Lectures on Grothendieck Duality

IV: A basic setup for duality.

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Introduction

The formalism of adjoint monoidal pseudofunctors was sketched in the preceding lecture. It will be used in this lecture as the foundation of a formal setup for duality theory.

Though we will illustrate mainly in the context of quasi-compact quasi-separated (e.g., noetherian) schemes, the axiomatic framework to be described has the usual advantage of underlying, and hence unifying, several distinct situations, such as affine schemes (local duality) or noetherian formal schemes.

Outline

1. Formal duality setup.

2. Projection isomorphisms.

3. Independent squares.

1. Formal duality setup

Let there be given, on a category **C**, a pair $({}^*, {}_*)$ of
adjoint monoidal closed-category-valued pseudofunctors.

Thus, to each object $X \in \mathbf{C}$ is associated a

closed category $\mathbf{D}_X$, with unit object $\mathcal{O}_X$;

and to each **C**-map $\psi\colon X \to Y$,

adjoint monoidal functors $\mathbf{D}_X \underset{\psi_*}{\overset{\psi^*}{\leftrightarrows}} \mathbf{D}_Y$.

There are also, as before, compatibilities—expressed by commutative diagrams—among adjunction, pseudofunctoriality, and monoidality.

The maps giving the monoidal structure on ψ_* are denoted

$$\mathbf{e}_\psi(E, E')\colon \psi_*E \otimes \psi_*E' \to \psi_*(E \otimes E') \qquad \big(E, E' \in \mathbf{D}_X\big),$$
$$\nu_\psi\colon \mathcal{O}_Y \to \psi_*\mathcal{O}_X\,.$$

Adjoint to the natural composition

$$F \otimes F' \to \psi_*\psi^*F \otimes \psi_*\psi^*F' \xrightarrow{\mathbf{e}} \psi_*(\psi^*F \otimes \psi^*F') \qquad (F, F' \in \mathbf{D}_Y)$$

(resp. to $\nu_\psi \colon \mathcal{O}_Y \to \psi_*\mathcal{O}_X$) we have maps

$$\mathbf{d}_\psi(F, F') \colon \psi^*(F \otimes F') \to \psi^*F \otimes \psi^*F',$$
$$\mu_\psi \colon \psi^*\mathcal{O}_Y \to \mathcal{O}_X.$$

For $E \in \mathbf{D}_X$ and $F \in \mathbf{D}_Y$ the composite map

$$\mathbf{p}_1(E, F) \colon \psi_*E \otimes F \xrightarrow{\text{natural}} \psi_*E \otimes \psi_*\psi^*F \xrightarrow{\mathbf{e}} \psi_*(E \otimes \psi^*F)$$

and the map deduced from it by application of the symmetry isomorphism

$$\mathbf{p}_2(F, E) \colon F \otimes \psi_*E \xrightarrow{\text{natural}} \psi_*\psi^*F \otimes \psi_*E \xrightarrow{\mathbf{e}} \psi_*(\psi^*F \otimes E)$$

are called projection maps.

Axioms

- For $X = Y$ and $\psi = \mathbf{1}_X$ the identity map of X,

$$(\mathbf{1}_X)_* \text{ is the identity functor of } \mathbf{D}_X.$$

- The map μ_ψ is an isomorphism

$$\psi^*\mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X.$$

- For all $F, G \in \mathbf{D}_Y$, the map $\mathbf{d}_\psi$ is an isomorphism

$$\psi^*(F \otimes G) \xrightarrow{\sim} \psi^*F \otimes \psi^*G.$$

- For all $E \in \mathbf{D}_X$ and $F \in \mathbf{D}_Y$ the projection maps are isomorphisms

$$\mathbf{p}_1\colon \psi_*E \otimes F \xrightarrow{\sim} \psi_*(E \otimes \psi^*F), \qquad \mathbf{p}_2\colon F \otimes \psi_*E \xrightarrow{\sim} \psi_*(\psi^*F \otimes E).$$

- The functor $\psi_*\colon \mathbf{D}_X \to \mathbf{D}_Y$ has a right adjoint $\psi^\#$.

So there is a **duality isomorphism**

$$\mathrm{Hom}_{\mathbf{D}_Y}(\psi_*E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_X}(E, \psi^\#F) \qquad (E \in \mathbf{D}_X,\ F \in \mathbf{D}_Y).$$

Example: Commutative algebra

$\mathbf{C} :=$ *opposite* of the category of commutative rings.

For $R \in \mathbf{C}$, $\mathbf{D}_R := \{R\text{-modules}\}$, with the obvious closed structure: $\otimes$ is the usual tensor product, and $[E, F] := \mathrm{Hom}_R(E, F)$.

For $\psi\colon S \to R$ (i.e., a ring-homomorphism $R \to S$),
$\psi_*\colon \mathbf{D}_S \to \mathbf{D}_R$ is restriction of scalars:
for any S-module E, $\psi_* E$ is the naturally resulting R-module E; and $\mathbf{e}_\psi\colon E \otimes_R E' \to E \otimes_S E'$ the natural map.

$\psi^*\colon \mathbf{D}_S \to \mathbf{D}_R$ is extension of scalars:
for any R-module F, $\psi^* F$ is the S-module $S \otimes_R F$.

One verifies that

$$\mu_\psi\colon S \otimes_R R \xrightarrow{\ \sim\ } S,$$

$$\mathbf{d}_\psi\colon S \otimes_R (F \otimes_R G) \xrightarrow{\ \sim\ } (S \otimes_R F) \otimes_S (S \otimes_R G)$$

are the usual isomorphisms; and that $\mathbf{p}_1$ is the natural R-isomorphism

$$E \otimes_R F \xrightarrow{\ \sim\ } E \otimes_S (S \otimes_R F) \qquad (E \in \mathbf{D}_S,\ F \in \mathbf{D}_R).$$

Finally, a right adjoint $\psi^\#$ of ψ_* is given by $\psi^\# F := \mathrm{Hom}_R(S, F)$.

Remarks

In the preceding example, one can substitute derived categories and functors for ordinary ones. Then, at least in the noetherian case, the existence of the right adjoint $\psi^{\#}$ is a consequence of the local duality isomorphism from Lecture 2, with J the unit ideal:

$$\mathrm{Hom}_{\mathbf{D}(R)}(\psi_* \mathbf{R}\Gamma_J E, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S)}(E, \psi_J^{\#} G).$$

One can deal with arbitrary J in a similar way, but at the cost of further elaborating the basic setup.
Globalizing (as we are about to do in the unit-ideal case) then leads to *duality over formal schemes.*
Thus we have a common framework for local and global duality.

Such "topological" generalizations are beyond the scope of the present lectures. But several papers dealing with formal schemes are available at **math.purdue.edu/~lipman**

Globalization: Noetherian schemes

$\mathbf{C} :=$ category of noetherian schemes.[1]

For any $X \in \mathbf{C}$, $\mathbf{D}_X := \mathbf{D}_{\mathrm{qc}}(X)$, the full subcategory of $\mathbf{D}(X)$ whose objects are complexes with quasi-coherent homology. Together with the derived tensor product, this is a monoidal category.

To make it closed, set $[E, F] := \mathrm{Q}_X \mathbf{R}\mathrm{Hom}(E, F)$, where Q_X is a right adjoint to the inclusion functor $\mathbf{D}_X \hookrightarrow \mathbf{D}(X)$.

(*Existence* of such a right adjoint—a *derived quasi-coherator*—is a very special case of the duality theorem to be discussed later.)

Indeed, using derived adjoint associativity, one has, for all $E, F, G \in \mathbf{D}_X$,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}_X}(E \underline{\otimes} F, G) &= \mathrm{Hom}_{\mathbf{D}(X)}(E \underline{\otimes} F, G) \\
&\cong \mathrm{Hom}_{\mathbf{D}(X)}(E, \mathbf{R}\mathcal{H}om(F, G)) \\
&\cong \mathrm{Hom}_{\mathbf{D}_X}(E, \mathrm{Q}_X \mathbf{R}\mathcal{H}om(F, G)).
\end{aligned}$$

[1] Much of what follows applies, with some elaborations, to arbitrary quasi-compact quasi-separated schemes.

For each $f\colon X \to Y$ in $\mathbf{C}$, one shows that $\mathbf{R}f_*\mathbf{D}_{\mathrm{qc}}(X) \subset \mathbf{D}_{\mathrm{qc}}(Y)$;[2] we denote the resulting functor simply by $f_*\colon \mathbf{D}_X \to \mathbf{D}_Y$.

Also, one shows (easily) that $\mathbf{L}f^*\mathbf{D}_{\mathrm{qc}}(Y) \subset \mathbf{D}_{\mathrm{qc}}(X)$; we denote the resulting functor simply by $f^*\colon \mathbf{D}_Y \to \mathbf{D}_X$.

As in the main example of the preceding lecture, this gives us an adjoint pair of closed-category-valued pseudofunctors.

The first three of the above axioms are easy to check.

The fourth, that the projection maps are isomorphisms, will be discussed below.

The fifth, *one of the basic facts of duality theory*, is the existence of a right adjoint for $\mathbf{R}f_*$,
to be discussed in a subsequent lecture.

[2] Showing that $E \in \mathbf{D}_{\mathrm{qc}}(X) \implies \mathbf{R}f_*E \in \mathbf{D}_{\mathrm{qc}}(Y)$ involves only standard arguments when $H^iE = 0$ for all $i \ll 0$, but is somewhat trickier otherwise.

2. Projection isomorphisms

Theorem

Let $f\colon X \to Y$ *be a map of noetherian schemes,* $F \in \mathbf{D}_{\mathrm{qc}}(X)$, $G \in \mathbf{D}_{\mathrm{qc}}(Y)$. *Then the projection maps are isomorphisms*

$$\mathbf{p}_1\colon (\mathbf{R}f_*F) \underline{\underline{\otimes}}\, G \xrightarrow{\sim} \mathbf{R}f_*(F \underline{\underline{\otimes}}\, \mathbf{L}f^*G), \quad \mathbf{p}_2\colon G \underline{\underline{\otimes}}\, \mathbf{R}f_*F \xrightarrow{\sim} \mathbf{R}f_*(\mathbf{L}f^*G \underline{\underline{\otimes}}\, F).$$

Sketch of proof

A *key fact* is that $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}_{\mathrm{qc}}(Y)$ is a bounded-above functor: there is an integer d such that for all $E \in \mathbf{D}_{\mathrm{qc}}(X)$ and all $n \in \mathbb{Z}$,

$$H^i(E) = 0 \text{ for all } i \geq n \implies H^i(\mathbf{R}f_*E) = 0 \text{ for all } i \geq n + d.$$

This is shown by induction on the least number of affines covering X and Y.

Turning to the theorem, we treat only $\mathbf{p}_1$. ($\mathbf{p}_2$ can be handled similarly, or by symmetry.)
The question is local on Y, so we may assume Y affine.

Suppose first that both F and G are bounded-above complexes. Then boundedness of $\mathbf{R}f_*$ implies that the source and target of

$$\mathbf{p}_1\colon (\mathbf{R}f_*F)\underset{=}{\otimes} G \xrightarrow{\ \sim\ } \mathbf{R}f_*(F \underset{=}{\otimes} \mathbf{L}f^*G)\,,$$

are, for fixed F, bounded-above functors of G.

This allows us to use inductive "way-out" methods to reduce the question to where G is a single *free* $\mathcal{O}_Y$-module G^0, whence $\mathbf{L}f^*G$ is isomorphic to the free $\mathcal{O}_X$-module f^*G^0.

One verifies that everything in sight commutes with direct sums, so we have a further reduction to the case $G = \mathcal{O}_Y$.

In that case, $\mathbf{p}_1$ is isomorphic to the identity map of $\mathbf{R}f_*F$.

The unbounded case requires additional considerations, omitted here. (Full details in the reference notes.)

Remark: An example in the reference notes shows that quasi-coherence of homology is necessary for the theorem to hold.

3. Independent squares

We describe a certain class of commutative squares which will play an important role later on, in connection with a fundamental base-change theorem for the right adjoint of $\mathbf{R}f_*$.

Recall that to a commutative $\mathbf{C}$-square

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

one associates the map $\quad \theta = \theta_\sigma \colon u^* f_* \to g_* v^*,$

adjoint to the natural composition

$$f_* \to f_* v_* v^* \xrightarrow{\sim} u_* g_* v^*.$$

Similarly, one has the map

$$\theta'_\sigma \colon f^* u_* \to v_* g^*$$

Example

In the commutative algebra situation, σ corresponds to a commutative square of ring-maps

$$\begin{array}{ccc} S' & \xleftarrow{\;\bar v\;} & S \\ {\scriptstyle \bar g}\uparrow & \bar\sigma & \uparrow{\scriptstyle \bar f} \\ R' & \xleftarrow[\;\bar u\;]{} & R \end{array}$$

and θ_σ is the usual functorial map, for S-modules M,

$$R' \otimes_R M \to S' \otimes_S M,$$

while θ'_σ is the usual functorial map, for R'-modules N,

$$S \otimes_R N \to S' \otimes_{R'} N.$$

In the more significant scheme-theoretic context, with u^* standing for $\mathbf{L}u^*$, f_* for $\mathbf{R}f_*\dots$, one replaces M and N by q-flat quasi-coherent complexes.

Künneth map

For a commutative **C**-square
$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

setting $h := fv = ug$, define the functorial Künneth map

$$\eta_\sigma(E,F)\colon u_*E \otimes f_*F \to h_*(g^*E \otimes v^*F) \qquad (E \in \mathbf{D}_{Y'}, F \in \mathbf{D}_X)$$

to be the natural composition

$$u_*E\otimes f_*F \to h_*h^*(u_*E\otimes f_*F) \to h_*(g^*u^*u_*E\otimes v^*f^*f_*F) \to h_*(g^*E\otimes v^*F).$$

Example

1. When $X = X' = Y$, and v, g are identity maps (so that $u = f$), then
$$\eta = \mathbf{e}_f\colon f_*E \otimes f_*F \to f_*(E \otimes F).$$
2. When $X = Y$, $X' = Y'$, and f, g, are identity maps (so that $u = v$),
$$\eta = \mathbf{p}_1\colon u_*E \otimes F \to u_*(E \otimes u^*F).$$

Example

In the commutative algebra situation, σ corresponds to a commutative square of ring-maps

$$\begin{array}{ccc} S' & \xleftarrow{\bar v} & S \\ {\scriptstyle \bar g}\uparrow & \bar\sigma & \uparrow{\scriptstyle \bar f} \\ R' & \xleftarrow[\bar u]{} & R \end{array}$$

and η_σ is the usual functorial map, for R'-modules M, and S-modules N,

$$M \otimes_R N \to (M \otimes_{R'} S') \otimes_{S'} (S' \otimes_S N).$$

In the corresponding scheme-theoretic context, with u^* standing for $\mathbf{L}u^*$, f_* for $\mathbf{R}f_*$. . . , one replaces M and N by q-flat quasi-coherent complexes.

Equivalent definitions of independence

Theorem

Let

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

be a fiber square of quasi-compact quasi-separated schemes (i.e., σ commutes and the associated map $X' \to Y' \times_Y X$ is an isomorphism). Set $h := fv = gu$. The following conditions are equivalent—and when they hold we say that σ is an independent square:

(i) *For all $E \in \mathbf{D}_{\mathrm{qc}}(X)$, θ_σ is an isomorphism*

$$\mathbf{L}u^*\mathbf{R}f_*E \xrightarrow{\ \sim\ } \mathbf{R}g_*\mathbf{L}v^*E.$$

(i)$'$ *For all $F \in \mathbf{D}_{\mathrm{qc}}(Y')$, θ'_σ is an isomorphism*

$$\mathbf{L}f^*\mathbf{R}u_*E \xrightarrow{\ \sim\ } \mathbf{R}v_*\mathbf{L}g^*E.$$

(ii) *For all $E \in \mathbf{D}_{\mathrm{qc}}(X)$ and $F \in \mathbf{D}_{\mathrm{qc}}(Y')$, η_σ is an isomorphism*

$$\mathbf{R}u_*E \otimes \mathbf{R}f_*F \xrightarrow{\ \sim\ } \mathbf{R}h_*(\mathbf{L}g^*E \otimes \mathbf{L}v^*F).$$

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

(iii) The square σ is tor-independent, that is, for all pairs of points $y' \in Y'$, $x \in X$ such that $y := u(y') = f(x)$,

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y',y'}, \mathcal{O}_{X,x}) = 0 \quad \text{for all } i > 0.$$

or, equivalently, for any affine open neighborhood $\mathrm{Spec}(A)$ of y and affine open sets $\mathrm{Spec}(A') \subset u^{-1}\mathrm{Spec}(A)$, $\mathrm{Spec}(B) \subset f^{-1}\mathrm{Spec}(A)$,

$$\mathrm{Tor}_i^A(A', B) = 0 \quad \text{for all } i > 0.$$

Remarks. (a) Condition (iii) holds if either f or u is flat.

(b) When f and g are identity maps, then of course (iii) holds, and so the implication (iii) $\Rightarrow$ (ii) amounts to saying that the projection map $\mathbf{p}_1$ is an isomorphism. But actually this latter fact is used in proving (iii) $\Rightarrow$ (ii).

Outline of proof

That either (i) or (i)$'$ implies (ii) results from commutativity of the following natural diagram, for any $E \in \mathbf{D}_{\mathrm{qc}}(Y')$ and $F \in \mathbf{D}_{\mathrm{qc}}(X)$, the proof of which is a formal exercise on adjoint monoidal pseudofunctors:

$$
\begin{array}{ccccc}
u_*(E \otimes u^* f_* F) & \xleftarrow[\mathbf{p}_1]{\sim} & u_*E \otimes f_*F & \xrightarrow[\mathbf{p}_2]{\sim} & f_*(f^*u_*E \otimes F) \\
{\scriptstyle u_*(1\otimes\theta)}\downarrow & & & & \downarrow{\scriptstyle f_*(\theta'\otimes 1)} \\
u_*(E \otimes g_*v^*F) & & \downarrow{\scriptstyle \eta} & & f_*(v_*g^*E \otimes F) \\
{\scriptstyle u_*(\mathbf{p}_2)}\downarrow{\scriptstyle \simeq} & & & & {\scriptstyle \simeq}\downarrow{\scriptstyle f_*(\mathbf{p}_1)} \\
u_*g_*(g^*E \otimes v^*F) & \xrightarrow{\sim} & h_*(g^*E \otimes v^*F) & \xleftarrow{\sim} & f_*v_*(g^*E \otimes v^*F)
\end{array}
$$

For the rest, one first treats the case where all the schemes in σ are affine. To reduce to this case, by means of suitable affine covers, one needs to know that the conditions (i), (i)$'$, and (ii) are local. For this, one needs the behavior of independence under "concatenation of squares":

For each one of the following **C**-diagrams, assumed commutative,

$$
\begin{array}{ccccc}
X'' & \xrightarrow{\;v_1\;} & X' & \xrightarrow{\;v\;} & X \\
{\scriptstyle h}\big\downarrow & \sigma_1 & {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\
Y'' & \xrightarrow[\;u_1\;]{} & Y' & \xrightarrow[\;u\;]{} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
Z' & \xrightarrow{\;w\;} & Z \\
{\scriptstyle g_1}\big\downarrow & \sigma_1 & \big\downarrow{\scriptstyle f_1} \\
X' & \xrightarrow{\;v\;} & X \\
{\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\;u\;]{} & Y
\end{array}
$$

if σ and σ_1 satisfy (i) (resp. (i)$'$, resp. (ii)) then so does the rectangle σ_0 enclosed by the outer border.

Proof outline (continued)

This is shown via transitivity relations for θ, θ' and η. For instance, the θs for σ, σ_1 and σ_0 are related by commutativity, for any $G \in \mathbf{D}_X$, of the following $\mathbf{C}$-diagram, a formal consequence of previously stated axioms:

$$
\begin{array}{ccccc}
(uu_1)^* f_* G & & \xrightarrow{\quad \theta_{\sigma_0}(G) \quad} & & h_*(vv_1)^* G \\
\simeq \downarrow & & & & \downarrow \simeq \\
u_1^* u^* f_* G & \xrightarrow[u_1^*\theta_\sigma(G)]{} & u_1^* g_* v^* G & \xrightarrow[\theta_{\sigma_1}(v^*G)]{} & h_* v_1^* v^* G
\end{array}
$$

$$
\begin{array}{ccccc}
X'' & \xrightarrow{v_1} & X' & \xrightarrow{v} & X \\
h \downarrow & \sigma_1 & g \downarrow & \sigma & \downarrow f \\
Y'' & \xrightarrow[u_1]{} & Y' & \xrightarrow[u]{} & Y
\end{array}
$$