Lectures on Grothendieck Duality

V: Global Duality.

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1 Statement(s) of global duality.

Theorem 1. *Let $f\colon X \to Y$ be a map of concentrated (= quasi-compact, quasi-separated) schemes. Then the Δ-functor $\mathbf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}(Y)$ has a bounded-below right Δ-adjoint.*

1. If you wish, substitute "noetherian" for "concentrated."

2. A functor $\Phi\colon \mathbf{D}(Y) \to \mathbf{D}_{\mathsf{qc}}(X)$ is bounded below if there is an integer d such that for all $E \in \mathbf{D}(Y)$ and all $n \in \mathbb{Z}$,

$$\mathrm{H}^i(E) = 0 \text{ for all } i \le n \implies \mathrm{H}^i(\Phi E) = 0 \text{ for all } i \le n - d.$$

3. A right-Δ-adjoint of a Δ-functor Φ is a right adjoint Ψ such that the unit $\mathbf{1} \to \Psi\Phi$ of the adjunction is Δ-functorial; or equivalently, the counit $\Phi\Psi \to \mathbf{1}$ is Δ-functorial.

Corollary 2. *When restricted to concentrated schemes, the $\mathbf{D}_{\mathsf{qc}}$-valued pseudofunctor "derived direct image" has a* pseudofunctorial *right Δ-adjoint $^\times$.*

Proof. Choose for each $f\colon X \to Y$ a functor $f^\times$ right-adjoint to $\mathbf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}_{\mathsf{qc}}(Y)$, with $f^\times$ the identity functor whenever f is an identity map. Given $g\colon Y \to Z$, define $d_{f,g}\colon f^\times g^\times \to (gf)^\times$ to be the functorial map adjoint to the natural composition $\mathbf{R}(gf)_* f^\times g^\times \xrightarrow{\sim} \mathbf{R}g_*\mathbf{R}f_* f^\times g^\times \to \mathbf{R}g_* g^\times \to \mathbf{1}$.
This $d_{f,g}$ is an isomorphism, its inverse $(gf)^\times \to f^\times g^\times$ being the map adjoint to the natural composition

$$\mathbf{R}g_*\mathbf{R}f_*(gf)^\times \xrightarrow{\sim} \mathbf{R}(gf)_*(gf)^\times \to \mathbf{1}.$$

Verifying the Corollary is now straightforward. Q.E.D.

Elaboration

Derived category maps are isomorphisms iff they induce homology isomorphisms; and

$$\mathrm{H}^n\,\mathbf{R}\mathrm{Hom}^\bullet_X(C,D) = \mathrm{Hom}_{\mathbf{D}(X)}(C, D[n]) \qquad (n\in\mathbb{Z}).$$

Hence the following statement is equivalent to the global duality theorem:

Theorem 3. *For $f\colon X\to Y$ as above, there exists a bounded-below Δ-functor $f^\times\colon \mathbf{D}(Y)\to\mathbf{D}_{\mathsf{qc}}(X)$ and a Δ-functorial map $\tau\colon \mathbf{R}f_*f^\times\to\mathbf{1}$ such that for all $F\in\mathbf{D}_{\mathsf{qc}}(X)$ and $G\in\mathbf{D}(Y)$, the natural composite map (in the derived category of abelian groups)*

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_X(F, f^\times G) &\to \mathbf{R}\mathrm{Hom}^\bullet_X(Lf^*\mathbf{R}f_*F, f^\times G)\\ &\to \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^\times G)\\ &\xrightarrow{\tau} \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*F, G)\end{aligned}$$

is a Δ-functorial isomorphism.

Indeed, application of the functor H^0 to the preceding composite map yields the duality theorem; *and conversely,* if $(f^\times,\tau)$ is right Δ-adjoint to $\mathbf{R}f_*$, whence $\exists$ a functorial isomorphism

$$(f^\times G)[n]\cong f^\times(G[n]),$$

then one checks that for all $n\in\mathbb{Z}$, application of the functor H^n to the preceding composite map gives the natural composite map—an isomorphism by the duality theorem—

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}_{\mathsf{qc}}(X)}\big(F, f^\times(G[n])\big) &\to \mathrm{Hom}_{\mathbf{D}(Y)}\big(\mathbf{R}f_*F, \mathbf{R}f_*f^\times(G[n])\big)\\ &\to \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*F, G[n]).\end{aligned}$$

The preceding isomorphism can also be written as

$$\mathrm{Ext}^n_X(F, f^\times G) \xrightarrow{\sim} \mathrm{Ext}^n_Y(\mathbf{R}f_*F, G) \qquad \big(n\in\mathbb{Z},\ F\in\mathbf{D}_{\mathsf{qc}}(X)\ G\in\mathbf{D}(Y)\big).$$

Example: smooth maps, Serre duality

Notation for proper maps
For reasons to emerge in a while, when f is *proper* we set $f^! := f^\times$.

For a proper *smooth* map $f\colon X\to Y$, with (smooth) fibers of dimension, say, d, and a complex $G^\bullet$ of $\mathcal{O}_Y$-modules, $\exists$ a functorial isomorphism

$$f^*G^\bullet\otimes_{\mathcal{O}_X}\Omega^d_f[d] \xrightarrow{\sim} f^!G^\bullet$$

with $\Omega^d_f[d]$ the complex vanishing in all degrees except $-d$, where it is the sheaf of relative Kähler d-forms. (To be shown later.)

Then, if $Y=\mathrm{Spec}(k)$, k a field, Global Duality specializes to Serre Duality: the existence, for quasi-coherent $\mathcal{O}_X$-modules F, of natural isomorphisms

$$\mathrm{Hom}_k(\mathrm{H}^i(X,F),\,k) \xrightarrow{\sim} \mathrm{Ext}^{d-i}_X(F,\Omega^d_f)$$

Pseudofunctoriality, $f^!g^! \xrightarrow{\sim} (gf)^!$ reflects the standard isomorphism for smooth maps $X\xrightarrow{f}Y\xrightarrow{g}Z$ of respective relative dimensions d, e:

$$\Omega^d_f\otimes_{\mathcal{O}_X} f^*\Omega^e_g \xrightarrow{\sim} \Omega^{d+e}_{gf}$$

Remarks: abstract and concrete duality

The preceding example illustrates that there are two complementary aspects to duality theory—abstract and concrete.

Without the enlivening concrete interpretations, the abstract functorial approach can be rather austere—though when it comes to treating complex relationships, it can be quite advantageous.

While the theory can be based on either aspect (see e.g., Springer Lecture Notes by Hartshorne (no. 20) and Conrad (no. 1750) for concrete foundations), bridging the two aspects is not a trivial matter.

For example, as an instructive exercise: identify the pseudofunctoriality isomorphism given by the abstract theory with the above one for differential forms—even when $d = e = 0$ (i.e., f and g are finite étale maps)!

2 Neeman's proof.

The proof of Global Duality in the reference notes is an exposition of Deligne's proof in the appendix to Hartshorne's "Residues and Duality."
We will describe here a more recent approach, due to Neeman.
Until further notice, schemes are assumed to be concentrated.
Over a scheme X <, a complex $E \in \mathbf{D}(X)$ is perfect if each $x \in X$ has an open neighborhood U such that the restriction $E|_U$ is isomorphic in $\mathbf{D}(U)$ to a bounded complex of finite-rank free $\mathcal{O}_U$-modules.

Theorem 4 (Neeman, mid 90s, Bondal & van den Bergh, 2003)**.** *For a scheme X, the category $\mathbf{D}_{\mathsf{qc}}(X)$ has a perfect generator, that is, there is a perfect $E \in \mathbf{D}_{\mathsf{qc}}(X)$ such that for every nonzero $F \in \mathbf{D}_{\mathsf{qc}}(X)$,*

$$\mathrm{Hom}_{\mathbf{D}(X)}(E, F[n]) \neq 0 \ \textit{ for some } n \in \mathbb{Z}.$$

Neeman's proof, for quasi-compact *separated* X, uses nontrivial facts about extending perfect complexes from open sets to X. Bondal and van den Bergh adapted the argument for the quasi-separated case.

Example: If X is any affine scheme, then $\mathcal{O}_X$ is a perfect generator.

Adjoint functor theorem

Neeman pioneered the application of category-theoretical methods from homotopy theory to algebraic geometry. The following theorem is a corollary of his reworking of the Brown representability theorem.

First, some preliminary remarks. Let X be a scheme.

1. The usual $\oplus$ of complexes is a categorical direct sum in $\mathbf{D}(X)$ or $\mathbf{D}_{\mathsf{qc}}(X)$: for any $\mathbf{D}(X)$-family (E_λ),

$$E_\lambda \in \mathbf{D}_{\mathsf{qc}}(X)\ \forall\lambda \implies \oplus_\lambda E_\lambda \in \mathbf{D}_{\mathsf{qc}}(X);$$

and for any $E \in \mathbf{D}(X)$, the natural map is an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X)}(\textstyle\bigoplus_\lambda E_\lambda, E) \xrightarrow{\sim} \textstyle\prod_\lambda \mathrm{Hom}_{\mathbf{D}(X)}(E_\lambda, E).$$

2. For any categories $\mathbf{D}$, $\mathbf{D}'$, if a functor $\Phi\colon \mathbf{D} \to \mathbf{D}'$ has a right adjoint Ψ, then Φ transforms direct sums in $\mathbf{D}$ to direct sums in $\mathbf{D}'$: for any $\mathbf{D}$-family (E_λ), and E' in $\mathbf{D}'$, there are natural isomorphisms

$$\mathrm{Hom}_{\mathbf{D}'}\big(\Phi\textstyle\bigoplus_\lambda E_\lambda, E'\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}\big(\textstyle\bigoplus_\lambda E_\lambda, \Psi E'\big) \xrightarrow{\sim} \displaystyle\prod_\lambda \mathrm{Hom}_{\mathbf{D}}\big(E_\lambda, \Psi E\big) \xrightarrow{\sim} \prod_\lambda \mathrm{Hom}_{\mathbf{D}'}\big(\Phi E_\lambda, E'\big).$$

Conversely:

Theorem 5. *Let X and Y be schemes, and $\Phi\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}(Y)$ a Δ-functor.*
If Φ transforms direct sums in $\mathbf{D}_{\mathsf{qc}}(X)$ to direct sums in $\mathbf{D}(Y)$ then Φ has a right adjoint.

What remains then for establishing Global Duality is to show that for a scheme-map $f\colon X \to Y$,

$(*)$ $\mathbf{R}f_*$ transforms direct sums in $\mathbf{D}_{\mathsf{qc}}(X)$ to direct sums in $\mathbf{D}(Y)$.

Before doing this, we should remark that Neeman actually proves the adjoint functor theorem for an arbitrary Δ-functor from a "compactly generated" triangulated category to a triangulated category. Without explaining these terms, let us just note that consequently the theorem is widely applicable, yielding duality theorems in the contexts, for example, of formal schemes or diagrams of schemes or D-modules.

3 Derived direct image respects direct sums.

Proof of $(*)$: sketch

Boundedness of the restriction $\mathbf{R}f_*|_{\mathbf{D}_{\mathsf{qc}}}$ allows a reduction to the case of a $\mathbf{D}_{\mathsf{qc}}$-family (E_λ) which is *uniformly bounded-below,* i.e, there is an n_0 such that $H^nE_\lambda = 0$ for all λ and $n < n_0$.
Indeed, what is required is that for all n the homology functor H^n transforms the natural map

$$\textstyle\bigoplus_\lambda \mathbf{R}f_*E_\lambda \to \mathbf{R}f_* \bigoplus_\lambda E_\lambda$$

into an isomorphism; and boundedness of $\mathbf{R}f_*$ implies that nothing changes in degree n when each E_λ is altered by nullifying terms in all degrees $< n_0 := n-d$ (and suitably modifying E_λ^{n-d}) for fixed $d \gg 0$.

In the category of bounded-below $\mathcal{O}_X$-complexes E, construct canonical flasque resolutions $E \to F$ as follows: for each $q \in \mathbb{Z}$, let $0 \longrightarrow E^q \to F^{0q} \to F^{1q} \to F^{2q} \to \dots$ be the (flasque) Godement resolution of E^q,
set $F^{pq} := 0$ if $p < 0$, and let F be "totalization" of the double complex F^{pq}, i.e., $F^m := \oplus_{p+q=m}F^{pq}$, etc.
Then F^m is flasque, and a standard argument (using that E is bounded below) shows that the family of natural maps $E^m \to F^{0m} \subset F^m$ gives a *quasi-isomorphism* $E \to F$.
There results a quasi-isomorphism

$$\textstyle\bigoplus_\lambda E_\lambda \to \bigoplus_\lambda F_\lambda := \mathcal{F}$$

with each F_λ flasque. Since X is concentrated, a result of Kempf shows that $\mathcal{F}$ is a bounded-below complex of flasque sheaves, and hence (well-known) there are natural isomorphisms

$$\textstyle\mathbf{R}f_* \bigoplus_\lambda E_\lambda \xrightarrow{\ \sim\ } \mathbf{R}f_*\mathcal{F} \xleftarrow{\ \sim\ } f_*\mathcal{F}.$$

Another result of Kempf gives the second of the isomorphisms

$$\textstyle H^n \bigoplus_\lambda \mathbf{R}f_*E_\lambda \xrightarrow{\ \sim\ } H^n \bigoplus_\lambda f_*F_\lambda \xrightarrow{\ \sim\ } H^n f_*\mathcal{F} \xrightarrow{\ \sim\ } H^n\mathbf{R}f_* \bigoplus_\lambda E_\lambda. \qquad \text{QED}$$

4 Sheafified duality—preliminary form.

We move toward a more general *sheafified* version of duality. This amounts to the behavior of $f^\times$ vis-à-vis open immersions $U \hookrightarrow Y$, a special case of *tor-independent base change* (next lecture).
Let $f\colon X \to Y$, $f^\times$ and τ be as before.
The duality map $\delta(f, F, G)$ $(F \in \mathbf{D}(X), G \in \mathbf{D}(Y))$ is the composition

$$\begin{aligned}
\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^\times G) &\longrightarrow \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*\mathbf{R}f_*F, f^\times G) && \text{(via unit of } \mathbf{L}f^*\text{-}\mathbf{R}f_* \text{ adjunction)}\\
&\xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, \mathbf{R}f_*f^\times G) && \text{(sheafifed } \mathbf{L}f^*\text{-}\mathbf{R}f_* \text{ adjunction)}\\
&\longrightarrow \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G) && \text{(via } \tau).
\end{aligned}$$

Theorem 6. *For any $E \in \mathbf{D}_{\mathsf{qc}}(Y)$, $F \in \mathbf{D}_{\mathsf{qc}}(X)$ and $G \in \mathbf{D}(Y)$, the map*

$$\mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^\times G)) \xrightarrow{\delta(F,G)} \mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G))$$

is an isomorphism.

Corollary 7. *If both* $\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^!G)$ *and* $\mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G)$ *are in* $\mathbf{D}_{\mathsf{qc}}(X)$ *then the duality map* $\delta(F,G)$ *is an isomorphism*

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^\times G) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G).$$

The hypotheses in the Corollary are needed because, in the Theorem, $E \in \mathbf{D}_{\mathsf{qc}}(Y)$. Eventually, we'll prove this Corollary under considerably weaker hypotheses.

Example of sheafified duality

Meanwhile, here is a situation where the hypotheses hold.

(Fairly simple): $F \in \mathbf{D}^-_{\mathsf{c}}(X),\ G \in \mathbf{D}^+_{\mathsf{qc}}(X) \implies \mathbf{R}\mathcal{H}om^\bullet(F,G) \in \mathbf{D}^+_{\mathsf{qc}}(X).$

(Basic theorem): *If* $f\colon X \to Y$ *is a proper map of noetherian schemes then* $\mathbf{R}f_*$ *preserves coherence of homology.*

Thus, $\mathbf{R}f_*|_{\mathbf{D}_{\mathsf{c}}}$ being bounded, $\mathbf{R}f_*\mathbf{D}^-_{\mathsf{c}}(X) \subset \mathbf{D}^-_{\mathsf{c}}(Y)$.
From these facts, and the preceding Corollary, one deduces:

Corollary 8. *If* $f\colon X \to Y$ *is a proper map of noetherian schemes then for all* $F \in \mathbf{D}^-_{\mathsf{c}}(X)$ *and* $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$, *the duality map* $\delta(F,G)$ *is an isomorphism*

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^!G) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G).$$

Proof of theorem: sketch

Despite the particular notation to be used, the proof can be given entirely in terms of axioms of the basic duality setup.
Adjunctions forming part of the axioms, as well as the projection isomorphism $\mathbf{p}_2$, yield isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{D}(Y)}\big(E, \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(F, f^\times G)\big) &\xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}(X)}\big(\mathbf{L}f^*E, \mathbf{R}\mathcal{H}om^\bullet_X(F, f^\times G)\big)\\
&\xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}(X)}\big(\mathbf{L}f^*E \underline{\underline{\otimes}} F, f^\times G)\big)\\
&\xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}(Y)}\big(\mathbf{R}f_*(\mathbf{L}f^*E \underline{\underline{\otimes}} F), G)\big)\\
&\xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}(Y)}\big(E \underline{\underline{\otimes}} \mathbf{R}f_*F, G)\big)\\
&\xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}(Y)}\big(E, \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*F, G)\big).
\end{aligned}$$

Is this composed map is the same as the one in the theorem?

Of course, yes (QED for theorem), *but this must be shown*—one of many tedious duality-setup exercises which arise as the theory develops.