MA 385 FINAL EXAM

May 4, 2004

NAME ____

• **No calculators, books, notes, or any other aids may be used.**

• **Don't work on this cover page. If necessary, use the back of any page.**

• **Pages 5–6 carry 15 points altogether, and page 9 carries 15 points. The other five pages carry 10 points each, for a total of 80 points, (which works out to 10 points for every 15 minutes of exam time—so budget accordingly.)**

5 x 2pt

1. In (a), (b), and (c), translate the bulleted formulas into *colloquial* English, interpreting the predicates as indicated. (That is, express the meaning as you would in ordinary conversation.)

NOTE There are usually many different correct answers

(a) • $\exists x(Cx \wedge \exists! y \forall z((Rz \wedge Exz) \to Sxy))$.

Cx : x is a candidate (for some political office).

Rx : x is a political rally.

Exy : y is an event where x speaks.

Suv : u delivers the speech v.

Some candidates give the same speech whenever they address a political rally.

(b) • $Pz \leftrightarrow \forall x \forall y(z = x.y \to (x = 1 \vee y = 1))$.

(The domain of discourse is the set of positive integers.)

Pz : z is a prime number. A positive integer is prime if it is not the product of two integers unless one of them is 1.

Remarks 1. The condition $z \neq 1$ should have been part of Pz (1 is not considered to be prime).
2. Strictly speaking, "if" should be "if and only if". In the context of a definition, this abuse of language is usually allowed (conversationally).

(c) • $Lxy \leftrightarrow \exists z(y = x + z)$.

(The domain of discourse is the set of positive integers.)

Luv : u is less than v.

(conversationally understood)

u is less than v if [and only if] v is the sum of u and another number.

(d) Using the above predicates P and L, express the following sentence as a wff in the Predicate Calculus. (You may *not* use the abbreviation $\exists^{!2}$.)

• There are exactly two prime numbers less than 4.

$$\exists x \exists y [x \neq y \wedge \forall z ((Pz \wedge Lz4) \leftrightarrow (z = x \vee z = y))]$$

(e) Express the following sentence as a wff in the Predicate Calculus, using the given abbreviations.

• Every student solves some problem on each exam he/she takes.

Sx : x is a student.

Eu : u is an exam. (This should have been given as a binary predicate

Pxy : x is a problem on exam y. Euv : u is an exam taken by v)

Cxz : x solves z.

$$\forall x \forall y ((Sx \wedge Ey) \to \exists z (Pzy \wedge Cxz))$$

2. A sentence S containing just three atoms A, B and C has the following truth table:

5 pt

A	B	C	S	
T	T	T	T	
T	T	F	T	
T	F	T	F	$\leftarrow A\bar{B}C$
T	F	F	T	
F	T	T	T	
F	T	F	F	$\leftarrow \bar{A}B\bar{C}$
F	F	T	T	
F	F	F	T	

Write a disjunctive-normal-form sentence equivalent to $\overline{S}$ ($= \neg S$); and use that to decide which of the following conjunctive-normal-form sentences is equivalent to S. (Circle the letter in front of the correct answer.)

$\bar{S} \equiv A\bar{B}C \vee \bar{A}B\bar{C}$

$\neg\bar{S} \equiv (\bar{A} \vee B \vee \bar{C}) \wedge (A \vee \bar{B} \vee C)$

(a) $(A \vee B \vee C) \wedge (\overline{A} \vee \overline{B} \vee \overline{C})$.

(b) $(A \vee \overline{B} \vee C) \wedge (\overline{A} \vee B \vee \overline{C})$.

(c) $(\overline{A} \vee B \vee \overline{C}) \wedge (A \vee B \vee \overline{C})$.

(d) $(\overline{A} \vee B) \wedge (A \vee C) \wedge (B \vee \overline{C})$.

(e) None of the above.

3. For the formula $\exists x R(x,y) \rightarrow \exists x(P(x) \vee \neg\exists y Q(x,y))$, construct an equivalent formula in prenex form.

5 pt

$\neg \exists x R(x,y) \vee \exists u (P(u) \vee \forall z \neg Q(u,z))$

$\forall x \neg R(x,y) \vee \exists u \forall z (P(u) \vee \neg Q(u,z))$

$\forall x \exists u \forall z \underbrace{(\neg R(x,y) \vee (P(u) \vee \neg Q(u,z)))}_{\text{(alternatively)}}$

$\ldots\ldots (R(x,y) \rightarrow (Q(u,z) \rightarrow P(u)))$

4. For any x, abbreviate Dodec(x) to Dx and Cube(x) to Cx. Write down a formal (Fitch-style) proof whose premises are

1. Cb $\leftrightarrow$ (Ca $\leftrightarrow$ Cc),
2. Db $\rightarrow$ $\neg$Cb,

and whose conclusion is

Db $\rightarrow$ a$\neq$c.

Number your lines, and indicate whenever you can which inference rule you are using, as well as the appropriate lines or subproofs referred to by the rule.

1	$Cb \leftrightarrow (Ca \leftrightarrow Cc)$	
2	$Db \rightarrow \neg Cb$	
3	Db	
4	$\neg Cb$	$\{\rightarrow$ Elim, 2, 3$\}$
5	$a = c$	
6	Ca	
7	Cc	$\{=$ Elim, 5, 6$\}$
8	Cc	
9	Ca	$\{=$ Elim, 5, 8$\}$
10	$Ca \leftrightarrow Cc$	$\{\leftrightarrow$ Intro, 6–7, 8–9$\}$
11	Cb	$\{\leftrightarrow$ Elim, 1, 10$\}$
12	$\perp$	$\{\perp$ Intro, 4, 11$\}$
13	$a \neq c$	$\{\neg$ Intro, 5–12$\}$
14	$Db \rightarrow a \neq c$	$\{\rightarrow$ Intro, 3–13$\}$

$Db \rightarrow a \neq c$

10 pt

5. (a) The following formal proof has one faulty step. Say which one, and explain why it is invalid.

Step 6. When $\exists$ Elim is used, the letter introduced in the preceding subproof (in this case, $\boxed{b}$) cannot be taken out of the subproof.

(3pt)

1. $\forall x\, \exists y\, (Tet(x) \to Tet(y))$	
2. $\boxed{a}$ Tet(a)	
3. $\exists y\, (Tet(a) \to Tet(y))$	$\forall$ Elim 1
4. $\boxed{b}$ $Tet(a) \to Tet(b)$	
5. Tet(b)	$\to$ Elim 2,4
6. Tet(b)	$\exists$ Elim 3,4-5
7. $\forall x\, (Tet(x) \to Tet(b))$	$\forall$ Intro 2-6
8. $\exists y\, \forall x\, (Tet(x) \to Tet(y))$	$\exists$ Intro 7

(b) Nevertheless, the conclusion, Sentence 8, is a first-order validity. Explain why (informally).

(3pt)

<u>NOTE</u> Many people (mis)read this question as stating that Sentence 1 $\leftrightarrow$ Sentence 8 is a FO validity. In fact, only sentence 8 is addressed here.

<u>Solution</u> Discussed in class: break it into two cases, according as there exists a Tet or there doesn't.... This is independent of the meaning of "Tet".

<u>Better solution</u> (From one exam paper). Sentence 8 is the prenex form of $\exists x\, T(x) \to \exists y\, T(y)$, which is obviously an FO validity! (The same holds for Sentence 1.)

(c) Write down a formal (Fitch-style) proof whose premise is $\exists y \forall x\, T(x,y)$ and whose conclusion is $\forall x \exists y\, T(x,y)$. ($T$ is any binary predicate.) Number your lines, and indicate whenever you can which inference rule you are using, as well as the appropriate lines or subproofs referred to by the rule.

5 pt

1	$\exists y \forall x\ T(x,y)$	
2	$\boxed{a}$	
3	$\boxed{b}\ \forall x\ T(x,b)$	
4	$T(a,b)$	$\forall$ elim, (3)
5	$\exists y\ T(a,y)$	$\exists$ intro, (4)
6	$\exists y\ T(a,y)$	$\exists$ elim (1, 3-5)
7	$\forall x \exists y\ T(x,y)$	$\forall$ intro (2-6)

OR

1	$\exists y \forall x\ T(x,y)$	
2	$\boxed{a}\ \forall x\ T(x,a)$	
3	$\boxed{b}$	
4	$T(b,a)$	$\forall$ Elim, 2
5	$\exists y\ T(b,y)$	$\exists$ Intro, 4
6	$\forall x \exists y\ (T(x,y))$	$\forall$ Intro 3-5
7	$\forall x \exists y\ T(x,y)$	$\exists$ Elim, 2-6

(d) Why is $\forall x \exists y\, T(x,y) \rightarrow \exists y \forall x\, T(x,y)$ not a first-order validity?

Fails for the interpretation of $T(x,y)$ as $x=y$ on the set $\{1,2\}$.

4 pt

6. In the following statements, a and b are sets, and $\wp x$ is the powerset of the set x. For each of the statements, explain (informally) why it is always true or sometimes false, as the case may be.

1. $b \in \wp b$.
2. $b \subset \wp b$.
3. $\wp(a \cup b) = \wp a \cup \wp b$.
4. $\wp(a \cap b) = \wp a \cap \wp b$.

2pt 1. True: b is a subset of b, and so is a member of $\wp b$.

2pt 2. False (usually): Take b to be a set consisting of one dog. That dog is not a member of $\wp b$ (which consists of <u>sets</u>)

3pt 3. False (usually): If a has a member x not in b, and b has a member y not in a, then $\{x, y\}$ [or $a \cup b$] is a member of $\wp(a \cup b)$, but not of $\wp a \cup \wp b$. (Example: $a = \{1\}$, $b = \{2\}$).

3pt 4. True: A subset of $a \cap b$ is a subset both of a and of b, so $(*)$ $\wp(a \cap b) \subset \wp a \cap \wp b$.

A set in $\wp a \cap \wp b$ is a subset of a and of b, hence of $a \cap b$ (because all its elements are in a and in b).

Thus $(**)$ $\wp a \cap \wp b \subset \wp(a \cap b)$.

Together, $(*)$ and $(**)$ show that $\wp(a \cap b) = \wp a \cap \wp b$

7. The Fibonacci sequence

$$F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, \ldots = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

is defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1).$$

Prove by induction that for all $n \geq 1$,

10 pt

$$F_n^2 = F_{n+1}F_{n-1} - (-1)^n.$$

(Hint: Expand $F_{n+1}F_{n-1}$ using $F_{n+1} = F_n + F_{n-1}$.)

Base case $\quad F_2 F_0 - (-1)^1 = 0 + 1 = 1 = F_1^2$

Inductive step $\quad$ Assume true for $n-1$ $(n \geq 2)$, prove for n.

(following hint) $\quad F_{n+1}F_{n-1} = (F_n + F_{n-1})F_{n-1} = F_n F_{n-1} + F_{n-1}^2$

$$= F_n F_{n-1} + F_n F_{n-2} - (-1)^{n-1}$$

(by case $n-1$)

$$= F_n(F_{n-1} + F_{n-2}) + (-1)^n$$

$$= F_n^2 + (-1)^n.$$

Subtract $(-1)^n$ from both sides to get conclusion.

REMARK $\quad$ Many people tried the method of "backwards proof" where you start with the desired conclusion and make transformations until you reach a sentence which is clearly true. This may sometimes suggest a real proof, which consists in running the backwards proof from its end to its beginning. But to do so, you must check that every step in the proof is reversible. Otherwise, you could get something like the following "proof" that $1 = -1$.

$1 = -1$

$1^2 = (-1)^2$

$1 = 1$ ✓

8. $\mathcal{T}$ denotes a set of sentences of Propositional Logic (Sentential Calculus).

ϕ denotes the empty set.

$\mathcal{T}$ is *tt-satisfiable* if there is a truth assignment making all sentences in $\mathcal{T}$ true.

For a sentence S write $\mathcal{T} \vdash S$ as an abbreviation for "there is a formal proof whose premises are among the members of $\mathcal{T}$ and whose conclusion is S."

Write $\mathcal{T} \models S$ as an abbreviation for "S is a tautological consequence of $\mathcal{T}$." (Remember that this notion is defined in terms of truth assignments.)

S is a *tautology* if $\phi \models S$. (In other words, every truth assignment makes S true).

3x5pt

Prove the following statements:

(a) $\mathcal{T} \models S$ if and only if the set $\mathcal{T} \cup \{\neg S\}$ is not tt-satisfiable.

(b) For any sentence T, $T \to S$ is a tautology if and only if $\{T\} \models S$.

(c) If $\mathcal{T} \vdash S$ then $(\mathcal{T} \cup \{\neg S\}) \vdash \bot$.

(a) $\mathcal{T} \models S$ means that every truth assignment making all sentences in $\mathcal{T}$ true also makes S true, i.e., makes $\neg S$ false. This is exactly the same as no truth assignment making everything in $\mathcal{T}$ true and also $\neg S$ true. In other words it says that $\mathcal{T} \cup \{\neg S\}$ is not tt-satisfiable

(b) "$T \to S$" being a tautology means that every truth assignment makes it true. In other words (according to the truth table for $\to$) whenever T is true then S is true. That's equivalent to $\{T\} \models S$.

(c) If $\mathcal{T} \vdash S$ then there are sentences $T_1, \dots, T_n$ in $\mathcal{T}$ and a proof.

T_1
$\vdots$
T_n
———
$\vdots$
S

Putting $\neg S$ at the top and $\bot$ at the bottom, you obviously have a proof that has premises $\neg S, T_1, \dots, T_n$ and conclusion $\bot$ (use $\bot$ intro at the last step)

Thus $\mathcal{T} \cup \{\neg S\} \vdash \bot$.