Lectures on Grothendieck Duality

VIII: Some flesh on the skeleton.

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Introduction

The fundamental existence and base-change theorems for the twisted inverse image pseudofunctor provide only the skeleton of a living, breathing theory with many interesting concrete manifestations.

In this lecture we touch on some of the more down-to-earth aspects of Grothendieck Duality.

All schemes are assumed to be noetherian, and all scheme-maps are assumed to be finite-type and separated.

We write $\otimes$ in place of $\underline{\underline{\otimes}}$; and $\mathcal{H}om$ in place of $\mathcal{H}om^\bullet$.

For a scheme-map h, we use the abbreviations

$$h^* := \mathbf{L}h^*, \qquad h_* := \mathbf{R}h_*.$$

And, as should have been noted before:

if $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is an *exact* functor, then its extension takes quasi-isomorphisms in $\mathbf{K}(\mathcal{A}_1)$ to quasi-isomorphisms in $\mathbf{K}(\mathcal{A}_2)$, whence to isomorphisms in $\mathbf{D}(\mathcal{A}_2)$. So the natural map is an *isomorphism* $Q_2\phi \xrightarrow{\ \sim\ } \mathbf{R}\phi Q_1$. In brief: an exact functor is its own derived functor.

Outline

1. Twisted inverse image of derived $\mathcal{H}om$ and $\otimes$.
2. Perfect maps.
3. Pseudo-Cohen-Macaulay maps.
4. Finite maps.
5. Regular immersions; fundamental local isomorphism.
6. Smooth maps
7. The concrete approach to duality.

1. Twisted inverse image of derived $\mathcal{H}om$ and $\otimes$

For a scheme-map $f\colon X \to Y$, and certain $E,\ F \in \mathbf{D}_{\mathrm{qc}}(X)$, we'll describe canonical pseudofunctorial maps

$$\chi(f,E,F)\colon f^*E \otimes_X f^!F \to f^!(E \otimes_Y F),$$
$$\zeta(f,E,F)\colon \mathbf{R}\mathcal{H}om_X(f^*E, f^!F) \to f^!\mathbf{R}\mathcal{H}om_Y(E,F).$$

Recall that the twisted inverse image is defined only on $\mathbf{D}^+_{\mathrm{qc}}$.
So for χ to be meaningful, one needs $F \in \mathbf{D}^+_{\mathrm{qc}}(Y)$ and $E \otimes_X F \in \mathbf{D}^+_{\mathrm{qc}}(Y)$.

Similarly, for ζ one needs $F \in \mathbf{D}^+_{\mathrm{qc}}(Y)$ and $\mathbf{R}\mathcal{H}om_Y(E,F) \in \mathbf{D}^+_{\mathrm{qc}}(Y)$.
(The latter holds whenever $F \in \mathbf{D}^+_{\mathrm{qc}}(Y)$ and $E \in \mathbf{D}^-_{\mathrm{c}}(Y)$.)

Note: Both χ and ζ have the form $\mathbf{T}(f^*E, f^!F) \to f^!\mathbf{T}(E,F)$.

I don't know a good reason for this parallelism between $\underline{\underline{\otimes}}$ and $\mathbf{R}\mathcal{H}om$ to hold.

Transitivity of χ and ζ

Pseudofunctoriality of χ and ζ means that for any $X \xrightarrow{f} Y \xrightarrow{g} Z$, the following natural "transitivity" diagrams should commute:

$$
\begin{tikzcd}
(gf)^*E \otimes (gf)^!F \arrow[rr, "\sim"] \arrow[d, "\chi"'] & & f^*g^*E \otimes f^!g^!F \arrow[d, "\chi"] \\
(gf)^!(E \otimes F) \arrow[r, "\sim"] & f^!g^!(E \otimes F) & f^!(g^*E \otimes g^!F)) \arrow[l, "\chi"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{R}\mathcal{H}om\big((gf)^*E, (gf)^!F\big) \arrow[rr, "\sim"] \arrow[d, "\zeta"'] & & \mathbf{R}\mathcal{H}om\big(f^*g^*E, f^!g^!F\big) \arrow[d, "\zeta"] \\
(gf)^!\mathbf{R}\mathcal{H}om\big(E, F\big) \arrow[r, "\sim"] & f^!g^!\mathbf{R}\mathcal{H}om\big(E, F\big) & f^!\mathbf{R}\mathcal{H}om\big(g^*E, g^!F\big) \arrow[l, "\zeta"]
\end{tikzcd}
$$

Defining χ and ζ

For f an *open immersion* (so $f^! = f^*$), χ and ζ are the obvious isomorphisms

$$f^*E \otimes f^*F \xrightarrow{\sim} f^*(E \otimes F), \quad \mathbf{R}\mathcal{H}om(f^*E, f^*F) \xrightarrow{\sim} f^*\mathbf{R}\mathcal{H}om(f^*E, f^*F)$$

For f *proper,* χ is defined to be adjoint to the natural composition (where the isomorphism comes from *projection*):

$$f_*(f^*E \otimes f^!F) \xrightarrow{\sim} E \otimes f_*f^!F \to E \otimes F;$$

and ζ is defined to be adjoint to the natural composition (where the isomorphism comes from *sheafified f^*-f_* duality*):

$$f_*\mathbf{R}\mathcal{H}om(f^*E, f^!F) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(E, f_*f^!F) \to \mathbf{R}\mathcal{H}om(E, F).$$

Now for an arbitrary f, with compactification $f = \overline{f} \circ u$, the above transitivity diagrams (with u in place of f and $\overline{f}$ in place of g) determine $\chi(f, \dots)$ and $\zeta(f, \dots)$.

Of course one must show—and this is not trivial—that the result is independent of the choice of compactification, and pseudofunctorial.

When is ζ an isomorphism?

Proposition

If $E \in \mathbf{D}_{\mathrm{c}}^{-}(Y)$ *and* $F \in \mathbf{D}_{\mathrm{qc}}^{+}(Y)$ *then* $\zeta(f, E, F)$ *is an isomorphism*

$$\mathbf{R}\mathcal{H}om_X(f^*E, f^!F) \xrightarrow{\sim} f^!\mathbf{R}\mathcal{H}om_Y(E, F).$$

Proof (partial)

It suffices to treat the case where f is proper.

Use "axiomatic" notation, $[A, B] := \mathbf{R}\mathcal{H}om(A, B)$, etc.

$$\{E \in \mathbf{D}_{\mathrm{c}}^{-}(Y),\ F \in \mathbf{D}_{\mathrm{qc}}^{+}(Y)\} \Longrightarrow \{f^*E \in \mathbf{D}_{\mathrm{c}}^{-}(X),\ f^!F \in \mathbf{D}_{\mathrm{qc}}^{+}(X)\} \Longrightarrow [f^*E, f^!F] \in \mathbf{D}_{\mathrm{qc}}(X)$$

$\exists$ natural isomorphisms, for any $G \in \mathbf{D}_{\mathrm{qc}}(X)$:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(X)}(G, [f^*E, f^!F]) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(X)}(G \otimes f^*E, f^!F) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(f_*(G \otimes f^*E), F) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(f_*G \otimes E, F) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(f_*G, [E, F]) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(X)}(G, f^![E, F]).
\end{aligned}$$

There results an isomorphism $[f^*E, f^!F] \xrightarrow{\sim} f^![E, F]$.

Is this isomorphism the same as ζ?

Interlude: coherence in categories

Answering the preceding question is a highly recommended exercise. One has to show that some big diagram commutes.

Doing this will reveal some of the wealth of the axiomatic setup—a setup that was based on Grothendieck's notion of six operations.

And the tedium involved may foster appreciation for one of the intriguing (at least to me) questions arising out of the theory. Namely,

Is there a "coherence" theorem guaranteeing that all diagrams of a certain form, built up from the axioms, must commute?

Or an algorithm for deciding whether or not such diagrams commute?

Or, at least, could one train a computer to become an expert assistant in the task?

When is χ an isomorphism?

The story for χ is more complex.

First, some definitions:

For $f\colon X \to Y$, an $\mathcal{O}_X$-complex E is f-perfect if it has coherent homology and is $\mathbf{D}(f^{-1}\mathcal{O}_Y)$-isomorphic to a bounded flat $f^{-1}\mathcal{O}_Y$-complex. ($f^{-1}\mathcal{O}_Y$ is the sheaf of rings on X whose stalk at $x \in X$ is $\mathcal{O}_{Y,fx}$.)

E is perfect if it is $\mathbf{1}_X$-perfect. This turns out to be equivalent to saying that each $x \in X$ has an open neighborhood U such that the restriction $E|_U$ is $\mathbf{D}(U)$-isomorphic to a bounded complex of finite-rank free $\mathcal{O}_U$-modules.

One shows that:

The map f has finite tor-dimension $\iff$ $\mathcal{O}_X$ is f-perfect.

So (in the noetherian context) maps of finite tor-dimension are often called perfect maps.

The next result says that "χ an isomorphism" characterizes perfect maps.

2. Perfect maps

Recall: schemes are noetherian, and scheme-maps are separated, of finite type.

Theorem

For any scheme-map $f\colon X \to Y$, the following conditions are equivalent.

(i) *The map f is perfect.*

(ii) *The complex $f^!\mathcal{O}_Y$ is f-perfect.*

(iii) $f^!\mathcal{O}_Y \in \mathbf{D}_{\mathrm{c}}^{-}(X)$, *and* $\forall E \in \mathbf{D}_{\mathrm{qc}}^{+}(Y)$, $\chi(f, E, \mathcal{O}_Y)$ *is an* isomorphism

$$f^*E \otimes f^!\mathcal{O}_Y \xrightarrow{\ \sim\ } f^!E.$$

(iii)$'$ *For every perfect $F \in \mathbf{D}(Y)$, $f^!F$ is f-perfect; and $\forall E, F \in \mathbf{D}(Y)$ such that F and $E \otimes F$ are in $\mathbf{D}_{\mathrm{qc}}^{+}(Y)$, $\chi(f, E, F)$ is an* isomorphism

$$f^*E \otimes f^!F \xrightarrow{\ \sim\ } f^!(E \otimes F).$$

(iv) *The functor $f^!\colon \mathbf{D}_{\mathrm{qc}}^{+}(Y) \to \mathbf{D}_{\mathrm{qc}}^{+}(X)$ is bounded.*

A proof is in the reference notes. (Theorem 4.9.4).

Perfect maps and $f^!\mathcal{O}$

Using transitivity, one sees that on the category of perfect maps, $\chi(f, E, \mathcal{O})$ gives a pseudofunctorial isomorphism

$$f^{\#}E := f^*E \otimes f^!\mathcal{O}_Y \xrightarrow{\sim} f^!E \qquad \big(E \in \mathbf{D}^+_{\mathrm{qc}}(Y)\big),$$

where, for a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of perfect maps, the canonical isomorphism $f^{\#}g^{\#}F \xrightarrow{\sim} (gf)^{\#}F$ $(F \in \mathbf{D}^+_{\mathrm{qc}}Z)$ is the natural composition

$$\begin{aligned} f^*(g^*F \otimes g^!\mathcal{O}_Z) \otimes f^!\mathcal{O}_Y &\xrightarrow{\sim} f^*g^*F \otimes (f^*g^!\mathcal{O}_Z \otimes f^!\mathcal{O}_Y) \\ &\xrightarrow[1\otimes\chi]{\sim} f^*g^*F \otimes f^!g^!\mathcal{O}_Z \xrightarrow{\sim} (gf)^*F \otimes (gf)^!\mathcal{O}_Z \end{aligned}$$

So the pseudofunctor $f^{\#}$ is determined by the family of f-perfect complexes $f^!\mathcal{O}_Y$ (f a perfect map) plus a family of isomorphisms $f^*g^!\mathcal{O}_Z \otimes f^!\mathcal{O}_Y \xrightarrow{\sim} (gf)^!\mathcal{O}_Z$ (that satisfies, with respect to triple compositions, a condition left to the reader).

The point is that for perfect maps the theory of $f^!$ is determined by f^* and the behavior of complexes of the form $f^!\mathcal{O}$.

For example, for any independent square

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

the identification, via χ, of $f^!$ and $f^\#$ transforms the base-change map β_σ to

$$v^*f^\#E = v^*f^*E \otimes v^*f^!\mathcal{O}_Y \to g^*u^*E \otimes g^!u^*\mathcal{O}_Y = g^*u^*E \otimes g^!\mathcal{O}_{Y'} = g^\#u^*E,$$

a map which is determined by the base-change map $v^*f^!\mathcal{O}_Y \to g^!u^*\mathcal{O}_Y$.

3. Pseudo-Cohen-Macaulay maps

With all this in mind we will now examine some specific kinds of perfect maps (Cohen-Macaulay, Gorenstein, regular immersion, smooth).

Suppose for simplicity that schemes are *connected.*

For a scheme-map $f\colon X \to Y$, the lowest-degree nonvanishing cohomology of $f^!\mathcal{O}_Y$ is called the canonical sheaf of f.
This coherent $\mathcal{O}_X$-module is denoted ω_f.

As an instance of a general fact about derived categories, one has:

If $\omega_f = H^d(f^!\mathcal{O}_Y)$ is the only nonvanishing cohomology of $f^!\mathcal{O}_Y$ then there is a natural $\mathbf{D}(X)$-isomorphism $f^!\mathcal{O}_Y \xrightarrow{\sim} \omega_f[-d]$.

As before, for perfect f the theory of the pseudofunctor $f^!$ reduces to the theory of $f^!\mathcal{O}_Y$. This leads one to focus on the following class of maps.

Definition

A scheme-map $f\colon X \to Y$ is pseudo-Cohen-Macaulay if f is perfect and for some d, $f^!\mathcal{O}_Y$ is $\mathbf{D}(X)$-isomorphic to $\omega_f[-d]$.

Example (Cohen-Macaulay maps)

f is pseudo-Cohen-Macaulay and $\mathcal{O}_X$ and ω_f are both $(f^{-1}\mathcal{O}_Y)$-flat $\iff$ f is flat and the fibers of f are Cohen-Macaulay varieties.

Such maps are called Cohen-Macaulay.

Example (Gorenstein maps)

f is called Gorenstein if f is pseudo-Cohen-Macaulay and ω_f is an *invertible* $\mathcal{O}_X$-module.

A flat map f is Gorenstein $\iff$ the fibers of f are Gorenstein varieties.

So flat Gorenstein maps are Cohen-Macaulay.

Example (Smooth maps)

Smooth maps, being flat, with smooth fibers, are Gorenstein.

For a smooth map f, with d-dimensional fibers, $\omega_f \cong \Omega_f^d$, the d-th exterior power of the sheaf of relative Kähler differentials (see below).

4. Finite maps

Let X be a scheme, and $\mathcal{A}_{\mathrm{qc}}(X)$ the abelian category of quasi-coherent $\mathcal{O}_X$-modules. There is a natural functor

$$\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X)) \to \mathbf{D}_{\mathrm{qc}}(X),$$

which is in fact an equivalence of categories.

For a scheme-map $f\colon X \to Y$, therefore, one can think of $f^!$ as being a right adjoint of $\mathbf{R}f_*\colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X)) \to \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(Y))$.

$f^!$ for finite maps

Any ringed-space map $f\colon (X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ factors naturally as

$$(X,\mathcal{O}_X)\xrightarrow{\bar f}\overline{Y}:=(Y,f_*\mathcal{O}_X)\xrightarrow{\psi}(Y,\mathcal{O}_Y).$$

If f is a finite scheme-map, then $\bar f_*$ is an exact functor that induces an equivalence of categories from $\mathcal{A}_{\mathrm{qc}}(X)$ to the category $\mathcal{A}_{\mathrm{qc}}(\overline{Y})$ of quasi-coherent $f_*\mathcal{O}_X$-modules. Hence, $\bar f$ being flat, $\bar f^* = \mathbf{L}\bar f^*$ is right-adjoint to $\mathbf{R}\bar f_* = \bar f_*\colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))\to\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(\overline{Y}))$.

So if Ψ is a right adjoint of $\mathbf{R}\psi_*\colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(\overline{Y}))\to\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(Y))$ then $\bar f^*\circ\Psi$ is a right adjoint of $\mathbf{R}f_* = \mathbf{R}\psi_*\mathbf{R}\bar f_*\colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))\to\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(Y))$.

Such a $\Psi =: \mathbf{R}\mathcal{H}om_\psi(f_*\mathcal{O}_X,-)$ is gotten by right-deriving the functor $\mathcal{H}om_\psi(f_*\mathcal{O}_X,-)$ that takes a quasi-coherent $\mathcal{O}_Y$-module F to the quasi-coherent $\mathcal{O}_{f_*\mathcal{O}_X}$-module $\mathcal{H}om_Y(f_*\mathcal{O}_X,F)$.

This assertion is local, where it just means (as in Lecture 2) that for a ring-homomorphism $\varphi\colon R\to S$, the derived restriction-of-scalars functor φ_* has the right adjoint $\mathbf{R}\mathrm{Hom}_\varphi(S,-)$.

5. Regular immersions; fundamental local isomorphism

Preliminaries: Let Y be a scheme and $\mathcal{I}$ a quasi-coherent $\mathcal{O}_Y$-ideal. Any exact $\mathcal{O}_Y$-sequence $P \to \mathcal{O}_Y \to \mathcal{O}_Y/\mathcal{I} \to 0$ gives rise to an isomorphism $P/\mathcal{I}P \xrightarrow{\sim} \mathcal{I}/\mathcal{I}^2$. Taking P to be flat, one gets a natural isomorphism

$$\psi\colon \mathcal{T}or_1^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{I}) \xrightarrow{\sim} \mathcal{I}/\mathcal{I}^2.$$

Tensor product of resolutions makes $\oplus_{i\ge 0}\mathcal{T}or_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{I})$ into an alternating graded $\mathcal{O}_Y$-algebra; so by the universal property of exterior algebras, one deduces from ψ^{-1} a canonical graded-algebra homomorphism

$$\oplus_{i\ge 0}\bigwedge^i(\mathcal{I}/\mathcal{I}^2) \longrightarrow \oplus_{i\ge 0}\mathcal{T}or_i^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{I}).$$

Locally, for a quasi-coherent $\mathcal{O}_Y$-projective resolution $P_\bullet \to \mathcal{O}_Y/\mathcal{I}$, $\exists$ natural maps

$$\begin{aligned}
\mathcal{E}xt^i(\iota_*\mathcal{O}_X, \mathcal{O}_Y) &\xrightarrow{\sim} H^i(\mathcal{H}om(P_\bullet, \mathcal{O}_Y)) \\
&\xrightarrow{\sim} H^i(\mathcal{H}om(P_\bullet \otimes \mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{I})) \\
&\longrightarrow \mathcal{H}om(H^i(P_\bullet \otimes \mathcal{O}_Y/\mathcal{I}), \mathcal{O}_Y/\mathcal{I}) \\
&\xrightarrow{\sim} \mathcal{H}om(\mathcal{T}or_i^{\mathcal{O}_Y}(\mathcal{O}_Y/P, \mathcal{O}_Y/P), \mathcal{O}_Y/\mathcal{I}) \\
&\longrightarrow \mathcal{H}om(\bigwedge^i(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}) \qquad (i \ge 0).
\end{aligned}$$

Fundamental homomorphisms

This composed map is independent of the choice of $P_\bullet$. Hence one can paste to get natural *global* maps, the fundamental homomorphisms

$$\lambda^i\colon \mathcal{E}xt^i_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}\big(\textstyle\bigwedge^i(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}\big) \qquad (i \geq 0).$$

Regular immersions

A regular immersion $\iota\colon X \hookrightarrow Y$, of codimension n, is an isomorphism of X onto a closed subscheme of Y defined by an ideal $\mathcal{I}$ that is locally generated by a *regular sequence of length n*. Note that then $\iota_*\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$, and $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_Y/\mathcal{I}$-module of rank n.

Using (locally) the Koszul complex (concentrated in degrees from $-n$ to 0) on such a generating sequence as a projective $\mathcal{O}_Y$-resolution of $\mathcal{O}_Y/\mathcal{I}$, one sees that ι is a perfect map, with

$$\iota^!\mathcal{O}_Y = \iota^*\mathbf{R}\mathcal{H}om_Y(\iota_*\mathcal{O}_X, \mathcal{O}_Y) \cong \iota^*\mathcal{E}xt^n_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{O}_Y)[-n] =: \omega_\iota[-n].$$

Moreover, the above fundamental homomorphism λ^n is an *isomorphism*

$$\mathcal{E}xt^n_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{O}_Y) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}\big(\textstyle\bigwedge^n(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}\big).$$

This is often called the fundamental local isomorphism.

In conclusion:

Any regular immersion $f\colon X \hookrightarrow Y$ of codimension n is a Gorenstein map, with invertible canonical sheaf

$$\omega_f \cong f^*\mathcal{H}om_{\mathcal{O}_Y}\big(\textstyle\bigwedge^n(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}\big)$$

where $\mathcal{I}$ is the kernel of the natural surjection $\mathcal{O}_Y \twoheadrightarrow f_*\mathcal{O}_X$.

6. Smooth maps

Theorem

Let $f: X \to Y$ be smooth, of relative dimension n, i.e., flat, with fibers that are n-dimensional nonsingular varieties.
Then f is a Gorenstein map, with

$$f^!\mathcal{O}_Y \cong \omega_f[n] \cong \Omega_f^n[n],$$

the invertible sheaf Ω_f^n being the n-th exterior power of the sheaf Ω_f of relative Kähler differentials.

Proof (Verdier)

Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\delta"] & X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

with π_1, π_2 the projections, and δ the diagonal (so that $\pi_i\delta = \mathbf{1}_X$).
The map δ is a regular immersion of codimension n; and if $\mathcal{I}$ is the kernel of $\mathcal{O}_{X\times_Y X} \twoheadrightarrow \delta_*\mathcal{O}_X$ then $\Omega_f = \delta^*\big(\mathcal{I}/\mathcal{I}^2\big)$. Flat base change, and the preceding results on regular immersions, yield isomorphisms

$$\begin{aligned}\mathcal{O}_X = (\pi_2\delta)^!\mathcal{O}_X \cong \delta^!\pi_2^! f^*\mathcal{O}_Y \cong \delta^!\pi_1^* f^!\mathcal{O}_Y &\cong \delta^!\mathcal{O}_{X\times_Y X} \otimes \delta^*\pi_1^* f^!\mathcal{O}_Y \\ &\cong \delta^*\mathcal{H}om_{\mathcal{O}_Y}\big(\textstyle\bigwedge^n\big(\mathcal{I}/\mathcal{I}^2\big), \mathcal{O}_Y/\mathcal{I}\big)[-n] \otimes \delta^*\pi_1^* f^!\mathcal{O}_Y \\ &\cong \delta^*\mathcal{H}om_{\mathcal{O}_X}\big(\Omega_f^n, \mathcal{O}_X\big)[-n] \otimes f^!\mathcal{O}_Y.\end{aligned}$$

Tensoring throughout with $\Omega_f^n[n]$ gives the desired conclusion.

Concrete pseudofunctoriality

For smooth maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, of respective relative dimensions n and m, there are canonical $\mathbf{D}(X)$-isomorphisms

$$f^!\mathcal{O}_Y \otimes f^*g^!\mathcal{O}_Z \xrightarrow{\sim} f^!g^!\mathcal{O}_Z \xrightarrow{\sim} (gf)^!\mathcal{O}_Z.$$

Via the above isomorphisms $f^!\mathcal{O}_Y \xrightarrow{\sim} \Omega_f^n[n]$ and $g^!\mathcal{O}_Z \xrightarrow{\sim} \Omega_g^m[m]$, one deduces an isomorphism

$$\Omega_f^n \otimes f^*\Omega_g^m \xrightarrow{\sim} \Omega_{gf}^{n+m}.$$

There is also a concrete isomorphism with the same source and target, that looks locally—for suitable x_i and y_j—like

$$dx_1 dx_2 \cdots dx_n \otimes dy_1 dy_2 \cdots dy_m \mapsto dx_1 dx_2 \cdots dx_n dy_1 dy_2 \cdots dy_m.$$

Are these two isomorphisms the same?

Yes, but why?

6. The concrete approach to duality

A proof based on a different approach to duality theory is sketched in Hartshorne's classic **Residues and Duality**, starting on p. 388. Filling in all the details could prove to be a formidable task.

His approach is "bottom up" (concrete $\longrightarrow$ abstract) rather than "top down" (abstract $\longrightarrow$ concrete). This is Grothendieck's initial way, worked out in Residues and Duality, with many clarifications and corrections in Conrad's **Grothendieck Duality and Base Change** (SLN 1750).

One starts with the above concrete realizations of $f^!$ for smooth and for finite maps, proves the appropriate duality theorems for them, then pastes together locally using factorizations of arbitrary maps as smooth $\circ$ finite; and finally pastes together locally defined duality data—not directly, which isn't always possible in derived categories, but via the theory of dualizing complexes, a theory we have not had time to discuss.

Carrying this program out requires the verification of a very large number of nontrivial compatibilities among diverse maps. Working through it all reveals many riches hidden behind the unifying abstract facade.

Conclusion

As indicated by the discussion of “concrete pseudofunctoriality” for smooth maps, passing between the concrete and abstract theories, in either direction, is challenging—and rewarding. Negotiating that passage adds much to the fascination of duality theory.