Lectures on Grothendieck Duality

II: Derived Hom-Tensor adjointness. Local duality.

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1 Left-derived functors. Tensor and Tor.

1. Derived functors.

$Q_{\mathcal{A}}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ denotes the canonical functor from the homotopy category of an abelian category $\mathcal{A}$ to its derived category.
Let $\mathcal{A}_1$, $\mathcal{A}_2$ be abelian categories, and set $Q_i := Q_{\mathcal{A}_i}$.
Let $\gamma\colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{K}(\mathcal{A}_2)$ be a Δ-functor.

A right-derived functor $(\mathbf{R}\gamma, \zeta)$ of γ consists of a Δ-functor $\mathbf{R}\gamma\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ and a Δ-functorial map $\zeta\colon Q_2\gamma \to \mathbf{R}\gamma Q_1$ such that:
every Δ-functorial map $Q_2\gamma \to \Gamma$ where $\Gamma\colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ takes quasi-isomorphisms to isomorphisms, factors uniquely as

$$Q_2\gamma \xrightarrow{\zeta} \mathbf{R}\gamma Q_1 \to \Gamma.$$

In other terms, omitting Qs, in the category whose objects are functorial $\mathbf{D}(\mathcal{A}_2)$-maps of the form

$$\gamma(E) \to \Gamma(E) \qquad \big(E \in \mathbf{K}(\mathcal{A}_1)\big),$$

with fixed source $Q_2\gamma$, and target Γ as above, the map $\zeta(E)\colon \gamma(E) \to \mathbf{R}\Gamma(E)$ is an *initial object*—and thus it is *unique up to canonical isomorphism.*
Dually: A left-derived functor $(\mathbf{R}\gamma, \xi)$ of γ consists of a Δ-functor $\mathbf{L}\gamma\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ and a Δ-functorial map $\xi\colon \mathbf{L}\gamma Q_1 \to Q_2\gamma$ such that:
every Δ-functorial map $\Gamma \to Q_2\gamma$ where $\Gamma\colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ takes quasi-isomorphisms to isomorphisms, factors uniquely as

$$\Gamma \to \mathbf{L}\gamma Q_1 \xrightarrow{\xi} Q_2\gamma.$$

Here ξ is a *final object* in the appropriate category of functorial maps.

Tensor product

We've already seen some right-derived functors, $\mathbf{R}\Gamma_I(-)$ and $\mathbf{R}\mathrm{Hom}(-,-)$.
Describe next an important example of a left-derived functor.
The tensor product $C \otimes_R D$ of two R-complexes is such that

$$(C \otimes_R D)^n = \oplus_{i+j=n} C^i \otimes_R D^j,$$

the differential $\delta^n \colon (C \otimes_R D)^n \to (C \otimes_R D)^{n+1}$ being determined by

$$\delta^n(x \otimes y) = d_C^i x \otimes y + (-1)^i x \otimes d_D^j y \qquad (x \in C^i,\ y \in D^j).$$

Fixing D, we get a functor $\gamma_D := - \otimes_R D \colon \mathbf{K}(R) \to \mathbf{K}(R)$, that, together with $\theta :=$ the identity map of $C[1] \otimes_R D = (C \otimes_R D)[1]$, is a Δ-functor.

There is an isomorphism $\rho\colon \gamma'_C(D) := C \otimes_R D \xrightarrow{\ \sim\ } D \otimes_R C = \gamma_C D$ taking $x \otimes y$ to $(-1)^{ij} y \otimes x$.

There is a then a unique Δ-functor (γ'_C, θ') such that ρ is Δ-functorial.

The map $\theta' \colon \gamma'_C(D[1]) \xrightarrow{\ \sim\ } \gamma'_C(D)[1]$ is *not the identity:* its restriction to $C^i \otimes_R D^j$ is multiplication by $(-1)^i$.

q-flat resolutions

One gets a left-derived functor $- \underline{\underline{\otimes}}_R D$ of γ_D as follows:

An R-complex F is q-flat if for every exact R-complex E (i.e., $\mathrm{H}^i E = 0$ for all i), $F \otimes_R E$ is exact too.

Equivalently: *the functor $F \otimes_R -$ preserves quasi-isomorphism.*
(By the exactness of the homology sequence of a triangle, a map of complexes is a quasi-isomorphism iff its cone is exact, and tensoring with F "commutes" with forming cones.)

For example, any bounded-above (i.e., cohomology vanishing above some degree) flat complex is q-flat.

Every R-complex C has a q-flat resolution, i.e., a q-flat complex F plus a quasi-isomorphism $F \to C$.
This can be constructed as a $\varinjlim$ of bounded-above flat resolutions of truncations of C.

For example, a flat resolution of an R-module M

$$\cdots \to F^{-2} \to F^{-1} \to F^0 \to M \to 0$$

can be viewed as a q-flat resolution of M (as a complex).

Left-derived tensor product

After choosing for each C a q-flat resolution $F_C \to C$, one shows that there exists a left-derived functor $- \underline{\underline{\otimes}}_R D$ of γ_D with

$$C \underline{\underline{\otimes}}_R D = F_C \otimes_R D$$

If $F_D \to D$ is a q-flat resolution, there are natural $\mathbf{D}(R)$-isomorphisms

$$C \otimes_R F_D \xleftarrow{\ \sim\ } F_C \otimes_R F_D \xrightarrow{\ \sim\ } F_C \otimes_R D,$$

so any of these complexes could be used to define $C \underline{\underline{\otimes}}_R D$. Using $F_C \otimes_R F_D$ one can, as before, make $C \underline{\underline{\otimes}}_R D$ into a Δ-functor of both variables C and D.

As such, it has a universal mapping property as above, but with respect to two-variable functors.

Taking homology produces the (hyper)tor functors $\boxed{\mathrm{Tor}_i(C,D) = \mathrm{H}^{-i}(C \underline{\underline{\otimes}}_R D).}$

2 Hom-Tensor adjunction.

Relations between Ext and Tor—in particular, as we'll see, Local Duality—are neatly encapsulated by a derived-category upgrade of the basic adjoint associativity relation between Hom and $\otimes$.

For R-modules E, F, G, adjoint associativity is the isomorphism

$$\mathrm{Hom}_R(E \otimes_R F,\, G) \xrightarrow{\ \sim\ } \mathrm{Hom}_R\big(E, \mathrm{Hom}_R(F, G)\big)$$

that takes $\phi\colon E \otimes_R F \to G$ to $\phi'\colon E \to \mathrm{Hom}_R(F, G)$ where

$$[\phi'(e)](f) = \phi(e \otimes f) \qquad (e \in E,\ f \in F).$$

More generally, with $\varphi\colon R \to S$ a homomorphism of commutative rings, E, F, S-complexes and G an R-complex, there is an isomorphism of S-complexes

$$\mathrm{Hom}^\bullet_R(E \otimes_S F,\, G) \xrightarrow{\ \sim\ } \mathrm{Hom}^\bullet_S\big(E, \mathrm{Hom}^\bullet_R(F, G)\big) \tag{adj}$$

that in degree n takes a family $\big(\phi_{ij}\colon E^i \otimes_S F^j \to G^{i+j+n}\big)$ to the family $\big(\phi'_i\colon E^i \to \mathrm{Hom}^{i+n}_R(F, G)\big)$ with $\phi'_i(e) = \big(\phi'_{ij}(e)\colon F_j \to G^{i+j+n}\big)$ where

$$[\phi'_{ij}(e)](f) = \phi_{ij}(e \otimes f) \qquad (e \in E^i,\ f \in F^j).$$

Derived adjoint associativity

With $\varphi\colon R \to S$ as before, let $\varphi_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ denote the obvious restriction of scalars functor.
For a fixed S-complex E, the functor $\mathrm{Hom}^\bullet_R(E, G)$ from R-complexes G to S-complexes has a right-derived functor from $\mathbf{D}(R)$ to $\mathbf{D}(S)$ (gotten via q-injective resolution of G), denoted $\mathbf{R}\mathrm{Hom}^\bullet_\varphi(E, G)$.
If we replace G in (adj) by a q-injective resolution, and F by a q-flat one, then the S-complex $\mathrm{Hom}^\bullet_R(F, G)$ is easily seen to become q-injective; and consequently (adj) gives a $\mathbf{D}(S)$-isomorphism

$$\alpha(E, F, G)\colon \mathbf{R}\mathrm{Hom}^\bullet_\varphi(E \mathbin{\underline{\underline{\otimes}}}_S F,\, G) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_S\big(E, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(F, G)\big)$$

The map α is Δ-functorial. Showing this requires some additional grinding.

Derived Hom-Tensor adjunction

α does not depend on the choices of resolutions made above: it's canonically characterized by commutativity, for all E, F and G, of the following otherwise natural $\mathbf{D}(S)$-diagram (where $\mathrm{H}^\bullet$ stands for $\mathrm{Hom}^\bullet$):

$$\begin{array}{ccccc}
\mathrm{H}^\bullet_R(E \otimes F, G) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_R(\varphi_*(E \otimes F), G) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_R(\varphi_*(E \mathbin{\underline{\underline{\otimes}}} F), G) \\
{\scriptstyle (\mathrm{adj})}\big\downarrow \simeq & & & & \simeq \big\downarrow {\scriptstyle \alpha} \\
\mathrm{H}^\bullet_S\big(E, \mathrm{H}^\bullet_R(F, G)\big) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_S\big(E, \mathrm{H}^\bullet_R(F, G)\big) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_S\big(E, \mathbf{R}\mathrm{H}^\bullet_R(\varphi_* F, G)\big)
\end{array}$$

Application of the functor H^0 to α yields a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}\big(\varphi_*(E \mathbin{\underline{\underline{\otimes}}}_S F),\, G\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}\big(E, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(F, G)\big),$$

Thus, for fixed $F \in \mathbf{D}(S)$, there is a natural adjunction between the functors

$$\varphi_*(- \mathbin{\underline{\underline{\otimes}}}_S F)\colon \mathbf{D}(S) \to \mathbf{D}(R) \quad \text{and} \quad \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* F, -)\colon \mathbf{D}(R) \to \mathbf{D}(S).$$

3 Abstract local duality.

Recall briefly the connection between $\mathbf{R}\Gamma_I$ and Koszul complexes.
R is a commutative *noetherian* ring; $\otimes := \otimes_R$.
$\mathbf{t} = (t_1, \dots, t_m)$ is a sequence in R, generating the ideal $I := \mathbf{t}R$.

For $t \in R$, let $\mathcal{K}(t)$ be the complex that in degrees 0 and 1 is the usual map from R to the localization R_t, and that vanishes elsewhere.
For any R-complex C, define the "stable" Koszul complex

$$\mathcal{K}(\mathbf{t}) := \mathcal{K}(t_1) \otimes \cdots \otimes \mathcal{K}(t_m), \qquad \mathcal{K}(\mathbf{t}, C) := \mathcal{K}(\mathbf{t}) \otimes C.$$

Since the complex $\mathcal{K}(\mathbf{t})$ is flat and bounded, hence q-flat, therefore $\mathcal{K}(\mathbf{t}, -)$ takes quasi-isomorphisms to quasi-isomorphisms, and so may—and will—be regarded as a functor from $\mathbf{D}(R)$ to $\mathbf{D}(R)$.
Given a q-injective resolution $C \to E_C$ we have for $E = E_C^j$ $(j \in \mathbb{Z})$,

$$\Gamma_I E = \ker\big(\mathcal{K}^0(\mathbf{t}, E) = E \to \oplus_{i=1}^m E_{t_i} = \mathcal{K}^1(\mathbf{t}, E)\big),$$

whence a $\mathbf{D}(R)$-map

$$\delta(C)\colon \mathbf{R}\Gamma_I C = \Gamma_I E_C \hookrightarrow \mathcal{K}(\mathbf{t}, E_C) \cong \mathcal{K}(\mathbf{t}, C).$$

The following proposition is a key to many properties of Γ_I.
(Details in §3 of "Lectures on Local Cohomology…")

Proposition 1. *The* $\mathbf{D}(R)$*-map* $\delta(C)$ *is a functorial* isomorphism

$$\mathbf{R}\Gamma_I C \xrightarrow{\sim} \mathcal{K}(\mathbf{t}, C).$$

Since $\mathcal{K}(\mathbf{t}, C) = \mathcal{K}(\mathbf{t}, R) \otimes C$ and $\mathcal{K}(\mathbf{t}, R) \cong \mathbf{R}\Gamma_I R$ is q-flat, therefore:

Corollary 2. *There is a functorial* $\mathbf{D}(R)$ *isomorphism*

$$\mathbf{R}\Gamma_I C \xrightarrow{\sim} (\mathbf{R}\Gamma_I R) \underline{\otimes} C.$$

Taking homology, one gets $\boxed{\mathrm{H}_I^i(C) = \mathrm{H}^i\mathbf{R}\Gamma_I C \cong \mathrm{Tor}_{-i}(\mathbf{R}\Gamma_I R, C) \qquad (i \in \mathbb{Z}).}$

Local duality

Let J be an S-ideal. Let $\varphi_J^\#\colon \mathbf{D}(R) \to \mathbf{D}(S)$ be the functor

$$\begin{aligned}\varphi_J^\#(G) &:= \mathbf{R}\mathrm{Hom}_\varphi^\bullet(\mathbf{R}\Gamma_J S, G)\\ &\cong \mathbf{R}\mathrm{Hom}_S^\bullet\big(\mathbf{R}\Gamma_J S, \mathbf{R}\mathrm{Hom}_\varphi^\bullet(S, G)\big) \qquad \big(G \in \mathbf{D}(R)\big),\end{aligned}$$

The isomorphism results from setting $E = \mathbf{R}\Gamma_J S$ and $F = S$ in the derived adjoint associativity isomorphism

$$\alpha(E, F, G)\colon \mathbf{R}\mathrm{Hom}_\varphi^\bullet(E \underline{\otimes}_S F, G) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_S^\bullet\big(E, \mathbf{R}\mathrm{Hom}_\varphi^\bullet(F, G)\big).$$

For $E \in \mathbf{D}(S)$ and $G \in \mathbf{D}(R)$, one has then functorial $\mathbf{D}(S)$-isomorphisms

$$\mathbf{R}\mathrm{Hom}_\varphi^\bullet(\mathbf{R}\Gamma_J E, G) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\varphi^\bullet(E \underline{\otimes}_S \mathbf{R}\Gamma_J S, G) \underset{\alpha}{\xrightarrow{\sim}} \mathbf{R}\mathrm{Hom}_S^\bullet\big(E, \varphi_J^\# G\big).$$

Application of the functor $\mathrm{H}^0\varphi_*$ produces the local duality isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma_J E, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S)}(E, \varphi_J^\# G),$$

an adjunction between the functors $\varphi_*\mathbf{R}\Gamma_J$ and $\varphi_J^\#$.

4 Concrete local duality.

Henceforth, all rings are noetherian as well as commutative.

Concrete versions of local duality convey more information about $\varphi_J^\#$.

Suppose, e.g., that S is module-finite over R, and let $G \in \mathbf{D}_{\mathsf{c}}(R)$, i.e., each homology module of $G \in \mathbf{D}(R)$ is finitely generated.

Suppose also that $\mathrm{Ext}^i_R(S, G)$ is a finitely-generated R-module for all $i \in \mathbb{Z}$, i.e., $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* S, G) \in \mathbf{D}_{\mathsf{c}}(R)$. (This holds, e.g., if $\mathrm{H}^i G = 0$ for all $i \ll 0$.)

Then $\mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G) \in \mathbf{D}_{\mathsf{c}}(S)$, since, as is easily seen,

$$\varphi_* \mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G) \cong \mathbf{R}\mathrm{Hom}_R(\varphi_* S, G) \in \mathbf{D}_{\mathsf{c}}(R).$$

Now Greenlees-May duality (= Grothendieck duality for the natural map $\mathrm{Spec}(\hat{S}) \to \mathrm{Spec}(S)$, with $\hat{S}$ the J-adic completion of S), gives

$$\mathbf{R}\mathrm{Hom}^\bullet_S\big(\mathbf{R}\Gamma_J S, F\big) \cong F \otimes_S \hat{S} \qquad \big(F \in \mathbf{D}_{\mathsf{c}}(S)\big).$$

In particular:

$$\varphi_J^\# G = \mathbf{R}\mathrm{Hom}^\bullet_S\big(\mathbf{R}\Gamma_J S, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G)\big) \cong \mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G) \otimes_S \hat{S}.$$

In particular, for $S = R$ and $\varphi = \mathrm{id}$ (the identity map) one gets

$$\mathrm{id}_J^\# G = G \otimes_R \hat{R} \qquad \big(G \in \mathbf{D}_{\mathsf{c}}(R)\big).$$

Specialize further to where R is local, $\varphi = \mathrm{id}$, $J = \mathfrak{m}$, the maximal ideal of R, and $G \in \mathbf{D}_{\mathsf{c}}(R)$ is a normalized dualizing complex, so that in $\mathbf{D}(R)$, $\mathcal{I} := \mathbf{R}\Gamma_{\mathfrak{m}} G$ is an R-injective hull of $R/\mathfrak{m}$.
Then there is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, \mathcal{I}) = \mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, \mathbf{R}\Gamma_{\mathfrak{m}} G) \cong \mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, G)$$

Substitution into the local duality isomorphism gives, for all $E \in \mathbf{D}(R)$,

$$\boxed{\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_{\mathfrak{m}} E, \mathcal{I}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_R(E, \mathrm{id}_J^\# G) = \mathbf{R}\mathrm{Hom}^\bullet_R(E, G \otimes_R \hat{R}).}$$

For $E \in \mathbf{D}_{\mathsf{c}}(R)$ this is just classical local duality, modulo Matlis duality.

More familiar local duality

Applying homology H^{-i} one gets the duality isomorphism

$$\mathrm{Hom}_R(\mathrm{H}^i_{\mathfrak{m}} E, \mathcal{I}) \xrightarrow{\sim} \mathrm{Ext}^{-i}_R(E, G \otimes_R \hat{R}).$$

Suppose R Cohen-Macaulay, i.e., there's an $\mathfrak{m}$-primary ideal generated by an R-regular sequence of length $d := \dim(R)$. Then $\mathrm{H}^i_{\mathfrak{m}} R = 0$ for $i \neq d$.

Since $\hat{R}$ is R-flat, the preceding isomorphism now yields, for $i \neq d$,

$$0 = \mathrm{Ext}^{-i}_R(R, G \otimes_R \hat{R}) = \mathrm{H}^{-i}\mathbf{R}\mathrm{Hom}^\bullet(R, G \otimes_R \hat{R}) = \mathrm{H}^{-i}(G \otimes_R \hat{R}) = (\mathrm{H}^{-i} G) \otimes_R \hat{R}.$$

Hence $H^{-i} G = 0$, so there is a derived-category isomorphism

$$G \cong \omega[d], \text{ where } \omega := \mathrm{H}^{-d} G, \text{ a canonical module of } R.$$

Thus, when R is Cohen-Macaulay local duality takes the familiar form

$$\boxed{\mathrm{Hom}_R(\mathrm{H}^i_{\mathfrak{m}} E, \mathcal{I}) \xrightarrow{\sim} \mathrm{Ext}^{d-i}_R(E, \hat{\omega}).}$$

5 Residues and duality for power series rings.

Another situation in which $\varphi_J^{\#}$ can be described concretely is when φ is the inclusion of R into a power-series ring $S := R[[\mathbf{t}]] := R[[t_1, \dots, t_m]]$, and J is the ideal $\mathbf{t}S = (t_1, \dots, t_m)S$.

There exist an S-module $\hat{\Omega}_{S/R}$ and an R-derivation $d\colon S \to \hat{\Omega}_{S/R}$ such that $(dt_1, \dots, dt_m)$ is a free S-basis of $\hat{\Omega}_{S/R}$, characterized by the universal property that for any finitely-generated S-module M and R-derivation $D\colon S \to M$ there is a unique S-linear map $\delta\colon \hat{\Omega}_{S/R} \to M$ such that $D = \delta d$.

Let $\hat{\Omega}^m$ $(m > 0)$ be the m-th exterior power of $\hat{\Omega}_{S/R}$, a free rank-one S-module with basis $dt_1 \wedge dt_2 \cdots \wedge dt_m$. Then (fact) there is a canonical functorial isomorphism

$$\varphi_J^{\#} G \xrightarrow{\sim} G \otimes \hat{\Omega}^m[m] \qquad \big(G \in \mathbf{D}_{\mathsf{c}}(R)\big).$$

Residue map

There is a natural surjection

$$\pi\colon (\hat{\Omega}^m)_{t_1 t_2 \cdots t_m} = \mathcal{K}^m(\mathbf{t}, \hat{\Omega}^m) \twoheadrightarrow \mathrm{H}^m \mathcal{K}(\mathbf{t}, \hat{\Omega}^m) = \mathrm{H}_J^m \hat{\Omega}^m$$

For $\nu \in \hat{\Omega}^m$ and nonnegative integers $n_1, \dots, n_m$, set

$$\begin{bmatrix} \nu \\ t_1^{n_1}, \dots, t_m^{n_m} \end{bmatrix} := \pi \left(\frac{\nu}{t_1^{n_1} \cdots t_m^{n_m}} \right).$$

Theorem 3. *There is a canonical (i.e., depending only on the topological R-algebra S) residue map*

$$\mathrm{res}_{S/R}\colon \mathrm{H}_J^m \hat{\Omega}^m \to R,$$

such that

$$\mathrm{res}_{S/R} \begin{bmatrix} dt_1 \cdots dt_m \\ t_1^{n_1}, \dots, t_m^{n_m} \end{bmatrix} = \begin{cases} 1 & \text{if } n_1 = \cdots = n_m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Canonical local duality

As a concrete realization of the abstract local duality theorem, one has, in the preceding situation, an affine version of Serre duality:

Theorem 4. *There is, for S-modules E, a canonical functorial isomorphism*

$$\mathrm{Hom}_R(\mathrm{H}_J^m E, R) \xrightarrow{\sim} \mathrm{Hom}_S(E, \hat{\Omega}^m)$$

that for $E = \hat{\Omega}^m$ takes $\mathrm{res}_{S/R}$ to the identity map of $\hat{\Omega}^m$.

In other words:
The functor $\mathrm{Hom}_R(\mathrm{H}_J^m E, R)$ of S-modules E is represented by $(\hat{\Omega}^m, \mathrm{res}_{S/R})$.

Proofs of the foregoing statements are in "Lectures...," §5.

Wrap-up

It has been illustrated that duality theory is a (gold) coin with two faces, the abstract and the concrete.

Typically, concrete theorems are more striking, and harder to prove directly than their abstract counterparts; but passing from abstract to concrete is not easy—it is one of the most challenging aspects of the area.