Lectures on Grothendieck Duality

VII: The twisted inverse-image pseudofunctor.

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Contents

Introduction

We have treated the right adjoint $^{\times}$ of $\mathbf{R}(-)_*$ for quite general maps.
But for non-proper maps this pseudofunctor may be of limited interest.

Grothendieck Duality is basically about a $\mathbf{D}^+_{\mathsf{qc}}$-valued pseudofunctor $^!$ over the category of separated finite-type maps of noetherian schemes, agreeing with $^{\times}$ on proper maps, but, unlike $^{\times}$, agreeing with the usual inverse-image pseudofunctor * on open immersions (more generally, on separated étale maps); and also compatible in a suitable sense with flat base change.

The *existence and uniqueness* (*up to isomorphism*) of this remarkable twisted inverse-image pseudofunctor is the first fundamental theorem to be discussed in this lecture; and its *behavior vis-à-vis flat base change* is the second.

Conventions

As before, we assume throughout that all schemes are noetherian,

For a scheme-map h, we write the abbreviation h^* for $\mathbf{L}h^*$.

1 Nagata's compactification theorem.

A key point here is Nagata's compactification theorem:

Theorem 1. *Any finite-type separated map* $f\colon X \to Y$ *of noetherian schemes factors as* $f = \overline{f}u$ *with* $\overline{f}$ *proper and* u *an open immersion.*

Remarks. 1. Any such $\overline{f}$ is called a compactification of f.

2. For *quasi-finite* f, the theorem is essentially Zariski's Main Theorem.

3. Nagata's original paper appeared in 1962. The theorem is a hard one, and for a long time, his proof was not well-understood. But now there are several expositions, the most recent one, by Brian Conrad, having appeared in 2007 (J. Ramanujan Math. Soc., see references in reference notes.)

4. In 2008, Suresh Nayak extended Nagata's theorem—and hence the twisted inverse image—to essentially finite-type separated maps. (arXiv:0809.1201).

2 Characterizaton of the twisted inverse image.

Theorem 2. *On the category* $\mathbf{S_f}$ *of finite-type separated maps of noetherian schemes there is a* $\mathbf{D}^+_{\mathsf{qc}}$*-valued pseudofunctor* $^!$ *uniquely determined up to isomorphism by the following three properties.*

(i) *The pseudofunctor* $^!$ *restricts on the subcategory of proper maps to a right adjoint of the derived direct-image pseudofunctor.*

(ii) *The pseudofunctor* $^!$ *restricts on the subcategory of étale maps to the usual inverse-image pseudofunctor* *.

(iii) *For any fiber square in* $\mathbf{S_f}$

$$\begin{array}{ccc} \bullet & \xrightarrow{\ v\ } & \bullet \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array}$$

with f, g *proper, and* u, v *étale, the base-change map* β_σ *(previously defined) equals the natural composite isomorphism*

$$v^*f^! = v^!f^! \xrightarrow{\ \sim\ } (fv)^! = (ug)^! \xrightarrow{\ \sim\ } g^!u^! = g^!u^*.$$

Proof strategy

In view of (i) and (ii), the obvious way to define $f^!$ is to compactify f, say $f = \overline{f}u$, and set

$$f^! := \overline{f}^! \circ u^*.$$

The point is then to show, using flat base change for proper maps, that this definition is essentially independent of the chosen compactification, and that the result is pseudofunctorial and satisfies (iii).

The argument is based on a general method of Deligne for pasting together two pseudofunctors on subcategories of a given one.

To deal both with existence and (later) with base change for the twisted inverse image, a formalization of some basic features of base change will be useful.

3 Base-change setups.

A base-change setup $\mathcal{B}(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square})$ consists of the following data (a)–(d), subject to conditions (1)–(3):

(a) Subcategories $\mathbf{P}$ and $\mathbf{F}$ of a category $\mathbf{S}$, each containing every object of $\mathbf{S}$.

(b) Contravariant pseudofunctors ${}^!$ on $\mathbf{P}$ and * on $\mathbf{F}$, such that for all objects $X \in \mathbf{S}$, the categories $\mathbf{X}^!$ and $\mathbf{X}^*$ coincide.

(c) A class $\square$ of commutative $\mathbf{S}$-squares, called distinguished squares:

$$\begin{array}{ccc} \bullet & \xrightarrow{\ v\ } & \bullet \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array} \qquad (f, g \in \mathbf{P};\ u, v \in \mathbf{F}).$$

(d) For each distinguished $\sigma(u, g, f, v)$, an isomorphism of functors

$$\beta_\sigma \colon v^* f^! \xrightarrow{\ \sim\ } g^! u^*.$$

(1) If two commutative $\mathbf{S}$-squares

$$\begin{array}{ccc} \bullet & \xrightarrow{\ v\ } & \bullet \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array} \qquad\qquad \begin{array}{ccc} \bullet & \xrightarrow{\ v_1\ } & \bullet \\ {\scriptstyle g_1}\big\downarrow & \sigma_1 & \big\downarrow{\scriptstyle f_1} \\ \bullet & \xrightarrow[\ u_1\ ]{} & \bullet \end{array}$$

are *isomorphic,* then σ is distinguished $\Leftrightarrow$ σ_1 is distinguished.

(2) For every $\mathbf{P}$-map f (resp. $\mathbf{F}$-map u), the square

$$\begin{array}{ccc} \bullet & \xrightarrow{\ 1\ } & \bullet \\ {\scriptstyle f}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ 1\ ]{} & \bullet \end{array} \quad \text{resp.} \quad \begin{array}{ccc} \bullet & \xrightarrow{\ u\ } & \bullet \\ {\scriptstyle 1}\big\downarrow & \sigma & \big\downarrow{\scriptstyle 1} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array}$$

is distinguished.

(3) (Horizontal transitivity.) If the square $\sigma_0 = \sigma_2 \circ \sigma_1$ (with g deleted)

$$\begin{array}{ccccc} \bullet & \xrightarrow{\ v_1\ } & \bullet & \xrightarrow{\ v_2\ } & \bullet \\ {\scriptstyle h}\big\downarrow & \sigma_1 & {\scriptstyle g}\big\downarrow & \sigma_2 & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u_1\ ]{} & \bullet & \xrightarrow[\ u_2\ ]{} & \bullet \end{array}$$

as well as its constituents σ_2 and σ_1 are all distinguished, then the corresponding natural diagram of functorial maps commutes:

$$\begin{array}{ccccc} (v_2 v_1)^* f^! & & \xrightarrow{\qquad \beta_{\sigma_0} \qquad} & & h^!(u_2 u_1)^* \\ \simeq\big\downarrow & & & & \big\downarrow\simeq \\ v_1^* v_2^* f^! & \xrightarrow[\ v_1^*\beta_{\sigma_2}\ ]{} & v_1^* g^! u_2^* & \xrightarrow[\ \beta_{\sigma_1}\ ]{} & h^! u_1^* u_2^*. \end{array}$$

Similarly for transitivity vis-á-vis vertical juxtaposition of squares.

Examples.

(A) (*Pseudofunctors as base-change setups.*)

Let $\mathbf{S}$ be a category, take $\mathbf{P} = \mathbf{F} := \mathbf{S}$, let ${}^! = {}^*$ be a contravariant pseudofunctor on $\mathbf{S}$, let $\square :=$ {all commutative squares in $\mathbf{S}$}, and for any such square σ, let

$$\beta_\sigma \colon v^*f^* \xrightarrow{\ \sim\ } (fv)^* = (ug)^* \xrightarrow{\ \sim\ } g^*u^*$$

be the isomorphism naturally associated with the pseudofunctor *.
For this $\big(\mathbf{S}, \mathbf{P}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$, conditions (1)–(3) are easily checked.

(B) $\mathcal{B}\big(\mathbf{S_f}, \mathbf{P}, \mathbf{E}, {}^\times, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$, *with:*

- $\mathbf{S_f} :=$ {finite-type separated scheme-maps},
- $\mathbf{P} \subset \mathbf{S_f} :=$ {proper maps}, with $\mathbf{D}^+_{\mathsf{qc}}$-valued duality pseudofunctor ${}^!$;
- $\mathbf{E} \subset \mathbf{S_f} :=$ {étale maps}, with $\mathbf{D}^+_{\mathsf{qc}}$-valued duality pseudofunctor * where $u^* := \mathsf{L}u^*$ for any $\mathbf{E}$-map u;
- $\square :=$ {*fiber squares* $\sigma(u, g, f, v)$ with $f, g \in \mathbf{P}$, $(u, v) \in \mathbf{E}$};
- $\beta_\sigma \colon v^*f^\times \to g^\times u^* :=$ the corresponding base-change isomorphism.

More strategy

To be able to apply Deligne's pasting arguments to prove the theorem, we'll need to enlarge this last $\mathcal{B}$ to a setup $\mathcal{B}\big(\mathbf{S_f}, \mathbf{P}, \mathbf{E}, {}^\times, {}^*, (\beta'_\sigma)_{\sigma\in\square'}\big)$ where $\square'$ consists of *all* commutative $\mathbf{S_f}$-squares $\sigma(u, g, f, v)$ with f, g proper and u, v étale.
That means we have to extend β_σ to all $\sigma \in \square'$, while maintaining transitivity.
In fact we will extend to an even larger class of squares:

Admissible squares

In the category of schemes, an admissible square is a commutative square

$$(*) \qquad \begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array} \qquad \begin{cases} u, v \text{ flat;} \\ f, g \text{ finite-type separated} \end{cases}$$

such that in the associated diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\ i\ } & X \times_Y Y' & \xrightarrow{\ q_1\ } & X \\ & & {\scriptstyle q_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ & & Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

(where q_1, q_2 are the projections, $q_1 i = v$ and $q_2 i = g$) the map i is *étale.*

Example: Any commutative $(*)$ with u and v *étale* is admissible.
That's because a map which is étale remains so after any base change, and if qi and i are both étale then so is i, see [EGA IV, §17].

4 Flat base change for the twisted inverse image.

Theorem 3. *Let* $\mathbf{S}$ *be the category of separated maps of noetherian schemes.*

Let $\mathbf{S_f} \subset \mathbf{S}$ *be the subcategory of finite-type maps, and* $^!$ *the* $\mathbf{D}^+_{\mathsf{qc}}$*-valued twisted inverse-image pseudofunctor on* $\mathbf{S_f} \subset \mathbf{S}$*.*

Let $\mathbf{F} \subset \mathbf{S}$ *be the subcategory of flat maps, and* * *the usual* $\mathbf{D}^+_{\mathsf{qc}}$*-valued inverse-image pseudofunctor on* $\mathbf{F}$*.*

Let $\square$ *be the class of* admissible $\mathbf{S}$*-squares.*

Then there is a unique base-change setup $\mathcal{B}\big(\mathbf{S}, \mathbf{S_f}, \mathbf{F}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ *such that the following conditions hold for any* $\sigma(u, g, f, v) \in \square$*.*

(i) *If* σ *is a fiber square with* f *proper then* β_σ *is the base-change isomorphism.*

(ii) *If* f *(hence* g*)—is étale, so that* $f^! = f^*$ *and* $g^! = g^*$*, then* β_σ *is the natural isomorphism* $v^*f^* \xrightarrow{\sim} g^*u^*$*.*

(iii) *If* u *(hence* v*)—is étale, so that* $u^* = u^!$ *and* $v^* = v^!$*, then* β_σ *is the natural isomorphism* $v^!f^! \xrightarrow{\sim} g^!u^!$*.*

In other words:

The theorem says there is essentially one way to associate to each admissible square of noetherian schemes

$$\begin{array}{ccc} \bullet & \xrightarrow{\ v\ } & \bullet \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array}$$

(with f and g finite-type separated maps, u and v flat), a functorial isomorphism $\beta_\sigma \colon v^*f^! \to g^!u^*$ that satisfies horizontal and vertical transitivity, and that for certain special admissible squares is the isomorphism specified by conditions (i)—(iii).

5 Enlargement of base-change setups.

As has been indicated, the fundamental existence and base-change theorems for the twisted inverse image are proved via two kinds of abstract pasting theorems, one for pseudofunctors and one for base-change setups.

In general terms, pasting of data given on two subcategories of a category is done by making a fairly obvious construction which uses "compactifications" of maps—factorizations $f = \overline{f}u$ where $\overline{f}$ and u are in the respective subcategories, and then checking via numerous commutative diagrams that the result is independent of the choice of compactification, and has all the desired properties.

For example, to construct β_σ for an admissible $\sigma(u, g, f, v)$, compactify f to decompose σ as follows:

$$\begin{array}{ccccc} \bullet & \xrightarrow{\ i\ } & \bullet & \xrightarrow{\ q\ } & \bullet \\ & & {\scriptstyle u_2}\big\downarrow & \sigma_0 & \big\downarrow{\scriptstyle u_1} \\ & & \bullet & \xrightarrow{\ w\ } & \bullet \\ & & {\scriptstyle \overline{g}}\big\downarrow & \overline{\sigma} & \big\downarrow{\scriptstyle \overline{f}} \\ & & \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array} \qquad (\sigma)$$

where the two squares are fiber squares, $\overline{f}$ and $\overline{g}$ are proper, u_1 and u_2 are open immersions (hence étale), i is étale, and u, w, q are flat.

If the base-change theorem is to hold, then β_σ must be the natural composed isomorphism

$$g^!u^* \xrightarrow{\sim} i^!u_2^!\overline{g}^!u^* \xrightarrow[i^!u_2^!\beta_{\overline{\sigma}}]{\sim} i^!u_2^!w^*\overline{f}^! = i^*u_2^*w^*\overline{f}^! \xrightarrow{\sim} i^*q^*u_1^*\overline{f}^! \xrightarrow{\sim} v^*u_1^!\overline{f}^! \xrightarrow{\sim} v^*f^!.$$

Is this composed map independent of the choice of compactification, and transitive?
The answer is, of course, "yes". But for the most part, details of the proof are not very suitable for a lecture; they can be found in the reference notes.

Example of enlargement.

Just to illustrate, in the rest of this lecture, we'll sketch some techniques for enlarging base-change setups.
Recall the base-change setup $\mathcal{B} = \mathcal{B}\big(\mathbf{S_f}, \mathbf{P}, \mathbf{E}, {}^{\times}, {}^{*}, (\beta_\sigma)_{\sigma\in\square}\big)$ with:

- $\mathbf{S_f} :=$ {finite-type separated scheme-maps},
- $\mathbf{P} \subset \mathbf{S_f} :=$ {proper maps}, with $\mathbf{D}^+_{\mathsf{qc}}$-valued duality pseudofunctor ${}^{!}$;
- $\mathbf{E} \subset \mathbf{S_f} :=$ {étale maps}, with $\mathbf{D}^+_{\mathsf{qc}}$-valued duality pseudofunctor * where $u^* := \mathsf{L}u^*$ for any $\mathbf{E}$-map u;
- $\square :=$ {*fiber squares* $\sigma(u, g, f, v)$ with $f, g \in \mathbf{P}$, $(u, v) \in \mathbf{E}$};
- $\beta_\sigma \colon v^*f^\times \to g^\times u^* :=$ the corresponding base-change isomorphism.

The problem is to extend β_σ to a larger class of σ; more precisely, to *enlarge* $\mathcal{B}$ to a setup $\mathcal{B}\big(\mathbf{S_f}, \mathbf{P}, \mathbf{E}, {}^{\times}, {}^{*}, (\beta'_\sigma)_{\sigma\in\square'}\big)$ where $\square'$ consists of *all* commutative $\mathbf{S_f}$-squares $\sigma(u, g, f, v)$ with f, g proper and u, v étale.
We approach this problem *axiomatically.*

Special subcategories

For a setup $\mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{E}, {}^{!}, {}^{*}, (\beta_\sigma)_{\sigma\in\square}\big)$, a subcategory $\mathbf{A} \subset \mathbf{S}$ is special if for any maps $i \colon X \to Y$ in $\mathbf{A}$, $g \colon X' \to X$ in $\mathbf{P}$, and $v \colon X' \to X$ in $\mathbf{E}$, $\square$ contains the squares

$$\begin{array}{ccc} X' & \xrightarrow{\;1\;} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle ig} \\ X & \xrightarrow[\;i\;]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle i} \\ X' & \xrightarrow[\;iv\;]{} & Y \end{array}$$

Example 4. For the preceding example, the category $\mathbf{A}$ whose maps are all the open-and-closed immersions is special. Indeed, since i is a monomorphism, the above squares are fiber squares.

A simple deduction

After fixing a special subcategory $\mathbf{A}$, we call its maps special.
For any special map $i \colon X \to Y$, $\beta_i \colon i^! \xrightarrow{\;\sim\;} i^*$ is the isomorphism β_τ associated to the distinguished square

$$\begin{array}{ccc} X & \xrightarrow{\;1\;} & X \\ {\scriptstyle 1}\downarrow & \tau & \downarrow{\scriptstyle i} \\ X & \xrightarrow[\;i\;]{} & Y \end{array}$$

Proposition 5. *Let* $\mathbf{A}$ *be a special subcategory of* $\mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{E}, {}^{!}, {}^{*}, (\beta_\sigma)_{\sigma\in\square}\big)$. *Then the* β_i *give a pseudofunctorial isomorphism of the restrictions of* ${}^{!}$ *and* * *to* $\mathbf{A}$.

So, for instance, *if* i *is an open-and-closed immersion then* i^* *is right-adjoint—pseudofunctorially*—to $\mathbf{R}i_*$. (This fact can easily be shown directly; the foregoing relates it to base change.)

Proof. For pseudofunctoriality of the isomorphism β_i, apply (3) and (2) in the definition of setup to

$$\begin{array}{ccccc}
X & \xrightarrow{1} & X & \xrightarrow{1} & X \\
\Big\downarrow{\scriptstyle 1} & & {\scriptstyle i}\Big\downarrow & & \Big\downarrow{\scriptstyle i} \\
& & Y & \xrightarrow{1} & Y \\
& & {\scriptstyle 1}\Big\downarrow & & \Big\downarrow{\scriptstyle j} \\
X & \xrightarrow[i]{} & Y & \xrightarrow[j]{} & Z
\end{array}$$

to see that the left and right halves of the following diagram commute:

$$\begin{array}{ccccc}
(ji)^! & = & (ji)^! & \xrightarrow{\beta_{ji}} & (ji)^* \\
{\scriptstyle \simeq}\Big\downarrow & & {\scriptstyle \simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} \\
i^! j^! & \xrightarrow[i^!\beta_j]{} & i^! j^* & \xrightarrow[\beta_i]{} & i^* j^*
\end{array}$$

Additional conditions

For the basic enlargement result, we impose additional mild conditions on $\mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{E}, ^{\mathbf{!}}, ^{\mathbf{*}}, (\beta_\sigma)_{\sigma\in\square}\big)$ and its special subcategory $\mathbf{A}$. These conditions are easily verified for the (proper, ètale, fiber-square) setup we are most interested in at present.

(4) In the following $\mathbf{S}$-diagrams, suppose that $u_1 \in \mathbf{E}$ (resp. $f_1 \in \mathbf{P}$).

$$\begin{array}{ccccc}
\bullet & \xrightarrow{v_1} & \bullet & \xrightarrow{v_2} & \bullet \\
{\scriptstyle h}\Big\downarrow & \sigma_1 & {\scriptstyle g}\Big\downarrow & \sigma_2 & \Big\downarrow{\scriptstyle f} \\
\bullet & \xrightarrow[u_1]{} & \bullet & \xrightarrow[u_2]{} & \bullet
\end{array}
\qquad\qquad
\begin{array}{ccc}
\bullet & \xrightarrow{w} & \bullet \\
{\scriptstyle g_1}\Big\downarrow & \sigma_1 & \Big\downarrow{\scriptstyle f_1} \\
\bullet & \xrightarrow{v} & \bullet \\
{\scriptstyle g_2}\Big\downarrow & \sigma_2 & \Big\downarrow{\scriptstyle f_2} \\
\bullet & \xrightarrow[u]{} & \bullet
\end{array}$$

In either diagram, if σ_2 is a fiber square and the composed square $\sigma_2\sigma_1$ is in $\square$, then $\sigma_1 \in \square$.

(5) If the $\mathbf{S}$-square $\sigma(u, g, f, v)$

$$\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
{\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[u]{} & Y
\end{array}$$

is in $\square$, and u (resp. f) is special then so is v (resp. g).

(6) If the above $\sigma(u, g, f, v)$ is in $\square$ then so is any fiber square with the same u and f,

$$\begin{array}{ccc}
X'' & \longrightarrow & X \\
\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[u]{} & Y
\end{array}$$

and furthermore, the resulting map $X' \to X''$ is special.

Remark. Let $\mu\colon X' \to X''$ be an isomorphism and consider the following fiber squares, the first of which is, by (2), distinguished:

$$\begin{array}{ccc} X'' & \xrightarrow{\;1\;} & X'' \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ X'' & \xrightarrow[\;1\;]{} & X'' \end{array} \qquad\qquad \begin{array}{ccc} X' & \xrightarrow{\;\mu\;} & X'' \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ X'' & \xrightarrow[\;1\;]{} & X'' \end{array}$$

From (6) it follows that μ is special. Thus every isomorphism is special.

Proposition 6 (Enlargement)**.** *Under the preceding assumptions on* $\mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{E}, {}^!, {}^*, (\beta_\sigma)_{\sigma\in\square}\big)$ *and* $\mathbf{A}$, *there is a unique base-change setup* $\mathcal{B}' = \mathcal{B}'_{\mathbf{A}} = \mathcal{B}\big(\mathbf{S}, \mathbf{P}, \mathbf{E}, {}^!, {}^*, (\beta'_\sigma)_{\sigma\in\square'}\big)$ *such that*:

(i) *A commutative square*

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

is in $\square'$ *if and only if there is a fiber square in* $\square$

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

such that the resulting map $X' \to X''$ *is special.* (*Hence* $\square \subseteq \square'$; *and every fiber square in* $\square'$ *is in* $\square$.)

(ii) *For every* $\sigma \in \square \subseteq \square'$ *it holds that* $\beta_\sigma = \beta'_\sigma$.

This Proposition does not yet suffice for our purposes, since for the proper, étale, fiber-square $\mathcal{B}$, it only gives β_σ for diagrams which decompose as

$$\begin{array}{ccccc} \bullet & \xrightarrow{\;i\;} & \bullet & \xrightarrow{\;q_1\;} & \bullet \\ & & {\scriptstyle q_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ & & \bullet & \xrightarrow[\;u\;]{} & \bullet \end{array}$$

where the square is a fiber square and i an *open-and-closed immersion*, whereas we need the result more generally for where i is *étale.*

But the following Corollary provides a *second enlargement* to reach the desired situation:

Corollary 7. *In the Proposition, let* $\mathbf{A}'$ *be a subcategory of* $\mathbf{S}$ *such that for every map* $i\colon X \to Y \in \mathbf{A}'$ *the diagonal map* $\delta_i\colon X \to X \times_Y X$ *is in* $\mathbf{A}$.

Assume further that for any fiber square $\sigma_{v,f,g,u}$ *in* $\mathbf{S}$, *if* u *(resp.* f*) is in* $\mathbf{A}'$ *then so is* v *(resp.* g*). Then :*

(i) $\mathbf{A}'$ *is* $\mathcal{B}'$*-special; and conditions* (4)-(6) *hold for* $(\mathcal{B}', \mathbf{A}')$. *Thus it is meaningful to set* $\mathcal{B}'' := (\mathcal{B}')'_{\mathbf{A}'}$.

(ii) *If a fiber square* $\sigma(u, g, f, v)$ *with* $u \in \mathbf{A}'$ *is in* $\square$, *then any commutative* $\sigma_{v',f,g',u}$ *with* $v' \in \mathbf{A}'$ *and* $g' \in \mathbf{P}$ *is* $\mathcal{B}''$*-distinguished.*

The proofs of the Proposition and its Corollary consist mainly of verifying formally the commutativity of a number of suitably chosen diagrams, some of them rather large.

Example 8. The diagonal of a separated étale map is an open-and-closed immersion; and maps which are étale (resp. proper) remain so after arbitrary base change [**EGA IV,** §17].

Therefore the category $\mathbf{A}'$ of proper étale maps satisfies the hypotheses of the Corollary with respect to the (proper, étale, fiber-square) setup $\mathcal{B}$ and its special subcategory $\mathbf{A}$ of open-and-closed maps.

The resulting setup $\mathcal{B}''$ is then the sought-after unique enlargement of $\mathcal{B}$ (i.e., the one where all commutative $\sigma(u, g, f, v)$ with f, g proper and u, v étale are distinguished).

To review: Construct $\mathcal{B}'$, the enlargement of $\mathcal{B}$ via its special subcategory of open-and-closed immersions, then get $\mathcal{B}''$ as the enlargement of $\mathcal{B}'$ via its special subcategory of étale maps.