschild homology—is $\pm$ other well-known versions.

Details in preprint (with Alonso and Jeremías) at my home page.

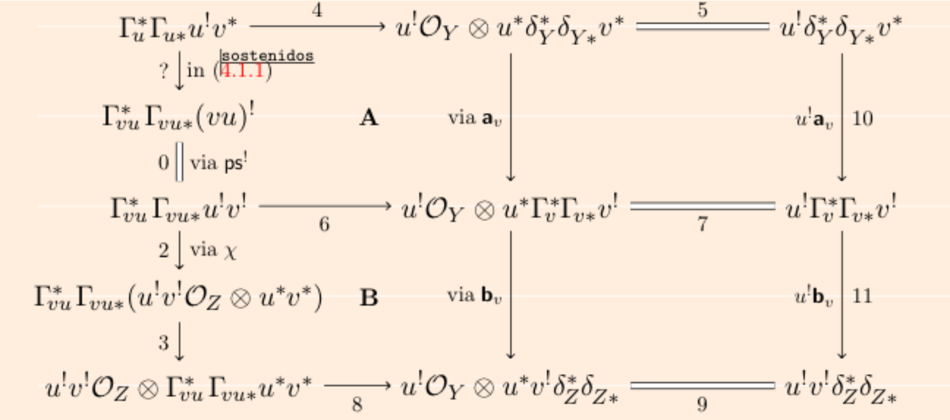
Transitivity of the fundamental class

A basic property of the fundamental class, in the abstract setup, is its transitivity w.r.t. a pair of flat maps, say of rings, $R \xrightarrow{v} S \xrightarrow{u} T$.
For example, this generalizes the well-known transitivity of trace maps.
Without any further explanation, the assertion is roughly that
the fundamental class of gf is obtained in the most obvious way it could be from those of g and f.

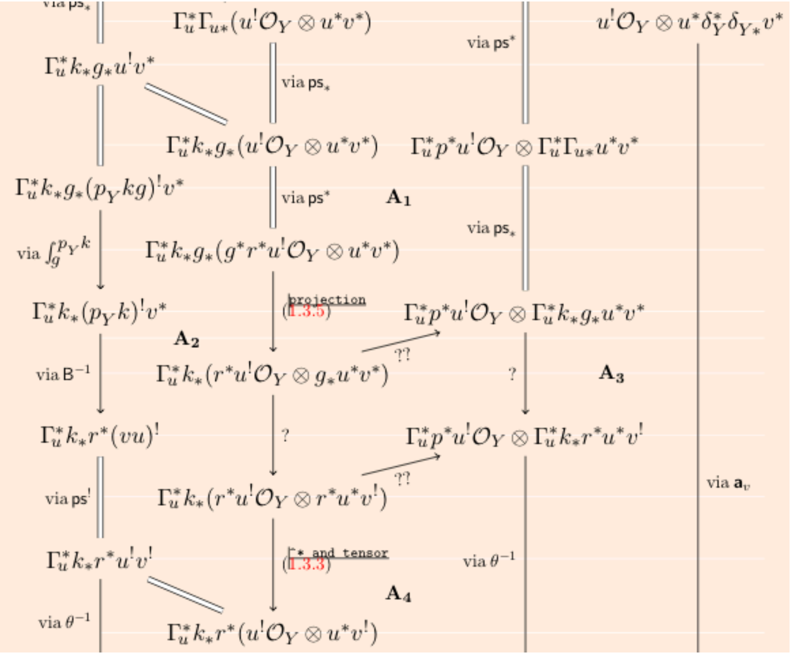
This can be rephrased by saying that a certain small diagram involving the three fundamental classes commutes. After factoring the fundamental class $\mathbf{c} = \mathbf{ba}$ according to its definition, the diagram looks like this (δ and Γ are certain graph maps; but never mind the symbols—just look at the overall appearance):

$$
\begin{array}{ccccccc}
\delta_X^*\delta_{X*}u^*v^* & \xrightarrow{\ \mathbf{a}_u\ } & \Gamma_u^*\Gamma_{u*}u^!v^* & \xrightarrow{\ \mathbf{b}_u\ } & u^!\delta_Y^*\delta_{Y*}v^* & \xrightarrow{u^!\mathbf{c}_v} & u^!v^!\delta_Z^*\delta_{Z*} \\
\text{via } \mathsf{ps}^* \Big\| & (\#) & \Big\downarrow ? & & (\#\#) & & \Big\| \text{via } \mathsf{ps}^! \\
\delta_X^*\delta_{X*}(vu)^* & \xrightarrow[\ \mathbf{a}_{vu}\ ]{} & \Gamma_{vu}^*\Gamma_{vu*}(vu)^! & & \xrightarrow[\qquad \mathbf{b}_{vu} \qquad]{} & & (vu)^!\delta_Z^*\delta_{Z*}
\end{array}
$$

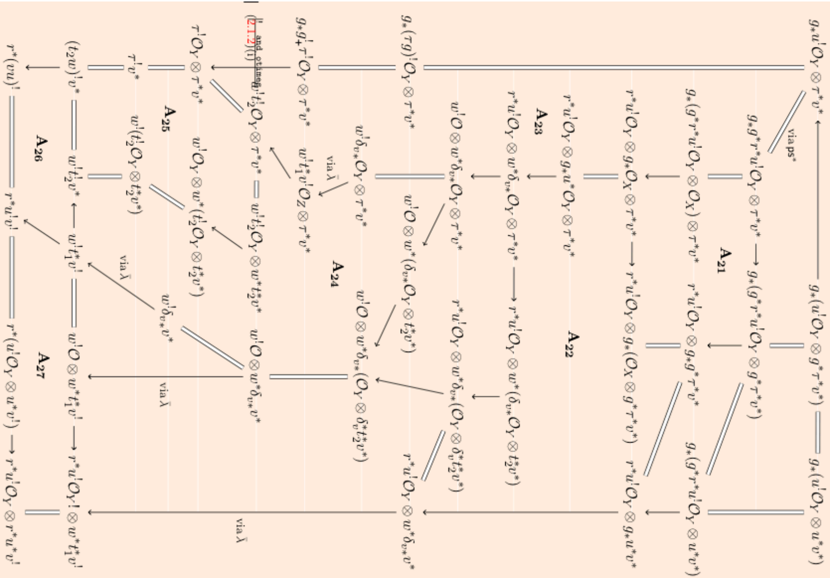
The subdiagram (##), for example, expands as



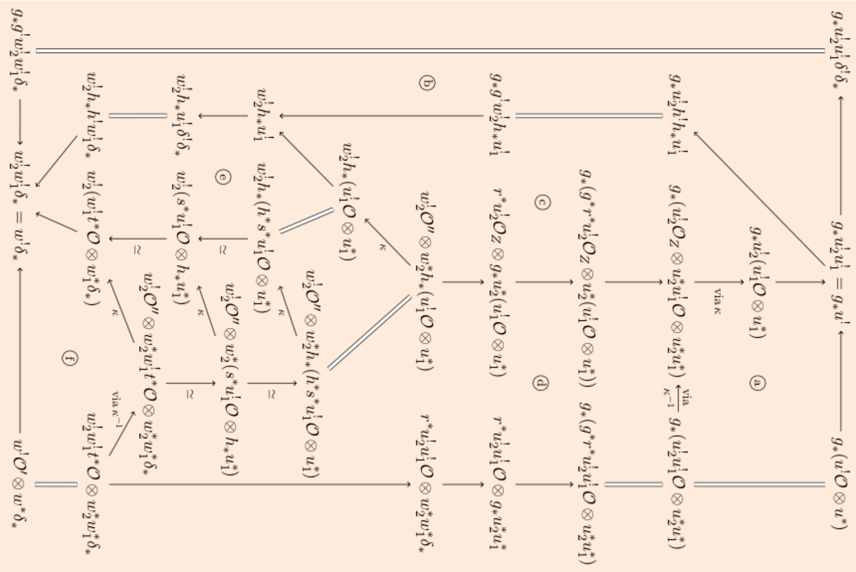
Subdiagram **A**, for example, expands further as (in part)



Subdiagram $\mathbf{A}_2$, for example, expands further as



Subdiagram $\mathbf{A}_{23}$, for example, leads further to diagrams such as



... and so on.

For this lecture, what matters is not the details of the proof (which can be found in the avove-mentioned preprint), but its general structure and complexity.

In the present absence of a conceptual approach to proving transitivity, it would clearly be of great value here if one had a practical way of testing for commutativity of diagrams arising from the 6-operation formalism. This might not be possible, any more than testing mathematical statements for provability. But an area which tries to do this, called Coherence in categories flourished several decades ago. I haven't kept up with all the developments, and I'm not aware of any recent breakthroughs. Anyway, one of the better-known results in the area is, roughly speaking:

8. Coherence theorem for closed categories

Theorem (Kelly-Mac Lane, J. pure and applied Algebra, 1971)

Given two functors built up from the basic data of a closed category, and two natural transformations α, β, between these functors, also built up from the basic data, we have $\alpha = \beta$ provided that in the construction of these transformations, there is no functor of the form $[T, R]$ with T a nonconstant functor.

Example (where T=identity functor and $\alpha \neq \beta$)

Fix a closed category $\overline{\mathbf{R}}$ as above. For $A \in \mathbf{R}$ there is a natural map $r_A\colon A \to [[A, R], R]$ corresponding under π to the "evaluation" map $A \otimes [A, R] \cong [A, R] \otimes A \to R$ (where the latter map is the unit of tensor-hom adjuction). The composition

$$[A, R] \xrightarrow{r_{[A,R]}} [[[A, R], R], R] \xrightarrow{[r_A, 1]} [A, R]$$

is NOT always the identity, e.g., for ∞-dim'l vector spaces over a field.

Example

Putting $D := [A \otimes B,\ C]$, one gets from the isomorphisms

$$\begin{aligned}\mathrm{Hom}\big(D,\ [A\otimes B,\ C]\big) &\xrightarrow{\pi} \mathrm{Hom}\big(D\otimes(A\otimes B),\ C\big) \xrightarrow{\alpha} \mathrm{Hom}\big((D\otimes A)\otimes B,\ C\big)\\ &\xrightarrow{\pi} \mathrm{Hom}\big(D\otimes A,\ [B,C]\big) \xrightarrow{\pi} \mathrm{Hom}\big(D,\ \big[A,[B,C]\big]\big)\end{aligned}$$

"internal tensor-hom adjunction" (pointless over rings, but not schemes):

$$\pi_i = \pi_i(A,B,C)\colon [A\otimes B,\ C] \xrightarrow{\ \sim\ } \big[A,[B,C]\big]\,.$$

Using the description via unit and counit of the tensor-hom adjunction in closed categories, one gets from the coherence theorem that the following functorial diagram commutes:

$$\begin{array}{ccccc}
[(A\otimes B)\otimes C, D] & \xrightarrow{\pi_i} & [A\otimes B,[C,D]]] & \xrightarrow{\pi_i} & [A,[B,[C,D]]]\\
\alpha\downarrow & & & & \downarrow[1,\pi_i]\\
[A\otimes(B\otimes C), D] & & \xrightarrow{\qquad\qquad\pi_i\qquad\qquad} & & [A,[B\otimes C,D]]
\end{array}$$

It is instructive to prove this commutativity directly from the axioms.

Desideratum, and challenge

One would like, ideally, to have a similar coherence theorem for the full formalism of adjoint monoidal closed-category pseudofunctors. This would obviously be very useful.

But though there have been some results beyond Kelley-Mac Lane, none—to my knowledge—can handle, say, the two earlier exercises involving adjoint functors between closed categories.

How hard can it be to do one diagram? To do a whole class of diagrams? This is the challenge.

9. Summary

• The formalism of Grothendieck's six operations (of which we considered only five), readily illustrated in the context of commutative rings, appears in many other contexts. For example, it forms the natural framework around which to build Grothendieck duality (for schemes with Zariski topology, or with étale topology, or classical topology.)

• The formalism is very rich, leading to *many* diagrams whose commutativity is basic to the various domains of application.

• Proving these diagrams commute usually means decomposing them, via definitions of the maps involved, into smaller diagrams which are known to commute. Ultimately this process has to lead back to the axioms of the formalism. Finding such a decomposition can be tedious, even for one diagram, let alone many—which soon tax the limits of human patience.

Summary (continued)

• Experience suggests that the level of complexity of this process is somewhere between that of solving Rubik's cube and proving theorems from axioms. Subjectively, it seems that the number of techniques used is rather small, suggesting that a computer could be taught how to do it, or, in case no definitive algorithm can exist, at least to be of significant use as an assistant.

However, despite some small efforts, I am unable to teach a computer how to find a solution even to the relatively simple exercise mentioned before, namely, to prove from the axioms of monoidal categories that the following diagrams commute:

$$\begin{array}{ccc} (R\otimes A)\otimes B & \xrightarrow{\ \alpha\ } & R\otimes(A\otimes B) \\ {\scriptstyle\lambda\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ A\otimes B & = & A\otimes B \end{array} \qquad \begin{array}{ccc} A\otimes(B\otimes R) & \xrightarrow{\ \alpha\ } & (A\otimes B)\otimes R \\ {\scriptstyle 1\otimes\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ A\otimes B & = & A\otimes B \end{array}$$

It is not hard to teach the computer the axioms, and the formal deduction rules. But then, in my (very limited) experience, the computer just starts to generate endless trivialities, without ever approaching a solution.
On the other hand, there has been significant work done in automated theorem-proving, so there may be some techniques for preventing such vacuous deduction. If there are any experts in the audience, I would be glad to hear from them.

• Best of all would be a coherence theorem, giving practical criteria for testing commutativity.

• I suspect that working seriously on this problem (more so than I have) could lead to substantial new results in category theory and/or logic, with applications to artificial intelligence.