## MA 385 FINAL EXAM

May 4, 2004

| NAME    |      |
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| 14 WIVI | <br> |

- No calculators, books, notes, or any other aids may be used.
- Don't work on this cover page. If necessary, use the back of any page.
- Pages 5-6 carry 15 points altogether, and page 9 carries 15 points. The other five pages carry 10 points each, for a total of 80 points, (which works out to 10 points for every 15 minutes of exam time—so budget accordingly.)

1. In (a), (b), and (c), translate the bulleted formulas into colloquial English, interpreting the predicates as indicated. (That is, express the meaning as you would in ordinary conversation.)

(a)  $\bullet \exists x (Cx \land \exists! y \forall z ((Rz \land Exz) \rightarrow Sxy)).$ 

Cx: x is a candidate (for some political office).

Rx:x is a political rally.

Exy: y is an event where x speaks.

Suv: u delivers the speech v.

Some candidates give the same speech whenever they address a political rally

(b) • 
$$Pz \leftrightarrow \forall x \forall y (z = x.y \rightarrow (x = 1 \lor y = 1)).$$

(The domain of discourse is the set of positive integers.)

Pz: z is a prime number. A positive integer is prime if it is not the product two integers unless one of them is 1

Remarks 1). The condition 2 = 1 should have been part of Pe (1 is not (c) •  $Lxy \leftrightarrow \exists z(y=x+z)$ . Context of a definition, this abuse of language is usually allowed (conversationally).

(conversationally understood)

u is less than v if [and only if] v is the sum of u and

- (d) Using the above predicates P and L, express the following sentence as a wff in the Predicate Calculus. (You may *not* use the abbreviation  $\exists^{!2}$ .)
  - There are exactly two prime numbers less than 4.

- (e) Express the following sentence as a wff in the Predicate Calculus, using the given abbreviations.
- Every student solves some problem on each exam he/she takes.

Sx:x is a student.

(This should have been given as a binary predicate Eu:u is an exam.

Eur: u is an exam taken by v) Pxy: x is a problem on exam y.

Cxz: x solves z.

2. A sentence S containing just three atoms A, B and C has the following truth table:



Write a disjunctive-normal-form sentence equivalent to  $\overline{S}$  (=  $\neg S$ ); and use that to decide which of the following conjunctive-normal-form sentences is equivalent to S. (Circle the letter in front of the correct answer.)

- (a)  $(A \lor B \lor C) \land (\overline{A} \lor \overline{B} \lor \overline{C})$ .
- $(\overline{b})(A \vee \overline{B} \vee C) \wedge (\overline{A} \vee B \vee \overline{C}).$ 
  - $(c) (\overline{A} \vee B \vee \overline{C}) \wedge (A \vee B \vee \overline{C}).$
  - (d)  $(\overline{A} \vee B) \wedge (A \vee C) \wedge (B \vee \overline{C})$ .
  - (e) None of the above.

3. For the formula  $\exists x R(x,y) \to \exists x (P(x) \lor \neg \exists y Q(x,y))$ , construct an equivalent formula in prenex form.

(5 pt)

42 (-R(x,y) v (P(u) v - Q(u,z)))

(alternatively)  
... 
$$(R(x,y) \rightarrow (Q(u,z) \rightarrow P(u)))$$

4. For any x, abbreviate Dodec(x) to Dx and Cube(x) to Cx. Write down a formal (Fitch-style) proof whose premises are

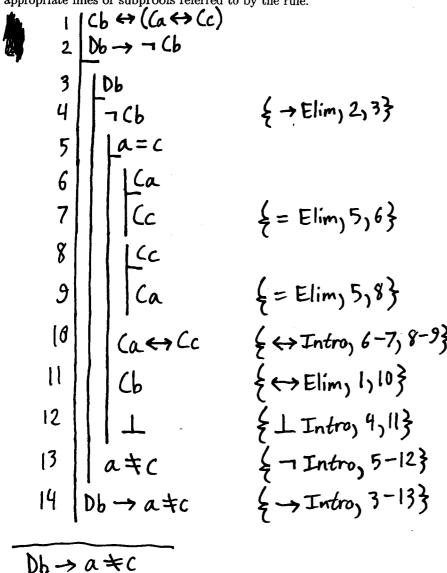
1. Cb 
$$\leftrightarrow$$
 (Ca  $\leftrightarrow$  Cc),

2. 
$$Db \rightarrow \neg Cb$$
,

and whose conclusion is

$$Db \rightarrow a \neq c$$
.

Number your lines, and indicate whenever you can which inference rule you are using, as well as the appropriate lines or subproofs referred to by the rule.



5. (a) The following formal proof has one faulty step. Say which one, and explain why it is invalid.

Step 6. When 3 Elim is used,
the letter introduced in the
preceding subproof (in this
case, [b]) cannot be taken
out of the subproof.

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1. \forall x \exists y (Tet(x) \rightarrow Tet(y))

2. a \forall Tet(a)
3. \exists y (Tet(a) \rightarrow Tet(y))
4. b \forall Tet(a) \rightarrow Tet(b)
5. Tet(b)
6. Tet(b)
7. \forall x (Tet(x) \rightarrow Tet(b))
8. \exists y \forall x (Tet(x) \rightarrow Tet(y))
9. Intro 26
9. a \exists y \forall x (Tet(x) \rightarrow Tet(y))
9. Intro 7
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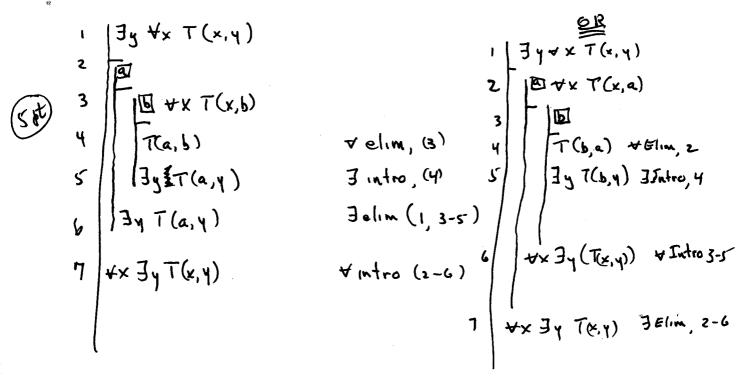
(b) Nevertheless, the conclusion, Sentence 8, is a first-order validity. Explain why (informally).

NOTE Many people (mis) read this question as stating that Sentence 1 +> Sentence 8 is a fo validity. In fact, only Sentence 8 is addressed here.

Solution Discussed in class: break it into two cases, according as there exists a Tet or there doesn't....
This is independent of the meaning of "Tet".

Better solution (From one exam paper). Sentence 8 is the prenex form of  $\exists x T(x) \rightarrow \exists y T(y)$ , which is obviously an FO validity! (The 5 same holds for Sentence 1.)

(c) Write down a formal (Fitch-style) proof whose premise is  $\exists y \forall x T(x,y)$  and whose conclusion is  $\forall x \exists y T(x,y)$ . (T is any binary predicate.) Number your lines, and indicate whenever you can which inference rule you are using, as well as the appropriate lines or subproofs referred to by the rule.



(d) Why is  $\forall x \exists y \, T(x,y) \to \exists y \forall x \, T(x,y)$  not a first-order validity?

Fails for the interpretation of T(x,y) as x=y on the sct {1,2}.



- 6. In the following statements, a and b are sets, and px is the powerset of the set x. For each of the statements, explain (informally) why it is always true or sometimes false, as the case may be.
  - 1.  $b \in \wp b$ .
  - 2.  $b \subset \wp b$ .
  - 3.  $\wp(a \cup b) = \wp a \cup \wp b$ .
  - 4.  $\wp(a \cap b) = \wp a \cap \wp b$ .
- 1. True: b is a subset of b, and so is a member of b.
- 2 False: Take b to be a set consisting of one dog.

  (usually)

  That dog is not a member of bb (which consists of sets)
- 3. False: If a has a member x not in b, and b has a (usually) member y not in a, then  $\{x,y\}$  [or aub] is a member of  $\{a,b\}$ , but not of  $\{a,b\}$ .

  (Example:  $a = \{i\}$ ,  $b = \{2\}$ ).
- Y. True: A subset of and is a subset both of a and of b, so (anb) = fangb.

  A set in fangb is a subset of a and of b, hence of anb (because all its elements are in a end in b).

  Thus gangb = f(anb).

  Together, and show that f(anb) = fangb

## 7. The Fibonacci sequence

$$F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, \ldots = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

is defined recursively by

$$F_0 = 0$$
,  $F_1 = 1$ ,  $F_{n+1} = F_n + F_{n-1}$   $(n \ge 1)$ .

Prove by induction that for all  $n \ge 1$ ,

$$F_n^2 = F_{n+1}F_{n-1} - (-1)^n.$$

(Hint: Expand  $F_{n+1}F_{n-1}$  using  $F_{n+1} = F_n + F_{n-1}$ .)

Inductive step Assume true for n-1 (n22), prove for n.

$$= F_{n}F_{n-1} + F_{n}F_{n-2} - (-1)^{n-1}$$
(by case n-1)

Subtract (-1) from both sides to get conclusion.

REMARK Many people tried the method of "backwards proof" where you start with the desired conclusion and make transformations until you reach a sentence which is clearly true. This may sometimes suggest a real proof, which consists in running the backwards proof from its end to its beginning. But to do so you must cleck that every stop in the proof is reversible. Otherwise, you could get something like the following "proof" that 1=-1.

1=1 /

8.  $\mathcal T$  denotes a set of sentences of Propositional Logic (Sentential Calculus).  $\phi$  denotes the empty set.

T is tt-satisfiable if there is a truth assignment making all sentences in T true.

For a sentence S write  $\mathcal{T} \vdash S$  as an abbreviation for "there is a formal proof whose premises are among the members of  $\mathcal{T}$  and whose conclusion is S."

Write  $\mathcal{T} \models S$  as an abbreviation for "S is a tautological consequence of  $\mathcal{T}$ ." (Remember that this notion is defined in terms of truth assignments.)

S is a tautology if  $\phi \models S$ . (In other words, every truth assignment makes S true).

315pt

Prove the following statements:

- (a)  $\mathcal{T} \models S$  if and only if the set  $\mathcal{T} \cup \{\neg S\}$  is not tt-satisfiable.
- (b) For any sentence  $T, T \to S$  is a tautology if and only if  $\{T\} \models S$ .
- (c) If  $\mathcal{T} \vdash S$  then  $(\mathcal{T} \cup \{\neg S\}) \vdash \bot$ .
- (a) T = S means that every truth assignment making all sentencer in T true also makes S true, i.e., makes 7 S false. This is exactly the same as no truth assignment making everything in T true and also 7 S true. In other words it says that TU {75} is not tt-satisfiable
  - (b) "T→S" being a tautology means that every truth assignment makes it true. In other words (according to the truth table for →) whenever T is true them S is true. That's equivalent to {T} = S.
  - (c) If The Sthen there are sentences Ti, The in T and a proof. It is at the bottom, you obviously have a proof that has premises 757,..., The and conclusion I (use I intro at the last step)

    Thus Tuf75} + I.