Lectures on Grothendieck Duality

I: Derived categories and functors.

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February 16, 2009

References

Details for the course, and many additional references, can be found in

Notes on Derived Functors and Grothendieck Duality,

in: Foundations of Grothendieck Duality for diagrams of schemes,
Lecture Notes in Mathematics #1960,
Springer, New York, 2009, 1–259.

A preprint, as well as all the slides from these lectures, is available now at

< http://www.math.purdue.edu/~lipman >.

Underlying history and philosophy are sketched in the notes' Introduction.
It is recommended to read those few pages for background and motivation.

More details for the first two lectures can be found in

Lectures on local cohomology and duality,

in: Local Cohomology and Its Applications, ed. G. Lyubeznik,
Lecture Notes in Pure and Applied Mathematics #226,
Marcel Dekker, New York, 2001, 39-89.

This is also available at the above web site.

Apology

There is time only for the theory, and not applications, such as:

- resolution of singularities of 2-dimensional schemes,
- Briançon-Skoda theorems,
- Cohen-Macaulayness in graded algebras,

+ many, many more. . .

Outline

1. Cohomology with supports.
2. Generalization to complexes.
3. Derived categories.
4. Triangles.
5. Right-derived functors. RHom and Ext.

1. Cohomology with supports (local cohomology).

- R: a commutative ring
- $\mathbf{M}(R)$: the category of R-modules.
- I: an R-ideal.

$$\Gamma_I M := \{\, m \in M \mid \text{for some } s > 0, I^s m = 0 \,\},$$

viewed as a subfunctor of the identity functor of $\mathbf{M}(R)$.

Choose for each $M \in \mathbf{M}(R)$ an injective resolution, i.e., a sequence of injective R-modules

$$E_M^\bullet : \quad \cdots \to 0 \to 0 \to E_M^0 \to E_M^1 \to E_M^2 \to \cdots$$

together with an R-homomorphism $M \to E_M^0$ such that the sequence

$$0 \to M \to E_M^0 \to E_M^1 \to E_M^2 \to \cdots$$

is exact.

Then define the local cohomology modules

$$\mathrm{H}^i_I M := \mathrm{H}^i(\Gamma_I E^\bullet_M) \qquad (i \in \mathbb{Z}).$$

Each H^i_I may be viewed as a functor from $\mathbf{M}(R)$ to $\mathbf{M}(R)$, a higher derived functor of $\Gamma_I \cong \mathrm{H}^0_I$.

To each "short" exact sequence of R-modules

$$(\sigma): \quad 0 \to M' \to M \to M'' \to 0$$

there are naturally associated connecting R-homomorphisms

$$\delta^i_I(\sigma)\colon \mathrm{H}^i_I M'' \to \mathrm{H}^{i+1}_I M' \qquad (i \in \mathbb{Z}),$$

varying functorially (in the obvious sense) with the sequence (σ), and such that the resulting "long" cohomology sequence

$$\cdots \to \mathrm{H}^i_I M' \to \mathrm{H}^i_I M \to \mathrm{H}^i_I M'' \xrightarrow{\delta^i} \mathrm{H}^{i+1}_I M' \to \mathrm{H}^{i+1}_I M \to \cdots$$

is exact.

A sequence of functors $(\mathrm{H}^i_*)_{i\geq 0}$, in which H^0_* is left-exact, together with connecting maps δ^i_* taking short exact sequences functorially to long exact sequences, as above, is called a cohomological functor.

Local cohomology is characterized up to canonical isomorphism as being a universal cohomological extension of Γ_I—there is a functorial isomorphism $\mathrm{H}^0_I \cong \Gamma_I$, and for any cohomological functor $(\mathrm{H}^i_*, \delta^i_*)$, every functorial map $\phi^0\colon \mathrm{H}^0_I \to \mathrm{H}^0_*$ extends uniquely to a map of cohomological functors, i.e., a family of functorial maps $(\phi^i\colon \mathrm{H}^i_I \to \mathrm{H}^i_*)_{i\geq 0}$ such that for any short exact (σ) as above,

$$\begin{array}{ccc} \mathrm{H}^i_I(M'') & \xrightarrow{\delta^i_I(\sigma)} & \mathrm{H}^{i+1}_I(M') \\ {\scriptstyle \phi^i(M'')}\downarrow & & \downarrow{\scriptstyle \phi^{i+1}(M')} \\ \mathrm{H}^i_*(M'') & \xrightarrow[\delta^i_*(\sigma)]{} & \mathrm{H}^{i+1}_*(M') \end{array}$$

commutes for all $i \geq 0$.

Like considerations apply to any left-exact functor on $\mathbf{M}(R)$.

Example (Hom and Ext)

For a fixed R-module N the functors

$$\mathrm{Ext}^i_R(N, M) := \mathrm{H}^i\mathrm{Hom}_R(N, E^\bullet_M) \qquad (i \ge 0)$$

with their standard connecting homomorphisms form a universal cohomological extension of $\mathrm{Hom}_R(N, M)$ (considered as a functor of M).

From $\Gamma_I E^\bullet_M = \varinjlim_{s>0} \mathrm{Hom}_R(R/I^s, E^\bullet_M)$ one gets the canonical identification of cohomological functors

$$\mathrm{H}^i_I M = \varinjlim_{s>0} \mathrm{Ext}^i_R(R/I^s, M).$$

Local cohomology has a global analogue:

Example (Cohomology with supports)

For an abelian sheaf M on a topological space X, and a closed $Z \subset X$, let $\Gamma_Z(X, M)$ be the sheaf associating to an open $U \subset X$ the group

$$\Gamma_Z(M)(U) := \{\, m \in M(U) \mid m \text{ vanishes on } U \setminus Z \,\}.$$

The universal cohomological extension of the functor Γ_Z is the sequence $(H^i_Z)_{i \geq 0}$—cohomology sheaves supported in Z.

When we restrict to quasi-coherent sheaves over schemes, with Z defined by the quasi-coherent ideal $\mathcal{I}$, then

$$\Gamma_Z(M)(U) = \{\, m \in M(U) \mid \text{for some } s > 0,\ \mathcal{I}^s m = 0 \,\}$$

2. Generalization to complexes.

Terminology:

An *R-complex* $C^\bullet = (C^\bullet, d^\bullet)$ is a sequence of R-module maps

$$\cdots \xrightarrow{d^{i-2}} C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots \qquad (i \in \mathbb{Z})$$

such that $d^i d^{i-1} = 0$ for all i. (The differential $d^\bullet$ is often omitted in the notation.)

The *i*-th cohomology $\mathrm{H}^i C^\bullet$ is $\ker(d^i)/\mathrm{im}(d^{i-1})$.

The translation (or suspension) $C[1]^\bullet$ of $C^\bullet$ is the complex such that

$C[1]^i := C^{i+1}$ and $d^i_{C[1]}\colon C[1]^i \to C[1]^{i+1}$ is $-d^{i+1}_C\colon C^{i+1} \to C^{i+2}$.

Clearly, $H^i(C[1]^\bullet) = H^{i+1}(C^\bullet)$.

Terminology (ct'd).

A map of R-complexes $\psi\colon (C^\bullet, d^\bullet) \to (C_*^\bullet, d_*^\bullet)$ is a family of R-homomorphisms $(\psi^i\colon C^i \to C_*^i)_{i\in\mathbb{Z}}$ such that $\psi^{i+1}d^i = d_*^i\psi^i$ for all i.

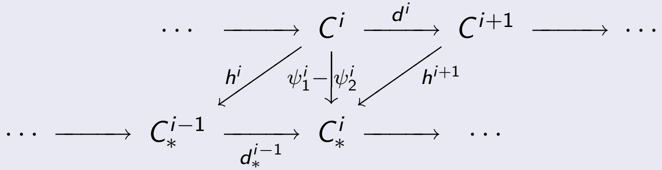
$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C^i & \xrightarrow{d^i} & C^{i+1} & \xrightarrow{d^{i+1}} & \cdots \\
 & & \Big\downarrow{\scriptstyle \psi^i} & & \Big\downarrow{\scriptstyle \psi^{i+1}} & & \\
\cdots & \longrightarrow & C_*^i & \xrightarrow[d_*^i]{} & C_*^{i+1} & \xrightarrow[d_*^{i+1}]{} & \cdots
\end{array}$$

Such a map induces R-homomorphisms $\mathrm{H}^i(\psi)\colon \mathrm{H}^iC^\bullet \to \mathrm{H}^iC_*^\bullet$.
ψ is a quasi-isomorphism if $\mathrm{H}^i(\psi)$ is an isomorphism for all $i \in \mathbb{Z}$.

Homotopy.

A homotopy between R-complex maps $\psi_1\colon C^\bullet \to C_*^\bullet$ and $\psi_2\colon C^\bullet \to C_*^\bullet$ is a family of R-homomorphisms $(h^i\colon C^i \to C_*^{i-1})$ such that

$$\psi_1^i - \psi_2^i = d_*^{i-1}h^i + h^{i+1}d^i \qquad (i \in \mathbb{Z}).$$



ψ_1 and ψ_2 are homotopic if such h^i exist. This is an equivalence relation, preserved by addition and composition of maps.

Hence the R-complexes are the objects of an additive category $\mathbf{K}(R)$ whose morphisms are the homotopy-equivalence classes.

Cohomology functors.

Homotopic maps induce identical maps on homology. So it is clear what a quasi-isomorphism in $\mathbf{K}(R)$ is.

Each H^i can be thought of as a functor from $\mathbf{K}(R)$ to $\mathbf{M}(R)$, taking quasi-isomorphisms to isomorphisms.

q-injective complexes.

An R-complex $C^\bullet$ is q-injective if
any quasi-isomorphism $\psi\colon C^\bullet \to C^\bullet_*$ has a left homotopy-inverse,
i.e., $\exists \psi_*\colon C^\bullet_* \to C^\bullet$ such that $\psi_*\psi$ is homotopic to the identity map of $C^\bullet$.

Equivalently:

(#): for any $\mathbf{K}(R)$-diagram

$$\begin{array}{ccc} C^\bullet_* & & \\ \text{quasi-isomorphism } \psi \Big\uparrow & & \\ X^\bullet & \xrightarrow[\phi]{\quad} & C^\bullet \end{array}$$

$\exists!\ \phi_*\colon C^\bullet_* \to C^\bullet$ such that $\phi_*\psi = \phi$.

q-injectivity is often called K-injectivity. ("q" connotes "quasi-isomorphism.")

Example

Any bounded-below injective complex $C^\bullet$ (i.e., C^i is an injective R-module for all i, and $C^i = 0$ for $i \ll 0$) is q-injective.

And if $C^\bullet$ vanishes in all but one degree, say $C^j \neq 0$, then $C^\bullet$ is q-injective $\iff$ this C^j is an injective R-module.

q-injective resolutions.

A q-injective resolution of an R-complex $C^\bullet$ is
a quasi-isomorphism $C^\bullet \to E^\bullet$ with $E^\bullet$ q-injective.

Such exists for any $C^\bullet$, with $E^\bullet$ the total complex of an injective Cartan-Eilenberg resolution of $C^\bullet$.

Example

An injective resolution of an R-module M is essentially the same as a q-injective resolution of the complex $M^\bullet$ such that $M^0 = M$ and $M^i = 0$ for all $i \neq 0$.

In fact a q-injective resolution exists for any complex in an arbitrary Grothendieck category, i.e., an abelian category with exact direct limits and having a generator. In this generality, Cartan-Eilenberg resolutions don't always suffice (for instance, in categories of abelian sheaves on topological spaces, where infinite direct products don't preserve exactness).

Local (hyper)cohomology

After choosing for each R-complex $C^\bullet$ a specific q-injective resolution $C^\bullet \to E^\bullet_C$, we can define the local cohomology modules of $C^\bullet$:

$$\mathrm{H}^i_I C^\bullet := \mathrm{H}^i(\Gamma_I E^\bullet_C) \qquad (i \in \mathbb{Z}).$$

(#) above $\implies$ *for any* $\mathbf{K}(R)$*-diagram with* ψ_1, ψ_2 *q-injective resolutions,*

$$\begin{array}{ccc} C_1^\bullet & \xrightarrow{\psi_1} & E^\bullet_{C_1} \\ {\scriptstyle\phi}\big\downarrow & & \\ C_2^\bullet & \xrightarrow[\psi_2]{} & E^\bullet_{C_2} \end{array}$$

there is a unique $\phi_*\colon E^\bullet_{C_1} \to E^\bullet_{C_2}$ *in* $\mathbf{K}(R)$ *such that* $\phi_*\psi_1 = \psi_2\phi$.

It follows that H^i_I can be viewed as a functor from $\mathbf{K}(R)$ to $\mathbf{M}(R)$, independent (up to canonical isomorphism) of choice of resolution, and taking quasi-isomorphisms to isomorphisms.

Long exact sequences

It will be explained below, in the context of derived categories, how a short exact sequence of complexes in $\mathbf{M}(R)$, i.e., a sequence $C_1^\bullet \to C^\bullet \to C_2^\bullet$ with $0 \to C_1^i \to C^i \to C_2^i \to 0$ exact for every i, gives rise functorially to connecting maps

$$H^i(C_2^\bullet) \longrightarrow H^{i+1}(C_1^\bullet) \qquad (i \in \mathbb{Z})$$

such that the resulting functorial cohomology sequence

$$\cdots \to \mathrm{H}_I^i C_1^\bullet \to \mathrm{H}_I^i C^\bullet \to \mathrm{H}_I^i C_2^\bullet \to \mathrm{H}_I^{i+1} C_1^\bullet \to \mathrm{H}_I^{i+1} C^\bullet \to \cdots$$

is exact.

Thus one has a *generalization of the basic features of local cohomology of modules to local (hyper)cohomology of complexes.*

(Hyper)Ext

Similar considerations lead to the definition of Ext functors of complexes:

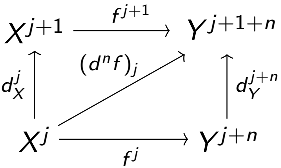
$$\mathrm{Ext}^i_R(D^\bullet, C^\bullet) := \mathrm{H}^i\mathrm{Hom}^\bullet_R(D^\bullet, E^\bullet_C) \qquad (i \in \mathbb{Z})$$

where for two R-complexes $(X^\bullet, d^\bullet_X)$, $(Y^\bullet, d^\bullet_Y)$,
the complex $\mathrm{Hom}^\bullet_R(X^\bullet, Y^\bullet)$ is given in degree n by

$\mathrm{Hom}^n_R(X^\bullet, Y^\bullet) := \{\text{families of } R\text{-homomorphisms } f = (f_j \colon X^j \to Y^{j+n})_{j\in\mathbb{Z}}\}$

with differential $d^n \colon \mathrm{Hom}^n_R(X^\bullet, Y^\bullet) \to \mathrm{Hom}^{n+1}_R(X^\bullet, Y^\bullet)$ specified by

$$(d^n f)_j := d^{j+n}_Y \circ f_j - (-1)^n f_{j+1} \circ d^j_X \qquad (j \in \mathbb{Z}).$$



Local cohomology and Ext (hyper)

As before, from $\Gamma_I E_C^\bullet = \varinjlim_{s>0} \mathrm{Hom}_R(R/I^s, E_C^\bullet)$ one gets the canonical identification, compatible with connecting maps,

$$\mathrm{H}^i_I C^\bullet = \varinjlim_{s>0} \mathrm{Ext}^i_R(R/I^s, C^\bullet)$$

where R/I^s is thought of as a complex vanishing outside degree 0.

3. Derived categories

View relations among homologies as shadows of a more basic reality involving complexes (cf. Plato). Leads to derived category $\mathbf{D}(R)$ of $\mathbf{M}(R)$ (or in general, of any abelian category):

1. Factor out homotopy (which respects homology), i.e., start with $\mathbf{K}(R)$.
2. Make quasi-isomorphisms into isomorphisms (since they "preserve" homology), by formally adjoining an inverse for each such map. (Cf. localization in commutative algebra.)

Get a new category $\mathbf{D}(R)$, same objects as $\mathbf{K}(R)$, but a morphism $C \to C'$ is an equivalence class, denoted f/s, of $\mathbf{K}(R)$-diagrams $C \xleftarrow{s} X \xrightarrow{f} C'$ with s a quasi-isomorphism, the equivalence relation being the least such that $f/s = fs_1/ss_1$ for all f, s, and quasi-isomorphisms $s_1 \colon X_1 \to X$.

For details, in particular how to compose "fractional morphisms," see, e.g., the reference mentioned at the beginning.

Characterization of $\mathbf{D}(R)$ by a universal property.

$\exists$ a canonical functor $Q\colon \mathbf{K}(R) \to \mathbf{D}(R)$ taking any complex to itself, and the $\mathbf{K}(R)$-map $f\colon C \to C'$ to the $\mathbf{D}(R)$-map $f/1_C$ ($1_C :=$ identity of C).

Q takes any quasi-isomorphism f to an isomorphism: $(f/1_C)^{-1} = 1_C/f$.

The pair $(\mathbf{D}(R), Q)$ is characterized up to isomorphism by the

Universal property: For any category $\mathbf{L}$, $F \mapsto F \circ Q$ is an isomorphism of the category of functors from $\mathbf{D}(R)$ to $\mathbf{L}$ onto the category of functors from $\mathbf{K}(R)$ to $\mathbf{L}$ that take quasi-isomorphisms to isomorphisms.

(If $F\colon \mathbf{K}(R) \to \mathbf{L}$ takes quasi-isomorphisms to isomorphisms then the corresponding functor $F_D\colon \mathbf{D}(R) \to \mathbf{L}$ satisfies $F_D(f/s) = F(f)\circ F(s)^{-1}$.)

$\mathbf{D}(R)$ has a unique additive-category structure such that Q is additive.

To add two maps f_1/s_1, f_2/s_2 with same source and target, rewrite them with a common denominator—always possible—then add the numerators.

The universal property of $(\mathbf{D}(R), Q)$ remains valid when restricted to additive functors into additive categories.

Cohomology functors from $\mathbf{D}(R)$ to $\mathbf{M}(R)$.

Example

The cohomology functors H^i take quasi-isomorphisms to isomorphisms and may therefore be viewed as additive functors from $\mathbf{D}(R)$ to $\mathbf{M}(R)$.

In accordance with the initial motivation, one shows (easily):

A $\mathbf{D}(R)$-map α is an isomorphism $\iff$
the homology maps $\mathrm{H}^i(\alpha)$ $(i \in \mathbb{Z})$ are all isomorphisms.

Example

If $T\colon \mathbf{K}(R) \to \mathbf{K}(R)$ is the functor taking C to $C[1]$, then T respects homotopy and takes quasi-isomorphisms to isomorphisms (since $\mathrm{H}^i C[1] = \mathrm{H}^{i+1} C$), whence QT takes quasi-isomorphisms to isomorphisms. The universal property ensures then that there is a functor $\overline{T}\colon \mathbf{D}(R) \to \mathbf{D}(R)$ taking C to $C[1]$ (i.e., $\overline{T}Q = QT$).

More examples

One can embed $\mathbf{M}(R)$ into $\mathbf{D}(R)$:

Example

The functor taking any R-module M to the R-complex that is M in degree zero and 0 elsewhere, and doing the obvious thing to R-module maps, is an equivalence of $\mathbf{M}(R)$ with the full subcategory of $\mathbf{D}(R)$ having as objects the complexes with homology vanishing in all nonzero degrees.

A quasi-inverse for this equivalence is given by the functor H^0.

Example

When R is a field, any R-complex $(C^\bullet, d^\bullet)$ is $\mathbf{D}(R)$-isomorphic to

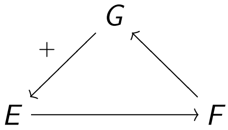
$$\cdots \xrightarrow{0} \mathrm{H}^{i-1}C \xrightarrow{0} \mathrm{H}^{i}C \xrightarrow{0} \mathrm{H}^{i+1}C \xrightarrow{0} \cdots$$

Hence, *the functor* $C \mapsto \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i C$ *from* $\mathbf{D}(R)$ *to graded* R*-vector spaces is an equivalence of categories.*

4. Triangles

As we've seen, exact sequences of complexes play an important role in the discussion of derived functors. But $\mathbf{D}(R)$ *is not an abelian category, so it does not support a notion of exactness.*

Instead, $\mathbf{D}(R)$ carries a supplementary structure given by certain diagrams of the form $E \to F \to G \to E[1]$, called triangles, occasionally represented in the typographically inconvenient form



Mapping cone

Specifically, triangles—in $\mathbf{K}(R)$ or $\mathbf{D}(R)$—are those diagrams which are isomorphic (in the obvious sense) to diagrams of the form

$$X \xrightarrow{\alpha} Y \hookrightarrow C_\alpha \twoheadrightarrow X[1]$$

with α an ordinary map of R-complexes and C_α the mapping cone of α: as a graded group, $C_\alpha := Y \oplus X[1]$, and the differential $C_\alpha^n \to C_\alpha^{n+1}$ is the sum of the differentials d_Y^n and $d_{X[1]}^n$, plus $\alpha^{n+1}\colon X^{n+1} \to Y^{n+1}$.

$$\begin{array}{ccccccc}
C_\alpha^{n+1} & = & Y^{n+1} & \oplus & & & X^{n+2} \\
{\scriptstyle d_{C_\alpha}}\uparrow & & \uparrow{\scriptstyle d_Y} & & \nwarrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle -d_X} \\
C_\alpha^n & = & Y^n & \oplus & & & X^{n+1}
\end{array}$$

Long exact sequence of a triangle

For any exact sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0 \qquad (\tau)$$

of R-complexes, the composite map of graded groups $C_\alpha \twoheadrightarrow Y \xrightarrow{\beta} Z$ turns out to be a quasi-isomorphism of complexes, and so becomes an isomorphism in $\mathbf{D}(R)$. Thus we get a $\mathbf{D}(R)$-triangle

$$X \to Y \to Z \to X[1].$$

Up to isomorphism, these are all the triangles in $\mathbf{D}(R)$.

Applying the i-fold translations T^i $(i \in \mathbb{Z})$ to a triangle

$$E \to F \to G \to E[1] \qquad (\triangle)$$

and then taking homology, one gets a long homology sequence

$$\cdots \to \mathrm{H}^i E \to \mathrm{H}^i F \to \mathrm{H}^i G \to \mathrm{H}^i E[1] = \mathrm{H}^{i+1} E \to \cdots$$

This sequence is *exact*, as one need only verify for mapping cones.
If $(\triangle)$ is the triangle coming from the exact sequence (τ), then this homology sequence is, after multiplication of the connecting maps $\mathrm{H}^i G \to \mathrm{H}^{i+1} E$ by -1, just the usual long exact sequence associated to (τ).

Triangle-preserving functors

Let $\mathcal{A}_1$, $\mathcal{A}_2$ be abelian categories. From these, one gets triangulated derived categories $\mathbf{D}(\mathcal{A}_1)$, $\mathbf{D}(\mathcal{A}_2)$, in the same way as $\mathbf{D}(R)$ from $\mathbf{M}(R)$. Denote the respective translation functors by T_1, T_2.

A Δ-functor $\Phi\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ is an additive functor which "preserves translation and triangles," in the following sense:

Φ comes equipped with a functorial isomorphism

$$\theta\colon \Phi T_1 \xrightarrow{\ \sim\ } T_2\Phi$$

such that for any triangle

$$E \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{w} E[1] = T_1 E$$

in $\mathbf{D}(\mathcal{A}_1)$, the corresponding diagram

$$\Phi E \xrightarrow{\Phi u} \Phi F \xrightarrow{\Phi v} \Phi G \xrightarrow{\theta\circ\Phi w} (\Phi E)[1] = T_2\Phi E$$

is a triangle in $\mathbf{D}(\mathcal{A}_2)$.

Summary

The derived-category functors that appear in what follows can always be equipped in some natural way with a θ making them into Δ-functors.

And Δ-functors associate long-exact homology sequences to short-exact sequences of complexes:

If $\Phi\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ is a Δ-functor, then to any short exact sequence of complexes in $\mathcal{A}_1$

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0 \qquad (\tau_1)$$

there is naturally associated a long exact homology sequence in $\mathcal{A}_2$

$$\cdots \to \mathrm{H}^i(\Phi X) \to \mathrm{H}^i(\Phi Y) \to \mathrm{H}^i(\Phi Z) \to \mathrm{H}^{i+1}(\Phi X) \to \cdots,$$

that is, the homology sequence of the triangle in $\mathbf{D}(\mathcal{A}_2)$ gotten by applying Φ to the triangle given by (τ_1).

Local cohomology as a Δ-functor

Here's an example of lifting focus from homology functors to Δ-functors.

Example

For each R-complex C, choose a q-injective resolution $q_C \colon C \to E_C$. Set

$$\mathbf{R}\Gamma_I C := \Gamma_I E_C .$$

Then

$$\mathrm{H}^i_I C := \mathrm{H}^i \Gamma_I E_C = \mathrm{H}^i \mathbf{R}\Gamma_I C .$$

The point is: $\mathbf{R}\Gamma_I$ can be made into a Δ-functor from $\mathbf{D}(R)$ to $\mathbf{D}(R)$, of which all the H^i_I, and their properties, are mere reflections.

For, q-injective resolutions are $\mathbf{D}(R)$-isomorphisms, whence any $\mathbf{D}(R)$-map $\phi/\psi \colon C \to C'$ is isomorphic to a $\mathbf{D}(R)$-map $\Phi/\Psi \colon E_C \to E_{C'}$;
and the characterization (#) of q-injectivity implies that
taking ϕ/ψ to $\Gamma_I\Phi/\Gamma_I\Psi \colon \mathbf{R}\Gamma_I C \to \mathbf{R}\Gamma_I C'$ is a well-defined operation.
This operation respects identities and composition, making $\mathbf{R}\Gamma_I$ into a functor.

The Δ-structure on $\mathbf{R}\Gamma_I$ is left to the reader.

Elaborating on the preceding example, extend Γ_I to a Δ-functor from $\mathbf{K}(R)$ to $\mathbf{K}(R)$. (Recall, triangles in $\mathbf{K}(R)$ are diagrams isomorphic to those coming from mapping cones, which are preserved by additive functors.) There is a Δ-functorial (commuting with the respective Δ-structures) map $\zeta\colon Q\Gamma_I \to \mathbf{R}\Gamma_I Q$ such that for $C \in \mathbf{K}(R)$, with $q_C\colon C \to E_C$ as before,

$$\zeta(C) = Q\Gamma_I(q_C)\colon \Gamma_I C \to \Gamma_I E_C\,.$$

The pair $(\mathbf{R}\Gamma_I, \zeta)$ is a right-derived Δ-functor of Γ_I, characterized up to canonical isomorphism by the initial-object property:

every Δ-functorial map $Q\Gamma_I \to \Gamma$, where $\Gamma\colon \mathbf{K}(R) \to \mathbf{D}(R)$ takes quasi-isomorphisms to isomorphisms, factors uniquely as $Q\Gamma_I \underset{\zeta}{\to} \mathbf{R}\Gamma_I Q \to \Gamma$.

Informally, *among such* Γ, $\mathbf{R}\Gamma_I Q$ *is the one nearest on the right to* $Q\Gamma_I$.

Similarly, one has via q-injective resolutions a right-derived Δ-functor $(\mathbf{R}F, \zeta_F)$ for *any* Δ-functor $F\colon \mathbf{K}(R) \to \mathbf{K}(R)$. Such F arise most often as extensions of additive functors from $\mathbf{M}(R)$ to $\mathbf{M}(R)$.

RHom and Ext

In the foregoing, Γ_I can be replaced by any additive functor between arbitrary abelian categories—though for existence of derived functors one needs further assumptions, for example that the source be a Grothendieck category.

Example

For any R-complex D one has the functor $\mathbf{R}\mathrm{Hom}^\bullet_R(D, -)$ with

$$\mathbf{R}\mathrm{Hom}^\bullet_R(D, C) = \mathrm{Hom}^\bullet_R(D, E_C)$$

and then,

$$\mathrm{Ext}^i_R(D, C) := \mathrm{H}^i\mathrm{Hom}^\bullet_R(D, E_C) = \mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet_R(D, C).$$

Remark: With some caution regarding signs, $\mathbf{R}\mathrm{Hom}^\bullet_R(D, C)$ can also be made into a contravariant Δ-functor in the first variable.

Exts as derived-category Homs

Another characterization of q-injectivity of an R-complex E is that

for every R-complex D the natural map is an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(R)}(D, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(R)}(D, E).$$

Example

There are simple natural isomorphisms

$$\mathrm{H}^i\mathrm{Hom}^\bullet(D, C) \cong \mathrm{H}^0\mathrm{Hom}^\bullet(D, C[i]) \cong \mathrm{Hom}_{\mathbf{K}(R)}(D, C[i]).$$

Replacing C by E_C, one gets natural isomorphisms

$$\begin{aligned}\mathrm{Ext}^i_R(D, C) = \mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet(D, C) &= \mathrm{H}^i\mathrm{Hom}^\bullet(D, E_C) \\ &\cong \mathrm{Hom}_{\mathbf{K}(R)}(D, E_C[i]) \\ &\cong \mathrm{Hom}_{\mathbf{D}(R)}(D, E_C[i]) \cong \mathrm{Hom}_{\mathbf{D}(R)}(D, C[i]).\end{aligned}$$