A bivariant theory of Hochschild homology

Joseph Lipman

Purdue University
Department of Mathematics

lipman@math.purdue.edu

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Outline

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1. Bivariant theories.

To extend the co/homology theory of manifolds, with its products (cup for cohomology, intersection for homology), and functorialities (contravariance for cohomology, covariance for homology—or vice-versa, via Gysin maps), to more general contexts, Fulton and Macpherson formulated and developed the notion of a Bivariant Theory.

In bivariant theories, homology and cohomology, which are much the same for manifolds, because of Poincaré duality, get separated into a pair of interacting functors, one contravariant and the other covariant.

The language of bivariant theories is useful in many contexts, for example in Intersection Theory and in Riemann-Roch-type theorems.

Presented today will be a bivariant theory relevant to the Hochschild co/homology theory on the category **S** of schemes that are flat, separated and essentially of finite-type over a fixed noetherian scheme S, a theory heavily involving Grothendieck duality.

Ingredients of a bivariant theory

1. An underlying category **C**.
   (In our case, **S**.)
2. A map T taking each **C**-map $X \to Y$ to a graded abelian group $T(X \to Y) = \oplus_{i \in \mathbb{Z}} T^i(X \to Y)$.
   (In our case, to be specified below.)
3. A class of maps in **C** called confined maps.
   (In our case, the proper **S**-maps.)
4. A class of oriented commutative squares in **C**, called independent squares

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

   (In our case, fiber squares with étale g and g'.)

Conditions on the data

1. The class of confined maps contains all identities; and it is stable under composition.

2. The class of independent squares contains all squares **d** such that $g = g' = \mathrm{id}$; and is stable for vertical and horizontal juxtaposition.

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \quad \& \quad \begin{tikzcd} X \arrow[r, "h'"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathbf{d}'"] & X'' \arrow[d, "f''"] \\ Y \arrow[r, "h"'] & Y'' \end{tikzcd} \quad \text{ind't} \implies \begin{tikzcd} X' \arrow[r, "h' \circ g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathbf{d}' \circ \mathbf{d}"] & X'' \arrow[d, "f''"] \\ Y' \arrow[r, "h \circ g"'] & Y \end{tikzcd} \quad \text{ind't.}$$

3. In an independent square **d**, if f (or g) is confined then so is f' (or g', respectively).

Operations for a bivariant theory

Product: For $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathbf{C}$, homomorphisms

$$T^i(X \xrightarrow{f} Y) \otimes T^j(Y \xrightarrow{g} Z) \longrightarrow T^{i+j}(X \xrightarrow{g\circ f} Z) \quad (i, j \in \mathbb{Z}).$$

Pushforth: For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{C}$, with f *confined*, homomorphisms

$$f_\star\colon\ T^i(X \xrightarrow{gf} Z) \longrightarrow T^i(Y \xrightarrow{g} Z) \quad (i \in \mathbb{Z}).$$

Pullback: For each independent square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

homomorphisms

$$g^\star\colon\ T^i(X \xrightarrow{f} Y) \longrightarrow T^i(X' \xrightarrow{f'} Y')$$

Required compatibilities among the operations

(A_1) Product is associative: For $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbf{C}$ and $\alpha \in T^i(f), \beta \in T^j(g), \gamma \in T^\ell(h)$, with $\cdot$ denoting product,

$$\boxed{(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)}$$

(A_2) pushforth is functorial: For $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbf{C}$, with f and g confined, and $\alpha \in T^i(hgf)$,

$$\boxed{(gf)_\star(\alpha) = g_\star f_\star(\alpha)}$$

(A_{12}) Product and pushforth commute: For $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathbf{C}$, with f confined, and $\alpha \in T^i(gf), \beta \in T^j(h)$,

$$\boxed{f_\star(\alpha \cdot \beta) = f_\star(\alpha) \cdot \beta}$$

Compatibilites (ct'd)

(A_3) Pullback is functorial: For independent squares

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array}$$

and $\alpha \in T^i(f'')$,

$$\boxed{(gh)^\star(\alpha) = h^\star g^\star(\alpha)}$$

Compatibilites (ct'd)

Consider the diagram of independent squares

$$\begin{array}{ccc}
X' & \xrightarrow{h''} & X \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[h']{} & Y \\
{\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
Z' & \xrightarrow[h]{} & Z
\end{array}$$

(A_{13}) Product and pullback commute:

For $\alpha \in T^i(f)$ and $\beta \in T^j(g)$,

$$\boxed{h^\star(\alpha \cdot \beta) = h'^\star(\alpha) \cdot h^\star(\beta)}$$

(A_{23}) Pushforth and pullback commute:

If f is confined, and $\alpha \in T^i(gf)$, then

$$\boxed{f'_\star h^\star(\alpha) = h^\star f_\star(\alpha)}$$

Compatibilites (ct'd)

($\mathbf{A}_{123}$) Projection formula: For

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "f"] & \\ Y' \arrow[r, "g"'] & Y \arrow[r, "h"'] & Z \end{tikzcd}$$

with **d** independent and f confined, and
$\alpha \in T^i(f), \beta \in T^j(hg)$,

$$\boxed{g'_\star(g^\star(\alpha) \cdot \beta) = \alpha \cdot g_\star(\beta)}$$

2. Basics of Grothendieck Duality

A contravariant pseudofunctor (aka 2-functor) on a category $\mathbf{C}$ assigns:

- to each $X \in \mathbf{C}$ a category $\mathbf{X}^\#$,
- to each map $f\colon X \to Y$ a functor $f^\#\colon \mathbf{Y}^\# \to \mathbf{X}^\#$ (with $\mathbf{1}^\# = \mathbf{1}$), and
- to each map-pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ a functorial transitivity isomorphism

$$d_{f,g}\colon f^\# g^\# \xrightarrow{\ \sim\ } (gf)^\#$$

satisfying $d_{\mathbf{1},g} = d_{g,\mathbf{1}} =$ identity, and a kind of associativity, namely,

for each triple of maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ the following commutes:

$$\begin{tikzcd}
(hgf)^\# & f^\#(hg)^\# \arrow[l, "d_{f,hg}"'] \\
(gf)^\# h^\# \arrow[u, "d_{gf h}"] & f^\# g^\# h^\# \arrow[l, "d_{f,g}"] \arrow[u, "d_{g,h}"']
\end{tikzcd}$$

Covariant pseudofunctor is similarly defined, with arrows reversed, i.e., it means contravariant functor on $\mathbf{C}^{\mathrm{op}}$.

Example (restriction of scalars (contravariant); extension of scalars (covariant))

$$\mathbf{C} := \text{category of rings}, \qquad \mathbf{X}^{\#} := (\text{category of left } X\text{-modules}).$$

For any $f\colon X \to Y$, $M \in \mathbf{Y}^{\#}$, $N \in \mathbf{X}^{\#}$,

$$f^{\#}M = M \ (\in \mathbf{X}^{\#}), \qquad \text{resp.} \qquad f_{\#}N := Y \otimes_X N.$$

Example (derived inverse-image (contravariant); derived direct-image (covariant))

$$\mathbf{C} := \text{category of ringed spaces}$$

$$\mathbf{X}^{\#} := \mathbf{D}(X) \quad (\text{derived category of } \mathcal{O}_X\text{-modules})$$

$$f^{\#} := \mathbf{L}f^{*}, \qquad \text{resp.} \qquad f_{\#} := \mathbf{R}f_{*}.$$

These pseudofunctors are adjoint:

For any $f\colon X \to Y$, $E \in \mathbf{D}(Y)$, $F \in \mathbf{D}(X)$,

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{L}f^{*}E, F) \cong \mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathbf{R}f_{*}F).$$

Base change and tor-independence

To a commutative square of ringed-space maps

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

associate the functorial map

$$\theta_\sigma \colon \mathbf{L}u^*\mathbf{R}f_* \to \mathbf{R}g_*\mathbf{L}v^*,$$

adjoint to the natural composition $\mathbf{R}f_* \to \mathbf{R}f_*\mathbf{R}v_*\mathbf{L}v^* \xrightarrow{\;\sim\;} \mathbf{R}u_*\mathbf{R}g_*\mathbf{L}v^*$.

If σ is a fiber square, then, with $\mathbf{D}_{\mathrm{qc}}$ the full subcategory of $\mathbf{D}$ whose objects are the complexes with quasi-coherent homology,

θ_σ is an isomorphism of functors on $\mathbf{D}_{\mathrm{qc}} \iff \sigma$ is tor-independent,

i.e., for all $x \in X$ and $y' \in Y'$ such that $f(x) = u(y') =:$ (say) y,

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}\big(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}\big) = 0 \quad \text{for all } i \neq 0.$$

Twisted inverse-image pseudofunctor

$\mathbf{D}^+_{\mathrm{qc}}$ denotes the full subcategory of $\mathbf{D}_{\mathrm{qc}}$ whose objects are the homologically bounded-below complexes with quasi-coherent homology.

Example (twisted inverse image pseudofunctor)

Grothendieck Duality is concerned with the twisted inverse-image, a $\mathbf{D}^+_{\mathrm{qc}}$-valued pseudofunctor $(-)^!$ on the category $\mathbf{C}_{\mathrm{f}}$ of essentially-finite-type separated maps of noetherian schemes, uniquely determined up to isomorphism by the following three properties:

Characteristic properties of twisted inverse image

(i) The pseudofunctor $(-)^!$ restricts on the subcategory of proper maps to a right adjoint of the derived direct-image pseudofunctor.
(ii) The pseudofunctor $(-)^!$ restricts on the subcategory of étale maps to the (derived or not) inverse-image pseudofunctor $(-)^*$.
(iii) For any fiber square

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "u"'] & \bullet \end{tikzcd} \qquad (f, g \text{ proper; } u, v \text{ étale}),$$

the base-change map $\beta_\sigma \colon v^*f^! \to g^!u^*$, defined as the adjoint to the natural composition

$$\mathbf{R}g_*v^*f^! \xrightarrow{\theta_\sigma^{-1}} u^*\mathbf{R}f_*f^! \longrightarrow u^*,$$

is the natural composite isomorphism

$$v^*f^! = v^!f^! \xrightarrow{\sim} (fv)^! = (ug)^! \xrightarrow{\sim} g^!u^! = g^!u^*.$$

Twisted inverse image and derived tensor

For any $\mathbf{C}_\mathsf{f}$-map $f\colon X \to Y$ there is a natural functorial map (defined via compactification and the "projection isomorphism")

$$\chi^f_E\colon f^!\mathcal{O}_Y \underset{=}{\otimes} \mathbf{L}f^*E \to f^!E \qquad \big(E \in \mathbf{D}^+_{\mathrm{qc}}(Y)\big);$$

and

$$\chi^f_E \textit{ iso for all } E \iff f \textit{ has finite tordim.}$$

It follows that f has finite tordim iff for all $E, F \in \mathbf{D}^+_{\mathrm{qc}}(Y)$, the natural map is an isomorphism

$$\chi^f_{E,F}\colon f^!E \underset{=}{\otimes} \mathbf{L}f^*F \xrightarrow{\ \sim\ } f^!(E \underset{=}{\otimes} F).$$

3. Relative fundamental class

Convention. As we will be dealing exclusively with derived-category functors, we lighten notation by omitting **L** and **R**. So f_* means $\mathbf{R}f_*$, etc.

The isomorphism $f^!\mathcal{O}_Y \underset{=}{\otimes} f^*E \xrightarrow{\sim} f^!E$ (for $f\colon X \to Y$ of finite tordim) indicates that understanding $f^!$ in the finite tordim case reduces to understanding $f^!\mathcal{O}_Y$. The standard example is when f is smooth, with all fibers of dimension n, in which case there is a natural $\mathbf{D}(X)$-isomorphism

$$\Omega^n_f[n] \xrightarrow{\sim} f^!\mathcal{O}_Y.$$

This is the key to the realization of abstract Grothendieck duality in concrete terms, such as Serre duality, via differentials.

More generally, for any flat essentially-finite-type separated $x\colon X \to S$, with diagonal $\delta\colon X \to X \times_S X$, and with $\mathcal{H}_x := \delta^*\delta_*\mathcal{O}_X$, the Hochschild complex, there is a natural derived-category map

$$\mathcal{H}_x \to x^!\mathcal{O}_S$$

whence natural compositions (not usually isomorphisms) for $n \geq 0$,

$$\Omega^n_x \to HH^x_n(\mathcal{O}_X) := H^{-n}\mathcal{H}_x \to H^{-n}(x^!\mathcal{O}_S).$$

Residue theorem

If x is equidimensional, with n-dimensional fibers, then the preceding map is equivalent to a derived-category map

$$c_x \colon \Omega_x^n[n] \to x^!\mathcal{O}_S.$$

If, moreover, x is proper, there results a natural composite map

$$x_*\Omega_x^n \xrightarrow{x_* c_x} x_*x^!\mathcal{O}_S \to \mathcal{O}_S,$$

which is is a global pasting together of local residues (which can be defined via Hochschild homology).

This last statement is a general form of the Residue Theorem (which for curves is the one you know).

Therefore, functorial properties of fundamental classes (such as those enshrined in the following bivariant theory) and functorial properties of residues will reflect each other

Relative fundamental class (ct'd)

Still more, let $f: X \to Y$ be a flat map of flat finite-type separated S-schemes, with respective diagonal maps δ and γ. Then there is a canonical map of functors, the relative fundamental class,

$$c_f : \delta^*\delta_* f^* \to f^!\gamma^*\gamma_*.$$

The functors are understood to be operating on complexes in $\mathbf{D}^+_{\mathrm{qc}}$.

This fundamental class is the principal protagonist of our story.

It is defined by combining (details omitted) a number of elementary maps arising formally from the adjointness of inverse and direct image, and—for closed immersions—of direct image and $^!$, as well as inverses of isomorphisms such as base-change and projection:

$$f_*E \underset{=}{\otimes} F \xrightarrow{\ \sim\ } f_*(E \underset{=}{\otimes} f^*F).$$

(As a *map,* projection is defined to be adjoint to the natural composition

$$f^*(f_*E \underset{=}{\otimes} F) \xrightarrow{\ \sim\ } f^*f_*E \underset{=}{\otimes} f^*F \to E \underset{=}{\otimes} f^*F.$$

Showing it to be an isomorphism takes some work.)

Transitivity of the fundamental class

Theorem

For flat **S***-morphisms* $X \xrightarrow{g} Y \xrightarrow{h} Z$, *with diagonals* $\delta\colon X \to X \times_S X$, $\gamma\colon Y \to Y \times_S Y$, $\beta\colon Z \to Z \times_S Z$, *the following diagram commutes:*

$$\begin{tikzcd}
\delta^*\delta_* g^* h^* \arrow[r, "c_g"] \arrow[d, "c_{hg}"'] & g^!\gamma^*\gamma_* h^* \arrow[d, "c_h"] \\
(hg)^!\beta^*\beta_* \arrow[r, "\sim"] & g^! h^! \beta^*\beta_*
\end{tikzcd}$$

This motivates the ensuing bivariant Hochschild homology theory.

It provides canonical orientations for the class of flat maps in that theory.

How deep is the theorem? The proof is simple in principle but excruciating in practice: expand the diagram according to the definitions involved, then decompose the resulting diagram into "elementary" commutative ones. The problem is to find a suitable decomposition (cf. Rubik's cube), which at present turns out to involve 50 or more subdiagrams.

3. Bivariant Hochschild theory

Recall: **S** is the category of schemes that are flat, separated and essentially of finite type over a fixed noetherian scheme S.

S-maps are separated and essentially of finite type, but need not be flat.

Confined maps are now *proper* **S**-maps.

Independent squares are now those oriented fiber squares in **S**

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

such that g—and hence g'—is *étale*.

The Hochschild complex

Our theory uses a derived category avatar of the Hochschild complex.

Again, the symbols **L** and **R** are dropped, but all functors are derived.
For a scheme X in **S** let $\delta\colon X \to X \times_S X$ be the diagonal embedding, (closed, since X is separated over S). Set

$$\mathcal{H}_{X|S} = \mathcal{H}_X := \delta^*\delta_*\mathcal{O}_X.$$

The Hochschild homology and cohomology groups of $X|S$, of degree $i \in \mathbb{Z}$, with coefficients in $M \in \mathbf{D}_{\mathrm{qc}}(X)$, are, respectively,

$$\begin{aligned}
\mathrm{HH}_i^{X|S}(M) := \mathrm{Tor}_i^{X|S}(\mathcal{H}_{X|S}, M) &= \mathrm{H}^{-i}(X, \mathcal{H}_{X|S} \underline{\otimes}_X M) \\
&\cong \mathrm{H}^{-i}(X \times_S X, \delta_*\mathcal{O}_X \underline{\otimes}_{X\times_S X} \delta_* M).
\end{aligned}$$

$$\begin{aligned}
\mathrm{HH}^i_{X|S}(M) := \mathrm{Ext}^i_{X|S}(\mathcal{H}_{X|S}, M) &= \mathrm{H}^i\left(X, \mathbf{R}\mathcal{H}om_X(\mathcal{H}_{X|S}, M)\right) \\
&\cong \mathrm{H}^i\left(X \times_S X, \mathbf{R}\mathcal{H}om_{X\times_S X}(\delta_*\mathcal{O}_X, \delta_* M)\right).
\end{aligned}$$

These definitions are compatible with classical ones for ordinary algebras.

Functoriality of the Hochschild complex

Let $x\colon X \to S$ and $y\colon Y \to S$ be objects in $\mathbf{S}$; and let δ_x, δ_y be the corresponding diagonal immersions. Let $f\colon X \to Y$ be an $\mathbf{S}$-morphism. From the resulting commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\delta_x"'] \arrow[dr, phantom, "\sigma"] & Y \arrow[d, "\delta_y"] \\ X \times_S X \arrow[r, "f \times f"'] & Y \times_S Y \end{tikzcd}$$

one gets the composite map

$$f^\sharp\colon f^*\mathcal{H}_Y = f^*\delta_y^*\delta_{y*}\mathcal{O}_Y \xrightarrow{\ \sim\ } \delta_x^*(f \times f)^*\delta_{y*}\mathcal{O}_Y$$
$$\xrightarrow[\theta_\sigma]{} \delta_x^*\delta_{x*}f^*\mathcal{O}_Y = \mathcal{H}_X.$$

Proposition

If $f\colon X \to Y$ is étale then $f^\sharp\colon f^*\mathcal{H}_Y \to \mathcal{H}_X$ is an isomorphism.

(Challenge: eliminate flatness)

There have recently appeared more sophisticated approaches to the Hochschild complex $\mathcal{H}_{X|S}$, based on Quillen's viewpoint toward derived Hochschild homology, via DG algebra resolutions. (Buchweitz-Flenner, Lowen-van den Bergh).

These approaches do not require X to be flat over S.

This suggests the possibility of extending the theory which follows to the nonflat case.

For example, functoriality still holds, but is considerably harder to establish.

Associated graded group

To each **S**-morphism $f\colon X \to Y$ assign the graded group

$$\begin{aligned}\oplus_{i\in\mathbb{Z}} E^i(f) &:= \oplus_{i\in\mathbb{Z}} \mathrm{HH}^i(f^!\mathcal{H}_Y)\\ &= \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}(X)}\big(\mathcal{H}_X, f^!\mathcal{H}_Y[i]\big).\end{aligned}$$

Remark. For flat f, the fundamental class

$$c_f(\mathcal{O}_Y)\colon \mathcal{H}_X = \delta_X^*\delta_{X*}f^*\mathcal{O}_Y \to f^!\delta_Y^*\delta_{Y*}\mathcal{O}_Y = f^!\mathcal{H}_Y$$

is an element of $E^0(f)$.

As mentioned before, *transitivity of the fundamental class* can be interpreted as meaning that this element is a *canonical orientation* for f (as a member of the class of flat **S**-maps.)

Such orientations are used in bivariant theories to construct, e.g., canonical Gysin maps.

Product; pushforth

For a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\alpha \in E^i(f)$, $\beta \in E^j(g)$, the product

$$\beta \cdot \alpha \in E^{i+j}(gf)$$

is the composition

$$\mathcal{H}_X \xrightarrow{\alpha} f^!\mathcal{H}_Y[i] \xrightarrow{\text{via } \beta} f^!g^!\mathcal{H}_Z[j][i] \xrightarrow{\text{nat'l}} (gf)^!\mathcal{H}_Z[i+j].$$

When f is proper, the pushforth

$$f_\star \colon E^i(gf) \to E^i(g)$$

associates to $\alpha \in E^i(gf)$, say

$$\alpha \colon \mathcal{H}_X \to (gf)^!\mathcal{H}_Z[i] \cong f^!g^!\mathcal{H}_Z[i]$$

the natural composition

$$f_\star\alpha \colon \mathcal{H}_Y \longrightarrow f_*\mathcal{H}_X \xrightarrow{f_*\alpha} f_*f^!g^!\mathcal{H}_Z[i] \longrightarrow g^!\mathcal{H}_Z[i].$$

Setup for pullback

The independent squares should form a class of fiber squares

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

for which there is a natural isomorphism

$$g'^*\mathcal{H}_X \underset{=}{\otimes} f'^*\mathcal{H}_{Y'|Y} \xrightarrow{\sim} \mathcal{H}_{X'} \qquad (\#)$$

Not true for all fiber squares. (Note: $f'^*\mathcal{H}_{Y'|Y} = \mathcal{H}_{X'|X}$, always).
Okay, e.g., if g is a projection $Y' := S' \times_S Y \to Y$, or if g is étale (in which case $\mathcal{H}_{Y'|Y} = \mathcal{O}_Y$.)

This is why independent squares were required to have an étale base.

If g is smooth, then g factors locally as an étale map followed by a projection from $\mathbb{P}^n_Y \to Y$, so at least locally over Y' we get what we want.
One would hope these local isomorphisms paste to a canonical global one, allowing one to expand the class of independent squares.

Pullback

If for any independent square $\exists$ natural iso

(#): $g'^*\mathcal{H}_X \underset{=}{\otimes} f'^*\mathcal{H}_{Y'|Y} \xrightarrow{\sim} \mathcal{H}_{X'}$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{(ind't)}"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

then for

$$\alpha\colon \mathcal{H}_X \to f^!\mathcal{H}_Y[i] \in E^i(f),$$

the pullback

$$g^\star\alpha \in E^i(f')$$

is the natural composition

$$\begin{aligned} \mathcal{H}_{X'} &\xrightarrow[(\#)]{\sim} g'^*\mathcal{H}_X \underset{=}{\otimes} f'^*\mathcal{H}_{Y'|Y} \\ &\xrightarrow[g'^*\alpha]{} g'^*f^!\mathcal{H}_Y[i] \underset{=}{\otimes} f'^*\mathcal{H}_{Y'|Y} \\ &\xrightarrow{\sim} f'^!g^*\mathcal{H}_Y[i] \underset{=}{\otimes} f'^*\mathcal{H}_{Y'|Y} \qquad \text{(base change iso)} \\ &\xrightarrow[\chi]{} f'^!\big(g^*\mathcal{H}_Y[i] \underset{=}{\otimes} \mathcal{H}_{Y'|Y}\big) \xrightarrow{\sim} f'^!\mathcal{H}_{Y'}[i] \end{aligned}$$

where χ was defined earlier, and the final iso comes from (#) for $f = 1$.

Verifying the axioms

Verification of the axioms $\mathbf{(A_?)}$ amounts to verifying commutativity of various diagrams involving combinations of maps (discussed above) coming from Grothendieck Duality.

This verification becomes more complicated as the number of integers in **?** increases. For $\mathbf{(A_{123})}$ one gets a diagram which has to be split up into 12 subdiagrams, whose commutativity takes several pages to establish.

Carrying all this out (not to mention the more complicated transitivity theorem) reveals some of the wealth of the formalism of Duality—a formalism based on Grothendieck's notion of six operations.

But the tedium involved drives me to keep raising the following questions:

Is there a "coherence" theorem guaranteeing that all diagrams of a certain form, built up from the axioms, must commute?

Or an algorithm for deciding whether or not such diagrams commute?

Or, at least, could one train a computer to become an expert assistant in the task?

Base change for the fundamental class

Theorem (Orientation and pullback)

The fundamental class is compatible with base change: Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be an independent square, with f flat. *Then*

$$g^{\star}(\mathbf{c}_f) = \mathbf{c}_{f'}.$$

The proof involves, again, many commutativities.

5. Bivariant Hochschild cohomology ring

"Bivariance" comes from associated contra- and covariant graded functors. In the present case, one has the Hochschild cohomology of $X \in \mathbf{S}$:

$$\mathrm{HH}^i(X) := \mathrm{HH}^i(X \xrightarrow{\mathrm{id}} X) = \mathrm{Ext}^i_X(\mathcal{H}_X, \mathcal{H}_X)$$

The cup product

$$\smile\colon \mathrm{HH}^i(X) \otimes \mathrm{HH}^j(X) \to \mathrm{HH}^{i+j}(X)$$

is the product associated to the composition $X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$. It is just the usual Yoneda product.

There are pull back homomorphisms

$$f^\star\colon \mathrm{HH}^i(X) \to \mathrm{HH}^i(X')$$

for every *étale* morphism $f\colon X' \to X$.

These operations give HH^* the structure of a graded-ring-valued contravariant functor for étale morphisms.

Bivariant Hochschild homology modules

The associated covariant functor, the Hochschild homology of $X \in \mathbf{S}$, is

$$\mathrm{HH}_i(X) := \mathrm{HH}^{-i}(X \xrightarrow{x} S) = \mathrm{Ext}_X^{-i}(\mathcal{H}_X, x^!\mathcal{O}_S).$$

The cap product

$$\frown\colon\ \mathrm{HH}^i(X) \otimes \mathrm{HH}_j(X) \to \mathrm{HH}_{j-i}(X)$$

is the product for the composition $X \xrightarrow{\mathrm{id}} X \xrightarrow{x} S$. It makes HH_* into a graded module over the graded ring HH^* .

Associated to the composition $X' \xrightarrow{f} X \xrightarrow{x} S$, with f proper, there are push forward homomorphisms

$$f_\star\colon\ \mathrm{HH}_i(X') \to \mathrm{HH}_i(X).$$

These make HH_* a covariant functor for proper maps.

The axioms imply additional relations here. For example, (A_{123}) gives

$$f_\star(f^\star(\beta) \frown \alpha) = \beta \frown f_\star(\alpha) \qquad (f \text{ proper}).$$

Smooth maps

For smooth $x\colon X \to S$, our HH_* agrees with the one studied by Căldăraru in recent years. His HH^* is a retract of the one here.
In the smooth case, the fundamental class is, as before, a canonical iso

$$\Omega_x^n \xrightarrow{\sim} x^!\mathcal{O}_S \qquad (n = \text{relative dimension}).$$

If, furthermore, $S = \operatorname{Spec} k$ with k a field of characteristic 0, then one has the Hochschild-Kostant-Rosenberg isomorphism

$$\mathcal{H}_X \cong \bigoplus_{p=0}^{n} \Omega_X^p[p]$$

A brief computation using these isos yields canonical isos (for $i \in \mathbb{Z}$)

$$\mathrm{HH}_i(X) \cong \bigoplus_{p-q=i} \mathrm{H}^p(X, \Omega_X^q)$$

Bivariant Hochschild homology and Hodge homology

(After Căldăraru.) Again,

$$\mathrm{HH}_i(X) = \bigoplus_{p-q=i} \mathrm{H}^p(X, \Omega_X^q).$$

$$\begin{array}{ccccc}
 & & \mathrm{H}^{0,0} & & \\
 & \mathrm{H}^{1,0} & & \mathrm{H}^{0,1} & \\
\mathrm{H}^{2,0} & & \mathrm{H}^{1,1} & & \mathrm{H}^{0,2} \\
 & \mathrm{H}^{2,1} & & \mathrm{H}^{1,2} & \\
 & & \mathrm{H}^{2,2} & &
\end{array}$$

Thus, with $\mathrm{H}^{p,q} = \mathrm{H}^p(X, \Omega_X^q)$,

- The sums of the columns of the Hodge diamond give Hochschild homology.
- The sums of the rows of the Hodge diamond give de Rham cohomology.