QUASI-ORDINARY SINGULARITIES OF EMBEDDED SURFACES

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Joseph Lipman

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PREFACE

The present work comprises two papers on topics in the local theory of singularities. The first, and longer, is an analysis of a special class of singularities of surfaces, while the second is concerned with a question about the derivation module of the local ring of an arbitrary point. The contents of the two papers are described at some length in their respective introductions.

It gives me great pleasure to acknowledge my gratitude to Professor Oscar Zariski, for his suggestion of the topics studied herein, for many inspiring conversations and access to unpublished notes on which much of the material is based, for his patient guidance and encouragement, and for his personal interest when it was most needed.

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Introduction

Since their introduction by Jung in 1908, the so called quasi-ordinary (or Jungian) singularities have played an important role in the problems of uniformization and resolution of singularities (Walker, Zariski, Abhyankar). A point of a surface in three-space is quasi-ordinary if some neighbourhood of the point can be projected into a plane in such a way that the branch curve has an ordinary double point at the image of the original point. Thus the quasi-ordinary points are in some sense those which are next in order of complexity to points at which a surface is equisingular along some curve C, i.e. to those singular points whose image in some (local) projection is a simple point of the branch locus (which is the projection of C).

The situation of equisingularity always comes about when some one-parameter family of plane curves having "equivalent" singularities at their respective origins sweeps out a surface S, C being the nonsingular curve traced on S by the origins. The original curves appear then as sections of S by planes transversal to C, and around any point P of C, S is more or less a product of C with such a section of S. Thus the singularity of S at P is essentially no more complicated than that of the transversal plane sections, and singularities of plane curves can be classified according to classical methods.

In spite of their importance, the quasi-ordinary singularities have not yet been classified. In the absence of a general theory of classification for singularities of embedded surfaces, an analysis of the special situation presented by quasi-ordinary points would seem to be worthwhile. Such an analysis is the theme of this paper.

It is well-known that the neighbourhood of an analytically irreducible quasi-ordinary singularity may be represented by an algebroid function ζ of two variables X and Y, which is actually a fractional power series in X and Y (i.e. $\zeta = H(X^{1/n}, Y^{1/n})$ for some power series H and some integer n). If ζ_1, ζ_2 are two conjugate branches of ζ, then it turns out that $\zeta_1 - \zeta_2 = M_{12}\varepsilon_{12}(X^{1/n}, Y^{1/n})$ where $M_{12} = X^{u/n}Y^{v/n}$ is a (fractional) monomial in X and Y, and $\varepsilon_{12}(0,0) \neq 0$ (cf. §1 for technical details). The monomials so obtained are the "characteristic monomials" of ζ. This parallels exactly the situation for irreducible branches of plane curves, where we deal with fractional power series and characteristic monomials in one variable X. In the case of curves, two suitably normalized fractional power series represent equivalent singularities if and only if they have the same characteristic monomials. (By "suitably normalized" we mean that the parameter X is chosen to be "transversal"). Thus we may hope that the characteristic monomials of branches which are somehow normalized (cf. §2) would provide the key to the classification of irreducible quasi-ordinary singularities. Of course it would be essential to show that two different branches representing the same point should have the same characteristic monomials. More generally, we should demand that two different branches representing points with isomorphic local rings (in the absolute sense, and not necessarily relative to the ground field) should have the same characteristic monomials.

By further analogy with the case of plane curves, another more intrinsic approach to the classification problem would be to resolve the singularity under consideration by quadratic and monoidal transformations, observing the behavior of the resulting singularities at each stage. The prerequisite for such an attempt is to show that quadratic and monoidal transforms of quasi-ordinary points are once again quasi-ordinary. We find in §§3-4 that this is indeed the case when the original point is analytically irreducible, and that, moreover, the monoidal transforms and certain "special" quadratic transforms of such points are again analytically irreducible.

The "non-special" quadratic transforms are points P' which are either simple points of the transformed surface S' or such that S' is equisingular along the exceptional curve at P'; such transforms can be analytically reducible. As we have pointed out, the singularity of S' at a non-special transform is determined by a plane section of S' transversal to the exceptional curve. Such plane sections also appear as "generic sections" in the local ring of certain special transforms, and so we may restrict our attention at each stage to the (finitely many) special transforms.

In the case of curves the two different approaches are related by the following result: the characteristic monomials of a branch representing a point with local ring A determine and are uniquely determined by the multiplicities of the successive quadratic transforms of A.

Our main result should therefore run along the following lines: the characteristic monomials of a branch representing a quasi-ordinary point with local ring A determine and are uniquely determined by certain invariants associated with the successive local rings appearing as special transforms in some kind of resolution of A. In the case of curves, the only invariant used is the multiplicity; in the present case we need more information, such as the multiplicities of multiple curves appearing in the successive transformations, the number of components of certain tangent cones etc. (cf. §5 for exact description). The required information about tangent cones and multiple curves at a quasi-ordinary point is developed in §2; the behavior of characteristic monomials under quadratic and monoidal transformations is given in §§3-4; the classification theorem is then stated and proved in §5.

In §6 we investigate the relation between the characteristic monomials associated with a quasi-ordinary point and the "generic plane sections" transversal to a given multiple curve through the point. The material of this section leads to another formulation of the classification theorem.

We work throughout in the context of complete local rings although the motivation (as given in this introduction) is always geometric. For more background on embedded surfaces we refer the reader to [7]; for equisingularity we refer to [8].

The proofs of all our principal results are purely computational and consequently quite tedious. It is to be hoped that a more conceptual approach can be found. We do take care at all times however to show that the computations being performed on a branch have a significance which is intrinsic to the local ring of the point represented by the branch.

[As we have indicated, we are restricting ourselves to the case of analytically irreducible, two dimensional local rings. The reason is that in other cases, the property of being quasi-ordinary is not preserved under quadratic transformations: for example, the reducible surface $(Z^2 - X^2Y)(Z^3 - XY) = 0$ becomes $(Z^2 - XY)(XZ^3 - Y) = 0$; the three dimensional variety $U^4 = X^3YZ$ becomes $U^4 = XYZ$ which in turn becomes $XU^4 = YZ$].

Although the notion of equivalent singularities is at present a vague and intuitive one, we believe that there is sufficient evidence presented in this paper to justify the claim that the classification of analytically irreducibl quasi-ordinary singularities by characteristic monomials is the "correct" one.

1. The Distinguished Pairs of a Quasi-ordinary Branch.

Let $R = k[[X,Y,Z]]$ be the power series ring in three variables over an algebraically closed field k of characteristic zero. An element f in R is said to be a pseudo-polynomial in Z (over $k[[X,Y]]$) if

$$f = Z^m + g_1 Z^{m-1} + \dots + g_m \qquad (m \geq 1)$$

where the g_i $(i = 1, 2, \dots, m)$ are non-units in $k[[X,Y]]$.

DEFINITION 1.1. A pseudo-polynomial f in Z is a quasi-ordinary polynomial if the discriminant of f (f being considered as a polynomial over $k[[X,Y]]$) is of the form $X^a Y^b \varepsilon$, with a,b non-negative integers, and ε a unit in $k[[X,Y]]$.

If f is of degree one (as a polynomial in Z) we shall consider that f has discriminant equal to the identity; thus in this case f is a quasi-ordinary polynomial if and only if the absolute term g_1 is a non-unit.

We may also speak of a quasi-ordinary polynomial in X (over $k[[Y,Z]]$) or in Y (over $k[[X,Z]]$), the definitions in each case being obtained from definition 1.1 by a suitable permutation of the letters "X", "Y", "Z".

In order to discuss the roots of quasi-ordinary polynomials, we recall the notion of fractional power series. Let n be a positive integer, and let Γ_n be the set of all non-negative rational numbers α such that $n\alpha$ is an integer. A fractional power series ζ in X and Y of order $\leq n$, with coefficients in k, is a formal sum

$$\zeta = \sum_{(\alpha,\beta)\in\Gamma_n\times\Gamma_n} c_{\alpha\beta}X^{\alpha}Y^{\beta} \qquad (c_{\alpha\beta} \in k)$$

If $\xi = \Sigma d_{\alpha\beta}X^{\alpha}Y^{\beta}$ is a second such sum, then we set

$$\zeta + \xi = \sum_{(\alpha,\beta)} (c_{\alpha\beta} + d_{\alpha\beta})X^{\alpha}Y^{\beta}$$

$$\zeta\xi = \sum_{(\alpha,\beta)} \Bigg(\sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} c_{\alpha'\beta'}d_{\alpha''\beta''} \Bigg) X^{\alpha}Y^{\beta}$$

In particular $(X^{\alpha})^{i} = X^{i\alpha}$, $(Y^{\beta})^{j} = Y^{j\beta}$ for any integers i,j. We verify that the set of all fractional power series form a ring $\mathfrak{P}$; that for fixed n, the fractional power series of order $\leq n$ form a subring Φ_n; that we may identify $k[[X,Y]]$ with Φ_1 in such a way that X (respectively Y) is identified with X^1 (respectively Y^1); and that

$$\Phi_n = k[[X,Y]]\,[X^{1/n},Y^{1/n}] = k[[X^{1/n},Y^{1/n}]]$$

is isomorphic to the power series ring in two variables over k.

Thus any power series ζ of order $\leq n$ can be written in the form $\zeta = H(X^{1/n},Y^{1/n})$ where $H = H(X,Y)$ is a power series in two variables with integral exponents. The set of conjugates of ζ over $k[[X,Y]]$ is then the set of fractional power series $\{H(\omega_1 X^{1/n},\omega_2 Y^{1/n})\}$ where ω_1,ω_2 run through all the n-th roots of unity. ζ is a unit in Φ if and only if $H(0,0) \neq 0$, and when this is so, ζ is a unit in Φ_n.

We can, of course, define fractional power series in any number of variables $X,Y,Z,\ldots.$ over k, and all the above remarks have obvious extensions to such situations. For example, if a,b,c are any integers, then there is a k-isomorphism between $k[[X^{1/a},Y^{1/b},Z^{1/c}]]$ and $k[[X,Y,Z]]$

taking $X^{1/a} \to X$, $Y^{1/b} \to Y$, $Z^{1/c} \to Z$. Tacit use of the existence of such an isomorphism is often made in later sections.

It is known that the roots of a quasi-ordinary polynomial are fractional power series [1; Theorem 3]; in other words, if f is a quasi-ordinary polynomial in Z over k[[X,Y]] then there is an integer n and power series $H_1, H_2, \ldots, H_m$ such that

$$f(X,Y,Z) = \prod_{i=1}^{m} [Z - H_i(X^{1/n},Y^{1/n})]$$

The discriminant of f is then the product

$$\prod_{i \neq j} [H_i(X^{1/n},Y^{1/n}) - H_j(X^{1/n},Y^{1/n})] = X^a Y^b \varepsilon$$

Since $\Phi_n = k[[X^{1/n},Y^{1/n}]]$ is a unique factorization domain in which $X^{1/n}$, $Y^{1/n}$ are irreducible elements, it follows that

$$H_i(X^{1/n},Y^{1/n}) - H_j(X^{1/n},Y^{1/n}) = M_{ij}\varepsilon_{ij}$$

where M_{ij} is a <u>monomial</u> in $X^{1/n}$, $Y^{1/n}$, (i.e. $M_{ij} = X^{u/n}Y^{v/n}$

for some integers u,v) and $\varepsilon_{ij} = \varepsilon_{ij}(X^{1/n}, Y^{1/n})$ is a unit in Φ_n (i.e. $\varepsilon_{ij}(0,0) \neq 0$).

Moreover, since $f(0,0,Z) = Z^m$ (by our definition of pseudo-polynomials) we have

$$Z^m = \prod_i [Z - H_i(0,0)]$$

whence $H_i(0,0) = 0$. Thus ζ and its conjugates are non-units in Φ, and $M_{ij} \neq 1$.

Conversely if ζ is a non-unit in Φ_n, and if for every conjugate $\zeta_i \neq \zeta$ of ζ (over k[[X,Y]]) we have $\zeta - \zeta_i = M_i \varepsilon_i$, where M_i is some monomial in $X^{1/n}$, $Y^{1/n}$, and ε_i is a unit in Φ_n, then for any two distinct conjugates ζ_1, ζ_2 of ζ we have

$$\zeta_1 - \zeta_2 = M_{12}\varepsilon_{12}$$

where M_{12} is a monomial and ε_{12} is a unit (for, $\zeta_1 - \zeta_2$ is conjugate to some element of the form $\zeta - \zeta_i$). It follows that the minimum polynomial $\prod_i [Z - \zeta_i]$ of ζ over k[[X,Y]] is a quasi-ordinary polynomial.

In general, if ζ is any fractional power series in X and Y over k, and if $\zeta = \zeta_1, \zeta_2, \ldots, \zeta_m$ are the distinct conjugates of ζ over k[[X,Y]], then we will call the element $\prod_{i=1}^{m} [Z - \zeta_i]$ of R the minimum polynomial of ζ.

DEFINITION 1.2. A fractional power series in X and Y with coefficients in k is said to be a quasi-ordinary branch if its minimum polynomial is a quasi-ordinary polynomial.

To summarize the previous remarks, then, we have:

PROPOSITION 1.3. The roots of a quasi-ordinary polynomial which lie in any algebraically closed field containing the ring Φ of all fractional power series are themselves fractional power series. A fractional power series $\zeta \in \Phi_n$ is a quasi-ordinary branch if and only if ζ is a non-unit in Φ_n and for each conjugate $\zeta_i \neq \zeta$ we have $\zeta - \zeta_i = M_i \varepsilon_i$, where M_i is a monomial in $X^{1/n}$, $Y^{1/n}$, and ε_i is a unit in Φ_n.

* * *

DEFINITION 1.4. If ζ is a quasi-ordinary branch, then the monomials $M_i = X^{\lambda_i} Y^{\mu_i}$ of proposition 1.3 are called the characteristic monomials of ζ. The ordered pairs (λ_i, μ_i) are called the distinguished pairs of ζ.

(We would prefer to say "characteristic pairs"; however, such usage would be in conflict with established terminology in the theory of plane curves).

We shall now give some properties of the characteristic monomials and distinguished pairs of a quasi-ordinary branch $\zeta \in \Phi_n$ in a series of remarks.

REMARK 1.4.0. ζ may have no characteristic monomials. This will occur if and only if $\zeta \in k[[X,Y]]$.

REMARK 1.4.1. Any conjugate of ζ has the same set of characteristic monomials as ζ.

REMARK 1.4.2. Recalling that any conjugate of ζ is obtained from ζ by a substitution of the type $X^{1/n} \to \omega_1 X^{1/n}$, $Y^{1/n} \to \omega_2 Y^{1/n}$ $(\omega_1^n = \omega_2^n = 1)$, we see that the characteristic monomials actually appear as terms (with non-vanishing coefficients) in the expression of ζ as a fractional power series.

If (λ,μ), (σ,τ) are two ordered pairs of rational numbers, we write $(\lambda,\mu) \leq (\sigma,\tau)$ to signify $\lambda \leq \sigma$, $\mu \leq \tau$. If $(\lambda,\mu) \leq (\sigma,\tau)$ and $(\lambda,\mu) \neq (\sigma,\tau)$, we write $(\lambda,\mu) < (\sigma,\tau)$.

REMARK 1.4.3. Let $M_1 = X^\lambda Y^\mu$, $M_2 = X^\sigma Y^\tau$ be two distinct characteristic monomials of ζ, and let ζ_1, ζ_2 be conjugates of ζ such that $\zeta - \zeta_1 = M_1\varepsilon_1$, $\zeta - \zeta_2 = M_2\varepsilon_2$, ε_1 and ε_2 being units in Φ_n. We have seen that

$$M_1\varepsilon_1 - M_2\varepsilon_2 = \zeta_1 - \zeta_2 = M_{12}\varepsilon_{12}$$

where M_{12} is a monomial and ε_{12} is a unit. It follows easily that either $(\lambda,\mu) < (\sigma,\tau)$ or $(\sigma,\tau) < (\lambda,\mu)$. Thus, the distinguished pairs of ζ are totally ordered.

REMARK 1.4.4. Let $M_1, M_2, \ldots, M_s$ be the distinct characteristic monomials of ζ. Let K be the quotient field of $k[[X,Y]]$ and let K_n be the quotient field of $k[[X^{1/n},Y^{1/n}]]$; K_n is a galois extension of K. If $c_{\lambda\mu}X^\lambda Y^\mu$ is any term appearing in ζ ($c_{\lambda\mu} \neq 0$), then any automorphism of K_n/K leaving ζ fixed leaves $X^\lambda Y^\mu$ fixed; thus $X^\lambda Y^\mu \in K(\zeta)$. On the other hand, proposition 1.3 shows that any automorphism leaving $M_1, M_2, \ldots, M_s$ fixed leaves ζ fixed. Hence (and in view of remark 1.4.2)

$$K(\zeta) = K(M_1, M_2, \ldots, M_s)$$

REMARK 1.4.5. Setting $M_i = X^{\lambda_i}Y^{\mu_i}$, we may assume, by remark 1.4.3, that $(\lambda_1,\mu_1) < (\lambda_2,\mu_2) < \ldots < (\lambda_s,\mu_s)$. Suppose that $c_{\lambda_\mu}X^\lambda Y^\mu$ is a term of ζ $(c_{\lambda_\mu} \neq 0)$ and that $M = X^\lambda Y^\mu$ lies in $K(M_1, M_2, \ldots, M_t)$ but not in $K(M_1, M_2, \ldots, M_{t-1})$. Then there is an automorphism θ which leaves $M_1, M_2, \ldots, M_{t-1}$ fixed, and which "moves" M (i.e. $\theta M \neq M$). Then θ must move M_t, and so both M and M_t appear with non-vanishing coefficients in the expression of $\zeta - \theta\zeta = M_r\varepsilon_\theta$ (where $M_r = X^{\lambda_r}Y^{\mu_r}$ is some characteristic monomial of ζ, and ε_θ is a unit). Thus $(\lambda_r,\mu_r) \leq (\lambda,\mu)$, and $(\lambda_r,\mu_r) \leq (\lambda_t,\mu_t)$. The second inequality shows that M_r is one of $M_1, M_2, \ldots, M_t$, and since θ moves M_r, M_r is necessarily M_t. Hence $(\lambda_r,\mu_r) = (\lambda_t,\mu_t)$, and so $(\lambda_t,\mu_t) \leq (\lambda,\mu)$.

REMARK 1.4.6. For any t, $1 \leq t \leq s$, there exists an automorphism ψ such that $\zeta - \psi\zeta = M_t\varepsilon_\psi$ (ε_ψ a unit). If $X^\lambda Y^\mu$ is moved by ψ then clearly $(\lambda_t,\mu_t) \leq (\lambda,\mu)$; hence $M_1, M_2, \ldots, M_{t-1}$ are left fixed by ψ. On the other hand, M_t is moved by ψ; thus M_t does not lie in $K(M_1, M_2, \ldots, M_{t-1})$.

REMARK 1.4.7. Let s now be any integer, and let $M_i = X^{\lambda_i}Y^{\mu_i}$ (i = 1, 2, ..., s) be <u>arbitrary</u> monomials with fractional exponents. Then a monomial $X^\lambda Y^\mu$ lies in

$K(M_1, M_2, \ldots, M_s) = K[M_1, M_2, \ldots, M_s]$ if and only if the "vector" (λ,μ) satisfies

$$(\lambda,\mu) \equiv q_1(\lambda_1,\mu_1) + q_2(\lambda_2,\mu_2) + \ldots + q_s(\lambda_s,\mu_s) \quad \text{modulo } \mathbb{Z}\times\mathbb{Z}$$

where $q_1, q_2, \ldots, q_s$ are integers, and $\mathbb{Z}$ is the set of all integers. The sufficiency is obvious, and the necessity is easily established if we consider an integer N such that $N\lambda$, $N\mu$, $N\lambda_i$, $N\mu_i$ $(i = 1, 2, \ldots, s)$ are all integers, and note that the monomials $X^{c/N}Y^{d/N}$ with $0 \leq c < N$, $0 \leq d < N$ form a basis of the vector space $K(X^{1/N}, Y^{1/N})$ over K.

The properties given in remarks 1.4.2 through 1.4.6 characterize the characteristic monomials of a quasi-ordinary branch. More precisely, we have the following criterion, which will prove useful.

PROPOSITION 1.5. Let $\zeta = \sum_{(\alpha,\beta)\in\Gamma_n\times\Gamma_n} c_{\alpha\beta}X^{\alpha}Y^{\beta}$ be a fractional power series. In order that ζ be a quasi-ordinary branch, it is necessary and sufficient that there exist pairs of rational numbers $(\lambda_1,\mu_1), (\lambda_2,\mu_2), \ldots, (\lambda_s,\mu_s)\in\Gamma_n\times\Gamma_n$ such that, with $(\lambda_0,\mu_0) = (0,0)$, we have

1) $(\lambda_0,\mu_0) < (\lambda_1,\mu_1) < \dots < (\lambda_s,\mu_s)$

2) $c_{\lambda_i\mu_i} \neq 0$ for $i = 1, 2, \dots, s$

3) If $c_{\lambda\mu} \neq 0$, then the pair (λ,μ) is a linear combination, with integer coefficients, modulo $\mathbb{Z}\times\mathbb{Z}$, of the pairs (λ_i,μ_i) $i = 0, 1, \dots, s$.

4) If $c_{\lambda\mu} \neq 0$, and if $\tau = \tau(\lambda,\mu)$ is the least of the integers $t \geq 0$ such that (λ,μ) is a linear combination, with integer coefficients, modulo $\mathbb{Z}\times\mathbb{Z}$, of $(\lambda_0,\mu_0), (\lambda_1,\mu_1), \dots, (\lambda_t,\mu_t)$, then $(\lambda,\mu) \geq (\lambda_\tau,\mu_\tau)$.

5) $\tau(\lambda_i,\mu_i) = i$ for $i = 1, 2, \dots, s$.

If such pairs exist, they are uniquely determined by ζ; in fact they are the distinguished pairs of ζ.

PROOF. Necessity follows from remarks 1.4.2 - 1.4.7 and the fact that no characteristic monomial of ζ is equal to 1.

To prove sufficiency, we set

$$H_i = \sum_{\tau(\alpha,\beta)=i} c_{\alpha\beta} X^\alpha Y^\beta \qquad i = 0, 1, \dots, s.$$

By conditions 4), 5), we have

$$H_i = X^{\lambda_i}Y^{\mu_i}G_i(X^{1/n},Y^{1/n}) = M_iG_i \qquad \text{for } i \geq 1.$$

where M_i is defined to be $X^{\lambda_i}Y^{\mu_i}$, and $G_i(0,0) \neq 0$. By condition 3), $\zeta = H_0 + H_1 + \ldots + H_s$.

Let θ be an automorphism of K_n/K, and let t be an integer $1 \leq t \leq s$. By remark 1.4.7, θ leaves all the elements $H_0, H_1, \ldots, H_t$ fixed if and only if θ leaves all the elements $M_1, M_2, \ldots, M_t$ fixed. Setting $t = s$ we see that if $\theta\zeta \neq \zeta$, then $\theta(M_j) \neq M_j$ for some index j; setting t equal to the least such index, we see that $t \geq 1$ and that θ leaves $H_0, H_1, \ldots, H_{t-1}$ fixed; it follows (condition 1)) that

$$\zeta - \theta\zeta = M_t\varepsilon_\theta$$

with $\varepsilon_\theta(0,0) \neq 0$. By proposition 1.3, this means that ζ is a quasi-ordinary branch.

We see moreover that the distinguished pairs of ζ must be among the pairs (λ_i,μ_i) $i = 1, 2, \ldots, s$. Conversely, condition 5) and remark 1.4.7 show that for any $i \geq 1$, M_i does not lie in the field $K(M_1, M_2, \ldots, M_{i-1})$ (for $i = 1$, this is intended to mean $M_1 \notin K$). Hence, there is an automorphism ψ of K_n/K leaving $M_1, M_2, \ldots, M_{i-1}$ fixed and moving M_i, and clearly $\zeta - \psi\zeta = M_i \varepsilon_\psi$ $(\varepsilon_\psi(0,0) \neq 0)$ for such a ψ. It follows that (λ_i, μ_i) is a distinguished pair of ζ for any $i \geq 1$. q.e.d.

(We could prove <u>uniqueness</u> directly from the conditions 1) - 5) by an inductive argument, using the fact that (λ_i, μ_i) must be the least of the pairs (λ, μ) such that $c_{\lambda\mu} \neq 0$ and $\tau(\lambda, \mu) = i$).

During the course of the proof we have established the following fact:

COROLLARY 1.6. Let $\zeta \in \Phi_n$ be a quasi-ordinary branch. Then either $\zeta \in k[[X,Y]]$ or

$$\zeta = H_0(X,Y) + X^\lambda Y^\mu H(X^{1/n}, Y^{1/n})$$

where $H_0(X,Y) \in k[[X,Y]]$, $H(0,0) \neq 0$, and (λ, μ) is the least distinguished pair of ζ.

2. The Tangent Cone and Singular Locus of a Quasi-ordinary Local Ring.

Let $R = k[[X,Y,Z]]$ be as in §1. We say that an element f in R is a defining polynomial of a given ring A if f is a pseudo-polynomial and if A is ring-isomorphic to $R/(f)$.

DEFINITION 2.1. A local ring A will be said to be quasi-ordinary if A has a defining polynomial which is an irreducible quasi-ordinary polynomial. If f is such a defining polynomial, and if ζ is any root of f (so that ζ is a quasi-ordinary branch) then we shall say that ζ represents A.

The requirement that A be an integral domain is a matter of convenience. We shall always assume (unless otherwise stated) that defining polynomials are pseudo-polynomials in Z. We note that a quasi-ordinary branch ζ represents a given quasi-ordinary ring A if and only if $k[[X,Y]][\zeta]$ is isomorphic to A.

Our first task is to isolate a certain subset N of the set of all quasi-ordinary branches such that each quasi-ordinary ring is represented by at least one member of N and such that the members of N are of a form which is amenable to some computations which we will have to carry out later on.

LEMMA 2.2. If ζ is a representing branch for a given quasi-ordinary ring A, then $\zeta - h$ represents A for any non-unit h in k[[X,Y]]. Moreover ζ and $\zeta - h$ have the same distinguished pairs.

PROOF. It follows immediately from lemma 1.3 and definition 1.4 that if ζ is a quasi-ordinary branch, then $\zeta - h$ is a quasi-ordinary branch having the same distinguished pairs. This being so, we have only to note that $k[[X,Y]][\zeta] = k[[X,Y]][\zeta - h]$. q.e.d.

If $\zeta = \Sigma c_{\alpha\beta} X^{\alpha} Y^{\beta}$ is a representing branch of A, then setting $h = \Sigma' c_{\alpha\beta} X^{\alpha} Y^{\beta}$ where Σ' denotes summation over those pairs (α,β) such that both α and β are integers, we obtain a representing branch $\zeta' = \zeta - h$ in which there appear no integral monomials. By corollary 1.6, we have either $\zeta' = 0$ or $\zeta' = X^{\lambda} Y^{\mu} H(X^{1/n}, Y^{1/n})$ where (λ,μ) is the least distinguished pair of ζ' and $H(0,0) \neq 0$.

If it happens that one of λ,μ vanishes while the other is less than unity, then we shall find it convenient to replace ζ' by another defining branch.

LEMMA 2.3. Let $\zeta = X^{u/n}H(X^{1/n},Y^{1/n})$ be a defining branch for a quasi-ordinary ring A, with $0 < u < n$, and $H(0,0) \neq 0$. Then A has a defining branch of the form $\zeta' = X^{n/u}H'(X^{1/u},Y^{1/n})$ (with $H'(0,0) \neq 0$).

PROOF. Let $f(X,Y,Z)$ be the minimum polynomial of ζ over $k[[X,Y]]$. The conjugates of ζ over $k[[X,Y]]$ are of the form $\zeta_i = X^{u/n}H_i(X^{1/n},Y^{1/n})$ with $H_i(0,0) \neq 0$ ($i = 1, 2, \ldots, m$, where m = degree of f in Z). Thus

$$f(X,Y,Z) = \prod_{i=1}^{m} [Z - X^{u/n}H_i(X^{1/n},Y^{1/n})]$$

and $f(X,0,0) = X^{mu/n}\,\tilde{H}(X^{1/n},0)$ with $\tilde{H}(0,0) \neq 0$. Hence mu/n is a positive integer, and by the Weierstrass preparation theorem [9; p. 145] there is a power series $E(X,Y,Z)$, $E(0,0,0) \neq 0$, such that $Ef = g$ where g is a pseudo-polynomial of degree mu/n in X over $k[[Y,Z]]$. Since $A \cong k[[X,Y,Z]]/(g)$, it will clearly be sufficient to show that g is a quasi-ordinary polynomial in X (over $k[[Y,Z]]$) and that g has a root of the form $Z^{n/u}H'(Z^{1/u},Y^{1/n})$, $H'(0,0) \neq 0$.

Let $\overline{H} = X^u H$, so that $\zeta = \overline{H}(X^{1/n}, Y^{1/n})$. We shall construct a power series G in two variables (over k), $G(0,0) \neq 0$, such that

$$\overline{H}(Z^{1/u}G(Z^{1/u}, Y^{1/n}), Y^{1/n}) = Z$$

Assuming that such a G exists we set $\xi = G(Z^{1/u}, Y^{1/n})$. Since $f(X, Y, \overline{H}(X^{1/n}, Y^{1/n})) = 0$, we have, upon substituting $Z^{1/u}\xi$ for $X^{1/n}$, $f(Z^{n/u}\xi^n, Y, Z) = 0$; hence $Z^{n/u}\xi^n$ is a root of g. Therefore, the discriminant of g is the product of all the conjugates over k[[Y,Z]] of the element $g_X(Z^{n/u}\xi^n, Y, Z)$. Now

$$g_X(Z^{n/u}\xi^n, Y, Z) \cdot \frac{\partial}{\partial Z}(Z^{n/u}\xi^n) + g_Z(Z^{n/u}\xi^n, Y, Z) = 0$$

But $\frac{\partial}{\partial Z}(Z^{n/u}\xi^n) = Z^{(n/u)-1}\varepsilon'(Z^{1/u}, Y^{1/n}) \qquad \varepsilon'(0,0) \neq 0$

and $g_Z(Z^{n/u}\xi^n, Y, Z) = E(Z^{n/u}\xi^n, Y, Z) f_Z(Z^{n/u}\xi^n, Y, Z)$.

The product of all the conjugates (over k[[X,Y]]) of the element $f_Z(X, Y, \zeta)$ is the discriminant of f, which is, by assumption, of the form $X^a Y^b \varepsilon(X,Y)$, $\varepsilon(0,0) \neq 0$; hence

$$f_Z(X, Y, \overline{H}(X^{1/n}, Y^{1/n})) = X^{c/n} Y^{d/n} \varepsilon''(X^{1/n}, Y^{1/n})$$

(c,d integers, $\varepsilon''(0,0) \neq 0$). Thus

$$g_Z(Z^{n/u}\xi^n, Y, Z) = E(Z^{n/u}\xi^n, Y, Z)Z^{c/u}\xi^c Y^{d/n}\varepsilon''(Z^{1/u}\xi, Y^{1/n})$$

$$= Z^{c/u}Y^{d/n}\varepsilon'''(Z^{1/u}, Y^{1/n}) \qquad \varepsilon'''(0,0) \neq 0.$$

Hence $\quad g_X(Z^{n/u}\xi, Y, Z)\cdot Z^{(n/u)-1}\varepsilon' + Z^{c/u}Y^{d/n}\varepsilon''' = 0$

and $\qquad g_X(Z^{n/u}\xi, Y, Z) = Z^{s/u}Y^{t/n}\varepsilon^*(Z^{1/u}, Y^{1/n})$

(s,t integers, $\varepsilon^*(0,0) \neq 0$). Hence the product of the conjugates of $g_X(Z^{n/u}\xi, Y, Z)$ is of the form

$$Z^pY^q.(\text{unit in } k[[Y,Z]])$$

where p and q are integers, necessarily non-negative since $g_X(Z^{n/u}\xi, Y, Z)$ is integral ofer $k[[Y,Z]]$. Thus g is a quasi-ordinary polynomial.

To prove the existence of G, we remark that if W is an indeterminate, then $\overline{H}(X,Y) - W^u = X^uH(X,Y) - W^u$ has a factor (in $k[[X,Y,W]]$) of the form $X\overline{G}(X,Y) - W$ with $\overline{G}(0,0) \neq 0$ (for $\overline{G}$ we take any power series such that $\overline{G}^u = H$). By the preparation theorem, there is a unit $\overline{E}(X,Y,W)$ such that

$$\overline{E}(X,Y,W)(X\overline{G}(X,Y) - W) = X - G'(W,Y)$$

and, setting $X = 0$, we see that $G'(W,Y) = W\overline{E}(0,Y,W)$.

Let $G(W,Y) = \overline{E}(0,Y,W)$; then $G(0,0) \neq 0$, and setting $X = G'(W,Y) = WG$ in the above relation, we have $WG.\overline{G}(WG,Y) - W = 0$ whence $\overline{H}(WG,Y) - W^{\mu} = 0$.

Our conclusion follows on substituting $Z^{1/u}$ for W and $Y^{1/n}$ for Y. q.e.d.

The preceding lemmas show that any quasi-ordinary local ring A is represented either by $\zeta = 0$ or by a quasi-ordinary branch of the form

$$\zeta = X^{\lambda}Y^{\mu}H(X^{1/n},Y^{1/n}) \qquad n\lambda, n\mu \text{ integers; } H(0,0) \neq 0$$

where λ,μ are such that

1) not both of λ, μ are integers
2) if $\lambda + \mu < 1$ then $0 < \lambda$ and $0 < \mu$.

If $\zeta = 0$ or if ζ is of the form just described, then we shall say that ζ is a <u>normalized quasi-ordinary branch</u>. Thus

PROPOSITION 2.4. Any quasi-ordinary local ring can be represented by a normalized quasi-ordinary branch.

* * *

We shall now describe the "tangent cone" and the "singular locus" of a given quasi-ordinary ring A, and demonstrate the fact that these objects are determined by the distinguished pairs of any normalized representing branch of A. Conversely, given these objects we can deduce certain things about normalized representing branches of A in a way which will prove most useful when we come to classifying such branches.

The _tangent cone_ of A is the affine scheme defined by the associated graded ring gr(A) of A with respect to its maximal ideal _m_. (cf. [9; p. 248]).

Let A be a quasi-ordinary local ring and let f be a defining polynomial of A. Then gr(A) is isomorphic to $k[X,Y,Z]/(f_I)$ where $f_I = f_I(X,Y,Z)$ is the _initial form_ of the power series $f(X,Y,Z)$ (i.e. f_I is the sum of all the terms of f of lowest total degree in X,Y,Z).

LEMMA 2.5. Let A be a quasi-ordinary local ring and let $\zeta = X^{\lambda}Y^{\mu}H(X^{1/n},Y^{1/n})$ ($n\lambda, n\mu$ integers; $H(0,0) \neq 0$) be a normalized representing branch of A. Let f be the minimum polynomial of ζ over K (the quotient field of k[[X,Y]]) and let f_I be the initial form of f (f being considered as a power series in X,Y,Z). Then one of the following statements is true:

1) $\lambda + \mu > 1$, and $f_I = Z^m$ where $m = [K(\zeta):K]$.

2) $\lambda + \mu = 1$ and $f_I = (Z^t - X^{t\lambda}Y^{t\mu})^r$, where $t = [K(X^{\lambda}Y^{\mu}):K]$ and $r = [K(\zeta):K(X^{\lambda}Y^{\mu})]$.

3) $\lambda + \mu < 1$ and $f_I = cX^{m\lambda}Y^{m\mu}$, where $c \in k$, $m\lambda > 0$, $m\mu > 0$ (again $m = [K(\zeta):K]$).

PROOF. The conjugates of ζ are of the form $\zeta_i = X^{\lambda}Y^{\mu}H_i(X^{1/n},Y^{1/n})$, $H_i(0,0) \neq 0$, $i = 1, 2, \ldots, m$. Thus

$$f(X,Y,Z) = \prod_{i=1}^{m}[Z - X^{\lambda}Y^{\mu}H_i(X^{1/n},Y^{1/n})].$$

If $\lambda + \mu < 1$, then clearly $f_I = cX^{m\lambda}Y^{m\mu}$ for some $c \in k$, and since ζ is normalized, $m\lambda > 0$ and $m\mu > 0$. Similarly, if $\lambda + \mu > 1$ it is clear that $f_I = Z^m$.

If $\lambda + \mu = 1$, then

$$f_I = \prod_{i=1}^{m} [Z - X^\lambda Y^\mu H_i(0,0)]$$

Since any automorphism of $K(X^{1/n}, Y^{1/n})/K$ leaving ζ fixed leaves $X^\lambda Y^\mu$ fixed, we have $X^\lambda Y^\mu \in K(\zeta)$. On the other hand, the automorphism θ_i of $K(\zeta)$ over K which takes ζ into $X^\lambda Y^\mu H_i(X^{1/n}, Y^{1/n})$ takes $X^\lambda Y^\mu$ into $X^\lambda Y^\mu H_i(0,0)$ (since θ_i is effected by a substitution of the type $X^{1/n} \to \omega_1 X^{1/n}$, $Y^{1/n} \to \omega_2 Y^{1/n}$, $\omega_1^n = \omega_2^n = 1$). It follows that the family of elements $\{X^\lambda Y^\mu H_i(0,0)\}$ $i = 1, 2, \ldots, m$ is a complete set of conjugates of $X^\lambda Y^\mu$, each conjugate being repeated r times. This completes the proof.

Let N be the ideal of nilpotent elements in the ring $gr(A)$.

THEOREM 2.6. Let A be a quasi-ordinary local ring. Then $gr(A)$ is determined up to isomorphism by the distinguished pairs of any normalized representing branch of A, and precisely one of the four following pairs of statements holds:

1) a) A is a regular local ring, and $gr(A)$ is a polynomial ring over k.

   b) $\zeta = 0$ is the only normalized representing branch of A.

2) a) $N \neq 0$ is a prime ideal and $gr(A)/N$ is a polynomial ring over k.

b) If ζ is any normalized representing branch of A, then $\zeta \neq 0$ and the least distinguished pair (λ,μ) of ζ is such that $\lambda + \mu > 1$.

3) a) N is a prime ideal and gr(A)/N is of the form $k[X,Y,Z]/(Z^t - X^cY^d)$ where t, c, d are integers with $c + d = t$.

b) The integer t and the unordered pair (c,d) depend only on A. If ζ is any normalized representing branch of A, then $\zeta \neq 0$, and if (λ,μ) is the least distinguished pair of ζ, then $\lambda + \mu = 1$ and the unordered pair (c,d) is identical with the pair $(t\lambda, t\mu)$.

4) a) N is not a prime ideal. gr(A)/N is of the form $k[X,Y,Z]/(X^cY^d)$ where c and d are positive integers.

b) The unordered pair (c,d) depends only on A. If ζ is any normalized representing branch of A, then $\zeta \neq 0$, and if (λ,μ) is the least distinguished pair of ζ then $\lambda + \mu < 1$ and the two numbers λ, μ are proportional to the two numbers c, d (in some order).

PROOF. Most of the statements follow directly from the lemma. In the first place, gr(A) is determined by the pair (λ,μ) and by the degree $m = [K(\zeta):K]$. Since $K(\zeta)$ is obtained by adjoining the characteristic monomials of ζ to K (remark 1.4.4). m is determined by the distinguished pairs of ζ.

If $\zeta = 0$ is a representing branch of A, then A is a regular local ring, gr(A) is a polynomial ring, and $N = (0)$ is a prime ideal.

If $\zeta \neq 0$ is a normalized representing branch of A, then according to lemma 2.5 gr(A) is of one of the three forms (i) $k[X,Y,Z]/Z^m$ $(m > 1)$, (ii) $k[X,Y,Z]/(Z^t - X^cY^d)^r$ (c,d positive integers, $t = c + d \geq 2$) (iii) $k[X,Y,Z]/X^cY^d$ (c,d positive integers). In cases (i) and (ii) N is a prime ideal, and in case (iii) N is not. In case (i) $N \neq (0)$. In case (ii) gr(A)/N is the co-ordinate ring of the irreducible affine surface $Z^t = X^cY^d$, which has a t-fold point at the origin $(t \geq 2)$; hence gr(A)/N is not a polynomial ring.

Thus we see that one and only one of the statements 1a), 2a), 3a), 4a) holds, and that gr(A) determines which one of them does.

1a) → 1b): If $\zeta \neq 0$ is a normalized representing branch of A, then gr(A) is not a polynomial ring, and A is not regular.

2a) → 2b): This follows from the lemma.

3a) → 3b): The singular locus of $Z^t = X^c Y^d$ either has two components, in which case these components have multiplicities c, d respectively, or has less than two components, in which case one of the integers c, d is 1, and the other is t - 1 (and t is the multiplicity of the origin). The rest follows from the lemma.

4a) → 4b): The total quotient ring of $k[X,Y,Z]/(X^c Y^d)$ is a direct sum of two artinian rings of lengths c, d respectively. The rest follows from the lemma. q.e.d.

* * *

The <u>singular</u> <u>locus</u> of A is the set of prime ideals p in A such that the local ring A_p is not a regular local ring. If f is a defining polynomial of A, and if P is the inverse image in $R = k[[X,Y,Z]]$ of p, then $f \in P$ and the multiplicity of A_p is the unique integer e such that f is in the e-th symbolic power $P^{(e)} (=(P^e R_p) \cap R)$ while $f \notin P^{(e+1)}$ [6; §40.2]. In particular, A_p is regular if and only if its multiplicity is unity. We note that by a theorem of Zariski-Nagata [6; §38.3] the integer e is always $\leq$ multiplicity of A.

We introduce some geometric language. Let p be a prime ideal in A. We say that p _has multiplicity_ e _in_ A if the ring A_p has multiplicity e. We say that p _is a plane curve with a_ ν-_fold point at its origin_ if A/p is a one-dimensional local ring of multiplicity ν whose maximal ideal has a basis of two (or fewer) elements. We note that p is a curve with a 1-fold point (or a simple point) at its origin if and only if the maximal ideal of A/p is principal; in this case we say that p _is a non-singular plane curve_. Two plane curves p and q _intersect transversally_ if p and q together generate the maximal ideal of A. Finally we say that p is a _component_ of the singular locus of A if p is a minimal member of the singular locus.

In this terminology, the singular locus and its connection with distinguished pairs are described by the following theorem.

THEOREM 2.7. Let A be a quasi-ordinary local ring. Then precisely one of the following four statements holds true:

1) A is a regular local ring (i.e. the singular locus of A is empty).

2) The maximal ideal $\underline{m}$ of A is the only member of the singular locus.

3) The singular locus of A has precisely one component $p \neq \underline{m}$ and p is a non-singular plane curve.

4) The singular locus of A has two components, both of which are plane curves; these two curves intersect transversally. If one of the components has a ν-fold point at its origin ($\nu > 1$), then this component has multiplicity $< e$ in A (where e = multiplicity of A), while the other component is non-singular and has multiplicity e in A.

Furthermore, the following data (all of which are intrinsic to A) are completely determined by the distinguished pairs of any normalized representing branch of A.

(i) Which one of the above four statements holds for A.

(ii) The multiplicity of A.

(iii) The multiplicities in A of the curves in the singular locus.

(iv) The multiplicity of the origin of each such curve.

PROOF. We may assume that $A = k[[X,Y]][\zeta]$ where $\zeta \neq 0$ is some normalized quasi-ordinary branch. As in lemma 2.5, the minimum equation of ζ has the form

$$f(X,Y,Z) = \prod_{i=1}^{m} [Z - X^{\lambda}Y^{\mu}H_i(X^{1/n},Y^{1/n})] \qquad H_i(0,0) \neq 0$$

Suppose $\lambda > 0$. Let P be the ideal (X,Z) in $k[[X,Y,Z]]$. Then we see easily that $P^{(a)} = P^a$ for any integer a, and that $f \in P^a$ if and only if $a \leq \min(m,m\lambda)$. Hence if q is the ideal (X,ζ) in A, then q is a nonsingular plane curve of multiplicity $\min(m,m\lambda)$ in A. Similarly we see that A itself has multiplicity $\min(m,m\lambda+m\mu)$.

Similar remarks hold if $\mu > 0$.

If $p \subsetneq A$ does not contain the discriminant $X^aY^b\varepsilon$ of f over $k[[X,Y]]$ then A_p is an unramified extension of the regular local ring $k[[X,Y]]_q$ (where now $q = p \cap k[[X,Y]]$) cf. [6; §41.3]; since $\dim.A_p = \dim.k[[X,Y]]_q$ [6; §10.14] and since the maximal ideal of A_p is generated by that of $k[[X,Y]]_q$, A_p is itself regular. Hence any prime ideal in the singular locus of A contains either X or Y. If we suppose that neither λ nor μ vanishes, then both X and Y

divide $f(X,Y,Z) - Z^m$ (in $k[[X,Y,Z]]$) so that both X and Y divide ζ^m in A. If p contains one of X,Y, then p contains ζ, and p is one of the ideals (X,ζ), (Y,ζ), (X,Y,ζ). Thus, (if neither λ nor μ vanishes), no prime ideal other than these three belongs to the singular locus of A, and we see that precisely one of the statements 1), 2), 3), 4) hold.

Suppose now that $\mu = 0$. Since ζ is normalized, $\lambda > 1$, and so the ideal (X,ζ) is a nonsingular curve whose multiplicity in A is equal to the multiplicity of A itself (viz. to m). Let p belong to the singular locus of A. If $X \in p$, then, as before, p is one of the ideals (X,ζ), (X,Y,ζ).

Suppose $X \notin p$. Then $Y \in p$. Since $f(X,Y,\zeta) = 0$, $f(X,0,\zeta) \in p$. Now $f(X,0,Z) = \prod_i [Z - X^\lambda H_i(X^{1/n},0)]$. We claim that $f(X,0,Z)$ is a power of the minimum polynomial $g(X,Z)$ of $\zeta_0 = X^\lambda H(X^{1/n},0)$ over $k[[X]]$ (or, what is the same thing, over $k[[X,Y]]$). This amounts to saying that the family of elements $\{X^\lambda H_i(X^{1/n},0)\}$ $i = 1, 2, \ldots, m$ is a complete set of conjugates of ζ_0, each conjugate being repeated r times where $r = [K(\zeta):K(\zeta_0)]$; the proof of the latter statement is the same as that given for lemma 2.5 (statement 2)). Thus $f(X,0,Z) = g(X,Z)^r$.

Since $f(X,0,\zeta) \in p$, also $g(X,\zeta) \in p$. On the other hand, since g is an irreducible element of $k[[X,Z]]$, the ideal $(Y,g(X,Z))$ is a prime ideal of $k[[X,Y,Z]]$. It follows that p is the ideal $(Y,g(X,\zeta))$ in A.

We are left, therefore, with the following question: if q is the ideal $(Y,g(X,\zeta))$, what is the multiplicity of A_q? As a first step in answering the question we note that $K(\zeta_0) = K(X^{1/u})$ for some integer u where K is now the quotient field $k\{\{X\}\}$ of $k[[X]]$ (for it is clear that ζ_0 is a quasi-ordinary branch, (in one variable), whose characteristic monomials are of the form $X^{\lambda_1}, X^{\lambda_2}, \ldots, X^{\lambda_t}$, the λ_i being rational numbers; and we have seen (remark 1.4.4) that then $K(\zeta_0) = K(X^{\lambda_1}, X^{\lambda_2}, \ldots, X^{\lambda_t}) = K(X^{1/u})$ where u = l.c.m. of the denominators of the λ_i when the λ_i are written as reduced fractions).

Now we consider the ring $B = A[X^{1/u}] = k[[X^{1/u},Y]][\zeta]$. The prime ideals of B which contain q are those which contain Y (since Y divides $g(X,\zeta)^r$). However, $B = k[[X^{1/u},Y,Z]]/h(X^{1/u},Y,Z)$ where $h(X^{1/u},Y,Z)$ is the minimum polynomial (in Z) of ζ over $k[[X^{1/u},Y]]$; we have

$h(X^{1/u},0,Z) = (Z - \zeta_0)^r$, and it follows that any prime ideal in B containing Y contains $(\zeta - \zeta_0)^r$, hence contains the ideal $\mathfrak{Q} = (Y, \zeta - \zeta_0)$.

The ideal $\mathfrak{Q}$ is a prime ideal in B, and since B is integral over A, $\mathfrak{Q}$ must be the only prime ideal in B lying over q. It follows that $B_q = B \otimes_A A_q$ is a local ring, whence $B_q = B_{\mathfrak{Q}}$. B_q is a finite A_q-module. The residue field of B_q is obtained from the residue field $k\{\{X\}\}(\zeta)$ of A_q by adjunction of an element whose u-th power is X. Since $k\{\{X\}\}(\zeta_0) = k\{\{X^{1/u}\}\}$, there is actually no residue field extension from A_q to B_q. Finally, since

$$g(X,\zeta) = \prod_{j=1}^{t}[\zeta - \zeta_0^{(j)}], \text{ where } \zeta_0 = \zeta_0^{(1)}, \zeta_0^{(2)}, \ldots, \zeta_0^{(t)}$$

are the conjugates of ζ_0 over $k[[X]]$, and since, for $j \neq 1$, $\zeta - \zeta_0^{(j)} \in B$ but not to $\mathfrak{Q}$ (otherwise $\zeta_0 - \zeta_0^{(j)}$, which is of the form $X^{v/u}$.(unit in B) lies in $\mathfrak{Q}$, whence $X^{1/u}$ lies in $\mathfrak{Q}$ and $\mathfrak{Q}$ is the maximal ideal of B, which is absurd) we see that in fact $\mathfrak{Q}B_q$ is generated by Y and $g(X,\zeta)$, so that B_q is unramified over A_q. The foregoing facts imply that $B_{\mathfrak{Q}} = B_q = A_q$ [6; §41.8].

It remains to determine the multiplicity of $B_{\mathfrak{Q}}$. Now B is isomorphic to $k[[X,Y]][\bar{\zeta}]$ where $\bar{\zeta}$ is obtained from $\zeta - \zeta_0$ through replacing X by X^u, and under the indicated isomorphism, Q becomes the ideal $(Y,\overline{\zeta})$. It follows from the fact that ζ is a quasi-ordinary branch that also $(\overline{\zeta})$ is a quasi-ordinary branch (prop. 1.3). Moreover, examining ζ in light of the proof of proposition 1.5 we find that either (i) $\overline{\zeta} = YG(X,Y) \in k[[X,Y]]$ or (ii) $\overline{\zeta}$ is of the form $YG(X,Y) + X^{\sigma}Y^{\tau}G_1(X^{u/n},Y^{1/n})$ with $G \in k[[X,Y]]$ and $G_1(0,0) \neq 0$, and where $(\sigma/u,\tau)$ is the least distinguished pair of ζ whose second member does not vanish.

The first case occurs only if all the distinguished pairs (λ_i,μ_i) of ζ are such that $\mu_i = 0$ (this means that there is "equisingularity" along the curve (X,ζ), cf. end of §4); in this case B_Q has multiplicity 1. In case (ii), we find by previous reasoning that B_Q has multiplicity $\min(r,r\tau)$. We have therefore determined the multiplicity of A_q in terms of the distinguished pairs of ζ.

Noting that $r < m$ = multiplicity of A, we see that statement 4) holds in this case. If $\lambda = 0$, a similar argument leads again to condition 4).

To complete the proof, we note that all the data mentioned in the statement of the second part of the theorem have been determined by the distinguished pairs of ζ, and by the degrees of certain field extensions, which are also determined by the distinguished pairs (remark 1.4.4). q.e.d.

COROLLARY 2.7.1. If p is a prime ideal of A, then A_p is analytically irreducible.

PROOF. We may, clearly, restrict our attention to prime ideals in the singular locus. The latter part of the theorem (in which we replace A_q by B_Q) shows that we may then assume that neither λ nor μ vanishes. We may therefore assume that $A_p = R_P/(f)$ where P is the ideal (X,Z) in R = k[[X,Y,Z]], and f is the minimum polynomial of ζ. The completion of R_P is clearly the ring k{{Y}}[[X,Z]]; since f is irreducible over k{{Y,X}}, we conclude that the completion of A_p is an integral domain. q.e.d.

We remark now that condition 2) of the theorem holds for A if and only if A is integrally closed and not regular (cf. corollary to proposition 2.1 in the paper on "Free Derivation Modules ..." which is part of this thesis; the corollary is applicable, since the normalization of A is a finite A-module, A being complete [6; 32.1]. Actually, the remark can also be deduced directly from the proof of theorem 2.7 by a slight modification of the proof of the following corollary). In particular, if A has a defining equation of the form $Z^m - XY$ (i.e. if A is isomorphic to $k[[X,Y]][X^{1/m}Y^{1/m}]$) then A is normal. The converse is also true:

COROLLARY 2.7.2. If A is normal, then A has a defining polynomial of the form $Z^m - XY$ for some integer m.

PROOF. If A is regular we take $m = 1$. Otherwise set $A = k[[X,Y]][\zeta]$ where $\zeta = X^{\lambda}Y^{\mu}H(X^{1/n},Y^{1/n})$ as in the proof of the theorem. Since A is normal, then A has no curves in its singular locus, and we conclude that $0 < \lambda < 1$, $0 < \mu < 1$; it is then clear that $m\lambda = m\mu = 1$, where m is the degree of the minimum equation of ζ (all this comes out of the earlier part of the proof of the theorem). Thus $X^{1/m}Y^{1/m}$ is a distinguished monomial of ζ, and since $m = [K(\zeta):K] = [K(X^{1/m}Y^{1/m}):K]$ (here K = quotient field of $k[[X,Y]]$) we have $K(\zeta) = K(X^{1/m}Y^{1/m})$ (cf. remark 1.4.4). Since A is the integral closure of $k[[X,Y]]$ in $K(\zeta)$, and since $k[[X,Y]][X^{1/m}Y^{1/m}]$ is integrally closed (as we have just seen) we see that $A = k[[X,Y]][X^{1/m}Y^{1/m}] = k[[X,Y,Z]]/(Z^m - XY)$. q.e.d.

3. Quadratic and Monoidal Transforms of Quasi-ordinary Local Rings.

We review some facts about quadratic and monoidal transformations. All the statements made are easy consequences of the elementary properties of "Proj" (cf.[3;§§2.4,2.8]). The reader who is willing to accept Proposition-Definition 3.1 may well skip this preliminary material.

Let R be a (noetherian) local ring, let S = Spec R, let M be the maximal ideal of R, and let k = R/M. For the moment, we make no assumptions about k. Let P be a prime ideal in R. The direct sum $\bigoplus_{n\geq 0} P^n$ is, in a natural way, a graded R-algebra; the S-scheme $T = \mathrm{Proj}(\bigoplus_{n\geq 0} P^n)$ is called the <u>monoidal transform of</u> S <u>with center</u> P and the structural morphism $\pi: T \to S$ is called the <u>monoidal transformation</u> of S with center P. If P = M we usually use the word "<u>quadratic</u>" instead of "monoidal" and omit all reference to the center M. (All these definitions hold, of course, in a more general context [3;§8.1]).

The fibre $\pi^{-1}(M)$ is the algebraic scheme $\mathrm{Proj}((R/M \otimes_R (\bigoplus_{n\geq 0} P^n)) = \mathrm{Proj}(\bigoplus_{n\geq 0} P^n/MP^n)$, in other words the projective k-scheme defined by the graded ring $k[x_1, x_2, \ldots, x_n]$, the x_i being the MP-residues of some system of generators of P, with degree of $x_i = 1$ for all x_i. As a topological space, $\pi^{-1}(M)$ is a closed subspace of T. We say that a ring R' is a <u>monoidal</u> <u>transform</u> of R (or <u>quadratic</u> <u>transform</u>, as the case may be) if R' is the local ring on T of some <u>closed</u> point of $\pi^{-1}(M)$.

In the special case when $R = k[[X,Y,Z]]$, k an algebraically closed field, and $P = (X,Y)R$, $\pi^{-1}(M)$ is defined by a <u>polynomial</u> <u>ring</u> $k[x,y]$, i.e. $\pi^{-1}(M)$ is the projective line over k. Similarly if $P = M = (X,Y,Z)R$, then $\pi^{-1}(M)$ is the projective plane over k. Thus, the closed points of $\pi^{-1}(M)$ are in one-one correspondence with the directions $(\alpha:\beta)$ (or $(\alpha:\beta:\gamma)$ as the case may be), $\alpha,\beta,\gamma \in k$ (with $(0:0)$, respectively $(0:0:0)$, being excluded from consideration).

If $P = (X,Y,Z)R$, then T is covered by the three affine schemes $T_X = \mathrm{Spec}(R[Y/X,Z/X])$, $T_Y = \mathrm{Spec}(R[X/Y,Z/Y])$, $T_Z = \mathrm{Spec}(R[X/Z,Y/Z])$. We find that the point with coordinates $(\alpha:\beta:\gamma)$ lies in T_X if and only if $\alpha \neq 0$; similar remarks hold for (T_Y,β) and for (T_Z,γ). Suppose then that η is a closed pointof T with $\pi(\eta) = M$, and that η has co-ordinates $(\alpha:\beta:\gamma)$, $\alpha \neq 0$, when η is considered as a point of the projective plane $\pi^{-1}(M)$. The local ring R' of η on T is of the form $R[Y/X,Z/X]_{M'}$ where M' is a maximal ideal in $R[Y/X,Z/X]$. The local ring of η on the projective plane $\pi^{-1}(M)$ is $R'/MR' = R'/XR'$. R'/XR' is a regular local ring of dimension two, whose maximal ideal is generated by $y' - (\beta/\alpha)$, $z' - (\gamma/\alpha)$; y', z' being the XR' residues of Y/X, Z/X. It follows that R' is a regular local ring of dimension three, and that $\{X, Y/X - \beta/\alpha, Z/X - \gamma/\alpha\}$ is a regular system of parameters for R'.

Let $f \neq 0$ be an element of R contained in P, let $\bar{R} = R/(f)$ and let $\bar{P} = P/(f)$. The natural epimorphism $\bigoplus_{n \geq 0} P^n \to \bigoplus_{n \geq 0} \bar{P}^n$ gives rise to a closed immersion φ of $\bar{T} = \mathrm{Proj}(\oplus \bar{P}^n)$ into $T = \mathrm{Proj}(\oplus P^n)$.

If x is the image of X in $\bar{R}$, then the natural map of R onto $\bar{R}$ extends to a map of rings of quotients $R_X \to \bar{R}_x$, which restricts to a map of $R[Y/X,Z/X]$ into $\bar{R}_x$; the image $\tilde{R}$ of this last map is the co-ordinate ring of the affine scheme $T_X \cap \bar{T}$. The kernel of the map $R[Y/X,Z/X] \to \tilde{R}$ is the ideal $fR_X \cap R[Y/X,Z/X]$, which is easily seen to be the principal ideal generated by f/X^s where s is the greatest integer such that $f \in P^s$. Thus the kernel of the epimorphism $R' \to \bar{R}'$ (the localization of $\tilde{R}$ with respect to the image of M') is generated by f', the image of f/X^s in R'. We say that f' is the <u>strict transform of</u> f <u>in</u> R'. f' is a non-unit in R' if and only if the point η is in the image of the closed immersion φ; when this is so, $\bar{R}'$ is a quadratic transform of $R/(f)$, and the epimorphism $R' \to \bar{R}'$ is the local map associated with φ.

Let $X' = X$, $Y' = Y/X - \beta/\alpha$, $Z' = Z/X - \gamma/\alpha$ so that X', Y', Z' are regular parameters for R'. Set $\beta' = \beta/\alpha$, $\gamma' = \gamma/\alpha$. If f is written in the form

$$f = f_s(X,Y,Z) + f_{s+1}(X,Y,Z) + \ldots \qquad (s > 0)$$

where the f_i are homogeneous forms of degree i, then, noting that $X = X'$, $Y = X'(Y' + \beta')$, $Z = X'(Z' + \gamma')$ we find that

$$f' = f/X^s = f_s(1,Y'+\beta',Z'+\gamma') + X'f_{s+1}(1,Y'+\beta',Z'+\gamma') + \ldots$$

Note that f' is a non-unit in R' if and only if $f_s(1,\beta',\gamma') = 0$, i.e. if and only if $f_s(\alpha,\beta,\gamma) = 0$. If this is so, and if $\hat{R}'$ is the completion of R', then $\hat{R}'/(f')$ is a complete two dimensional local ring, which is the completion of some quadratic transform of $R/(f)$.

In a similar way, we find that when $P = (X,Y)R$, the monoidal transform R'' of R corresponding to some point η with co-ordinates $(\alpha:\beta)$, $\alpha \neq 0$, is a regular local ring of dimension three, with regular parameters X, $Y/X - \beta/\alpha$, Z. We consider, as above, an element $0 \neq f \in P$, and we impose the additional assumption that if t is an integer such that $f \in M^t$, then $f \in P^t$ (in other words the local rings $R/(f)$ and $(R/(f))_{\overline{P}}$ have the same multiplicity). Then we can write

$$f = F_s(X,Y,Z) + F_{s+1}(X,Y,Z) + \ldots$$

where F_i is a form with coefficients in $k[[Z]]$ and of degree i in X and Y, and where, moreover, $F_s(X,Y,0) \neq 0$. Setting $X'' = X$, $Y'' = Y/X - \beta/\alpha = Y/X - \beta''$, $Z'' = Z$, so that X'', Y'', Z'' are regular parameters for R'', we find that

$$f'' = f/X^s = F_s(1,Y''+\beta'',Z'') + X''F_{s+1}(1,Y''+\beta'',Z'') + \ldots$$

f'' is a non-unit in R'' if and only if $F_s(1,\beta'',0) \neq 0$, i.e. if and only if $f_s(\alpha,\beta) \neq 0$, where $f_s(X,Y,Z) = f_s(X,Y)$ is the initial form of f (i.e. the form which is the sum of the terms of f of lowest total degree in X,Y,Z). If this is so, then $\hat{R}''/(f'')$ is the completion of some monoidal transform of $R/(f)$ with center $P/(f)$.

The significance of the foregoing remarks for our purposes is summarized in

PROPOSITION - DEFINITION 3.1. Let $R = k[[X,Y,Z]]$ be as in previous sections. Let $f = f_s(X,Y,Z) + f_{s+1}(X,Y,Z) + \ldots$ $(s > 0)$ be a power series in R, the f_i being forms of degree i in X,Y,Z.

a) Let $(\alpha,\beta,\gamma) \neq (0,0,0)$ be a triple of elements in k such that $f_s(\alpha,\beta,\gamma) = 0$. If $\alpha \neq 0$, we set

$$f'_{\underline{\alpha},\beta,\gamma} = f_s(1,Y+\beta/\alpha,Z+\gamma/\alpha) + Xf_{s+1}(1,Y+\beta/\alpha,Z+\gamma/\alpha) + \ldots$$

If $\beta \neq 0$ we set

$$f'_{\alpha,\underline{\beta},\gamma} = f_s(X+\alpha/\beta,1,Z+\gamma/\beta) + Yf_{s+1}(X+\alpha/\beta,1,Z+\gamma/\beta) + \ldots$$

If $\gamma \neq 0$, we set

$$f'_{\alpha,\beta,\underline{\gamma}} = f_s(X+\alpha/\gamma,Y+\beta/\gamma,1) + Zf_{s+1}(X+\alpha/\gamma,Y+\beta/\gamma,1) + \ldots$$

According to the preceding discussion, those of the elements $f'_{\underline{\alpha},\beta,\gamma}$, $f'_{\alpha,\underline{\beta},\gamma}$, $f'_{\alpha,\beta,\underline{\gamma}}$ which are defined are non-units in R, and all of those rings $R/f'_{\underline{\alpha},\beta,\gamma}$, $R/f'_{\alpha,\underline{\beta},\gamma}$, $R/f'_{\alpha,\beta,\underline{\gamma}}$ which are defined are isomorphic to the completed quadratic transform of R/(f) at the point $(\alpha:\beta:\gamma) \in \pi^{-1}(M)$ (π = quadratic transformation of R, M = maximal ideal of R). It makes sense therefore to call any one of those rings the _formal quadratic transform of the pair_ (R,f) _in the direction_ $(\alpha:\beta:\gamma)$. The collection of all formal quadratic transforms so defined (one for each suitable direction) depends only on the ring R/(f).

We shall use the term _formal quadratic transform of_ A to refer to the completion of any quadratic transform of the local ring A, i.e. to any formal quadratic transform of a pair (R,f) with R/(f) isomorphic to A.

b) Let P be a prime ideal in R generated by some two of X,Y,Z, say by X,Y. Suppose that, s being as above, f may be written as $f = F_s(X,Y,Z) + F_{s+1}(X,Y,Z) + \dots$ the F_i being forms with coefficients in k[[Z]] of degree i in X and Y. (Then $f \in P$, $F_s(X,Y,0) \neq 0$, and the local rings R/(f), $(R/(f))_{\overline{P}}$ (where $\overline{P} = P/(f)$) have the same multiplicity). Let $(\alpha,\beta) \neq (0,0)$ be a pair of elements in k such that $F_s(\alpha,\beta,0) = 0$ (or, what amounts to the same thing, $f_s(\alpha,\beta) = 0$ (f_s is independent of Z under our assumptions on f and P)). If $\alpha \neq 0$ we set

$$f''_{\underline{\alpha},\beta} = F_s(1,Y+\beta/\alpha,Z) + XF_{s+1}(1,Y+\beta/\alpha,Z) + \ldots$$

If $\beta \neq 0$ we set

$$f''_{\alpha,\underline{\beta}} = F_s(X+\alpha/\beta,1,Z) + YF_{s+1}(X+\alpha/\beta,1,Z) + \ldots$$

We then proceed as before to define *the formal monoidal transform of the pair* (R,f), *with center* P, *in the direction* $(\alpha:\beta)$.

If A is isomorphic to R/(f), and $\overline{P}$ is the image of P in A (so that $A/\overline{P}$ is regular), we shall use the term *formal monoidal transform of* A *with center* $\overline{P}$ to refer to any formal monoidal transform of the pair (R,f) with center P. The formal monoidal transforms of A with center $\overline{P}$ are precisely the completions of the monoidal transforms of the local ring A with center $\overline{P}$.

DEFINITION 3.2. If A is any local ring, we say that a prime ideal $\overline{P}$ in A is a *permissible center* if $A/\overline{P}$ is a regular local ring and if the rings A and $A_{\overline{P}}$ have the same multiplicity.

The maximal ideal of A is a permissible center. The assumption in part b) of 3.1 above may also be expressed as follows: whenever we speak of formal monoidal transforms of A, such transforms are taken with respect to a permissible center which is a curve in A.

* * *

Our goal now will be to determine the extent to which the property of being quasi-ordinary is preserved under quadratic and monoidal transformations. The next few pages are devoted to procedures for displaying the roots of transforms of quasi-ordinary polynomials as fractional power series, and, in the easier cases, to showing that these roots are actually quasi-ordinary branches.

First we remark that any formal transform (quadratic or monoidal) of a regular local ring is again regular.

From now on, therefore, we assume that we are dealing with a quasi-ordinary ring A which is not regular. Let

$$\zeta = X^{u/n}Y^{v/n}H(X^{1/n},Y^{1/n}) \qquad u,v \text{ integers}, \quad H(0,0) \neq 0$$

be a normalized quasi-ordinary branch which represents A. The conjugates $\zeta = \zeta_1, \zeta_2, \ldots, \zeta_m$ are of the form $\zeta_i = X^{u/n}Y^{v/n}H_i(X^{1/n},Y^{1/n})$ $i = 1, 2, \ldots, m$, and so

$$f(X,Y,Z) = \prod_{i=1}^{m}[Z - X^{u/n}Y^{v/n}H_i(X^{1/n},Y^{1/n})]$$

is a defining polynomial of A. We may, and shall, assume that $A = k[[X,Y]][\zeta]$.

CASE 3.3. Monoidal Transformations

Suppose there exists a permissible center in A which is a curve. Then we see (by the proof of theorem 2.7) that $u/n + v/n > 1$, and that either $u/n \geq 1$, in which case the prime ideal (X,ζ) is a permissible center, or $v/n \geq 1$, in which case (Y,ζ) is a permissible center. (We may, of course, have <u>both</u> $u/n \geq 1$ and $v/n \geq 1$).

Suppose $u/n \geq 1$ and let $P = (X,\zeta)$. The inverse image of P in $R = k[[X,Y,Z]]$ is $(X,Z)R$. The initial form of $f(X,Y,Z)$ is $f_m(X,Z) = Z^m$, so that $f_m(\alpha,\beta) = 0$ if and only if $\beta = 0$. There is, therefore, only one formal monoidal transform of (R,f) viz. the one in the direction (1:0). A defining polynomial for this transform is

$$f'(X,Y,Z) = [f(X,Y,XZ)]/X^m = \prod_{i=1}^{m} [Z - X^{(u-n)/n} Y^{v/n} H_i(X^{1/n},Y^{1/n})]$$

$$= \prod_{i=1}^{m} [Z - (\zeta_i/X)]$$

The elements ζ_i/X $i = 1, 2, \ldots, m$ are obviously all conjugates of each other over $k[[X,Y]]$, so that f' is irreducible. Moreover, $(\zeta_i/X) - (\zeta_j/X) = M_{ij}\varepsilon_{ij}/X$ where M_{ij} is a monomial in $X^{1/n},Y^{1/n}$ and ε_{ij} is a unit. It follows that the (ζ_i/X) are quasi-ordinary branches (prop. 1.3), and that if $\{(\lambda_j,\mu_j)\}$ $j = 1, 2, \ldots, s$ is the set of distinguished pairs of ζ, then $\{(\lambda_j-1,\mu_j)\}$ $j = 1, 2, \ldots, s$ is the set of distinguished pairs of ζ_i/X for any i.

Similar results hold when we assume $v/n \geq 1$ and take P to be (Y,ζ). Thus, any formal monoidal transform A' of a quasi-ordinary local ring A is again quasi-ordinary; for any normalized representing branch ζ of A, the branch ζ/X (or ζ/Y as the case may be) is a representing branch (not necessarily normalized) of A', whose distinguished pairs are determined by those of ζ.

CASE 3.4. Quadratic Transforms.

CASE 3.4.1. $u/n + v/n \geq 1$ ("Transversal" Case).

By theorem 2.6, the initial form of f is
$f_m(X,Y,Z) = Z^m$ when $u/n + v/n > 1$, or
$f_m(X,Y,Z) = (Z^t - X^aY^b)^r$ $(a + b = t)$ when $u/n + v/n = 1$.
We use the term <u>exceptional</u> <u>curve</u> of a quadratic transformation π of Spec(A) for the reduced scheme underlying $\pi^{-1}(M)$. $\pi^{-1}(M)$ is defined by the associated graded ring gr(A) of A with respect to its maximal ideal. Thus, in the present case, the exceptional curve is either a projective line (defined in the projective plane by $Z = 0$) or an irreducible projective curve (defined by $Z^t = X^aY^b$).

We always have $f_m(0,0,1) \neq 0$, so we may restrict our attention to quadratic transforms in a direction of the form $(1:\beta:\gamma)$ or of the form $(\alpha:1:\gamma)$; for reasons of symmetry, it is sufficient to consider a direction $(1:\beta:\gamma)$ with $f_m(1,\beta,\gamma) = 0$. The appropriate transform of f is then

$$f'(X,Y,Z) = [f(X,X(Y+\beta),X(Z+\gamma))]/X^m$$

Let G be a power series in one variable over k such that $[G(Y^{1/n})]^n = Y + \beta$ (if $\beta = 0$ we take $G(Y^{1/n}) = Y^{1/n}$; if $\beta \neq 0$, the binomial theorem shows that $G(Y^{1/n})$ is actually of the form $\overline{G}(Y)$). Let $\xi = G(Y^{1/n})$. We have

$$f(X, X(Y+\beta), X(Z+\gamma)) = f(X, (X^{1/n}\xi)^n, X(Z+\gamma))$$

$$= \prod_{i=1}^{m} [X(Z+\gamma) - X^{u/n}(X^{1/n}\xi)^v H_i(X^{1/n}, X^{1/n}\xi)]$$

so that

$$f'(X,Y,Z) = \prod_{i=1}^{m} [Z + \gamma - X^{(u+v-n)/n}\xi^v H_i(X^{1/n}, X^{1/n}\xi)]$$

Thus the roots of f' (which is still a polynomial in Z over k[[X,Y]]) are the fractional power series

$$\zeta_i' = -\gamma + X^{(u+v-n)/n}\xi^v H_i(X^{1/n}, X^{1/n}\xi) \qquad i = 1, 2, \ldots, m.$$

If $u + v > n$, or if $u + v = n$ and $\beta = 0$, then $\gamma = 0$ and ζ_i' is a non-unit in Φ_n for all i. However, if $u + v = n$ and $\beta \neq 0$, it can easily be seen that ζ_i' is a non-unit for precisely r values of i, where $r = [K(\zeta):K(X^{u/n}Y^{v/n})]$, (since $H_i(0,0)$ takes on

$[K(X^{u/n}Y^{v/n}):K]$ different values).

If ζ_i' is a non-unit, then so are all its conjugates, and a representing polynomial for the transform under consideration is $\Pi^*[Z - \zeta_i']$, where the product Π^* is taken only over those values of i for which ζ_i' is a non-unit. Now

$$\zeta_i' - \zeta_j' = X^{(u+v-n)/n}\xi^v[H_i(X^{1/n},X^{1/n}\xi) - H_j(X^{1/n},X^{1/n}\xi)].$$

But since

$$\zeta_i - \zeta_j = X^{u/n}Y^{v/n}[H_i(X^{1/n},Y^{1/n}) - H_j(X^{1/n},Y^{1/n})] = M_{ij}\varepsilon_{ij}$$

(M_{ij} a monomial in $X^{1/n}$, $Y^{1/n}$, $\varepsilon_{ij}(0,0) \neq 0$) it follows that $\zeta_i' - \zeta_j' = X^{s/n}\xi^t\varepsilon_{ij}(X^{1/n},X^{1/n}\xi)$ (s, t integers). Hence any ζ_i' which is a non-unit in Φ_n is a quasi-ordinary branch, and so the irreducible factors of f' are quasi-ordinary polynomials.

If $\beta = 0$ (in which case $\xi = Y^{1/n}$), then it is not hard to see that all the ζ_i' are conjugate to each other, i.e. that f' is irreducible. Thus the corresponding formal transform of A is again quasi-ordinary. Moreover we see that if $\{(\lambda_i, \mu_i)\}$ is the set of distinguished pairs of ζ, then $\{(\lambda_i + \mu_i - 1, \mu_i)\}$ is the set of distinguished pairs of any ζ_i'. As in the case of a monoidal transform, the distinguished pairs of ζ_i' are determined by those of ζ.

If $\beta \neq 0$, then f' may very well be reducible. For example if $f = Z^2 - X^2Y$, then $f' = Z^2 - X^2(Y + \beta)$. According to our definition the transform of A is not quasi-ordinary if f' is reducible. This is no loss, however, for when $\beta \neq 0$, ξ is a unit in $k[[Y^{1/n}]]$ and the discriminant of f' is essentially a power of X alone. This indicates a situation of "equisingularity" and the analysis of such a situation depends only on the theory of plane curves. (cf. further remarks at end of §4, and also §6).

CASE 3.4.2. $u/n + v/n < 1$ ("Non-transversal" Case)

Since ζ is normalized, we have $0 < u$, $0 < v$. The initial form of f is $cX^{mu/n}Y^{mv/n}$ with $c \in k$, so that the exceptional curve is the pair of lines in the projective plane defined by the equation $XY = 0$. For any point $(\alpha:\beta:\gamma)$ on the exceptional curve, either $\alpha = 0$ or $\beta = 0$. We may therefore restrict our attention to quadratic transforms in a direction having one of the forms $(1:0:\gamma)$, $(0:1:\gamma)$, $(0:0:1)$; by symmetry it will be sufficient to consider only the first and last of these forms.

CASE 3.4.2.a). The direction $(1:0:\gamma)$.

We have to consider the power series

$$f'(X,Y,Z) = [f(X,XY,X(Z+\gamma))]/X^{(mu+mv)/n}$$

It is clear that $f'(0,Y,0) = Y^{mv/n}.(\text{unit in } k[[Y]])$. Thus by the Weierstrass preparation theorem, there is a <u>unique</u> power series $g'(X,Y,Z)$ such that g' is a pseudo-polynomial of degree mv/n in Y, and such that $g' = f'.(\text{unit in } k[[X,Y,Z]])$. g' is called the distinguished pseudo-polynomial in Y associated with f'. Since $k[[X,Y,Z]]/(f') = k[[X,Y,Z]]/(g')$, it will be sufficient for our purposes to study the roots of g'.

Let $G(Z^{1/v})$, $G_i(X^{1/n},Y^{1/n})$ be such that

$$[G(Z^{1/v})]^v = Z + \gamma \quad [G_i(X^{1/n},Y^{1/n})]^v = H_i(X^{1/n},Y^{1/n}).$$

Let $\xi = G(Z^{1/v})$, $\xi_i = G_i(X^{1/n}, X^{1/n}Y^{1/n})$. Then

$$f'(X,Y,Z) = \prod_{i=1}^{m} \{[X^{(n-u-v)/nv}\xi]^v - [Y^{1/n}\xi_i]^v\}$$

$$= \pm \prod_{i=1}^{m} \prod_{j=1}^{v} \{\omega_j X^{(n-u-v)/nv}\xi - Y^{1/n}\xi_i\}$$

where ω_j runs through the v-th roots of unity.

Let T be an indeterminate and let $E_i(X,Y,T)$ be such that $E_i(0,0,0) \neq 0$, and

$$E_i(X,Y,T)(T - YG_i(X,XY)) = Y - T\bar{G}_i(X,T) \qquad \bar{G}_i(0,0) \neq 0.$$

Since $G_i(0,0) \neq 0$, the existence of E_i is guaranteed by the preparation theorem. Setting

$\varepsilon_i = E_i(X^{1/n}, Y^{1/n}, \omega_j X^{(n-u-v)/nv}\xi)$ and

$\bar{\xi}_i = \bar{G}_i(X^{1/n}, \omega_j X^{(n-u-v)/nv}\xi)$ we have

$$\varepsilon_i(\omega_j X^{(n-u-v)/nv}\xi - Y^{1/n}\xi_i) = Y^{1/n} - \omega_j X^{(n-u-v)/nv}\xi\bar{\xi}_i$$

Hence for some unit ε in $k[[X^{1/nv}, Y^{1/n}, Z^{1/v}]]$ we have

$$f'(X,Y,Z) = \varepsilon \prod_{i=1}^{m} \prod_{j=1}^{v} \{Y^{1/n} - \omega_j X^{(n-u-v)/nv}\xi\bar{\xi}_i\}$$

Now the product on the right is clearly the distinguished pseudo-polynomial in $Y^{1/n}$ associated with $f'(X,Y,Z)$ when $f'(X,Y,Z)$ is thought of as an element of $k[[X^{1/nv}, Y^{1/n}, Z^{1/v}]]$; but so also is $g'(X,Y,Z)$. By uniqueness we must have

$$g'(X,Y,Z) = \prod_{i=1}^{m} \prod_{j=1}^{v} \{Y^{1/n} - \omega_j X^{(n-u-v)/nv} \xi \bar{\xi}_i\}$$

It follows that the roots of g' (considered as a polynomial in Y over $k[[X,Z]]$) are the n-th powers of the fractional power series $\omega_j X^{(n-u-v)/nv} \xi \bar{\xi}_i$. Thus the roots of g' are fractional power series which are non-units.

It would be possible now, using methods similar to those of lemma 2.3, to show that the discriminant of g' is of the form $X^a \xi^b$.(unit in $k[[X,Z]]$) thereby establishing the fact that the roots of g' are quasi-ordinary branches (in the variables X, Z). However, this fact will also result (although more circuitously) from the study of distinguished pairs in the next section. We prefer to defer the proof until that time.

We would also like to prove, as in the transversal case, that for $\gamma = 0$, the corresponding formal transform is an integral domain, and that the distinguished pairs of any of the roots of g' depend only on the distinguished pairs of ζ. We have found it necessary to devote most of §4 to the carrying out of such proofs.

CASE 3.4.2.b). The direction (0:0:1).

Let $F_i(X,Y)$ be such that $[F_i(X,Y)]^{n-u-v} = H_i(X,Y)$, and let $\xi_i = F_i(X^{1/n}Z^{1/n}, Y^{1/n}Z^{1/n})$. We are interested in the power series

$$f'(X,Y,Z) = [f(XZ,YZ,Z)]/Z^{(mu+mv)/n}$$

$$= \prod_{i=1}^{m} \{[Z^{1/n}]^{n-u-v} - [X^{u/n(n-u-v)}Y^{v/n(n-u-v)}\xi_i]^{n-u-v}\}$$

$$= \prod_{i=1}^{m} \prod_{j=1}^{n-u-v} \{Z^{1/n} - \omega_j X^{u/n(n-u-v)}Y^{v/n(n-u-v)}\xi_i\}$$

where now ω_j runs through the (n-u-v)-th roots of unity. Several applications of the Weierstrass preparation theorem, as in 3.4.2.a), show that all the roots of the distinguished pseudo-polynomial in Z associated with f' are fractional power series. We will give the details of the latter steps in §4, when we actually calculate distinguished pairs in order to show that the new roots are quasi-ordinary branches. As in 3.4.2.a), we shall wait until this calculation is performed before we show that the new distinguished pairs depend only on the "old" ones, and that the formal transform under consideration is actually an integral domain.

* * *

In the sequel, the formal monoidal transforms of A and certain formal quadratic transforms will play a special role. This has been indicated in the preceding remarks, where it was pointed out that formal quadratic transforms in any direction other than (1:0:0), (0:1:0) and (0:0:1) have defining polynomials whose discriminant has only one irreducible factor. We will describe the special transforms in an ad hoc manner, relating this description to the behavior of the discriminant in §4 (cf. corollary 4.4.1 and remarks on equisingularity at end of §4).

Let A be a local ring, let P be a prime ideal in A and let A' be a formal monoidal (or quadratic) transform of A with center P. If Q is a prime ideal not containing P, then we say that <u>the proper transform of Q passes through</u> A' if there is a prime ideal Q' in A' which contracts to Q in A (relative to the canonical local homomorphism of A into A').

DEFINITION 3.5. Let A be a quasi-ordinary local ring. We say that A' is a special transform of A if A' is not a regular local ring and if one of the following conditions holds:

1) A has a permissible center P which is a curve, and A' is the formal transform of A with center P.
2) A has no permissible center which is a curve, A' is a formal quadratic transform of A, and there is a curve Q in the singular locus of A whose proper transform passes through A'.
3) A has no permissible center which is a curve, and A' is the completion of a quadratic transform of A which lies at a singular point of the exceptional curve of the quadratic transformation of Spec(A).

When A is represented by a normalized quasi-ordinary branch, then it is easily verified that any special quadratic transform of A must occur in one of the directions (1:0:0), (0:1:0), (0:0:1).

We anticipate the result, partially demonstrated so far, and completed in §4, that a special transform of a quasi-ordinary local ring is again a quasi-ordinary local ring. In §5, we will consider sequences $A = A_0, A_1, A_2, \ldots, A_t$ where $A = A_0$ is a quasi-ordinary local ring and for each $i = 1, 2, \ldots, t$ A_i is isomorphic to a special transform of A_{i-1}. We call such a sequence a partial resolution of A. As a consequence of the resolution theorem for embedded surfaces [7] we have:

PROPOSITION 3.6. For a given quasi-ordinary local ring A, there is an integer N such that any partial resolution of A has fewer than N members.

We review briefly the idea of the proof, which is based on induction on the multiplicity of A. If A has permissible centers, then after finitely many monoidal transformations, we obtain a quasi-ordinary local ring which either has no permissible centers, or which is of multiplicity less than A. In the latter case, we can apply an inductive hypothesis, and we are done.

Suppose, therefore, that A has no permissible centers. If the tangent cone of A is irreducible (transversal case) then after a finite bounded number of quadratic transformations (in either of the directions

(1:0:0), (0:1:0) at any stage) there is a drop in multiplicity. (As long as there is no such drop, we remain in the transversal case, and no permissible centers which are curves are created). If the tangent cone of A is reducible (non-transversal case) then the initial form of a defining polynomial will be of the form cX^aY^b ($c \in k$; a, b integers). A consideration of the proper transform of the defining polynomial shows that one quadratic transformation in either of the directions (1:0:0), (0:1:0) produces a drop in multiplicity, and that after a finite number of quadratic transformations in the direction (0:0:1), (under which the tangent cone remains reducible) we get a drop in multiplicity, or else we get into the transversal case. In any case, an inductive argument completes the proof.

(All the preceding statements may be verified on the basis of formulas which have been given in this section).

4. Effect of Various Transformations on Distinguished Pairs.

Our purpose now is to continue the investigation of what happens to the distinguished pairs of quasi-ordinary branches when such branches are subjected to the transformations discussed in the previous section and in lemma 2.3. We shall find it convenient at this stage to formulate some technical lemmas.

Let $F = \Sigma c_{ij} X^i Y^j$ be a unit in $k[[X,Y]]$. Let $\Lambda(F)$ be the additive submonoid of $\mathbb{Z} \times \mathbb{Z}$ generated by those (i,j) such that $c_{ij} \neq 0$; thus $\Lambda(F)$ consists of all finite sums of pairs (i,j) for which $c_{ij} \neq 0$.

LEMMA 4.1. Let $\rho \neq 0$ be a rational number, and let $G = F^\rho$. Then $\Lambda(G) = \Lambda(F)$.

PROOF. Since k is algebraically closed and of characteristic zero, we can find F^ρ by expanding $(c_{00} + T)^\rho$ according to the binomial theorem (T - an indeterminate) and then substituting $\sum_{(i,j)>(0,0)} c_{ij} X^i Y^j$ for T. It is therefore clear that if d_{st} is the coefficient of $X^s Y^t$ in F^ρ, then d_{st} is a sum of terms of the form

$e_{ij}(c_{i_1 j_1} c_{i_2 j_2} \dots c_{i_r j_r})$ where $e_{ij} \in k$ and

$(i_1,j_1) + (i_2,j_2) + \ldots + (i_r,j_r) = (s,t)$. It follows that $d_{st} = 0$ if $(s,t) \notin \Lambda(F)$; hence $\Lambda(G) \subseteq \Lambda(F)$. Similarly, since $F = G^{1/p}$, $\Lambda(F) \subseteq \Lambda(G)$. q.e.d.

LEMMA 4.2. Let $U = XF_1(X,Y)$, $V = YF_2(X,Y)$ where F_1, F_2 are units in $k[[X,Y]]$. Let $G_1(X,Y)$, $G_2(X,Y)$ be the unique units in $k[[X,Y]]$ such that $X = UG_1(U,V)$, $Y = VG_2(U,V)$. Then $\Lambda(G_1) + \Lambda(G_2) = \Lambda(F_1) + \Lambda(F_2)$.

PROOF. Set

$$U = XF_1(X,Y) = X(a_{00} + \sum_{(i,j)>(0,0)} a_{ij}X^iY^j) \qquad a_{00} \neq 0$$

$$V = YF_2(X,Y) = Y(b_{00} + \sum_{(i,j)>(0,0)} b_{ij}X^iY^j) \qquad b_{00} \neq 0$$

$$X = UG_1(U,V) = U(d_{00} + \sum_{(i,j)>(0,0)} d_{ij}U^iV^j) \qquad d_{00} \neq 0$$

Substituting the series for U and V into the series for X we find that $X = X(\sum_{i,j} \bar{d}_{ij}X^iY^j)$ where

$\bar{d}_{ij} = a_{00}^{i+1}b_{00}^{j}d_{ij}$ + terms of the form

$$e_{ij}d_{i_0j_0}a_{i_1j_1}a_{i_2j_2} \cdots a_{i_rj_r}b_{i_{r+1}j_{r+1}} \cdots b_{i_sj_s}$$

with $e_{ij} \in k$, $(i_0,j_0) < (i,j)$, and $(i_0,j_0) + (i_1,j_1) + \dots + (i_s,j_s) = (i,j)$. Since $\overline{d}_{00} = 1$ and $\overline{d}_{ij} = 0$ $((i,j) \neq (0,0))$, d_{ij} is a sum of terms of the above displayed form. It follows by induction that $d_{ij} = 0$ unless $(i,j) \in \Lambda(F_1) + \Lambda(F_2)$; thus $\Lambda(G_1) \subseteq \Lambda(F_1) + \Lambda(F_2)$. Similarly, $\Lambda(G_2) \subseteq \Lambda(F_1) + \Lambda(F_2)$ so that $\Lambda(G_1) + \Lambda(G_2) \subseteq \Lambda(F_1) + \Lambda(F_2)$. Since $k[[X,Y]] = k[[U,V]]$, the opposite inclusion holds by symmetry, and our conclusion follows. q.e.d.

We shall use lemma 4.2 in the form of the following corollaries.

COROLLARY 4.2.1. Let T be an indeterminate, let $F(X,Y)$ be a unit in $k[[X,Y]]$ and let $E(X,Y,T)$ be a unit in $k[[X,Y,T]]$ such that

$$E.(T - XF(X,Y)) = X - TG(T,Y) \qquad G(0,0) \neq 0.$$

Then $\Lambda(F) = \Lambda(G)$.

PROOF. If $U = XF(X,Y)$, then substituting U for T in the above relation we have $X - UG(U,Y) = 0$. In lemma 4.2 set $F_1 = F$, $F_2 = 1$. Then $G_1 = G$, $G_2 = 1$. Since

$\Lambda(1) = \{(0,0)\}$ and since $(0,0)$ belongs to both $\Lambda(F)$ and $\Lambda(G)$ our statement follows.

COROLLARY 4.2.2. Let T, W be indeterminates, let $F(X,Y)$ be a unit in $k[[X,Y]]$ and let $E(X,Y,W,T)$ be a unit in $k[[X,Y,W,T]]$ such that

$$E.(T - WF(XW,YW)) = W - \overline{G}(X,Y,T) \qquad \text{---(1)}$$

Then $\overline{G}(X,Y,T) = TG(XT,YT)$ where $G(X,Y)$ is a unit in $k[[X,Y]]$ such that $\Lambda(G) = \Lambda(F)$.

PROOF. Let $U = XF(X,Y)$
$V = YF(X,Y)$.

There exists a unique unit $G(X,Y)$ in $k[[X,Y]]$ such that

$$X = UG(U,V) \qquad \text{---(2)}$$

Clearly $G(U,V) = F(X,Y)^{-1}$ so that $Y = VG(U,V)$. Hence by lemma 4.2 (with $F_1 = F_2 = F$, $G_1 = G_2 = G$) $\Lambda(F) = \Lambda(G)$.

Substituting $\overline{G}$ for W in (1) we find that $T - \overline{G}F(X\overline{G},Y\overline{G}) = 0$, whence

$$XT = X\overline{G}.F(X\overline{G},Y\overline{G})$$

$$YT = Y\overline{G}.F(X\overline{G},Y\overline{G})$$

and so by (2) $X\overline{G} = XT.G(XT,YT)$ q.e.d.

As a further preparatory step, we recast proposition 1.5. We recall that $(\lambda,\mu) \leq (\sigma,\tau)$ means $\lambda \leq \sigma$ and $\mu \leq \tau$.

LEMMA 4.3. Let $F(X,Y)$ be a unit in $k[[X,Y]]$, let n be a positive integer and let $(u,v) \neq (0,0)$ be a pair of non-negative integers. In order that $X^{u/n}Y^{v/n}F(X^{1/n},Y^{1/n})$ be a quasi-ordinary branch it is necessary and sufficient that there be elements $\underline{u}_2 \leq \underline{u}_3 \leq \cdots \leq \underline{u}_s \in \Lambda(F)$ such that, with $\underline{u}_1 = (u,v)$, we have

(i) Any element $\underline{u}$ of $\Lambda(F)$ satisfies a relation

$$\underline{u} = q_1\underline{u}_1 + q_2\underline{u}_2 + \cdots + q_s\underline{u}_s + n\underline{v}$$

where $q_1, q_2, \ldots, q_s$ are integers (possibly negative) and $\underline{v}$ is some element of $\mathbb{Z}\times\mathbb{Z}$. (We refer to this relation by saying "$\underline{u}$ is a combination of $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_s$ mod. n").

(ii) If $\underline{u} \in \Lambda(F)$ and if $\tau = \tau(\underline{u})$ is the least number $t \geq 1$ such that $\underline{u}$ is a combination of $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_t$ mod. n, then $\underline{u} \geq \underline{u}_\tau$.

(iii) $\tau(\underline{u}_i) = i$ for $i = 2, 3, \ldots, s$.

If the above conditions hold then the distinguished pairs of the quasi-ordinary branch $X^{u/n}Y^{v/n}F(X^{1/n},Y^{1/n})$ are the pairs $\{(\underline{u}_i + \underline{u}_1)/n\}$ $i = 2, 3, \ldots, s$ along with the pair $\underline{u}_1/n = (u/n, v/n)$, unless both u/n, v/n are integers, in which case u/n, v/n is not a distinguished pair.

PROOF. Suppose $\zeta = X^{u/n}Y^{v/n}F(X^{1/n},Y^{1/n})$ is a quasi-ordinary branch. Let $(\lambda_2,\mu_2) < (\lambda_3,\mu_3) < \cdots < (\lambda_s,\mu_s)$ be those distinguished pairs of ζ which are not equal to $(u/n,v/n)$. If u/n, v/n are not both integers then it follows from proposition 1.5 that $(u/n,v/n)$ is the least distinguished pair of ζ. In any case we set $\lambda_1 = u/n$, $\mu_1 = v/n$.

Set $\underline{u}_1 = (u,v)$ and for $i = 2, 3, \ldots, s$ set $\underline{u}_i = (u_i,v_i)$ where $u_i = n(\lambda_i - \lambda_1)$ and $v_i = n(\mu_i - \mu_1)$. Note that $(\lambda_i,\mu_i) = \frac{1}{n}(\underline{u}_i + \underline{u}_1)$. Conditions 1) and 2) of proposition 1.5 show that $\underline{u}_2 \leq \underline{u}_3 < \cdots < \underline{u}_s$ and that $\underline{u}_i \in \Lambda(F)$.

If $\underline{u}$ is any element of $\mathbb{Z}\times\mathbb{Z}$, then we see that $\underline{u}$ is a combination of $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_t$ mod. n $(s \geq t \geq 1)$ if and only if

$$(\underline{u}+\underline{u}_1)/n = q_1(\underline{u}_1/n) + q_2(\underline{u}_2+\underline{u}_1)/n + \dots + q_t(\underline{u}_t+\underline{u}_1)/n + \underline{v} \quad ---(3)$$

where $q_1, q_2, \dots, q_t$ are integers and $\underline{v} \in Z\times Z$. It follows immediately that condition 5) of proposition 1.5 implies (iii) above. Moreover, condition 3) of proposition 1.5 shows that any $\underline{u}$ which arises from a non-vanishing coefficient of F is a combination of $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_s$ mod. n (whether or not (λ_1,μ_1) is a distinguished pair of ζ) and (i) follows.

Now if $\underline{u} \in \Lambda(F)$ then $\underline{u}$ is a sum $\underline{u} = \underline{w}_1 + \underline{w}_2 + \dots + \underline{w}_r$ where $\underline{w}_1, \dots, \underline{w}_r$ are associated with non-vanishing coefficients of F. Clearly $\tau(\underline{u}) \leq \max_j \tau(\underline{w}_j)$ and $\underline{u} \geq \underline{w}_j$ for any $j = 1, 2, \dots, r$. On the other hand, condition 4) of proposition 1.5 shows that if $\underline{w}_j$ does come from a non-zero coefficient of F, then

$$(\underline{w}_j + \underline{u}_1)/n \geq (\lambda_\tau,\mu_\tau) = (\underline{u}_\tau + \underline{u}_1)/n \qquad \text{if } \tau = \tau(\underline{w}_j) \geq 2.$$

Hence $\underline{w}_j \geq \underline{u}_\tau$ when $\tau = \tau(\underline{w}_j)$ (this follows from the preceding inequality if $\tau(\underline{w}_j) \geq 2$, and is obvious if $\tau(\underline{w}_j) = 1$), and condition (ii) results.

Conversely, let $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_s$ verify the conditions of lemma 4.3. Setting $(\lambda_i,\mu_i) = (\underline{u}_i + \underline{u}_1)/n$

$i = 2, 3, \ldots, s$ and $(\lambda_1, \mu_1) = (u/n, v/n)$, we have $(0,0) < (\lambda_1, \mu_1) \leq (\lambda_2, \mu_2) \leq \cdots \leq (\lambda_s, \mu_s)$ and (iii) shows that the " $\leq$ " signs may be replaced by " $<$ ".

To prove 2) of proposition 1.5, it is sufficient to show that for $i = 2, 3, \ldots, s$ the coefficient of $X^{u_i} Y^{v_i}$ in $F(X,Y)$ does not vanish (where $(u_i, v_i) = \underline{u}_i$). To see this we note that $\underline{u}_i = \underline{w}_1 + \underline{w}_2 + \cdots + \underline{w}_r$ where $\underline{w}_j$ $(j = 1, 2, \ldots, r)$ arises from a non-vanishing coefficient in $F(X,Y)$; since $\tau(\underline{u}_i) = i$, $\tau(\underline{w}_t) \geq i$ for some $t \leq r$; hence, by (ii), $\underline{w}_t \geq \underline{u}_i$; but clearly $\underline{u} \geq \underline{w}_t$; thus $\underline{u}_i = \underline{w}_t$.

Condition 3) of proposition 1.5 follows easily from (i) and the remark preceding equation (3) above. Similarly, 4) of proposition 1.5 follows from (ii).

Finally, 5) follows from (iii) unless u/n, v/n are integers; in the latter case, it is now easy to see that 1) - 5) of proposition 1.5 continue to hold if we set $(\lambda_i, \mu_i) = (\underline{u}_{i+1} + \underline{u}_1)/n$, $i = 1, 2, \ldots s-1$, (and replace s by $s-1$ in proposition 1.5). q.e.d.

* * *

As a typical application of the preceding lemmas to the problem of calculating distinguished pairs, we shall re-examine the case of a quadratic transformation in the direction (0:0:1) as treated in §3.4.2.b.

We start off with a quasi-ordinary branch

$$\zeta = X^{u/n}Y^{v/n}H(X^{1/n},Y^{1/n}) \qquad u + v < n$$

Then we choose F such that $[F(X,Y)]^{n-u-v} = H(X,Y)$. By lemma 4.1, $\Lambda(F) = \Lambda(H)$.

We are then led to expressions of the form

$$Z^{1/n} - \omega X^{u/n(n-u-v)}Y^{v/n(n-u-v)}F(X^{1/n}Z^{1/n},Y^{1/n}Z^{1/n})$$

Multiplication by $-[F(X^{1/n}Z^{1/n},Y^{1/n}Z^{1/n})]^{-1}$, which is a unit in $k[[X^{1/n},Y^{1/n},Z^{1/n}]]$, gives us

$$U - Z^{1/n}F^{-1}(X^{1/n}Z^{1/n},Y^{1/n}Z^{1/n})$$

where $U = \omega X^{u/n(n-u-v)}Y^{v/n(n-u-v)}$. Multiplication by a suitable fractional power series of the form $E(X^{1/n},Y^{1/n},Z^{1/n},U)$, $E(0,0,0,0) \neq 0$, gives us

$$Z^{1/n} - UG(X^{1/n}U,Y^{1/n}U)$$

where, by corollary 4.2.2, $\Lambda(G) = \Lambda(F^{-1})$. Thus, arguing as in §3.4.2.a, we find that our formal transform is represented by a polynomial whose roots are of the form

$$\zeta' = U^n[G(X^{1/n}U, Y^{1/n}U)]^n$$

$$= X^{nu/n(n-u-v)} Y^{nv/n(n-u-v)} \overline{H}(X^{1/n(n-u-v)}, Y^{1/n(n-u-v)})$$

Note that $\Lambda(G^n) = \Lambda(G) = \Lambda(F^{-1}) = \Lambda(F) = \Lambda(H)$, (lemma 4.1 and preceding remarks). A typical non-vanishing term of $\overline{H}(X,Y)$ has the form

$$c_{ij}(X^{n-v}Y^v)^i \ (X^uY^{n-v})^j = c_{ij}X^{i(n-v)+ju}Y^{iv+j(n-u)}$$

where $(i,j) \in \Lambda(G^n) = \Lambda(H)$. Thus, a typical element of $\Lambda(\overline{H})$ is of the form $(i(n-v) + ju,\ iv + j(n-u))$ with $(i,j) \in \Lambda(H)$, and this may also be written as $(i,j)D$ where D is the matrix $\begin{pmatrix} n-v & v \\ u & n-u \end{pmatrix}$.

We note that (i,j) is uniquely determined by $(i,j)D$ since determinant of $D = n(n-u-v) > 0$.

Lemma 4.3 can be applied to H with $\underline{u}_1 = (u,v)$ and with the integer n. We wish to deduce the conditions of lemma 4.3 for the power series $\overline{H}(X,Y)$, the integer $n(n-u-v)$ and the sequence

$$\underline{w}_1 = (nu,nv),\ \underline{w}_2 = \underline{u}_2 D,\ \ldots,\ \underline{w}_s = \underline{u}_s D$$

(the $\underline{u}$'s belong to H). If we do this, then lemma 4.3 shows that ζ' is a quasi-ordinary branch, and gives the distinguished pairs. In fact, the distinguished pairs of ζ' will be $(\underline{w}_i + \underline{w}_1)/n(n-u-v)$ $(i = 2, 3, \ldots, s)$, along with the pair $(nu/n(n-u-v),\ nv/n(n-u-v))$ provided that the latter is not a pair of integers.

Since $\underline{w}_i = \underline{u}_i D$ and $\underline{u}_i = n(\lambda_i, \mu_i) - (u,v)$ $(i = 2, 3, \ldots, s)$ where (λ_i, μ_i) is the i-th distinguished pair of $X^{u/n}Y^{v/n}H(X^{1/n}, Y^{1/n})$, and since $(u,v) = (n\lambda_1, n\mu_1)$, we have

$$(\underline{w}_i + \underline{w}_1)/n(n-u-v) = (\lambda_i(1-\mu_1) + \mu_i\lambda_1,\ \lambda_i\mu_1 + \mu_i(1-\lambda_1))/1-\lambda_1-\mu_1.$$

$(i = 2, 3, \ldots, s)$, and for $i = 1$, the formula on the right gives $(\lambda_1, \mu_1)/1-\lambda_1-\mu_1 = (nu,nv)/n(n-u-v)$. Thus <u>the distinguished pairs of the transformed branch depend only on those of the original branch</u>.

In order to verify the conditions of lemma 4.3, we remark that if $\underline{u}$ is a combination of $\underline{u}_1, \ldots, \underline{u}_t$ mod. n,

$$\underline{u} = q_1\underline{u}_1 + q_2\underline{u}_2 + \ldots + q_t\underline{u}_t + n(a,b) \qquad (a,b \text{ integers})$$

then $\underline{u}D = q_1\underline{u}_1D + n(a,b)D + q_2\underline{u}_2D + \dots + q_t\underline{u}_tD$

and a brief computation shows that

$$q_1\underline{u}_1D + n(a,b)D = (q_1+a+b)\underline{u}_1D + n(n-u-v)(a,b)$$

so that $\underline{u}D$ is a combination of $\underline{u}_1D, \dots, \underline{u}_tD$ mod. $n(n-u-v)$. Conversely, if $\underline{u}D$ is such a combination, then we can show that $\underline{u}$ is a combination of $\underline{u}_1, \dots, \underline{u}_t$ mod n.

In view of this, and in view of the fact that multiplication by D preserves order between pairs, the conditions for $\overline{H}$ can be established by routine verification.

* * *

Other transformations which we have given can be treated in a similar manner. We omit the details of the computations, the principles of which are adequately illustrated in the above example. The final results, along with those of §3, are set down in the following table.

In each of the indicated processes of transformation, we have started out with a normalized quasi-ordinary branch and ended up with a fractional power series which turns out, as in the above example, to be a quasi-ordinary branch.

We note that in 3.4.2.a we ended up with a quasi-ordinary branch in X and Z. For the sake of uniformity we may substitute the letter "Y" for Z to get a "standard" quasi-ordinary branch in X and Y (which still represents the formal quadratic transform being considered). Similarly, when considering a direction of the form $(0:1:\gamma)$, (non-transversal case) we end up with a branch in Y and Z; substituting "X" for Z we get a branch in Y and X, and then we can interchange X and Y, to get a "standard" form for the representing branch. It is such standard branches which are referred to in the table under the heading "Resulting Branch".

If (λ_i, μ_i) $(i = 1, 2, \ldots, s)$ are the distinguished pairs of the original branch, then the distinguished pairs of the "standard" transformed branch are those given in the table for $i = 1, 2, \ldots, s$ unless it happens that the first pair given (i.e. that pair for which $i = 1$) turns out to be a pair of integers; when that is so, the first pair must be omitted.

TABLE 4.4.

Transformation	Distinguished Pairs of Resulting Branch
LEMMA 2.3	$(\lambda_i + 1 - \lambda_1)/\lambda_1, \mu_i$
MONOIDAL TRANSFORMATION	
Center (X,ζ)	$\lambda_i - 1, \mu_i$
Center (Y,ζ)	$\lambda_i, \mu_i - 1$
QUADRATIC TRANSFORMATION	
Transversal Case	
Direction (1:0:0)	$\lambda_i + \mu_i - 1, \mu_i$
Direction (0:1:0)	$\lambda_i, \lambda_i + \mu_i - 1$
Non-Transversal Case	
Direction (1:0:0)	$\lambda_i+[(1+\mu_i)(1-\lambda_1)/\mu_1]-2, [(1+\mu_i)/\mu_1]-1$
Direction (0:1:0)	$\mu_i+[(1+\lambda_i)(1-\mu_1)/\lambda_1]-2, [(1+\lambda_i)/\lambda_1]-1$
Direction (0:0:1)	$(\lambda_i(1-\mu_1)+\mu_i\lambda_1)/1-\lambda_1-\mu_1, (\lambda_i\mu_1+\mu_i(1-\lambda_1))/1-\lambda_1-\mu_1$

We have given the distinguished pairs only for "special" directions in the case of quadratic transformations. This information can always be used to give us the distinguished pairs for "non-special" directions. For example if we want a set of distinguished pairs at the transform in a direction $(1:0:\gamma)$, $\gamma \neq 0$, in the non-transversal case, then we remark that the process given in §3 shows that the roots of a defining equation f_γ at $(1,0,\gamma)$ can be obtained from those of a certain defining equation f at (1:0:0) by substituting the (integral) power series $(Z + \gamma)^{1/v}$ for $Z^{1/v}$. The difference of any two roots of f_γ is obtained by the same substitution from the difference of some two roots of f. It follows that the distinguished pairs of any of the roots at $(1,0,\gamma)$ lie among the pairs $(\bar{\lambda}_i,0)$, $(i = 1, 2, \ldots, s)$, where $\bar{\lambda}_i$ denotes the first member of the pairs given in table 4.4 for the direction (1:0:0) (non-transversal case). The distinguished pairs in fact turn out to be those of the pairs $(\bar{\lambda}_i,0)$ such that, in the notation of proposition 1.5, $\tau(\bar{\lambda}_i,0) = i$; i.e. such that $\bar{\lambda}_i$ is not congruent, mod. $\mathbb{Z}$, to an integral combination of $\bar{\lambda}_0 = 0, \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{i-1}$. The proof, based on proposition 1.5, is straightforward and we omit it.

We remark that the preceding result can also be obtained by an argument based on the concepts of "equisingularity" and "generic sections" (cf. end of this section and also §6).

From table 4.4 we can deduce another description of "special transforms". (cf. definition 3.5)

COROLLARY 4.4.1. Let A be a quasi-ordinary local ring, and let A be represented by a normalized branch ζ as in §3. Assume that A has no permissible center which is a curve. Then any non-special formal quadratic transform of A in a direction $(1:\beta:\gamma)$ has a defining equation whose discriminant either is divisible by X alone, or is a unit.

REMARK. Corollary 4.4.1 has the following interpretation: A non-regular formal transform A' of A is non-special if and only if T_A (the quadratic transform of Spec(A)) is <u>equisingular</u> along the exceptional curve at the point whose local ring is A'. (cf. remarks at end of this section).

PROOF. Since the discriminant of a polynomial is a product of differences between its roots, considerations such as those following table 4.4 lead immediately to a proof, except for a transform in the direction (1:0:0).

Our assumption on A shows that ζ has the form $X^{\lambda}Y^{\mu}H(X^{1/n},Y^{1/n})$ $0 < \lambda < 1$, $0 < \mu < 1$. The proper transform of the ideal (Y,ζ) passes through the transform in the direction (1:0:0), and so if that transform is to be non-special, (Y,ζ) must have multiplicity 1 in A. Hence $\mu = 1/m$ where $m = [K(\zeta):K]$ (cf. proof of corollary 2.7.1). It follows that $X^{\lambda}Y^{\mu}$ is the only characteristic monomial of ζ. Then it is easy to check that the transform under consideration is actually a regular local ring. q.e.d

* * *

The formulas of table 4.4 allow us to show that <u>special transforms of quasi-ordinary local rings are integral domains</u>. First, however, we must make some preliminary remarks about subfields of $K_n = K(X^{1/n},Y^{1/n})$, K being the quotient field of $k[[X,Y]]$.

If ω is a primitive n-th root of unity, then any automorphism of K_n/K is effected by a substitution of the form $X^{1/n} \to \omega^a X^{1/n}$, $Y^{1/n} \to \omega^b Y^{1/n}$, $0 \leq a < n$, $0 \leq b < n$. Representing such a substitution by the pair (a,b) we get, clearly, an isomorphism between the Galois group G of K_n/K and the direct sum $\mathbb{Z}_n \oplus \mathbb{Z}_n$ ($\mathbb{Z}_n$ = integers mod. n). If K' is any subfield of K_n and G' is the subgroup of G leaving K' fixed, then $X^{u/n}Y^{v/n}$ lies in K' if and only if

$ua + vb \equiv 0 \bmod. n$ for all (a,b) in G'. It follows that the set $\tilde{G}$ of elements (u,v) $(0 \leq u < n,\ 0 \leq v < n)$ such that $X^{u/n}Y^{v/n} \in K'$ may be identified with the group $\mathrm{Hom}(G/G', \mathbb{Z}_n)$. Since G/G' is a direct sum of cyclic groups, we can easily show that $\mathrm{Hom}(G/G', \mathbb{Z}_n)$ is (non-canonically) isomorphic to G/G'.

Thus the total number of elements in $\tilde{G}$ is the order of G/G', which is also the degree $[K':K]$. Hence the monomials $X^{u/n}Y^{v/n}$ form a basis for the vector space K'/K, and so $(a,b) \in G'$ if and only if $au + bv \equiv 0 \bmod. n$ for all (u,v) in $\tilde{G}$. [Thus the duality between the subgroups of G and the subfields of K_n/K is a reflection of the perfect duality between $\mathbb{Z}_n$-submodules of $\mathbb{Z}_n \oplus \mathbb{Z}_n$ associated with the bilinear form $\{(u,v), (a,b)\} \to ua + vb$].

We see now by symmetry that the order d of G' is the order of $G/\tilde{G}$, and it is well-known [2; p.95] that the latter order is the g.c.d. of the 2X2 subdeterminants of the matrix

$$\begin{pmatrix} n & 0 & u_1 & \cdots & u_s \\ 0 & n & v_1 & \cdots & v_s \end{pmatrix}$$

where (u_1,v_1), ..., (u_s,v_s) is any set of generators of $\tilde{G}$. The degree $[K':K]$ is therefore n^2/d.

We shall now consider the case of the formal quadratic transform A' of a quasi-ordinary local ring A in the direction (0:0:1). Our object is to show that A' is an integral domain.

Let ζ be a normalized representing branch of A, and let f be the minimum polynomial of ζ over K. Let $(\lambda_i,\mu_i) = (u_i/n, v_i/n)$ $i = 1, 2, \ldots, s$ be the distinguished pairs of ζ, and let $\Gamma(\zeta)$ be the subgroup of $\mathbb{Z}_n \oplus \mathbb{Z}_n$ consisting of those pairs (u,v) $(0 \leq u < n,\ 0 \leq v < n)$ such that $X^{u/n}Y^{v/n} \in K(\zeta)$. We have seen (remark 1.4.4) that $K(\zeta)$ is generated over K by the characteristic monomials of ζ; it follows easily that $\Gamma(\zeta)$ is generated by the pairs (u_i,v_i) $i = 1, 2, \ldots, s$. Hence $[K(\zeta):K] = n^2/d$, d being the g.c.d. of the 2×2 subdeterminants of the above displayed matrix.

Let ζ' be a root of the formal transform of f in the direction (0:0:1), ζ' being obtained from ζ as in earlier parts of this section (and also 3.4.2.b.). Table 4.4 gives us the distinguished pairs of ζ' and we conclude, by the preceding argument, that $[K(\zeta'):K] = n^2(n-u-v)^2/d'$ where $u = u_1$, $v = v_1$, and d' is the g.c.d. of the 2X2 subdeterminants of the matrix

$$\begin{pmatrix} 0 & n(n-u-v) & nu & \cdots & u_i(n-v)+v_iu & \cdots \\ n(n-u-v) & 0 & nv & \cdots & u_iv+v_i(n-u) & \cdots \end{pmatrix} \quad \text{---(4)}$$

Now the degree of the formal transform $f' = f'_{(0,0,\underline{1})}$ of the polynomial f is easily seen to be (degree of f).$(n-u-v)/n$. Hence if $[K(\zeta'):K] \geq [K(\zeta):K].(n-u-v)/n$ then, ζ' being a root of f', we must actually have equality and f' is necessarily irreducible. It follows then that A' is an integral domain, as required.

The preceding inequality may be written as $n(n-u-v)d \geq d'$. In view of the definition of d', this inequality can be established by showing that $n(n-u-v)d$ divides all the 2X2 subdeterminants of (4). We check that any 2X2 subdeterminant of (4) has a value of one of the following forms:

(i) $n^2(n-u-v)^2$

(ii) $n(n-u-v)(nv_i + u_iv - v_iu)$

(iii) $[(n-v)(n-u) - uv][u_iv_j - v_iu_j]$

By the definition of d, we see that d divides n^2, nu, nv and therefore n(n-u-v)d divides (i). Similarly d divides nv_i and d divides $u_i v - v_i u$, whence n(n-u-v)d divides (ii). Finally d divides $u_i v_j - v_i u_j$ and (n-v)(n-u) - nv = n(n-u-v); so n(n-u-v)d divides (iii). q.e.d.

Formal quadratic transforms in the directions (1:0:0), (0:1:0) (non-transversal case) can be treated similarly. Details are left to the interested reader. The transversal and monoidal cases have already been treated in 3.3 and 3.4.1.

To summarize the main results of this section we have:

THEOREM 4.5. Let A be a quasi-ordinary local ring. Any special transform A' of A is again a quasi-ordinary local ring. If ζ is a normalized branch representing A, then, by one of the processes of §3, we can find a quasi-ordinary branch ζ' (not necessarily normalized) which represents A', and whose distinguished pairs depend only on those of ζ and on the process employed, the exact nature of the dependence being as in table 4.4.

*   *   *

We close this section with some informal remarks about non-special transforms and equisingularity. Corollary 4.4.1 and also the remarks preceding 3.4.2 show that for non-special transforms there is a defining polynomial whose discriminant has at most one irreducible factor. Such behavior for a "general" transform could have been predicted on the basis of the theory of equisingular families of curves, developed by Zariski in [8].

Roughly speaking, suppose P is a closed point on a surface S in three-space, let $\pi : S' \to S$ be the locally quadratic transformation of S with center P, and assume that the exceptional curve $\pi^{-1}(P)$ (reduced) is a multiple curve of S'. If P' is a point of $\pi^{-1}(P)$ whose complete local ring on S' is defined by a polynomial whose discriminant is a power of a linear form, then S' is <u>equisingular</u> along $\pi^{-1}(P)$ at P'. This means that P' is a simple point of $\pi^{-1}(P)$ and that the singularity which S' has at P' is "no worse" than the singularity which it has at the generic point of $\pi^{-1}(P)$; in topological terms, S' is locally the product of a line with a section of S' by a plane which is transversal to $\pi^{-1}(P)$.

This situation is clearly illustrated in the preceding computations. For example, when we are considering directions of the form $(1,0,\gamma)$, $\gamma \neq 0$, in the non-transversal case, we can write down defining equations of the corresponding transforms according to the procedure given in 3.4.2.a; if we then set $Z = 0$, we get a family of plane curves (i.e. of one-dimensional branches) in which γ appears as a parameter. These curves are cross sections of the total transform T of Spec(A) by planes which are transversal to the exceptional curve. As long as $\gamma \neq 0$, these curves are easily seen to be equivalent in the sense that the components of two different curves can be matched in such a way that intersection multiplicities are preserved and that corresponding irreducible branches have the same characteristic monomials (in one variable). (cf. also [8]).

It turns out in the next section that as far as the classification of quasi-ordinary local rings by quadratic and monoidal transformations is concerned, we may ignore non-special transforms. This is not surprising, since any special transform contains all the information about its neighboring non-special transforms in its "generic transversal sections" (cf. §6).

5. Strict Resolutions of Quasi-ordinary Local Rings.

Let ζ be a normalized representing branch of a quasi-ordinary local ring A, and let (λ_i, μ_i) $(i = 1, 2, \ldots, s)$ be the distinguished pairs of ζ. Interchanging X and Y in ζ, we get another normalized representing branch of A whose distinguished pairs are (μ_i, λ_i). We are ready now to show that, modulo this ambiguity, the distinguished pairs of any two representing branches of A are the same.

To eliminate the ambiguity, we shall say that a normalized quasi-ordinary branch ζ is strongly normalized if either $\zeta = 0$ or if the distinguished pairs (λ_i, μ_i) $(i = 1, 2, \ldots, s)$ of ζ are such that $(\lambda_1, \lambda_2, \ldots, \lambda_s) \geq (\mu_1, \mu_2, \ldots, \mu_s)$ in the lexicographic ordering. With any quasi-ordinary branch ζ we can associate a well-determined normalized branch ζ' according to the procedure at the beginning of §2. If ζ' is not strongly normalized, let $\bar{\zeta}$ be the quasi-ordinary branch obtained from ζ' by interchanging X and Y; otherwise let $\bar{\zeta} = \zeta'$. $\bar{\zeta}$ is strongly normalized, and we shall say that $\bar{\zeta}$ is the strongly normalized branch associated with ζ. $\bar{\zeta}$ and ζ represent the same (up to isomorphism) quasi-ordinary local ring.

We are going to relate the distinguished pairs of strongly normalized quasi-ordinary branches to certain resolutions of the rings which they represent (cf. proposition 3.6). First we introduce some more terminology. If A' is a formal monoidal or quadratic transform of A, with center P, then we say that a curve P' in A' is an *exceptional curve for the transformation from* A *to* A' if P', as a prime ideal, contracts to P in A. We see easily (say by the formulas of §3) that if A' is a formal monoidal or quadratic transform of a quasi-ordinary ring A, and if A' has two curves in its singular locus, then at least one of these curves is exceptional for the transformation from A to A'. Moreover, unless A has a reducible tangent cone and A' is the formal transform which lies at the intersection of the two components of the exceptional curve of the quadratic transformation of Spec(A_i) (cf. 3.4.1), then there is precisely one curve in A' which is exceptional for the transformation from A to A'.

DEFINITION 5.1. A *strict resolution* of a quasi-ordinary local ring A is a sequence $A = A_0, A_1, \ldots, A_t$ such that

1) A_t is either normal or such that it has no special transforms.

2) A_{i+1} $(0 \leq i < t)$ is a special transform of A_i, subject to the following "exactness" conditions:

(i) A_{i+1} contains precisely one exceptional curve (for the transformation from A_i to A_{i+1}).

(ii) If A_i $(0 < i < t)$ has two curves in its singular locus, neither of which is a permissible center, then A_{i+1} is the formal quadratic transform of A_i through which passes the proper transform of the exceptional curve for the transformation from A_{i-1} to A_i. (cf. remarks preceding 3.5).

(iii) If A_i $(0 < i < t)$ has two permissible centers which are curves, then A_{i+1} is a monoidal transform of A_i and the center for the transformation from A_i to A_{i+1} is exceptional for the transformation from A_{i-1} to A_i.

We see easily that any quasi-ordinary local ring has at most two strict resolutions (up to isomorphism). In fact there are at most two possible choices for A_1, and from proposition 3.6 and the remarks preceding definition 5.1, we find that there exists precisely one strict resolution of A for each such choice. We note also that if $A = A_0, A_1, \ldots, A_t$ is a strict resolution of A, then $A_r, A_{r+1}, \ldots, A_t$ is a strict resolution of A_r for any $r \leq t$.

We are going to attach certain bits of information to each ring A_i appearing in a strict resolution of A_0. These "bits" will depend only on the distinguished pairs of any strongly normalized representing branch of A_0, and conversely, the collection of all the "bits" associated with a given strict resolution will determine such distinguished pairs uniquely.

To each quasi-ordinary ring A, then, we attach a label [a;b,c;d;e;f] with six numbers:

a: the multiplicity of A

b,c: (i) If A has two curves in its singular locus then b is the greater of their multiplicities in A and c is the lesser.

(ii) If A has only one curve in its singular locus, then b is the multiplicity of that curve in A and c = 1.

(iii) If A is integrally closed then b = c = 1.

d: If there is a curve in the singular locus of A which is not itself a non-singular curve then d = 1; otherwise d = 0.

e,f: need further explanation, given below.

If A is a quasi-ordinary local ring, and $\zeta = X^{\lambda}Y^{\mu}H$ is a normalized representing branch of A, and if the tangent cone of A is <u>reducible</u>, then the degree $m = [K(\zeta):K]$ (K = quotient field of $k[[X,Y]]$) is not, a priori, uniquely determined by A (if A has an <u>irreducible</u> tangent cone, then m = multiplicity of A). To show that m is actually unique, we consider A_1, the formal quadratic transform of A lying at the intersection of the two components of the exceptional curve of the quadratic transformation of Spec(A); if A_1 has a reducible tangent cone, then we derive A_2 from A_1 in the same way we got A_1 from A; in this way we get a sequence $A_1, A_2, \ldots, A_r$ which terminates after finitely many steps; i.e. the tangent cone of A_r is irreducible. We leave it as an exercise for the reader to show that r is the least integer such that $\lambda + \mu \geq 1 - r(\lambda + \mu)$, and that $\overline{m}$ = (multiplicity of A)/(multiplicity of A_r) = $(\lambda+\mu)/[1-r(\lambda+\mu)]$. Since $\overline{m}$ and r are intrinsic to A, so is $\lambda + \mu$. On the other hand, we know (proof of theorem 2.7) that $m(\lambda + \mu)$ = multiplicity of A. Thus m is intrinsic to A. By remark 1.4.4, m is determined by the distinguished pairs of ζ.

Now we describe the number e: if A has a reducible tangent cone, then e is the degree over K of any normalized representing branch of A. If A has an irreducible tangent cone, then e = 0.

We go on to the description of f.

LEMMA 5.2. Let A be a quasi-ordinary local ring of multiplicity $m > 1$ and let ζ be a strongly normalized representing branch of A. Suppose that A has a permissible center which is a curve and that the corresponding formal monoidal transform is regular. Then ζ has only one distinguished pair, and that pair is either $(1, 1/m)$ or $(1 + 1/m, 0)$.

PROOF. The proof is an easy consequence of the information in table 4.4. q.e.d.

The problem which arises is to show that if one strongly normalized representing branch of A has the distinguished pair $(1, 1/m)$ then so do all other such branches. We shall need therefore to make some distinction between a ring which is represented by a quasi-ordinary branch having only the one distinguished pair (α): $(1 + 1/m, 0)$ and a ring which is represented by a quasi-ordinary branch having only the one distinguished pair (β): $(1, 1/m)$. [Both rings have the same numbers a,b,c,d,e in their label]. This may be done either by looking for equisingularity or by counting the number of components in a "generic transversal section" (cf. next section). However, these are concepts which we have not yet introduced formally, and for our purposes any intrinsic distinction will do. Here is one: by 3.3 we can see that each of the rings has precisely one formal quadratic transform which is not regular, and that these formal transforms are represented by branches with one distinguished pair of the form (α'): $(1 + 1/m, 1/m)$ in case (α), respectively (β'): $(1, 1/m)$ in case (β); in both cases there is one permissible center; in case (α') the corresponding monoidal transform is not regular, in case (β') the monoidal transform is regular.

Now we can describe the number f. If A does not satisfy the hypotheses of lemma 5.2, then we set $f = 0$. Otherwise we define f to be 0 in case (α) and 1 in case (β).

The above discussion and theorems 2.6, 2.7 show that <u>the label of a quasi-ordinary ring</u> A <u>is completely determined by the distinguished pairs of any normalized quasi-ordinary branch which represents</u> A.

DEFINITION 5.3. Let A, A' be quasi-ordinary local rings, let $A = A_0, A_1, \ldots, A_t$ be a strict resolution of A and let $A' = A'_0, A'_1, \ldots, A'_s$ be a strict resolution of A'. We say that the two resolutions are <u>equivalent</u> if $t = s$

and if for each $i = 0, 1, 2, \ldots, t$, A_i and A'_i have the same label.

An equivalence class of strict resolutions may be identified with a sequence of labels, viz. those which belong to the successive members of any resolution in the equivalence class. We shall call such a sequence a <u>composition</u> and say that a composition <u>belongs to</u> a given quasi-ordinary local ring A if it is the sequence of labels belonging to some strict resolution of A.

Now we can state the main result.

THEOREM 5.4. (CLASSIFICATION THEOREM). Let A, A' be quasi-ordinary local rings, and let ζ, ζ' be strongly normalized representing branches of A, A' respectively. If there exists a composition which belongs both to A and to A', then ζ and ζ' have the same distinguished pairs. Conversely if ζ and ζ' have the same distinguished pairs then the set of compositions belonging to A is identical with the set of compositions belonging to A'.

COROLLARY. If A and A' are isomorphic then ζ and ζ' have the same distinguished pairs.

PROOF OF COROLLARY. Since A and A' are both represented by ζ, the second part of the theorem says that their respective sets of compositions are identical (a fact which is clear a priori, since the definition of "composition of A" is intrinsic to A); hence the first part says that ζ and ζ' have the same distinguished pairs.

PROOF OF THEOREM. We consider that enough has been said by now (theorem 4.5, remark preceding definition 5.3, etc.) to show that the following problem can be successfully programmed for a computer: given the distinguished pairs of a strictly normalized quasi-ordinary representing branch of a quasi-ordinary local ring A, construct all compositions belonging to A.

The second part of the theorem results.

Now we turn to the first part of the theorem, which will occupy us for the remainder of the section. We are assuming that A and A' have equivalent strict resolutions. Let us first treat the case in which these resolutions have only one member. Since A and A' have the same label, A contains a permissible center which is a curve if and only if A' does. Lemma 5.2 is applicable (otherwise A has a special transform), and so ζ has precisely one distinguished pair, which is either $(1 + 1/m, 0)$ or $(1, 1/m)$ (m = multiplicity of A). The decision between these two is made according to whether the number "f" in the label of A is 0 or 1. Thus the distinguished pair of ζ is determined by the label of A, and consequently ζ and ζ' have the same distinguished pair.

If neither A nor A' has a permissible center which is a curve, then neither has any curve at all in its singular locus (otherwise one of them would be non-normal and also would have a special transform). Hence both A and A' are normal. Clearly A has multiplicity 1 if and only if A' does; then $\zeta = \zeta' = 0$. Otherwise, by corollary 2.7.2 and its proof, ζ has a single distinguished pair, that pair being of the form $(1/m, 1/m)$; we have $m = 2$ if and only if the tangent cone of A is irreducible, and if the tangent cone is reducible, then m is the number "e" appearing in the label of A. Since A and A' have the same label, ζ and ζ' have the same distinguished pair.

If the resolutions under consideration have more than one member, then let B, B' be the successors of A, A' in the respective resolutions. B and B' are represented by ξ and ξ', where ξ and ξ' are transforms of ζ, ζ' respectively, obtained by the procedures of §3 (and then "standardized", cf. remarks preceding table 4.4). Let $\bar{\xi}$, $\bar{\xi}'$ be the strongly normalized branches associated with ξ, ξ' respectively. B and B' have equivalent resolutions, with fewer members than the above resolutions of A, A'; hence, arguing by induction, we may assume that $\bar{\xi}$, $\bar{\xi}'$ have the same distinguished pairs, which we denote by (σ_i, τ_i).

Now we show how conditions (ii) and (iii) of definition 5.1 can be used. Let $\bar{\xi}$ be any strongly normalized quasi-ordinary branch such that $k[[X,Y]][\bar{\xi}]$ has two curves in its singular locus, with either <u>both</u> or <u>neither</u> being permissible centers. These curves are then the prime ideals $(X,\bar{\xi})$ and $(Y,\bar{\xi})$. We shall say that $(X,\bar{\xi})$ is the X-curve of $\bar{\xi}$ and that $(Y,\bar{\xi})$ is the Y-curve of $\bar{\xi}$.

LEMMA 5.5. Let $\bar{\xi}$ be a strongly normalized quasi-ordinary branch having distinguished pairs (σ_i, τ_i) $(i = 1, 2, \ldots, s)$ with $\sigma_i \neq \tau_i$ for at least one i. Suppose also that $\bar{B} = k[[X,Y]][\bar{\xi}]$ has two curves in its singular locus. Then

a) If both curves are permissible centers then the respective monoidal transforms of $\bar{B}$ are represented by strongly normalized branches having distinct sets of distinguished pairs.

b) If neither curve is a permissible center, and if B_1, B_2 are the special transforms of $\bar{B}$ through which pass the respective proper transforms of these curves, then B_1 and B_2 are represented by strongly normalized branches having distinct sets of distinguished pairs.

PROOF. Suppose to begin with that $\sigma_1 + \tau_1 < 1$. Let $\bar{\xi}_1$ ($\bar{\xi}_2$) be a strongly normalized representing branch of B_1 (B_2). By table 4.4, $\bar{\xi}_1$, $\bar{\xi}_2$ are the strongly normalized branches associated with other branches whose distinguished pairs are among

$$(\sigma_i + [(1 + \tau_i)(1 - \sigma_1)/\tau_1] - 2,\ [(1 + \tau_i)/\tau_1] - 1)$$

$$(\tau_i + [(1 + \sigma_i)(1 - \tau_1)/\sigma_1] - 2,\ [(1 + \sigma_i)/\sigma_1] - 1)$$

respectively. For $i = 1$ we get $((1 - \sigma_1 - \tau_1)/\tau_1, 1/\tau_1)$, $((1 - \sigma_1 - \tau_1)/\sigma_1, 1/\sigma_1)$ respectively.

If $1/\tau_1$ and $(1 - \sigma_1 - \tau_1)/\tau_1$ are not both integers, then, since $1 - \sigma_1 - \tau_1 < 1$, the pairs of $\bar{\xi}_1$ are those which are displayed above, with the order reversed; and similarly for $\bar{\xi}_2$. Hence, in this case, if $\bar{\xi}_1$, $\bar{\xi}_2$ have the same pairs, then $(1 + \tau_i)/\tau_1 = (1 + \sigma_i)/\sigma_1$ for $i = 1, 2, \ldots, s$, whence $\sigma_i = \tau_i$ for all i.

If both $1/\tau_1$, $(1 - \sigma_1 - \tau_1)/\tau_1$ are integers, while one of $1/\sigma_1$, $(1 - \sigma_1 - \tau_1)/\sigma_1$ is not an integer, then $\overline{\xi}_1$ has $s - 1$ distinguished pairs, while $\overline{\xi}_2$ has s distinguished pairs, so our conclusion certainly holds. If all four of these quantities are integers, then clearly $\sigma_1 = \tau_1 = \sigma$ (say). Comparing the above displayed expressions, we see that if $\overline{\xi}_1$, $\overline{\xi}_2$ have the same distinguished pairs, then either $(1 + \tau_i)/\sigma = (1 + \sigma_i)/\sigma$ for $i = 2, 3, \ldots, s$, in which case $\sigma_i = \tau_i$ for all i, or

$$(1 + \tau_i)/\sigma = \tau_i + [(1 + \sigma_i)(1 - \sigma)/\sigma] - 1$$

and
$$(1 + \sigma_i)/\sigma = \sigma_i + [(1 + \tau_i)(1 - \sigma)/\sigma] - 1$$

for $i = 2, 3, \ldots, s$. Subtracting one of these equations from the other, we get $2(\tau_i - \sigma_i)(\frac{1}{\sigma} - 1) = 0$ whence $\tau_i = \sigma_i$ for $i = 2, 3, \ldots, s$. In any case, the lemma holds.

We still have to consider several cases in which $\sigma_1 + \tau_1 \geq 1$. The arguments are similar in principle to the preceding ones, and somewhat simpler to carry out, and we omit them. (We note only that in treating the case $\sigma_1 + \tau_1 = 1$, lemma 2.3 must be used).

LEMMA 5.6. Let A, A', B, B', $\overline{\xi}$, $\overline{\xi}'$ be as above.

Suppose that among the characteristic pairs (σ_i, τ_i) of $\bar{\xi}$ (or, what is the same, of $\bar{\xi}'$) there is one for which $\sigma_i \neq \tau_i$. Suppose also that B has two curves in its singular locus, neither or both of which are permissible centers. Let ψ be an isomorphism of B with $k[[X,Y]][\bar{\xi}]$, and let p be the curve in B which is exceptional for the transformation from A to B; let ψ', p' be defined similarly with respect to B'. Then $\psi(p)$ is the X-curve of $\bar{\xi}$ if and only if $\psi'(p')$ is the X-curve of $\bar{\xi}'$.

PROOF. We treat the case when B has no permissible centers which are curves, the other case being similar. As previously remarked, $\bar{\xi}$, $\bar{\xi}'$ have the same distinguished pairs; in particular B' also has no permissible centers which are curves.

Let C, C' be the formal quadratic transforms of B, B' through which pass the proper transforms of p, p' respectively. By (ii) of definition 5.1, C, C' are the successors of B, B' in the equivalent resolutions of A, A' which we have assumed to exist. By an inductive argument, we may assume that any two strongly normalized branches, one of which represents C, and the other of which represents C', have the same distinguished pairs. Since also $\bar{\xi}$, $\bar{\xi}'$ have the same distinguished pairs, our conclusion follows easily from lemma 5.5. q.e.d.

To carry out the proof of the theorem, it will be sufficient to show that the following data determine uniquely the distinguished pairs (λ_i, μ_i) of ζ:

1) The label of A.
2) The distinguished pairs (σ_i, τ_i) of $\bar{\xi}$.
3) When the conditions of lemma 5.6 are all fulfilled (a situation whose existence (or non-existence) depends only on (σ_i, τ_i)), whether the image of the unique exceptional curve in B under some isomorphism of B with $k[[X,Y]][\bar{\xi}]$ (and therefore, by lemma 5.6, under <u>any</u> such isomorphism) is the X-curve or the Y-curve of $\bar{\xi}$.

For, in view of the foregoing considerations, 1), 2), 3) do not change when A, B, $\bar{\xi}$ are replaced by A', B', $\bar{\xi}'$, and so we will be able to conclude that ζ and ζ' have the same distinguished pairs.

The rest of the proof is, again, a program for a computer, giving instructions for calculating (λ_i, μ_i) from 1), 2), 3) above. There is a subdivision into a number of cases and subcases, and of course the subdivision must depend only on the data 1), 2), 3); the reader will easily verify at each "fork in the road" that this requirement is adhered to.

We shall maintain the above notations:
$A, B, \xi, \bar{\xi}, (\lambda_i, \mu_i), (\sigma_i, \tau_i)$.

CASE I. Monoidal Transformations of A.
The distinguished pairs of ξ are either (i) $\{(\lambda_i-1, \mu_i)\}$ or (ii) $\{(\lambda_i, \mu_i-1)\}$.

I.1. A has two permissible centers.
In this case ξ is normalized.

I.1.α. $\sigma_1 \geq 1 > \tau_1$.
In this case $(\sigma_i, \tau_i) = (\mu_i, \lambda_i-1)$ or $(\sigma_i, \tau_i) = (\lambda_i, \mu_i-1)$.
Thus we have one of

I.1.α.(i). $(\ldots, \tau_i+1, \ldots) \geq (\ldots, \sigma_i, \ldots)$ lexicographically. In this case $(\lambda_i, \mu_i) = (\tau_i+1, \sigma_i)$.

I.1.α.(ii). $(\ldots, \tau_i+1, \ldots) < (\ldots, \sigma_i, \ldots)$ lexicographically. In this case $(\lambda_i, \mu_i) = (\sigma_i, \tau_i+1)$.

I.1.β. $\sigma_1 \geq \tau_1 \geq 1$, $\sigma_i \neq \tau_i$ for some i.

I.1.β.a. Exceptional curve is the X-curve of $\bar{\xi}$. In this case we have either (i) above, and then $(\sigma_i, \tau_i) = (\lambda_i - 1, \mu_i)$ for all i, or (ii) occurs and then $(\sigma_i, \tau_i) = (\mu_i - 1, \lambda_i)$ for all i. Thus we have one of

I.1.β.a.(i). $(\ldots, \sigma_i+1, \ldots) \geq (\ldots, \tau_i, \ldots)$ lexicographically. In this case $(\lambda_i, \mu_i) = (\sigma_i+1, \tau_i)$.

I.1.β.a.(ii). $(\ldots, \sigma_i+1, \ldots) < (\ldots, \tau_i, \ldots)$ lexicographically. In this case $(\lambda_i, \mu_i) = (\tau_i, \sigma_i+1)$.

I.1.β.b. Exceptional curve is the Y-curve of $\bar{\xi}$. Here we may proceed as in I.1.β.a.

I.1.γ. $\sigma_i = \tau_i \geq 1$ for all i.
In this case $(\lambda_i, \mu_i) = (\sigma_i+1, \tau_i)$ for all i.

We have now disposed of case I.1. From now on we will be a little less mechanical in our description of the program.

I.2. A has precisely one permissible center which is a curve.

I.2.α. There is no curve in the singular locus of A other than the permissible center.

In this case the proof of theorem 2.7 shows that the set of distinguished pairs of ζ is of one of the following kinds.

(i) $(\lambda_1, 1/m)$ where $\lambda_1 \geq 1$ and m is the multiplicity of A.

(ii) $(\lambda_1,0), (\lambda_2,0), \ldots, (\lambda_s,0)$ $\quad s \geq 1, \lambda_1 > 1.$

(iii) $(\lambda_1,0), (\lambda_2,0), \ldots, (\lambda_{s-1}, 0), (\lambda_s, 1/r)$ $\quad s>1, \lambda_1 > 1+1/m, 1<r<m.$

By definition 5.1, B is not a regular local ring. If (i) occurs, then ξ has one distinguished pair $(\lambda_1-1, 1/m)$. Since $\lambda_1 > 1$ (otherwise B is regular), $\bar{\xi}$ will have precisely one distinguished pair, and 1/m will be a member of that pair. If (ii) occurs, then every distinguished pair of $\bar{\xi}$ will have at least one vanishing member. If (iii) occurs, then the largest distinguished pair of $\bar{\xi}$ will have two members both of which are $> 1/m$. Thus we can distinguish between cases (i), (ii), (iii) by examining the distinguished pairs of $\bar{\xi}$.

In case (i), it is clear that $\lambda_1 - 1 \geq 1/m$, and hence $(\lambda_1, \mu_1) = (\sigma_1 + 1, \tau_1)$.

For cases (ii) and (iii), we remark that the multiplicity of B is $\min(m, m\lambda_1 - m)$. This implies, in the first place, that the multiplicity of B equals the multiplicity of A if and only if $\lambda_1 > 2$. In this case $\xi = \bar{\xi}$ and $(\lambda_i, \mu_i) = (\sigma_i + 1, \tau_i)$.

Suppose mult. of B < mult. of A. Then $\lambda_1 < 2$ and $1/(\lambda_1 - 1) = $ (mult. of A)/(mult. of B), so λ_1 is determined by the label of A and by the (σ_i, τ_i). Since $\lambda_1 - 1 < 1$, we must treat ξ as in lemma 2.3. By table 4.4, the (σ_i, τ_i) will be

$$(\lambda_i/(\lambda_1 - 1) - 1, \mu_i)$$

(where $\mu_i = 0$ for $i = 1, 2, \ldots, s-1$, and $\mu_s = 0$ or $\mu_s = 1/r$ according as we have case (ii) or (iii)), unless $\lambda_1/(\lambda_1 - 1)$ is an integer (or, equivalently, $1/(\lambda_1 - 1)$ is an integer). Since λ_1 is now known, it is clear how to recover (λ_i, μ_i) from (σ_i, τ_i).

I.2.β. A has two curves in its singular locus, precisely one of which is a permissible center.

Let P be the permissible center, and let Q be the other curve in the singular locus of A.

I.2.β.a. Q is a non-singular curve.

In this case $\lambda_1 > \mu_1 > 0$ and we can calculate μ_1 from the label of A, since μ_1 = (mult. of Q in A)/(mult. of A). We note that multiplicity of B = $\min(m, m\lambda_1+m\mu_1-m)$, (m = mult. of A). Thus, the multiplicity of B equals that of A if and only if $\lambda_1 + \mu_1 \geq 2$. When $\lambda_1 + \mu_1 \geq 2$, then ξ is normalized. Thus $\{(\sigma_i,\tau_i)\} = \{(\lambda_i-1,\mu_i)\}$ or $\{(\mu_i,\lambda_i-1)\}$. If $\sigma_1 = \mu_1$ and $\tau_1 \neq \mu_1$, then $(\lambda_i,\mu_i) = (\tau_i+1,\sigma_i)$ for all i; if $\tau_1 = \mu_1$, and $\sigma_1 \neq \mu_1$ then $(\lambda_i,\mu_i) = (\sigma_i+1,\tau_i)$ for all i.

Finally, if $\sigma_1 = \tau_1 = \mu_1$, then either $\sigma_i = \tau_i$ for all i, in which case $(\lambda_i,\mu_i) = (\sigma_i+1,\tau_i)$ or else B satisfies the conditions of lemma 5.6. In the latter case, we can use type 3) data and say that $\{(\lambda_i,\mu_i)\} = \{(\sigma_i+1,\tau_i)\}$ or $\{(\lambda_i,\mu_i)\} = \{(\tau_i+1,\sigma_i)\}$ according as the exceptional curve is the X-curve or the Y-curve of $\bar{\xi}$.

Suppose now that B has smaller multiplicity than A. Then (mult. of A)/(mult. of B) = $\lambda_1 + \mu_1 - 1$ and so λ_1 is determined. If $\lambda_1 > 1$, then ξ is normalized and we can proceed as in the case $\lambda_1 + \mu_1 \geq 2$.

If $\lambda_1 = 1$, then ξ must be treated as in lemma 2.3, and by table 4.4 we have $(\sigma_i, \tau_i) = ((\mu_i+1)/\mu_1-1,\ \lambda_i-1)$ for all i unless $1/\mu_1$ is an integer, in which case the pair $(1/\mu_1,\ \lambda_1-1)$ must be omitted. If $1/\mu_1$ is not an integer, then (λ_i, μ_i) is easily found. If $1/\mu_1$ is an integer, then we have one of

(i) $$(\sigma_i, \tau_i) = ((\mu_{i+1}+1)/\mu_1-1,\ \lambda_{i+1}-1)$$

(ii) $$(\sigma_i, \tau_i) = (\lambda_{i+1}-1,\ (\mu_{i+1}+1)/\mu_1-1).$$

First we check to see whether $\tau_1 < 1$. If so then we are in case (i) above, since $(\mu_2+1)/\mu_1 - 1 > 1$. If $\tau_1 > 1$, then B satisfies lemma 5.6 (unless $\sigma_i = \tau_i$ for all i, in which case (i) holds); if then the exceptional curve is the X-curve we have (ii) above, and if the exceptional curve is the Y-curve we have (i). Once we have decided whether (i) or (ii) holds, the (λ_i, μ_i) are easily obtained.

I.2.β.b. Q has a multiple point at its origin.

In this case $\mu_1 = 0$. We leave this case to the reader, since no arguments are needed which have not already been used in the above cases.

This completes the program for the monoidal case.

CASE II. Quadratic Transformations of A.

A quadratic transformation of A is called for if and only if A has no permissible centers which are curves. In this case we have $0 < \lambda_1 < 1$, $0 < \mu_1 < 1$. If A has an irreducible tangent cone then $\lambda_1 + \mu_1 \geq 1$; if a, b, c are the first three numbers in the label of A, then, by theorem 2.7, $a\lambda_1 = b$, $a\mu_1 = c$. If A has a reducible tangent cone, and e is the fifth number in the label of A, then $e\lambda_1 = b$, $e\mu_1 = c$. In either case, λ_1 and μ_1 are determined by the label of A.

II.1. $\lambda_1 + \mu_1 > 1$.

In this case the distinguished pairs of $\bar{\xi}$ are either $\{(\lambda_i, \lambda_i+\mu_i-1)\}$ or $\{(\mu_i, \lambda_i+\mu_i-1)\}$. Hence $\{(\lambda_i, \mu_i)\} = \{(\sigma_i, \tau_i+1-\sigma_i)\}$ if $(\ldots, \sigma_i, \ldots) \geq (\ldots, \tau_i+1-\sigma_i, \ldots)$ lexicographically; otherwise $\{(\lambda_i, \mu_i)\} = \{(\tau_i+1-\sigma_i, \sigma_i)\}$.

II.2. $\lambda_1 + \mu_1 = 1$.

In this case ξ has to be treated according to lemma 2.3; the resulting pairs are either

(i) $((\mu_i+1-\mu_1)/\mu_1,\ \lambda_i+\mu_i-1)$ or

(ii) $((\lambda_i+1-\lambda_1)/\lambda_1,\ \lambda_i+\mu_i-1)$

Since $\lambda_2 + \mu_2 > \lambda_1 + \mu_1$, $\overline{\xi}$ has at most one pair in which 0 appears. If there is such a pair, then the other member of the pair is either $1/\mu_1$ or $1/\lambda_1$. Then we find that $\{(\sigma_i+1-\sigma)/\sigma_1, \tau_i+1-(\sigma_i+1-\sigma_1)/\sigma_1)\}$ is either $\{(\lambda_i,\mu_i)\}$ or $\{(\mu_i,\lambda_i)\}$; we can easily decide which one by lexicographic order as in II.1.

If $\overline{\xi}$ has no pair in which 0 appears then at least one of $1/\lambda_1$, $1/\mu_1$ is an integer. If $1/\lambda_1$ is an integer, while $1/\mu_1$ is not, we must be dealing with case (ii) above. Since $(\lambda_2+1-\lambda_1)/\lambda_1 \geq 1/\lambda_1 \geq 2$ we check to see whether $\tau_1 < 2$. If so, then $\overline{\xi}$ has the pairs displayed in (ii) for i = 2, 3 ... and the (λ_i,μ_i) are easily found. If $\tau_1 \geq 2$, then we use the (by now) familiar arguments about exceptional curves being X-curves or Y-curves of $\overline{\xi}$ to distinguish between the case when the pairs of $\overline{\xi}$ are as displayed in (ii) and the case in which the pairs are in reverse order; then the (λ_i,μ_i) are easily found.

If $1/\mu_1$ is an integer and $1/\lambda_1$ is not, then we proceed as above. If both $1/\lambda_1$ and $1/\mu_1$ are integers, then, since $\lambda_1 + \mu_1 = 1$, $\lambda_1 = \mu_1 = 1/2$. A combination of the foregoing considerations about exceptional curves and lexicographic order easily indicates the procedure in this case.

II.3. $\lambda_1 + \mu_1 < 1$.

In this case we have (again) two possible sets of pairs for ξ, namely those given in table 4.4 for the directions (1:0:0), (0:1:0) respectively, in the non-transversal case; the first of the two members of each of these pairs is always the one associated with the exceptional curve.

For i = 1 we get one of the pairs

$$((1-\lambda_1-\mu_1)/\mu_1,\ 1/\mu_1) \qquad ((1-\lambda_1-\mu_1)/\lambda_1,\ 1/\lambda_1)$$

Once again, we are led to different situations, according to how many of $1/\mu_1$, $1/\lambda_1$, $(1-\lambda_1-\mu_1)/\mu_1$, $(1-\lambda_1-\mu_1)/\lambda_1$ are integers. The reader who has come this far will have no trouble filling in the details.

This completes the proof of theorem 5.4.

6. Generic Transversal Sections.

In our remarks about equisingularity (cf. introduction and also end of §4) we have indicated the importance of considering a section of a surface S by a plane which is transversal to a given curve C on S at some simple point P of C. If P is a variable point of C which approaches a given point Q of C, we can think of a corresponding family of continuously varying transversal sections. The members of the family are curves with "equivalent" singularities at their origins, although in the limiting curve (i.e. when P becomes Q) there may be some kind of degeneracy. At any rate S is equisingular along C in any sufficiently small neighborhood of Q (excluding Q) and the "generic" plane section transversal to C tells us everything about the nature of S along C near Q. Thus, such a generic section conveys a certain amount of information about the singularity which S has at Q itself.

The notions of equivalence of plane curves and of transversal sections are given in [8]. We shall study the connection between the distinguished pairs of a quasi-ordinary branch ζ and the generic sections transversal to curves in the singular locus of the quasi-ordinary ring represented by ζ, and then give a different version of the classification theorem.

We shall now define generic transversal sections in a general setting. We first reproduce definition 19.8.1 of [5; p. 109]:

Let A, B be noetherian local rings, let M be the maximal ideal of A, and let ω: A → B be a local homomorphism making B into an A-algebra. We say that B is a Cohen A-algebra if

(i) B is complete

(ii) B is a flat A-module

(iii) MB is the maximal ideal of B

(iv) B is residually separable over A (i.e. B/MB is a separable field extension of A/M).

If B is a Cohen A-algebra and if "gr" denotes associated graded ring with respect to the maximal ideal, then there is an isomorphism $(B/MB)\otimes_{A/M}gr(A) \cong gr(B)$ [4; §10.2.2, p. 18]; hence A and B have the same multiplicity and dimension.

DEFINITION 6.1. Let A be a noetherian local ring containing a field of characteristic zero, and let P be a prime ideal in A. We say that a local ring B is a generic section of A transversal to P if B is a Cohen A_P-algebra whose residue field is the algebraic closure of the residue field of A_P.

The existence and uniqueness (up to (not necessarily canonical) isomorphism) of generic sections of A transversal to P are given in theorem 19.8.2 of [5; p. 110]. In view of the uniqueness we may speak of the generic section transversal to P (as long as we are concerned only with the structure of B as a ring).

Now let R = k[[X,Y,Z]] be as usual, let P be the ideal (X,Z) in R, let f be an element of R contained in P and set A = R/(f), $\overline{P}$ = P/(f), so that $A_{\overline{P}} = R_P/fR_P$. Let L = k{{Y}} and let L^* be the algebraic closure of L. We have $R \subsetneq R_P \subseteq L[[X,Z]] \subsetneq L^*[[X,Z]]$ and L[[X,Z]] is the completion of R_P. Clearly $gr(L^*[[X,Z]]) = L^* \otimes_L gr(R_P)$ whence [4; §10.2.2, p. 18] $L^*[[X,Z]]$ is a flat R_P-module. It follows that $L^*[[X,Z]]$ is a Cohen R_P-algebra, so that $L^*[[X,Z]] \otimes_{R_P} (R_P/fR_P)$ is a Cohen (R_P/fR_P)-algebra. In other words, if f^* is f considered as a power series in $L^*[[X,Z]]$, then $L^*[[X,Z]]/(f^*)$ is the generic section of A transversal to $\overline{P}$.

$L^*[[X,Z]]/(f^*)$ is an algebroid plane curve in the sense of [8]. In the case that f is a quasi-ordinary polynomial we shall be particularly interested in the equivalence class of this curve. This equivalence class depends only on the ring structure of $L^*[[X,Z]]/(f^*)$, and so, by our previous remarks, depends only on A and P.

Let $\zeta \in \Phi_n$ be a strongly normalized quasi-ordinary branch, with conjugates $\zeta_1, \zeta_2, \ldots, \zeta_m$, and let $f = \prod_i [Z - \zeta_i]$ be the minimum polynomial of ζ; since ζ is strongly normalized, f is contained in $(X,Z)R$. The ζ_i can be thought of as fractional power series in X with coefficients in L^*, and when we do this, we write ζ_i^* instead of ζ_i. Clearly $f^* = \prod_i [Z - \zeta_i^*]$. We denote by C the algebroid curve defined over L^* by f^*, and by $C_1, C_2, \ldots$, the irreducible components of C. We say that ζ_i^* (or ζ_i) represents C_j if ζ_i^* is a root of the irreducible factor of f^* corresponding to C_j.

THEOREM 6.2. In the preceding situation, the equivalence class of C is determined by the distinguished pairs of ζ.

PROOF. It can be shown that two algebroid plane curves are equivalent if and only if there exists a 1-1 correspondence between their irreducible components such that corresponding components are equivalent and such that intersection multiplicities of pairs of components are preserved. We shall first discuss the equivalence classes of the irreducible components of C.

Everything we have said about distinguished <u>pairs</u> of quasi-ordinary branches applies (in simpler form) to fractional power series in one variable over an algebraically closed field of characteristic zero; we then have characteristic monomials in one variable instead of two, and distinguished <u>numbers</u> instead of distinguished <u>pairs</u>. In particular, we can prove as in previous sections (but with considerably less effort) the well-known fact that the distinguished numbers of a fractional power series in one variable determine the multiplicities of the successive quadratic transforms of the one-dimensional local domain represented by the power series. (If the least distinguished number is <1, we must first "normalize" as in lemma 2.3). It follows that the equivalence class of the one-dimensional local domain is specified by the sequence of distinguished numbers of the power series.

It is easy to see that the conjugates of ζ^* over $L^*[[X]]$ are those ζ_i^* such that $\zeta_i = \theta\zeta$ with θ in the galois group of $K(\zeta)/[K(\zeta)\cap K(Y^{1/n})]$. Thus if (λ_i, μ_i) $(i = 1, 2, \ldots, s)$ are the distinguished pairs of ζ, then the distinguished numbers of ζ^* are among the numbers $\lambda_1, \lambda_2, \ldots, \lambda_s$. We leave it to the reader to verify that the distinguished numbers of ζ^* are the elements λ_i such that λ_i is not an integral linear combination of $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}$ modulo $\mathbb{Z}$ (i.e., in the notation of proposition 1.5, such that $\tau(\lambda_i) = i$). The proof is a straightforward application of the final statement of proposition 1.5.

We conclude therefore that all irreducible branches of C belong to the same equivalence class, and that the equivalence class to which they belong is determined by the distinguished pairs of ζ.

We turn now to the question of intersection multiplicities of different components. Let G_0 be the galois group of $K(\zeta)/K$. G_0 operates in an obvious way in the set of irreducible components of C. Let H be the subgroup of G_0 which leaves all the irreducible components fixed. As we have indicated above, H is the galois group of $K(\zeta)/[K(\zeta)\cap K(Y^{1/n})]$.

Let $M_1, M_2, \ldots, M_s$ be the characteristic monomials of ζ, the numbering being such that M_i divides M_j in Φ whenever $i < j$ (cf. remark 1.4.3). For $i = 1, 2, \ldots, s$ let G_i be the galois group of $K(\zeta)/K(M_1, M_2, \ldots, M_i)$. Each G_i, being a subgroup of G_0, operates in the set of irreducible components of C. The orbits under G_i are in 1-1 correspondence with the cosets of $G_i H$ in G_0, and the number of elements in any orbit is the number $|G_i H|/|H| = |G_i|/|G_i \cap H|$ (where "$|\ |$" denotes "cardinality of"). For any two irreducible components C_1, C_2 of C^* let $I = I(C_1, C_2)$ be the least number i such that C_1 and C_2 do not lie in a single orbit under G_i. For various choices of C_1, C_2, I may be any one of those i such that $G_i H < G_{i-1} H$; we shall refer to these i as the <u>critical values</u>. (There always exist critical values unless C is irreducible). We have now:

LEMMA 6.2.1. The intersection multiplicity $(C_1 \cdot C_2)$ of C_1 and C_2 depends only on the number $I = I(C_1, C_2)$.

PROOF. Let ζ_1, ζ_2 be conjugates of ζ which represent C_1, C_2 respectively. Let $\theta_1, \theta_2, \ldots, \theta_r$ be the various elements of H. $(C_1 \cdot C_2)$ is then the order in X of the power series

$$\prod_{i,j=1}^{r}[\theta_i\zeta_1 - \theta_j\zeta_2] = \prod_{i=1}^{r}\theta_i\Big(\prod_{j=1}^{r}[\zeta_1 - \theta_j\zeta_2]\Big)$$

which is r times the order in X of $\Pi_{j=1}^{r}[\zeta_1 - \theta_j\zeta_2]$. For each t, $1 \leq t \leq s$, let q_t be the number of different j such that $\zeta_1 - \theta_j\zeta_2 = M_t\varepsilon_t$ ($\varepsilon_t(0,0) \neq 0$). Then $(C_1 \cdot C_2) = r(\Sigma_{t=1}^{s} q_t\lambda_t)$. Thus it will be sufficient to show that the sequence $q_1, q_2, \ldots, q_s$ depends only on I.

It is not hard to see that C_1, C_2 lie in the same orbit of G_i if and only if there is a θ_j such that M_{i+1} divides $\zeta_1 - \theta_j\zeta_2$ in Φ. Therefore $q_t = 0$ for $t > I$, and $q_t \neq 0$ for $t = I$. Thus for $t \leq I$ there exists θ_j such tha M_t divides $\zeta_1 - \theta_j\zeta_2$. Then if $\theta \in H$, M_t divides $\zeta_1 - \theta\theta_j\zeta_2$ if and only if θ leaves $M_1, M_2, \ldots, M_{t-1}$ invariant, i.e. if and only if θ lies in $G_{t-1} \cap H$. Thus we obtain the following rule for computing q_t:

$$q_t = |G_{t-1} \cap H| - |G_t \cap H| \qquad \text{if } 1 \leq t < I$$

$$q_t = |G_{t-1} \cap H| \qquad \text{if } t = I.$$

The lemma is therefore proved.

To complete the proof of the theorem, we prove:

LEMMA 6.2.2. Let $i_1, i_2, \ldots, i_t, \ldots$ be the critical values. For $t = 0, 1, 2, \ldots$ let n_t be the number of elements in any orbit under $G_{i_{t+1}}$ and let v_t be the intersection multiplicity of any two branches C_1, C_2 such that $I(C_1,C_2) = i_{t+1}$. Then the equivalence class of C is specified by the equivalence class of its irreducible branches and by the finite sequence (n_t,v_t) $t = 0, 1, \ldots$.

PROOF. All the irreducible components of C belong to the same equivalence class; moreover, by the proof of the preceding lemma, C has the property that for any integer v, the binary relation $[R_v\colon (C_i \cdot C_j) \geq v]$ is an equivalence relation on the set of components of C, and the number n(v) of components in each equivalence class for R_v is the same. (We are using the term "equivalence class" both for equivalence of algebroid curves and for the relation R_v; the two distinct meanings of the term should not, hopefully, lead to confusion). The above properties must hold for any algebroid curve which is equivalent to C.

For any algebroid curve enjoying such properties, let $\{v_t\}$ $(t = 0, 1, 2, \ldots)$ be the set of those v such that $n(v_t+1) < n(v_t)$; let $n_t = n(v_t)$. We check that this notation is consistent with that given for C in the statement of the lemma. It is clear that two equivalent such algebroid curves C_1, C_2 have the same sequence (n_t, v_t), and also that the equivalence class of algebroid curves to which the irreducible components of C_1 belong is the same as the class to which those of C_2 belong.

Conversely, two curves C_1, C_2 for which these data coincide are equivalent. In fact, we easily prove the following by induction on t: let E_i be the set of all equivalence classes of components of C_1 under the relation R_i, and let $\cup_{i=1}^{t} E_i$ be ordered by inclusion; let $\cup_{i=1}^{t} E_i'$ be the ordered set defined similarly with respect to C_2. Then there is a map F_t from $\cup_{i=1}^{t} E_i$ onto $\cup_{i=1}^{t} E_i'$ such that F_t is an isomorphism of ordered sets. For large enough t, each member of E_t consists of a single element, and it follows immediately that F_t then induces an equivalence between C_1 and C_2. This completes the proof of the lemma and of theorem 6.2.

REMARK 6.3. In connection with the formulae for q_t, we note that (i): $G_{t-1} \cap H = G_t \cap H$ if and only if (ii): λ_t is an integral linear combination, modulo Z, of $\lambda_1, \lambda_2, \ldots, \lambda_{t-1}$; for it is easy to see that (i) and (ii) both mean that $K(Y^{1/n})(M_1, M_2, \ldots, M_{t-1}) = K(Y^{1/n})(M_1, M_2, \ldots, M_t)$.

The next proposition allows us to identify a curve in the singular locus of a given quasi-ordinary ring by means of the distinguished pairs of any normalized representing branch.

PROPOSITION 6.4. Let $\zeta \in \Phi_n$ be a normalized quasi-ordinary branch with distinguished pairs (λ_i, μ_i) $(i = 1, 2, \ldots, s)$, and let $A = k[[X,Y]][\zeta]$. Suppose A has two curves P_1, P_2 in its singular locus, and that the generic sections of A transversal to P_1, P_2 respectively are equivalent. Then $\lambda_i = \mu_i$ for all i.

(The converse statement follows from theorem 6.2).

PROOF. P_1 and P_2 must both have the same multiplicity in A, since the corresponding generic transversal sections have the same multiplicity. Hence neither λ_1 nor μ_1 vanishes, and if either λ_1 or μ_1 is < 1, then (theorem 2.7) $\lambda_1 = \mu_1$. The methods used in the proof of theorem 6.2 are applicable now. The generic sections are represented by ζ considered respectively as a fractional power series in X and as a fractional power series in Y.

We have already stated that the distinguished numbers of a fractional power series determine the equivalence class of the one-dimensional local domain D represented by that series; conversely we can prove as in §5 (and again with much less effort) that the sequence of multiplicities of the successive quadratic transforms of D determines the original distinguished numbers uniquely, provided that the least distinguished number is > 1. If we start off with a power series whose least distinguished number is < 1, then we first normalize as in lemma 2.3, thereby obtaining a fractional power series which represents D and whose least distinguished number is > 1. The distinguished numbers of the normalized branch, along with the least distinguished number of the original branch together determine all the other distinguished numbers of the original branch. (To see this set all the $\mu_i = 0$ in the formula given in table 4.4 for lemma 2.3, and work with the resulting formula).

In view of these remarks, and the above mentioned fact that $\lambda_1 = \mu_1$ if either λ_1 or μ_1 is < 1, our assumption implies that the distinguished numbers of ζ as a fractional power series in X and those of ζ as a fractional power series in Y coincide.

Let the groups G_i be as in the proof of theorem 6.2, let H be the galois group of $K(\zeta)/[K(\zeta)\cap K(Y^{1/n})]$ and let H' be the galois group of $K(\zeta)/[K(\zeta)\cap K(X^{1/n})]$. Since $|G_0|/|H|$ is the number of irreducible components of one generic section and $|G_0|/|H'|$ is the number of components in the other generic section, $|H| = |H'|$. Set $\lambda_0 = \mu_0 = 0$. Then:

LEMMA 6.4.1. If $\lambda_i = \mu_i$ for $i = 0, 1, \ldots, t$, then for $i = 0, 1, \ldots, t$, $|G_i \cap H| = |G_i \cap H'|$ and $G_i H = G_i H' = G_0$.

PROOF. It will be sufficient to prove the second assertion since

$$|G_i \cap H||G_i H| = |G_i||H| = |G_i||H'| = |G_i \cap H'||G_i H'|.$$

If $i = 0$, $G_i H = G_0 = G_i H'$. Otherwise, $G_i H$ is the galois group of $K(\zeta)/[K(M_1, M_2, \ldots, M_i) \cap K(Y^{1/n})]$. The denominator is generated over K by terms Y^λ such that $(0,\lambda)$ is a linear combination, with integer coefficients, modulo $Z \times Z$, of (λ_1,μ_1), (λ_2,μ_2), $\ldots$, (λ_i,μ_i). Since we are assuming $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, $\ldots$, $\lambda_i = \mu_i$, the denominator must in fact be K. A similar argument shows that also $G_i H'$ is the galois group of $K(\zeta)/K$. q.e.d.

Now let $1 \leq t$ and suppose $\lambda_i = \mu_i$ for

$i = 0, 1, 2, \ldots, t-1$. We wish to show $\lambda_t = \mu_t$, since then the proposition follows by induction. We have three possibilities:

1) λ_t is an integral linear combination of $\lambda_1, \lambda_2, \ldots, \lambda_{t-1}$ mod. $\mathbb{Z}$, and μ_t is such a combination of $\mu_1, \mu_2, \ldots, \mu_{t-1}$.

In this case remark 6.3 along with lemma 6.4.1 shows that $G_t \cap H = G_{t-1} \cap H = G_{t-1} \cap H' = G_t \cap H'$; it follows, for one thing, that $G_t H < G_{t-1} H$ and that $G_t H' < G_{t-1} H'$; thus t is a critical value for both generic sections (in fact t is the least critical value in each case). By lemma 6.2.2, the corresponding intersection multiplicities v_0, v_0' are the same. The formulas for q_t given in the proof of theorem 6.2, along with lemma 6.4.1, show then that $|G_t \cap H|\lambda_t = |G_t \cap H'|\mu_t$, whence $\lambda_t = \mu_t$.

2) Neither λ_t nor μ_t is an integral combination ...

In this case λ_t is the t-th distinguished number of ζ as a fractional power series in X and μ_t is the t-th distinguished number of ζ as a fractional power series in Y, and so $\lambda_t = \mu_t$.

3) λ_t is an integral combination ... while μ_t is not.

In this case μ_t is the t-th distinguished number of ζ as a fractional power series in Y while λ_t is less than the t-th distinguished number of ζ as a fractional power series in X, and so $\mu_t > \lambda_t$. Also, as in 1), t is the least critical value for the generic section represented by ζ as a fractional power series in Y; let $u \geq t$ be the least critical value for the other generic section. (Such a u exists since both generic sections have the same number of components). Again by the proof of theorem 6.2, and lemma 6.4.1,

$$0 = v_0' - v_0 = (|G_{t-1} \cap H'| - |G_t \cap H'|)\mu_t + \dots + (|G_u \cap H|)\mu_u - (|G_{t-1} \cap H|)\lambda_t$$

$$\geq |G_{t-1} \cap H'|\mu_t - |G_{t-1} \cap H|\lambda_t > 0$$

since $|G_{t-1} \cap H'| = |G_{t-1} \cap H|$ (lemma 6.4.1). Thus case 3) cannot occur, and proposition 6.4 is proved.

PROPOSITION 6.5. Let B be a special transform of A, let P be a curve in the singular locus of A which does not contain the center of the transformation from A to B, and let Q be a prime ideal in B which contracts to P. Then Q is the only such prime ideal and the generic sections of A transversal to P and B transversal to Q are equivalent.

PROOF. The uniqueness of Q can be verified by examining the formulas of §3 in the light of the proof of theorem 2.7. Let B' be the quadratic or monoidal transform of A of which B is the completion, and let $Q' = Q \cap B$. It is easily seen that $B'_{Q'} \cong A_P$. Moreover, since Q' is analytically unramified (B' being pseudo-geometric [6; p. 112, p. 134]), B_Q is a Cohen $B'_{Q'}$-algebra. If L is the algebraic closure of B_Q/QB_Q, then the unique Cohen B_Q-algebra D whose residue field is L [5; p. 110] is also a Cohen $B'_{Q'}$-algebra. The uniqueness implies that if L' is the algebraic closure of the residue field of $B'_{Q'}$ and D' is the Cohen $B'_{Q'}$-algebra with residue field L', then D is (isomorphic to) the Cohen D'-algebra with residue field L.

The proposition states that D and D' are equivalent algebroid curves (although they have different residue fields). This follows easily from the last statement in the preceding paragraph since L' and L are both algebraically closed. q.e.d.

We also mention, without proof, the following proposition about monoidal transformations:

PROPOSITION 6.6. Let A be a quasi-ordinary local ring and let P be a permissible center in A which is a curve. Let B be a formal monoidal transform of A with center P and let Q be a prime ideal in B which contracts to P in A. Then Q is the only such ideal in B, and the generic section of B transversal to Q is equivalent to any one of the connected components of the quadratic transform of the generic section of A transversal to P.

* * *

Using the properties of generic sections, we can reformulate the classification theorem.

A <u>resolution</u> of A will be a sequence $A = A_0, A_1, \ldots, A_t$ such that

1) A_{i+1} is a special transform of A_i (for all $i, 0 \leq i \leq t-2$)
2) If for some i, $0 \leq i \leq t-1$, A_i has a reducible tangent cone, then A_{i+1} is the formal transform of A which lies at the intersection of the two components of the exceptional curve.
3) Suppose A_{t-1} has an irreducible tangent cone. If there is a permissible center in the singular locus of A_{t-1} then A_t is a monoidal transform of A_{t-1}; if there is no permissible center in A_{t-1}, then every quadratic transform of A_{t-1} is regular.

4) A_t is a regular quasi-ordinary local ring.

The labels which we now attach to a quasi-ordinary ring A will have five numbers [a;b,c;d;e] where a, b, c, d are as in §5 and e is the number of irreducible components of the tangent cone of A (e = 1 or 2).

Two resolutions $A = A_0, A_1, \ldots, A_t$; $A' = A'_0, A'_1, \ldots, A'_s$; will be said to be equivalent if $s = t$, if for each $i = 0, 1, 2, \ldots, t$, A_i and A'_i have the same label and if, in addition, the following conditions hold:

(i) If for some i, $0 \leq i < t$, A_{i+1} is a formal monoidal transform of A_i, in which case A'_{i+1} is necessarily a formal monoidal transform of A'_i, then the generic section of A_i transversal to the center of the transformation from A_i to A_{i+1} is equivalent to the generic section of A'_i transversal to the center of the transformation from A'_i to A'_{i+1}.

(ii) Suppose that for some i, $0 \leq i < t$, A_i has an irreducible tangent cone, and that A_i has two curves in its singular locus, neither of which is a permissible center. Let C be the generic section transversal to the curve in the singular locus of A_i whose proper transform passes through A_{i+1}; let C' be similarly defined for A_i' (A_i' satisfies the same conditions as A_i since both have the same label); then C and C' are equivalent.

An equivalence class of resolutions may therefore be identified with a sequence of labels, along with certain equivalence classes of algebroid plane curves, one for each label which describes a situation to which (i) or (ii) applies. We call such a sequence of augmented labels a <u>composition</u> of A (without fear of confusion with the compositions of §5, which will not appear in this section).

THEOREM 6.7. Theorem 5.4 continues to hold when the word "composition" is interpreted as above.

PROOF. We prove the second part of the theorem by induction. Suppose ζ and ζ' have the same distinguished pairs. Suppose further that A has an irreducible tangent cone and that A has two curves in its singular locus, neither of which is a permissible center (all other cases can be treated in a way similar to that which follows). Clearly A' satisfies the same assumptions as A.

By theorem 6.2, the sections of A, A', transversal to the X-curve of ζ and the X-curve of ζ' respectively, are equivalent, and similarly for the respective Y-curves. Moreover (theorem 4.5) the transforms B_1, B_1' of A, A' through which pass the proper transforms of the respective X-curves are represented by strongly normalized branches having the same distinguished pairs. By an inductive hypothesis, we may assume that B_1, B_1' have identical sets of associated compositions. We proceed similarly for the special transforms B_2, B_2' through which pass the proper transforms of the Y-curves. It then becomes clear that A and A' have identical sets of associated compositions.

The proof of the first part of the theorem runs along lines similar to those of §5. We shall point out the places where the reasoning is different.

We are assuming that A and A' have equivalent resolutions (in the sense of this section). In the first place, if these resolutions have only one member, then we need only remark that A and A' are regular, and so $\zeta = \zeta' = 0$.

We can then pass directly to lemma 5.6. In the present context, that lemma is needed only under the additional hypothesis that A has an irreducible tangent cone; the proof, using the notation of lemma 5.6, is then as follows: there are associated with B two generic sections (transversal to the two curves in the singular locus of B) which are not equivalent (proposition 6.4), and which are determined by the ordered pairs (σ_i, τ_i) (theorem 6.2); therefore it is only necessary to show that the generic sections transversal to the non-exceptional curves in the singular loci of B, B' respectively are equivalent; however, because of our definition of equivalent resolutions, this follows easily from proposition 6.5.

The proof for monoidal transformations now continues exactly as in §5; to handle case I.2.α we must note however that if A (and therefore A') is such that the hypotheses of lemma 5.2 hold, then we can tell whether ζ has the distinguished pair $(1,1/m)$ or $(1+1/m,0)$; for in the first case the generic section transversal to the permissible center has m irreducible components, while in the second case, the generic section is irreducible; by our definition of equivalent resolutions, ζ and ζ' must have the same distinguished pair.

For quadratic transformations, the proof is still the same as in §5 if A has an irreducible tangent cone.

If A has a reducible tangent cone then we use the following approach (notation as in proof of theorem 5.4 except that B is now the formal transform of A lying at the intersection of the components of the exceptional curve).

We cannot calculate λ_1, μ_1 as in §5; however the ratio $\lambda_1:\mu_1$ is still the same as the ratio b:c where b, c are the second and third numbers in the label of A. Setting $\lambda_1/(1-\lambda_1-\mu_1) = x$, $\mu_1/(1-\lambda_1-\mu_1) = y$, we have $\lambda_1 = x/(1+x+y)$, $\mu_1 = y/(1+x+y)$. If x and y are both integers, then clearly $\lambda_1 + \mu_1 > 1/2$. Now (mult. of B) $= \min(m\lambda_1+m\mu_1,\ m(1-\lambda_1-\mu_1))$, $(m = [K(\zeta):K])$; thus mult. of B < mult. of A if and only if $\lambda_1 + \mu_1 > 1/2$. Hence if mult. of B = mult. of A, then x and y are not both integers, $(\sigma_1,\tau_1) = (x,y)$ (table 4.4), and λ_1, μ_1 can be found from (σ_1,τ_1). If mult.B < mult.A then (mult.B)/(mult.A) $= 1 - \lambda_1 - \mu_1$. Knowing the ratio $\lambda_1:\mu_1$ we can then calculate λ_1, μ_1.

We have $(1-\lambda_1)/(1-\lambda_1-\mu_1) = 1+y$, $(1-\mu_1)/(1-\lambda_1-\mu_1) = 1+x$. Table 4.4 tells us to examine the numbers $\alpha_i = (1+x)\lambda_i + \mu_i$, $\beta_i = y\lambda_i + (1+y)\mu_i$ $(i = 1, 2, \ldots, s)$. Note that $\alpha_1 = x \geq y = \beta_1$. We have $\alpha_i - \beta_i = (\lambda_i-\mu_i) + (x-y)(\lambda_i+\mu_i)$. It is therefore clear that $(\alpha_1,\alpha_2,\cdots,\alpha_s) \geq (\beta_1,\beta_2,\cdots,\beta_s)$ lexicographically. If x and y are both integers, then it is also clear that $(\alpha_2,\ldots,\alpha_s) \geq (\beta_2,\ldots,\beta_s)$ lexicographically. Hence

$$(1+x)\lambda_i + \mu_i = \sigma_i \text{ and } y\lambda_i + (1+y)\mu_i = \tau_i \quad (i = 1,2,\ldots,s)$$

unless x and y are both integers, in which case we replace σ_i by σ_{i-1}, τ_i by τ_{i-1} and consider only the values $i = 2, 3, \ldots, s$.

In any case, the equations can be solved, and so λ_i, μ_i are determined, as required.

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