Dualizing complexes and cousin complexes on formal schemes

Joseph Lipman

Segovia, August 16, 2006

Outline.

1. Grothendieck Duality.
2. Formal schemes.
3. Dualizing complexes and Grothendieck Duality.
4. A Cousin-complex pseudofunctor on formal schemes.

1. Grothendieck Duality.

First, a couple of definitions:

For a scheme X, $\mathbf{D}^+_{\mathsf{c}}(X)$ (resp. $\mathbf{D}^+_{\mathsf{qc}}(X)$) is the full subcategory of the derived category $\mathbf{D}(X)$ with objects those complexes whose homology sheaves are coherent (resp. quasi-coherent) and are zero in sufficiently large negative degree.

A (*contravariant*) *pseudofunctor* (or 2-*functor*) *on a category* $\mathbf{S}$ assigns to each $X \in \mathbf{S}$ a category $\mathbf{X}^{\#}$, to each map $f\colon X \to Y$ a functor $f^\sharp\colon \mathbf{Y}^{\#} \to \mathbf{X}^{\#}$ (with $1^\sharp = 1$), and to each map-pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ a functorial isomorphism

$$d_{f,g}\colon f^\sharp g^\sharp \to (gf)^\sharp$$

satisfying $d_{1,g} = d_{g,1} =$ identity, and "associativity": for any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, the following diagram commutes.

$$\begin{array}{ccc}
(hgf)^\sharp & \xleftarrow{d_{f,hg}} & f^\sharp(hg)^\sharp \\
{\scriptstyle d_{gf,h}}\uparrow & & \uparrow{\scriptstyle f^\sharp d_{g,h}} \\
(gf)^\sharp h^\sharp & \xleftarrow[d_{f,g}]{} & f^\sharp g^\sharp h^\sharp.
\end{array}$$

<u>Example</u>: $\mathbf{S}$=rings, $\mathbf{X}^{\#}$=X-modules, $f^\sharp$=restriction of scalars.

Duality Theorem. *On the category* $\mathbf{S}$ *of finite-type maps of separated noetherian schemes,* $\exists$ *a* $\mathbf{D}^+_{\mathsf{qc}}$*-valued pseudofunctor* $(-)^!$ *(i.e.,* $X^! := \mathbf{D}^+_{\mathsf{qc}}(X)\ \forall X \in \mathbf{S}$*) which is uniquely determined up to isomorphism by the properties that it restricts to the usual inverse-image pseudofunctor* $(-)^*$ *on the subcategory of étale maps, that if* p *is a proper map the functor* $p^!$ *is right-adjoint to* $\mathbf{R}p_*$*—pseudofunctorially: for proper* $X \xrightarrow{q} Y \xrightarrow{p} Z$*,* $q^!p^! \xrightarrow{\sim} (pq)^!$ *is adjoint to the natural composition*

$$\mathbf{R}(pq)_*q^!p^! \xrightarrow{\sim} \mathbf{R}p_*\mathbf{R}q_*q^!p^! \to \mathbf{R}p_*p^! \to 1,$$

and that for any fiber square σ *in* $\mathbf{S}$

$$\begin{array}{ccc} X' & \xrightarrow{j'} & X \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y' & \xrightarrow[j]{} & Y \end{array}$$

with j *étale and* p *proper, the natural functorial isomorphism*

$$(\flat)\qquad j'^*p^! = j'^!p^! \xrightarrow{\sim} (pj')^! = (jp')^! \xrightarrow{\sim} p'^!j^! = p'^!j^*$$

is the base-change map β_σ *associated to* σ*, i.e., the map adjoint (see above) to the natural composition*

$$\mathbf{R}p'_*j'^*p^! \xrightarrow{\sim} j^*\mathbf{R}p_*p^! \to j^*.$$

Steps in Proof. First show that $\mathbf{R}p_*$ has a right adjoint $p^!$, see Deligne [**6**] (uses special adjoint functor thm.), or Neeman [**13**] (uses Brown Representability). Since $\mathbf{R}(-)_*$ is a pseudofunctor on the category opposite to $\mathbf{S}$, one makes this adjunction pseudofunctorial by defining $q^!p^! \xrightarrow{\sim} (pq)^!$ as in the statement. Can factor finite-type map f as $f = pj$, p proper, j an open immersion (Nagata's compactificaton theorem), and then set $f^! := j^*p^!$.

Check: $f^!$ independent (up to canonical isomorphism) of factorization, and has all asserted properties. One needs to show that β_σ is an isomorphism, at least when j is an open immersion, a statement which is equivalent to *sheafified proper duality*: for $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and $F \in \mathbf{D}_{\mathsf{qc}}(Y)$, the natural composite map

$$\begin{aligned} \mathbf{R}p_*\mathbf{R}\mathcal{H}om_X(E, p^!F) &\to \mathbf{R}\mathcal{H}om(\mathbf{R}p_*E, \mathbf{R}p_*p^!F) \\ &\to \mathbf{R}\mathcal{H}om(\mathbf{R}p_*E, F) \end{aligned}$$

is an isomorphism; this provides the glue for pasting together the two—a priori unrelated—pseudofunctors $(-)^!$ (for proper maps) and $(-)^*$ (for open immersions, and *a posteriori* for étale maps).

A complete proof requires category-theoretic verification of *many* commutativities.

Grothendieck's first strategy for treating arbitrary proper maps was different: it was based on *pseudofunctorial properties of dualizing complexes.* We will return to this.

Remarks.

1. The fundamental *base-change theorem* states that β_σ *is an isomorphism whenever the maps* p *and* j *are tor-independent.*

For a proof, without noetherian hypotheses, see [**8,** Chap. 4]).

2. The approach in [**7**] to duality for *projective* maps is to treat separately several distinctive special situations, such as smooth maps, finite maps, and regular immersions (local complete intersections), where $f^!$ has a nice explicit description; and then to do the general case by pasting together special ones via some remarkable compatibilities, involving e.g., differential forms (fundamental local homomorphism, residue isomorphism, ...). These concrete manifestations of duality motivate and enliven the subject; but there's no time for them today.

3. Various approximations to the Duality Theorem (covering most important situations) have been known for decades, see, e.g., [**7,** p. 383, 3.4] (many missing or incorrect details of which are treated in [**3**].) At present, however, the proofs of the theorems, as stated here, seems to need, among other things, a compactification theorem of Nagata, that any finite-type separable map of noetherian schemes factors as an open immersion followed by a proper map, a fact whose lengthy proof was not well-understood before the appearance of [**10**] and [**4**]; and even modulo that theorem, I am not aware of any really complete, detailed exposition in print of the proofs of duality or of base change (in full generality) prior to the recent one by Nayak [**11**].

2. Formal schemes, quick review.

A, noetherian ring, complete w.r.t. ideal I. *Formal spectrum*

$$\mathrm{Spf}(A, I) := \{\text{prime ideals } p \supset I\}$$

Zariski topology, structure sheaf $\mathcal{O}$ (sheaf of topological rings) such that for any basic open subset

$$D(f) = \{p \in \mathrm{Spf}(A, I) \mid f \notin p\}, \qquad (f \in A)$$

one has

$$\Gamma\big(D(f), \mathcal{O}\big) = I\text{-adic completion of the localization } A_f.$$

EXAMPLES.

- If A is a complete local ring, maximal ideal $\mathfrak{m}$, then $\mathrm{Spf}(A, \mathfrak{m})$ is a one-point space, with structure sheaf A.
- $\mathrm{Spf}\big(A, (0)\big)$ is just the usual $\mathrm{Spec}(A)$.

Formal scheme: glue together finitely many formal spectra, i.e., it's a noetherian topological space with a sheaf of topological rings, locally isomorphic to a formal spectrum.

Grothendieck's upgrade of Zariski's "holomorphic functions."

Examples.

- An ordinary noetherian scheme $(X, \mathcal{O}_X)$, $\mathcal{O}_X$ regarded with discrete topology.

- Completion of such an $(X, \mathcal{O}_X)$ w.r.t. a coherent $\mathcal{O}_X$-ideal $\mathcal{I}$: topological space is $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$, structure sheaf is $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$.

The topology of the structure sheaf of a formal scheme X is determined by an *ideal of definition* $\mathcal{I}$ (unique up to radical) which pastes together all the local I's defining the topologies on the open formal spectra $\subset X$.

A map of formal schemes $f\colon X \to Y$ is a map of topologically ringed spaces—so if $\mathcal{I}$ and $\mathcal{J}$ are ideals of definition of X and Y, respectively, then $\mathcal{J}\mathcal{O}_X \subset \sqrt{\mathcal{I}}$; and there results a map of *ordinary* schemes $f_0\colon (X, \mathcal{O}_X/\mathcal{I}) \to (Y, \mathcal{O}_Y/\mathcal{J})$.

For example, if X and Y are ordinary schemes, then $\mathcal{I} = (0)$, $\mathcal{J} = (0)$, and $f_0 = f$.

<u>Terminology</u>:

f is *adic* if $\sqrt{\mathcal{J}\mathcal{O}_X} = \sqrt{\mathcal{I}}$.
f is *pseudo-xxx* if f_0 is xxx. (xxx=proper, or finite type, or ...).
f is *xxx* if it is pseudo-xxx and adic.

For example, the natural map of one-point formal schemes $\operatorname{Spf}\big(\mathbf{C}[[t]], (t)\big) \to \operatorname{Spf}\big(\mathbf{C}, (0)\big)$ is pseudo-proper, but not proper.

Motivation for formal schemes:

(A) Cf. role of complete rings in commutative algebra.

(B) Unifies local and global duality in one context (see below).

Proper duality for formal schemes. *For any proper map $f\colon X \to Y$ of formal schemes, $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathsf{c}}(X) \to \mathbf{D}^+_{\mathsf{c}}(Y)$ has a right adjoint.*

This is not a straightforward consequence of duality for ordinary schemes. It comes out of a more general pseudo-proper duality theorem [**1**, §8] via "Greenlees-May Duality," a generalization of local duality which first arose in algebra, but turns out to be closely related to duality for the canonical (pseudo-proper) map of formal schemes $\hat{X} \to X$ where $\hat{X}$ is a completion of a formal scheme X along a closed subscheme.

G-M duality, treated quite generally in [**2**], is used above in the following special form [**1**, Prop. 6.2.1]: on a formal scheme X, consider the *torsion functor* associating to any $\mathcal{O}_X$-module its submodule of sections annihilated by some open $\mathcal{O}_X$-ideal, and denote its derived functor by $\boldsymbol{\varGamma}$. (When X is an ordinary scheme, (0) is the only open ideal, so $\boldsymbol{\varGamma}$ is the identity functor). Then:

For all $E \in \mathbf{D}(X)$ and $F \in \mathbf{D}_{\mathsf{c}}(X)$, the natural map $\boldsymbol{\varGamma}E \to E$ induces an isomorphism

$$\mathbf{R}\mathcal{H}om(E, F) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\boldsymbol{\varGamma}E, F).$$

Problem. Don't know if can extend, as for ordinary schemes, to a pseudofunctor over *all* pseudo-finite-type separable maps, because not known if every such map compactifies, i.e., factors as an open immersion followed by a pseudo-proper map.

But at least one can extend to the category of *all composites of* (*any number of*) *compactifiable maps and étale maps.* This has been done by Nayak [**11**, §7], who needed to invent pasting techniques different than those of Deligne (which require compactifications of maps).

3. Dualizing complexes and Grothendieck Duality.

For a formal scheme X, a *dualizing* $\mathcal{O}_X$-complex R is one s.t.

(i) R is $\mathbf{D}(X)$-isomorphic to a bounded injective complex, with coherent homology, and

(ii) the natural map is an isomorphism $\mathcal{O}_X \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(R, R)$.

For ordinary schemes, this is just the classic definition. More generally, if $\kappa\colon \hat{X} \to X$ is a completion map and R is dualizing on X, then κ^*R is dualizing on $\hat{X}$.

Not every formal scheme has a dualizing complex. For example, a homologically bounded $R \in \mathbf{D}_{\mathsf{c}}(X)$ is dualizing iff X has finite dimension and $\forall x \in X$, R_x is dualizing on $\operatorname{Spec}(\mathcal{O}_{X,x})$—whence $\mathcal{O}_{X,x}$ is a homomorphic image of a Gorenstein ring.

But any finite-dimensional formal scheme admitting a compactifiable map to a finite-dimensional locally Gorenstein formal scheme does have a dualizing complex.

If R is dualizing then the dualizing functor $\mathcal{D}_R := \mathcal{H}om(-, R)$ is an *involutive auto-antiequivalence of* $\mathbf{D}_{\mathsf{c}}(X)$.

What this amounts to is that for any $F \in \mathbf{D}_{\mathsf{c}}(X)$—not just for $F = \mathcal{O}_X$ as in the above definition—the natural map is an isomorphism

$$F \xrightarrow{\sim} \mathcal{D}_R\mathcal{D}_R F.$$

Also have “Affine Duality”: with $\mathcal{D}_R^t(-) := \mathbf{R}\mathcal{H}om(-, \boldsymbol{\varGamma} R)$, *for any* $F \in \mathbf{D}_{\mathsf{c}}(X)$ *the natural map is an isomorphism*

$$F \xrightarrow{\sim} \mathcal{D}_R^t\mathcal{D}_R^t F.$$

For example, if $X = \operatorname{Spf}(A, \mathfrak{m})$ is, as before, the formal spectrum of a complete local ring, so that $\mathcal{O}_X$-modules are just A-modules, then a dualizing $\mathcal{O}_X$-complex is just a dualizing A-complex in the usual commutative-algebra sense.

Moreover, $\boldsymbol{\varGamma} R$ is an injective hull of the residue field $A/\mathfrak{m}$, and affine duality is just Matlis duality.

The study of dualizing complexes on formal schemes was started, I believe, by Yekutieli—with a technically different formulation—in the mid 90s, and further developed in [**1**, §2.5].

Relation to Duality Theorem.

The relation between dualizing complexes and the duality pseudofunctor $(-)^!$ is rooted in:

Proposition. *Let $f\colon X\to Y$ be a pseudo-proper map of formal schemes or a finite-type map of separable noetherian schemes, and let R be a dualizing $\mathcal{O}_Y$-complex. Then with $R_f := f^!R$,*

(i) *R_f is a dualizing $\mathcal{O}_X$-complex.*

(ii) *There is a functorial isomorphism*

$$f^!E \xrightarrow{\;\sim\;} \mathcal{D}_{R_f}\mathbf{L}f^*\mathcal{D}_R E \qquad \big(E\in\mathbf{D}^{+}_{\mathsf{c}}(Y)\big).$$

This suggests the idea behind Grothendieck's first strategy for approaching duality for not-necessarily-projective maps (see p. 4). Indeed, the main thrust of [**7**, Chap. 6 and first half of Chap. 7] (many details of which are filled in and clarified in [**3**, §§3.1–3.4]) is to prepare for the proof of a (somewhat restricted) Duality Theorem by constructing a "coherent system" of dualizing complexes, in the following sense:

Definition. A *Dualizing Complex* on a formal scheme Y is a map associating to each formal-scheme map $f\colon X \to Y$ a dualizing complex R_f on X, to each open immersion $u\colon U \to X$ a $\mathbf{D}(U)$-isomorphism $\beta_{f,u}\colon u^*R_f \xrightarrow{\ \sim\ } R_{fu}$, and to each proper map $g\colon X' \to X$ a $\mathbf{D}(X)$-map $\tau_{f,g}\colon \mathbf{R}g_*R_{fg} \to R_f$, subject to:

(a) (Transitivity for β) If $v\colon V \to U$ is an open immersion, then the following natural diagram commutes:

$$\begin{tikzcd} v^*u^*R_f \arrow[r, "\sim"] \arrow[d, "v^*\beta_{f,u}"'] & (uv)^*R_f \arrow[d, "\beta_{f,uv}"] \\ v^*R_{fu} \arrow[r, "\beta_{fu,v}"] & R_{fuv} \end{tikzcd}$$

(b) (Duality) The pair $(R_{fg}, \tau_{f,g})$ represents the functor

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}g_*E, R_f)\colon \mathbf{D}^+_{\mathsf{c}}(X') \to \mathbf{D}^+_{\mathsf{c}}(X),$$

that is, the natural composite map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(X')}(E, R_{fg}) &\longrightarrow \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}g_*E, \mathbf{R}g_*R_{fg}) \\ &\xrightarrow{\text{via } \tau} \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}g_*E, R_f) \end{aligned}$$

is an isomorphism. Further (transitivity for τ), if $h\colon X'' \to X'$ is proper then the following diagram commutes:

$$\begin{tikzcd} \mathbf{R}g_*\mathbf{R}h_*R_{fgh} \arrow[r, "\sim"] \arrow[d, "\mathbf{R}g_*\tau_{fg,h}"'] & \mathbf{R}(gh)_*R_{fgh} \arrow[d, "\tau_{f,gh}"] \\ \mathbf{R}g_*R_{fg} \arrow[r, "\tau_{f,g}"] & R_f \end{tikzcd}$$

(c) (Local nature of τ.) In a fiber square

$$\begin{tikzcd} V \arrow[r, "v"] \arrow[d, "h"'] & Z \arrow[d, "g"] \\ U \arrow[r, "u"'] & X \end{tikzcd}$$

if g (hence h) is proper and u (hence v) is an open immersion, then the following natural diagram commutes:

$$\begin{tikzcd} u^*\mathbf{R}g_*R_{fg} \arrow[rr, "\sim"] \arrow[d, "u^*\tau_{f,g}"'] & & \mathbf{R}h_*v^*R_{fg} \arrow[d, "\simeq"', "\mathbf{R}h_*\beta_{fg,v}"] \\ u^*R_f \arrow[r, "\sim", "\beta_{f,u}"'] & R_{fu} & \mathbf{R}h_*R_{fuh} = \mathbf{R}h_*R_{fgv} \arrow[l, "\tau_{fu,h}"] \end{tikzcd}$$

Example. If R is a dualizing $\mathcal{O}_Y$-complex, associate to each map $f\colon X \to Y$ the dualizing $\mathcal{O}_X$-complex $R_f := f^!R$, to each open immersion $u\colon U \to X$ the identity map of $u^*R_f = R_{fu}$, and to each proper map $g\colon X' \to X$ the natural composite map $\tau = \tau_{f,g}\colon \mathbf{R}g_*(fg)^!R \xrightarrow{\ \simeq\ } \mathbf{R}g_*g^!f^!R \to f^!R$.

The idea is to “reverse this example.”

Theorem. *Let* $\mathbf{S}$ *be a subcategory of the category of ordinary schemes such that if* $Y \in \mathbf{S}$ *then* Y *has a Dualizing Complex. Then there is on* $\mathbf{S}$ *a* $\mathbf{D}_{\mathsf{c}}^{+}$*-valued pseudofunctor* $^!$ *which has the properties listed in the Duality Theorem, with* $\mathbf{D}_{\mathsf{c}}^{+}$ *in place of* $\mathbf{D}_{\mathsf{qc}}^{+}$*.*

Moreover, with this $^!$*, any Dualizing Complex* (R, β, τ) *on* Y *for which* R_{1_Y} *is isomorphic to a given dualizing* $\mathcal{O}_Y$*-complex* R *is isomorphic* (*in the obvious sense*) *to the one in the preceding example.*

Idea of proof: Define $f^!$ for $f\colon X \to Y$ by

$$f^! E = \mathcal{D}_f^Y \mathbf{L} f^* \mathcal{D}_1^Y E \qquad \big(E \in \mathbf{D}_{\mathsf{c}}^{+}(Y)\big)$$

where, after choosing a Dualizing Complex (R^Y, β^Y, τ^Y) on each Y, we set $\mathcal{D}_f^Y(-) := \mathcal{H}om_X(-, R_f^Y)$. Then check!!

The second assertion is based on a standard uniqueness property of dualizing complexes [**7**, p. 266, Thm. 3.1].

Remarks. This theorem says less than the Duality Theorem: the restriction to $\mathbf{S}$ of the pseudofunctor in that theorem satisfies this one. But for formal schemes, a similar approach does give results not as yet otherwise obtainable (see below).

The real difficulty in this approach lies in the construction of Dualizing Complexes, as in [**7**, Chaps. 6 and 7].

Recently, Yekutieli and Zhang have exploited the notion of "rigid dualizing complex," first used by Van den Bergh in non-commutative algebra, to produce an elegant new approach to the construction of Dualizing Complexes on schemes of finite type over a fixed finite-dimensional regular one (see ArXiv).

4. A Cousin-complex pseudofunctor on formal schemes.

In [**7**], the first major step (Chapter 6) toward existence of Dualizing Complexes is the construction of a pseudofunctor on finite-type scheme-maps of bounded fiber dimension, taking values in categories of *residual complexes.* The second (Chapter 7) establishes the duality property (b) of Dualizing Complexes.

A residual $\mathcal{O}_X$-complex is one isomorphic to a direct sum of the injective hulls of the residue fields at the points of $x \in X$, and having coherent homology. More precisely, any $\mathcal{O}_{X,x}$-module is to be regarded as a sheaf constant on the closure of x, and 0 elsewhere.

On a formal scheme which has a dualizing complex, *if* $\mathcal{R}$ *is a residual complex then* $R := \mathbf{R}\mathcal{H}om(\boldsymbol{\mathit{\Gamma}}\mathcal{O}_X, \mathcal{R})$ *is dualizing, and in* $\mathbf{D}(X)$, $\mathcal{R} \cong \boldsymbol{\mathit{\Gamma}} R$. There results an equivalence of categories

$$Q\colon \{\text{Residual complexes}\} \xrightarrow{\approx} \{\text{Dualizing complexes}\}.$$

A quasi-inverse, the *Cousin functor* E, will now be described.

A *codimension function* on a formal scheme X is a function $d\colon X \to \mathbb{Z}$ such that $d(x') = d(x)+1$ whenever x' is an immediate specialization of x (i.e., the closure $\overline{\{x'\}}$ is a maximal irreducible subscheme of $\overline{\{x\}}$). For example, if $\mathcal{R}$ is a residual complex, and $x \in X$ has residue field $k(x)$, there is a unique integer $d(x)$ such that for all $i \neq d(x)$, $\operatorname{Ext}^i(k(x), \mathcal{R}_x) = 0$; and this defines a codimension function.

A formal scheme having a codimension function must be catenary. And any catenary formal scheme which is biequidimensional does have a codimension function.

W.r.t. a cod'n function d, a *Cousin complex* $\mathcal{G}^\bullet$ is one such that in each degree n there is an isomorphism

$$\mathcal{G}^n \xrightarrow{\sim} \oplus_{d(x)=n}\, \mathcal{G}(x)$$

where each $\mathcal{G}(x)$ is an $\mathcal{O}_{X,x}$-module (regarded as a sheaf, see above) supported at the maximal ideal. The subtlety lies in the boundary maps, which are made up of "residues."

For example, any residual complex is a Cousin complex w.r.t. its associated codimension function.

To each $\mathcal{O}_X$-complex $C \in \mathbf{D}(X)$ there is associated, functorially, a Cousin complex $E(C)$ such that $E(C)(x) = H_x^{d(x)}(C)$, where H_x is the derived functor of the functor of sections with support in the closure of x. (In the cases of most interest here, this turns out to be the same as algebraic local cohomology.) More precisely, $E(C)$ is the E^1 term of the spectral sequence associated to the filtration of X defined by d.

If C is a Cousin complex, there is a natural isomorphism $C \xrightarrow{\sim} EQ(C)$.

Reference [**9**] is devoted to the construction (based ultimately on local properties of residues), and the initial study of, a *pseudofunctor* $(-)^{\#}$ *on the category* $\mathbb{F}_{\mathrm{c}}$ *of pseudo-finite-type maps of formal schemes with codimension functions, taking values in the corresponding categories of Cousin complexes.*

This generalizes [**7**, Chap. 6] in several ways: it makes sense for all Cousin complexes, but does take residual complexes to residual complexes; and it is defined for maps in $\mathbb{F}_{\mathrm{c}}$, not just for maps (with bounded fiber dimension) of ordinary schemes.

The pseudofunctor $(-)^{\#}$ is an ordinary-complex approximation to the derived-category torsion duality functor $\boldsymbol{\Gamma}(-)^{!}$. It is got by gluing local candidates, whose definition uses that maps are locally (smooth)$\,\circ\,$(closed immersion). For smooth maps, one "Cousinifies" the relation $f^{!}E = \mathbf{L}f^{*}E \otimes \Omega_f^{\mathrm{top}}$, and for closed immersions $g\colon X \hookrightarrow Y$, one Cousinifies the relation $g^{!}E = \mathbf{R}\mathcal{H}om(g_{*}\mathcal{O}_X, E)$.

Details (100 pages or more) omitted here.

The relation between $(-)^{\#}$ and $\boldsymbol{\Gamma}(-)^!$ is nailed down by Sastry in [**14**]. He proves:

1. For any composite $f\colon X \to Y$ of compactifiable maps, and any Cousin $\mathcal{O}_Y$-complex C, there is a canonical pseudofunctorial map $Qf^{\#}C \to \boldsymbol{\Gamma} f^!C$, which becomes an isomorphism upon application of the Cousin functor E; and which is itself an isomorphism if f is flat or C is injective.

2. (Duality for Cousin complexes.) For a pseudo-proper $f\colon X \to Y$, and a Cousin $\mathcal{O}_Y$-complex F, $f^{\#}F$ represents the functor $\mathrm{Hom}(f_*C, F)$ of Cousin $\mathcal{O}_X$-complexes C.

3. (Recall that $(-)^{\#}$ takes residual complexes to residual complexes, so, via equivalence, can be thought of as taking dualizing complexes to dualizing complexes.)

On the category of pseudo-finite-type maps of formal schemes admitting a dualizing complex there is a pseudofunctor $(-)^{(!)}$ such that (modulo technicalities):

(a) For $f\colon X \to Y$, if R is any dualizing complex on Y then

$$f^{(!)} = \mathcal{D}_{f^{\#}R}\mathbf{L}f^*\mathcal{D}_R.$$

(b) If f is a composite of compactifiable maps, then $f^{(!)} \cong f^!$, where the latter is Nayak's version of the dualizing functor.

It was to the preceding item **3** that I referred earlier when I said that the Dualizing-Complexes approach affords duality results not otherwise obtainable—Sastry's results can be applied to some maps which are not composites of compactifiable ones.

All is properly explained in Contemporary Math 375.

Finally, here is a striking result due to Yekutieli and Zhang for ordinary schemes (where $\boldsymbol{\varGamma}=$ identity), generalized to formal schemes in [**9**] and [**12**]:

On a formal scheme which has a dualizing complex, let $\mathcal{R}$ be a residual complex, with corresponding codimension function d. Then duality w.r.t. $\mathcal{R}$, i.e., the functor $\mathcal{H}om^\bullet(-,\mathcal{R})$, is an *antiequivalence* from the category $\mathcal{A}_{\mathrm{c}}(X)$ of coherent $\mathcal{O}_X$-modules to the category of d-Cousin complexes $\mathcal{G}$ such that $\mathbf{R}\mathcal{H}om(\boldsymbol{\varGamma}\mathcal{O}_X,\mathcal{G})$ has coherent homology. A quasi-inverse is the functor $H^0\mathcal{H}om^\bullet(-,\mathcal{R})$.

References

1. L. Alonso Tarrío, A. Jeremías López, J. Lipman, *Duality and flat base change on formal schemes*, Contemp. Math. **244** (1999), 3–90.

2. ———, *Greenlees-May duality on formal schemes*, Contemp. Math. **244** (1999), 93–112.

3. B. Conrad, *Grothendieck Duality and Base Change*, Lecture Notes in Math., **1750** (2000).

4. ———, *Deligne's notes on Nagata compactifications*, item **5** on <`http://www-math.mit.edu/~dejong/#brian`>.

5. P. Deligne, *Cohomologie à supports propres*, (SGA 4) Tome 3, Lecture Notes in Math., **305** (1973), pp. 250–461.

6. ———, *Cohomologie à support propre et construction du foncteur $f^!$*, Residues and Duality, Lecture Notes in Math., **20** (1966), pp. 404–421.

7. R. Hartshorne, *Residues and Duality*, Lecture Notes in Math., no. 20, Springer-Verlag, New York, 1966.

8. J. Lipman, *Notes on derived functors and Grothendieck duality*, <`http://www.math.purdue.edu/~lipman/Duality.pdf`> 99% DONE.

9. J. Lipman, S. Nayak, P. Sastry, *Pseudofunctorial behavior of Cousin complexes on formal schemes*, Contemporary Math. **375** (2005), 3–133.

10. W. Lütkebohmert, *On compactification of schemes*, Manuscr. Math. **80** (1993), 95-111.

11. S. Nayak, *Pasting pseudofunctors*, Contemp. Math. **375** (2005), 195–271.

12. S. Nayak and P. Sastry, *Applications of duality theory to Cousin complexes*, arXiv.math.AG/0512105 v1, 5 Dec. 2005.

13. A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), 205–236.

14. P. Sastry, *Duality for Cousin complexes*, Contemp. Math. **375** (2005), 137–192.