Lectures on Grothendieck Duality

VI: Tor-independent base change; sheafified duality.

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Contents

Introduction

The existence of a right adjoint for $\mathbf{R}f_*$—Global Duality—is the first basic theorem in Grothendieck duality theory.

The second, Tor-independent Base Change, has to do with the behavior of this right adjoint with respect to certain independent fiber squares.

For simplicity, we assume throughout that all schemes are noetherian.

The following abbreviations will be used, for a scheme-map h or a scheme Z:

$$\begin{aligned} h_* &:= \mathbf{R}h_*\,, & h^* &:= \mathbf{L}h^*, \\ \mathcal{H}_Z &:= \mathbf{R}\mathcal{H}om^{\bullet}_Z\,, & \mathbf{H}_Z &:= \mathbf{R}\mathrm{Hom}^{\bullet}_Z\,, \\ \otimes_Z &:= \underline{\underline{\otimes}}_Z\,, & \mathbf{\Gamma}_Z(-) &:= \mathbf{R}\Gamma(Z,-). \end{aligned}$$

As in the global duality theorem, $h^{\times}(= h^!$ when h is proper) is right-adjoint to h_*, and $\tau\colon h_*h^{\times} \to \mathbf{1}$ is the canonical map.

1 Base change map for independent squares.

Recall that a commutative square σ of scheme-maps

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

is (tor-)independent if σ is a fiber square (i.e., the natural map is an isomorphism $X' \xrightarrow{\sim} X \times_Y Y'$) such that the functorial map $\theta_\sigma \colon u^* f_* \to g_* v^*$ adjoint to the natural composition $f_* \to f_* v_* v^* \xrightarrow{\sim} u_* g_* v^*$ is an isomorphism.

Remark: Formally, σ is an *ordered* 4-tuple (u, g, f, v) with $ug = fv$.
That is *not the same* as the ordered 4-tuple $\sigma' := (f, v, u, g)$.
σ' is the "reflection of σ in the upper-left to lower-right diagonal."
But, as we've seen, σ is independent $\iff$ so is σ'.

Definition: base change map

For an independent $\sigma = (u, g, f, v)$, the functorial base change map

$$\beta_\sigma \colon v^* f^\times \to g^\times u^*$$

is the map adjoint to the natural composition

$$g_* v^* f^\times \xrightarrow{\theta_\sigma^{-1}} u^* f_* f^\times \xrightarrow{u^*\tau} u^*.$$

Here is another way of getting β_σ, via "conjugate base change."

Exercise. Let $\sigma = (u, g, f, v)$ be a fiber square. Show that the map

$$\phi_\sigma \colon v_* g^\times \to f^\times u_*$$

(between functors from $\mathbf{D}_{\mathsf{qc}}(Y')$ to $\mathbf{D}_{\mathsf{qc}}(X)$) that is adjoint to the composition $f_* v_* g^\times \xrightarrow{\sim} u_* g_* g^\times \to u_*$, is right-conjugate to θ_σ.
Deduce that σ is independent $\iff$ ϕ_σ (or $\phi_{\sigma'}$) is an isomorphism.

(b) Show that when σ is independent the map β_σ is adjoint to the composition

$$f^\times \xrightarrow{\text{natural}} f^\times u_* u^* \xrightarrow{\text{via } \phi_\sigma^{-1}} v_* g^\times u^*.$$

2 Base change theorem

Definition 1. A scheme-map $u \colon Y' \to Y$ has finite tor-dimension or finite flat dimension if the functor $\mathbf{L}u^* \colon \mathbf{D}(Y) \to \mathbf{D}(Y')$ is bounded, i.e., $\exists d \in \mathbb{Z}$ such that
for every $\mathcal{O}_Y$-complex F and $n \in \mathbb{Z}$ such that $H^i F = 0$ for all $i > n$,
it holds that $H^j \mathbf{L}u^* F = 0$ for all $j > n + d$.

Equivalently (exercise): For each $y \in Y'$, $\exists$ an exact $\mathcal{O}_{Y,u(y)}$-module sequence

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \mathcal{O}_{Y',\,y} \to 0$$

with P_i flat over $\mathcal{O}_{Y,u(y)}$ $(0 \le i \le d)$.

Theorem 2 (Base change—BC)**.** *Suppose one has an independent square of maps of noetherian schemes*

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

with f (hence g) proper and u of finite tor-dimension.
Then for any $G \in \mathbf{D}^{+}_{\mathsf{qc}}(Y)$, the base change map is an isomorphism

$$\beta_\sigma\colon v^*f^!G \xrightarrow{\ \sim\ } g^!u^*G.$$

The case of Theorem BC where u (hence v) is an affine map (resp. an open immersion) is labeled $\mathrm{BC}^{\mathsf{af}}$ (resp. BC°).

Counterexample for unbounded G

BC° need not hold for unbounded G. Neeman gave a simple counterexample, with

$$X := \mathrm{Spec}(\mathbb{Z}[T]/T^2), Y := \mathrm{Spec}(\mathbb{Z}), Y' := \mathrm{Spec}(\mathbb{Z}[\tfrac{1}{2}]),\ G := \textstyle\prod_{n=0}^{\infty} \mathbb{Z}[n],$$

based on noncompatibility of localization and infinite products.

3 Base change and sheafified duality.

Recall from Lecture 3 the map

$$\nu(f,F,G)\colon f_*[F,G] \to [f_*F, f_*G].$$

It is a formal exercise to show that this map factors naturally as

$$f_*[F,H] \to f_*[f^*f_*F, H] \xrightarrow{\ \sim\ } [f_*F, f_*H].$$

In the latter form, ν appeared in Lecture 5 as part of the duality map for proper f:

$$\delta(f,F,G)\colon f_*\mathcal{H}_X(F, f^!G) \xrightarrow{\nu} \mathcal{H}_Y(f_*F, f_*f^!G) \xrightarrow{\tau} \mathcal{H}_Y(f_*F, G).$$

Remark. With $\mathbf{\Gamma}_Y$ the derived global section functor, the map

$$\mathbf{\Gamma}_Y\delta\colon \mathbf{\Gamma}_Y f_*\mathcal{H}_X(F, f^!G) \to \mathbf{\Gamma}_Y\mathcal{H}_Y(f_*F, G)$$

can be identified with the global duality map

$$\mathbf{H}_X(F, f^!G) \to \mathbf{H}_Y(f_*F, G).$$

In the course of the proof of BC, BC° will be shown equivalent to:

Theorem 3 (Sheafified duality—SD)**.** *If $f\colon X \to Y$ is a proper scheme-map then for all $F \in \mathbf{D}_{\mathsf{qc}}(X)$ and $G \in \mathbf{D}^{+}_{\mathsf{qc}}(Y)$, the duality map $\delta(f,F,G)$ is an isomorphism*

$$f_*\mathcal{H}_X(F, f^!G) \xrightarrow{\ \sim\ } \mathcal{H}_Y(f_*F, G).$$

The version of this theorem in Lecture 5 had the restriction $F \in \mathbf{D}^{-}_{\mathsf{c}}(X)$. That version is labeled SD_{c}.
SD_{c} having been proved, the strategy is to establish the following implications:

$$\mathrm{SD}_{\mathsf{c}} \implies (\mathrm{BC}^{\mathsf{af}} + \mathrm{BC}^{\circ}) \implies \mathrm{BC} \Longrightarrow \mathrm{BC}^{\circ} \Longleftrightarrow \mathrm{SD}.$$

Some indications of how to do this follow. The aim is to suggest the flavor of what's involved, and in particular to bring out the role played by purely formal considerations—that is, arguments based solely on the axioms of basic duality setups. (Full details are in the reference notes.)

4 Proof of base change.

We begin with the implication $\mathsf{SD_c} \implies (\mathsf{BC^{af}} + \mathsf{BC^{o}})$.

In the formal context of Lecture 3, we saw a canonical map

$$\rho(u, A, B)\colon u^*[A, B] \to [u^*A, u^*B].$$

Proposition 4. *Let $u\colon Y' \to Y$ be a scheme-map of finite tor-dimension, let $E \in \mathbf{D}^-_{\mathsf{c}}(Y)$ and $H \in \mathbf{D}^+(Y)$. Then the map $\rho(u, E, H)$ is an isomorphism*

$$u^*\mathcal{H}_Y(E,\, H) \xrightarrow{\ \sim\ } \mathcal{H}_{Y'}(u^*E,\, u^*H).$$

If u is an open immersion the same holds (more or less trivially) for any $E, H \in \mathbf{D}(Y)$.

We'll also need the following relation among the maps ρ, ν and θ.

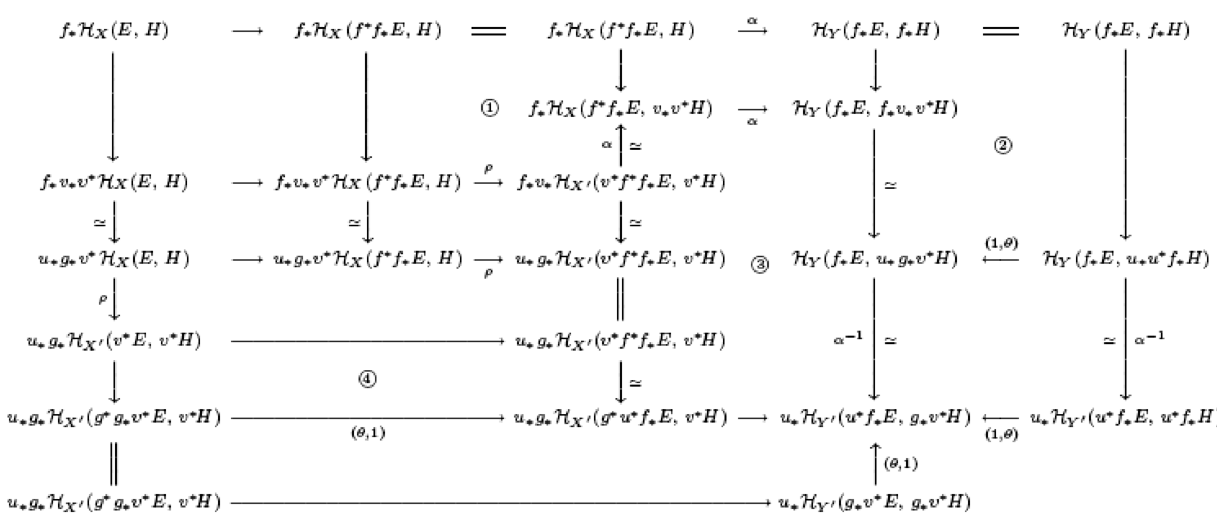
For any commutative diagram of scheme-maps

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

and $E,\, H \in \mathbf{D}(X)$, the following diagram commutes:

$$\begin{array}{ccccc} u^*f_*\mathcal{H}_X(E, H) & & \xrightarrow{\qquad\qquad \nu \qquad\qquad} & & u^*\mathcal{H}_Y(f_*E,\, f_*H) \\ {\scriptstyle \theta_\sigma}\big\downarrow & & & & \big\downarrow{\scriptstyle \rho} \\ g_*v^*\mathcal{H}_X(E, H) & & & & \mathcal{H}_{Y'}(u^*f_*E,\, u^*f_*H) \\ {\scriptstyle \rho}\big\downarrow & & & & \big\downarrow{\scriptstyle \theta_\sigma} \\ g_*\mathcal{H}_{X'}(v^*E, v^*H) & \xrightarrow[\ \nu\ ]{} & \mathcal{H}_{Y'}(g_*v^*E,\, g_*v^*H) & \xrightarrow[\ \theta_\sigma\ ]{} & \mathcal{H}_{Y'}(u^*f_*E,\, g_*v^*H) \end{array}$$

The proof of commutativity is formal: with patience, one finds a decomposition of the diagram into ones which are sufficiently small that their commutativity is given by axioms, or by other previously established commutativities. (It would really be nice to get a computer to take over such tedium.)



In the preceding, after replacing E by G and H by $f^!H$, one derives, formally, the commutative diagram

$$\begin{array}{ccc}
u^*f_*\mathcal{H}_X(G,\, f^!H) & \xrightarrow{u^*(\delta)} & u^*\mathcal{H}_Y(f_*G,\, H)\\
{\scriptstyle\theta_\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\rho}\\
g_*v^*\mathcal{H}_X(G,\, f^!H) & & \mathcal{H}_Y(u^*f_*G,\, u^*H)\\
{\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\theta_\sigma}\\
g_*\mathcal{H}_{X'}(v^*G,\, v^*f^!H) & & \mathcal{H}_{Y'}(g_*v^*G,\, u^*H)\\
{\scriptstyle\nu}\big\downarrow & & \big\uparrow{\scriptstyle\tau}\\
\mathcal{H}_{X'}(g_*v^*G,\, g_*v^*f^!H) & \xrightarrow[\theta_\sigma^{-1}]{} & \mathcal{H}_{X'}(g_*v^*G,\, u^*f_*f^!H)
\end{array}$$

If $G \in \mathbf{D}^-_{\mathsf{c}}(X)$, so that, f being proper, $f_*G \in \mathbf{D}^-_{\mathsf{c}}(Y)$, then by the above proposition, the maps labeled ρ are isomorphisms, as is $u^*(\delta)$ by the assumption SD_{c}, whence so is $\lambda := \tau\theta_\sigma^{-1}\nu$.

To be proven is that the base change map β_σ is an isomorphism when u (hence v) is either an open immersion or an affine map; in both these cases it is easily seen to suffice that $v_*\beta$ be an isomorphism.

Let $G \in \mathbf{D}^-_{\mathsf{c}}(X)$, so that $v^*G \in \mathbf{D}^-_{\mathsf{c}}(X')$. From the definition of β, one derives, formally, a commutative diagram

$$\begin{array}{ccc}
f_*\mathcal{H}_X(G,\, v_*v^*f^!H) & \xrightarrow{\text{via } v_*\beta} & f_*\mathcal{H}_X(G,\, v_*g^!u^*H)\\
{\scriptstyle f_*\widetilde{\alpha}^{-1}}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle f_*\widetilde{\alpha}^{-1}}\\
f_*v_*\mathcal{H}_{X'}(v^*G,\, v^*f^!H) & \xrightarrow{\text{via } \beta} & f_*v_*\mathcal{H}_{X'}(v^*G,\, g^!u^*H)\\
\big\downarrow\simeq & & \simeq\big\downarrow\\
u_*g_*\mathcal{H}_{X'}(v^*G,\, v^*f^!H) & \underset{u_*\lambda}{\xrightarrow{\ \sim\ }} & u_*\mathcal{H}_{Y'}(g_*v^*G,\, u^*H)
\end{array}$$

where the isomorphism $\widetilde{\alpha}$ is the sheafified expression of v^*–v_* adjointness (see Lecture 3), and the right column is an isomorphism by SD_{c} (for g). Thus the top horizontal map "via $v_*\beta$" is an isomorphism.

The desired implication results then from the following Key Fact (proof omitted):

If $f\colon X \to Y$ is a finitely presented scheme-map, then a $\mathbf{D}^+_{\mathsf{qc}}(X)$-map ϕ is an isomorphism if and only if so is the $\mathbf{D}(Y)$-map $f_*\mathcal{H}_X(G,\, \phi)$ for every $G \in \mathbf{D}^-_{\mathsf{c}}(X)$.

The implication $(\mathrm{BC}^{\circ} + \mathrm{BC}^{\mathsf{af}}) \implies \mathrm{BC}$ results from a simple formal "transitivity" property of θ (hence β) with respect to horizontal composition of fibre squares, a property which, along with BC°, allows a reduction to the case where Y and Y'—and hence u—are affine, so that $\mathrm{BC}^{\mathsf{af}}$ applies.

As for the implication $\mathrm{BC}^{\circ} \implies \mathrm{SD}$, when u is an open immersion, the columns of the following—previously derived—diagram are isomorphisms:

$$\begin{array}{ccc}
u^*f_*\mathcal{H}_X(G, f^!H) & \xrightarrow{u^*(\delta)} & u^*\mathcal{H}_Y(f_*G, H) \\
{\scriptstyle\theta_\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\
g_*v^*\mathcal{H}_X(G, f^!H) & & \mathcal{H}_Y(u^*f_*G, u^*H) \\
{\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\theta_\sigma} \\
g_*\mathcal{H}_{X'}(v^*G, v^*f^!H) & \xrightarrow[\lambda]{} & \mathcal{H}_{Y'}(g_*v^*G, u^*H)
\end{array}$$

Moreover, one checks that $\lambda = \delta(g_*, v^*G, v^*f^!H) \circ g_*\mathcal{H}_{X'}(v^*G, \beta)$, where β is, by BC°, an isomorphism. Then *since* $\mathbf{\Gamma}_{Y'}\delta$ *can be identified with a global duality isomorphism,* one sees that $\mathbf{\Gamma}_{Y'}\lambda$ is an isomorphism. Hence $\mathbf{\Gamma}_{Y'}u^*(\delta)$ is an isomorphism for all open immersions $u\colon Y' \to Y$; and SD follows. QED

Using the "Key Fact" one deduces similarly that $\mathrm{SD} \implies \mathrm{BC}^{\circ}$.