

Lectures on Grothendieck Duality

VI: Tor-independent base change; sheafified duality.

Joseph Lipman

Purdue University
Department of Mathematics
lipman@math.purdue.edu

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Introduction

The existence of a right adjoint for $\mathbf{R}f_*$ —**Global Duality**—is the first basic theorem in Grothendieck duality theory.

The second, **Tor-independent Base Change**—the subject of this lecture—has to do with the behavior of this right adjoint, with respect to certain independent fiber squares, at least when f is *proper*.

As a special case of Base Change, we'll get **sheafified duality** (in more general form than before), containing not only Global Duality, but also its behavior w.r.t. restriction to open subsets of the target of a proper map f).

For simplicity, we assume throughout that **all schemes are noetherian**.

The following abbreviations will be used, for a scheme-map h or a scheme Z :

$$\begin{aligned} h_* &:= \mathbf{R}h_* , & h^* &:= \mathbf{L}h^* , \\ \mathcal{H}_Z &:= \mathbf{R}\mathcal{H}om_Z^\bullet , & \mathbf{H}_Z &:= \mathbf{R}\mathrm{Hom}_Z^\bullet , \\ \otimes_Z &:= \bigotimes_Z , & \Gamma_Z(-) &:= \mathbf{R}\Gamma(Z, -) . \end{aligned}$$

As in the global duality theorem, $h^\times (= h^!$ when h is proper) is right-adjoint to h_* , and $\tau: h_* h^\times \rightarrow \mathbf{1}$ is the canonical (counit) map.

Outline

- 1 Base change map for independent squares.
- 2 Base change theorem
- 3 Base change and sheafified duality.
- 4 Proof of base change.

1. Base change map for independent squares

Recall that a commutative square σ of scheme-maps

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \sigma & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

is **(tor-)independent** if σ is a **fiber square** (i.e., the natural map is an isomorphism $X' \xrightarrow{\sim} X \times_Y Y'$) such that the functorial map

$$\theta_\sigma: u^* f_* \rightarrow g_* v^*$$

adjoint to $f_* \rightarrow f_* v_* v^* \xrightarrow{\sim} u_* g_* v^*$ is an **isomorphism**.

Any fiber square with u flat (e.g., u an open immersion) is independent.

Remark: Formally, σ is an *ordered* 4-tuple (u, g, f, v) with $ug = fv$. That is *not the same* as the ordered 4-tuple $\sigma' := (f, v, u, g)$. σ' is the “reflection of σ in the upper-left to lower-right diagonal.”

But, as we’ve seen, σ is independent \iff so is σ' .

Definition: base change map

For an independent $\sigma = (u, g, f, v)$, the functorial **base change map**

$$\beta_\sigma: v^* f^\times \rightarrow g^\times u^*$$

is the map adjoint to the natural composition

$$g_* v^* f^\times \xrightarrow{\theta_\sigma^{-1}} u^* f_* f^\times \xrightarrow{u^* \tau} u^*.$$

Here is another way of getting β_σ , via “conjugate base change.”

Exercise. Let $\sigma = (u, g, f, v)$ be a fiber square. Show that the map

$$\phi_\sigma: v_* g^\times \rightarrow f^\times u_*$$

(between functors from $\mathbf{D}_{\text{qc}}(Y')$ to $\mathbf{D}_{\text{qc}}(X)$) that is adjoint to the composition $f_* v_* g^\times \xrightarrow{\sim} u_* g_* g^\times \rightarrow u_*$, is right-conjugate to θ_σ .

Deduce that σ is independent iff ϕ_σ (or $\phi_{\sigma'}$) is an isomorphism.

(b) Show that when σ is independent the map β_σ is adjoint to the composition

$$f^\times \xrightarrow{\text{natural}} f^\times u_* u^* \xrightarrow{\text{via } \phi_\sigma^{-1}} v_* g^\times u^*.$$

2. Base change theorem

Definition

A scheme-map $u: Y' \rightarrow Y$ has **finite tor-dimension** or **finite flat dimension** if the functor $\mathbf{L}u^*: \mathbf{D}(Y) \rightarrow \mathbf{D}(Y')$ is bounded, i.e., $\exists d \in \mathbb{Z}$ such that for every \mathcal{O}_Y -complex F and $n \in \mathbb{Z}$ such that $H^i F = 0$ for all $i > n$, it holds that $H^j \mathbf{L}u^* F = 0$ for all $j > n + d$.

Equivalently (exercise): For each $y \in Y'$, \exists an exact $\mathcal{O}_{Y,u(y)}$ -module sequence

$$0 \rightarrow P_d \rightarrow P_{d-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathcal{O}_{Y',y} \rightarrow 0$$

with P_i flat over $\mathcal{O}_{Y,u(y)}$ ($0 \leq i \leq d$).

For example, a *flat* map has finite flat dimension (with $d = 0$).

Theorem (Base change—BC)

Suppose one has an independent square of maps of noetherian schemes

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \sigma & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

with f (hence g) proper and u of finite tor-dimension.

Then for any $G \in \mathbf{D}_{\text{qc}}^+(Y)$, the base change map is an isomorphism

$$\beta_{\sigma}: v^* f^! G \xrightarrow{\sim} g^! u^* G.$$

The case of Theorem BC where u (hence v) is an affine map (respectively, an open immersion) is labeled BC^{af} (respectively, BC°).

Counterexample for unbounded G

BC° need not hold for unbounded G .

Neeman gave a simple [counterexample](#), with

$$X := \operatorname{Spec}(\mathbb{Z}[T]/T^2), Y := \operatorname{Spec}(\mathbb{Z}), Y' := \operatorname{Spec}(\mathbb{Z}[\frac{1}{2}]), G := \prod_{n=0}^{\infty} \mathbb{Z}[n],$$

based on noncompatibility of localization and infinite products.

3. Base change and sheafified duality

Recall from Lecture 3 the map

$$\nu(f, F, H): f_*[F, H] \rightarrow [f_*F, f_*H].$$

It is a formal exercise to show that this map factors naturally as

$$f_*[F, H] \rightarrow f_*[f^*f_*F, H] \xrightarrow{\sim} [f_*F, f_*H].$$

In the latter form, ν appeared in Lecture 5 as part of the **duality map** for proper f :

$$\delta(f, F, G): f_*\mathcal{H}_X(F, f^!G) \xrightarrow{\nu} \mathcal{H}_Y(f_*F, f_*f^!G) \xrightarrow{\tau} \mathcal{H}_Y(f_*F, G).$$

Remark. With Γ_Y the derived global section functor, the map

$$\Gamma_Y\delta: \Gamma_Y f_*\mathcal{H}_X(F, f^!G) \rightarrow \Gamma_Y\mathcal{H}_Y(f_*F, G)$$

can be identified with the *global duality map*

$$\mathbf{H}_X(F, f^!G) \rightarrow \mathbf{H}_Y(f_*F, G).$$

In the course of the proof of BC, BC° will be shown equivalent to:

Theorem (Sheafified duality—SD)

If $f: X \rightarrow Y$ is a proper scheme-map then for all $F \in \mathbf{D}_{\text{qc}}(X)$ and $G \in \mathbf{D}_{\text{qc}}^+(Y)$, the duality map $\delta(f, F, G)$ is an isomorphism

$$f_* \mathcal{H}_X(F, f^! G) \xrightarrow{\sim} \mathcal{H}_Y(f_* F, G).$$

The version of this theorem in Lecture 5 had the restriction $F \in \mathbf{D}_c^-(X)$. That version is labeled SD_c .

SD_c having been proved, the **strategy** is to establish the implications

$$\text{SD}_c \implies (\text{BC}^{\text{af}} + \text{BC}^\circ) \implies \text{BC} \implies \text{BC}^\circ \iff \text{SD}.$$

Some indications of how to do this follow. The aim is to suggest the flavor of what's involved, and in particular to bring out the role played by purely formal considerations—that is, arguments based solely on the axioms of basic duality setups. (Full details are in the reference notes.)

4. Proof of base change

We begin with the implication $\mathrm{SD}_c \implies (\mathrm{BC}^{\mathrm{af}} + \mathrm{BC}^{\circ})$.

In the formal context of Lecture 3, we saw a canonical map

$$\rho(u, A, B): u^*[A, B] \rightarrow [u^*A, u^*B].$$

Proposition

Let $u: Y' \rightarrow Y$ be a scheme-map of finite tor-dimension, let $E \in \mathbf{D}_c^-(Y)$ and $H \in \mathbf{D}^+(Y)$. Then the map $\rho(u, E, H)$ is an isomorphism

$$u^*\mathcal{H}_Y(E, H) \xrightarrow{\sim} \mathcal{H}_{Y'}(u^*E, u^*H).$$

If u is an open immersion the same holds (more or less trivially) for any $E, H \in \mathbf{D}(Y)$.

We'll also need the following relation among the maps ρ , ν and θ .

For any commutative diagram of scheme-maps

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ g \downarrow & \sigma & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

and $E, H \in \mathbf{D}(X)$, the following diagram commutes:

$$\begin{array}{ccccc} u^* f_* \mathcal{H}_X(E, H) & \xrightarrow{\nu} & & u^* \mathcal{H}_Y(f_* E, f_* H) & \\ \theta_\sigma \downarrow & & & \downarrow \rho & \\ g_* v^* \mathcal{H}_X(E, H) & & & \mathcal{H}_{Y'}(u^* f_* E, u^* f_* H) & \\ \rho \downarrow & & & \downarrow \theta_\sigma & \\ g_* \mathcal{H}_{X'}(v^* E, v^* H) & \xrightarrow{\nu} & \mathcal{H}_{Y'}(g_* v^* E, g_* v^* H) & \xrightarrow{\theta_\sigma} & \mathcal{H}_{Y'}(u^* f_* E, g_* v^* H) \end{array}$$

The proof of commutativity is formal: one finds a decomposition of the diagram into ones which are sufficiently small that their commutativity is given by axioms, or by other previously established commutativities.

$$\begin{array}{ccccccc}
 f_* \mathcal{H}_X(E, H) & \longrightarrow & f_* \mathcal{H}_X(f^* f_* E, H) & = & f_* \mathcal{H}_X(f^* f_* E, H) & \xrightarrow{\alpha} & \mathcal{H}_Y(f_* E, f_* H) & = & \mathcal{H}_Y(f_* E, f_* H) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 f_* v_* v^* \mathcal{H}_X(E, H) & \longrightarrow & f_* v_* v^* \mathcal{H}_X(f^* f_* E, H) & \xrightarrow{\rho} & f_* v_* \mathcal{H}_{X'}(v^* f^* f_* E, v^* H) & & \mathcal{H}_Y(f_* E, f_* v_* v^* H) & & \mathcal{H}_Y(f_* E, f_* v_* v^* H) \\
 \cong \downarrow & & \cong \downarrow & & \downarrow \cong & \uparrow \alpha & \downarrow \cong & & \downarrow \cong \\
 u_* g_* v^* \mathcal{H}_X(E, H) & \longrightarrow & u_* g_* v^* \mathcal{H}_X(f^* f_* E, H) & \xrightarrow{\rho} & u_* g_* \mathcal{H}_{X'}(v^* f^* f_* E, v^* H) & \xrightarrow{\alpha} & \mathcal{H}_Y(f_* E, u_* g_* v^* H) & \xleftarrow{(1, \theta)} & \mathcal{H}_Y(f_* E, u_* u^* f_* H) \\
 \rho \downarrow & & \downarrow & & \parallel & & \downarrow \alpha^{-1} \cong & & \downarrow \cong \\
 u_* g_* \mathcal{H}_{X'}(v^* E, v^* H) & \longrightarrow & u_* g_* \mathcal{H}_{X'}(v^* f^* f_* E, v^* H) & & u_* g_* \mathcal{H}_{X'}(v^* f^* f_* E, v^* H) & & \downarrow \alpha^{-1} \cong & & \downarrow \cong \\
 \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \alpha^{-1} \cong & & \downarrow \cong \\
 u_* g_* \mathcal{H}_{X'}(g^* g_* v^* E, v^* H) & \xrightarrow{(\theta, 1)} & u_* g_* \mathcal{H}_{X'}(g^* u^* f_* E, v^* H) & \longrightarrow & u_* \mathcal{H}_Y(u^* f_* E, g_* v^* H) & \xleftarrow{(1, \theta)} & u_* \mathcal{H}_Y(u^* f_* E, u^* f_* H) & & u_* \mathcal{H}_Y(u^* f_* E, u^* f_* H) \\
 \parallel & & \downarrow \cong & & \uparrow (\theta, 1) & & & & \\
 u_* g_* \mathcal{H}_{X'}(g^* g_* v^* E, v^* H) & \longrightarrow & u_* g_* \mathcal{H}_{X'}(g^* u^* f_* E, v^* H) & \longrightarrow & u_* \mathcal{H}_Y(g_* v^* E, g_* v^* H) & & & &
 \end{array}$$

In the preceding, after replacing E by G and H by $f^!H$, one derives, formally, the commutative diagram

$$\begin{array}{ccc}
 u^* f_* \mathcal{H}_X(G, f^!H) & \xrightarrow{u^*(\delta)} & u^* \mathcal{H}_Y(f_* G, H) \\
 \theta_\sigma \downarrow \simeq & & \downarrow \rho \\
 g_* v^* \mathcal{H}_X(G, f^!H) & & \mathcal{H}_Y(u^* f_* G, u^* H) \\
 \rho \downarrow & & \simeq \downarrow \theta_\sigma \\
 g_* \mathcal{H}_{X'}(v^* G, v^* f^!H) & \xrightarrow{\lambda} & \mathcal{H}_{Y'}(g_* v^* G, u^* H) \\
 \nu \downarrow & & \uparrow \tau \\
 \mathcal{H}_{X'}(g_* v^* G, g_* v^* f^!H) & \xrightarrow[\theta_\sigma^{-1}]{\sim} & \mathcal{H}_{X'}(g_* v^* G, u^* f_* f^!H)
 \end{array}$$

If $G \in \mathbf{D}_c^-(X)$, so that, f being proper, $f_* G \in \mathbf{D}_c^-(Y)$, then by the above proposition, the maps labeled ρ are isomorphisms, as is $u^*(\delta)$ by the assumption SD_c , whence so is $\lambda := \tau \theta_\sigma^{-1} \nu$.

To be proven is that the base change map β_σ is an isomorphism when u (hence v) is either an open immersion or an affine map; in both these cases it is easily seen to suffice that $v_*\beta$ be an isomorphism.

Let $G \in \mathbf{D}_c^-(X)$, so that $v^*G \in \mathbf{D}_c^-(X')$. From the definition of β , one derives, formally, a commutative diagram

$$\begin{array}{ccc}
 f_*\mathcal{H}_X(G, v_*v^*f^!H) & \xrightarrow{f_*\mathcal{H}_X(G, v_*\beta)} & f_*\mathcal{H}_X(G, v_*g^!u^*H) \\
 f_*\tilde{\alpha}^{-1} \downarrow \simeq & & \simeq \downarrow f_*\tilde{\alpha}^{-1} \\
 f_*v_*\mathcal{H}_{X'}(v^*G, v^*f^!H) & \xrightarrow{\text{via } \beta} & f_*v_*\mathcal{H}_{X'}(v^*G, g^!u^*H) \\
 \downarrow \simeq & & \simeq \downarrow \\
 u_*g_*\mathcal{H}_{X'}(v^*G, v^*f^!H) & \xrightarrow[u_*\lambda]{\sim} & u_*\mathcal{H}_{Y'}(g_*v^*G, u^*H)
 \end{array}$$

where the isomorphism $\tilde{\alpha}$ is the sheafified expression of v^*-v_* adjointness (see Lecture 3), and the right column is an isomorphism by SD_c (for g , since $f_*v_* = u_*g_*$). Thus the above map $f_*\mathcal{H}_X(G, v_*\beta)$ is an isomorphism.

The desired implication results then from the following **Key Fact** (proof omitted):

If $f: X \rightarrow Y$ is a finitely presented scheme-map, then a $\mathbf{D}_{\text{qc}}^+(X)$ -map ϕ is an isomorphism iff so is the $\mathbf{D}(Y)$ -map $f_*\mathcal{H}_X(G, \phi)$ for every $G \in \mathbf{D}_c^-(X)$.

The implication $(\text{BC}^\circ + \text{BC}^{\text{af}}) \implies \text{BC}$ results from a simple formal “transitivity” property of θ (hence β) with respect to horizontal composition of fibre squares, a property which, along with BC° , allows a reduction to the case where Y and Y' —and hence u —are affine, so that BC^{af} applies.

As for the implication $\text{BC}^\circ \implies \text{SD}$, when u is an open immersion, the columns of the following—previously derived—diagram are isomorphisms:

$$\begin{array}{ccc}
 u^* f_* \mathcal{H}_X(G, f^! H) & \xrightarrow{u^*(\delta)} & u^* \mathcal{H}_Y(f_* G, H) \\
 \theta_\sigma \downarrow & & \downarrow \rho \\
 g_* v^* \mathcal{H}_X(G, f^! H) & & \mathcal{H}_Y(u^* f_* G, u^* H) \\
 \rho \downarrow & & \downarrow \theta_\sigma \\
 g_* \mathcal{H}_{X'}(v^* G, v^* f^! H) & \xrightarrow{\lambda} & \mathcal{H}_{Y'}(g_* v^* G, u^* H)
 \end{array}$$

Moreover, one checks that λ factors as

$$g_* \mathcal{H}_{X'}(v^* G, v^* f^! H) \xrightarrow{g_* \mathcal{H}_{X'}(v^* G, \beta)} g_* \mathcal{H}_{X'}(v^* G, g^! u^* H) \xrightarrow{\delta} \mathcal{H}_{Y'}(g_* v^* G, u^* H)$$

where β —and hence $g_* \mathcal{H}_{X'}(v^* G, \beta)$ —is, by BC° , an isomorphism.

Then since $\Gamma_{Y'} \delta$ can be identified with a global duality isomorphism, one sees that $\Gamma_{Y'} \lambda$ is an isomorphism.

Thus $\Gamma_{Y'} u^*(\delta)$ is an isomorphism for all open immersions $u: Y' \rightarrow Y$;
and hence (exercise) δ *itself is an isomorphism*, i.e., SD holds. QED.

Using the “Key Fact” one deduces similarly that $\text{SD} \implies \text{BC}^\circ$.