PICARD SCHEMES OF FORMAL SCHEMES; APPLICATION TO
RINGS WITH DISCRETE DIVISOR CLASS GROUP

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Introduction.

We are going to apply scheme-theoretic methods - originating
in the classification theory for codimension one subvarieties of
a given variety - to questions which have grown out of the
problem of unique factorization in power series rings.

Say, with Danilov [D2], that a normal noetherian ring A
has discrete divisor class group (abbreviated DCG) if the canonical map of divisor class groups \( i: \text{C}(A) \to \text{C}(A[[T]]) \) is bijective(2). In §1, a proof (due partially to J.-F. Boutot) of the following theorem is outlined:

**THEOREM 1.** Let \( A \) be a complete normal noetherian local ring with algebraically closed residue field. If the divisor class group \( \text{C}(A) \) is finitely generated (as an abelian group), then \( A \) has DCG.

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(2) For the standard definition of \( \cdot \), cf. [AC, ch. 7, §1.10].
(Note that the formal power series ring \( A[[T]] \) is noetherian [AC, ch. 3, §2.10, Cor. 6], integrally closed [AC, ch. 5, §1.4], and flat over \( A \) [AC, ch. 3, §3.4, Cor. 3]).

The terminology DCG is explained by the fact that in certain cases (cf. [B];[SGA 2, pp. 189-191]) with \( A \) complete and local, \( \text{C}(A) \) can be made into a locally algebraic group over the residue field of \( A \), and this locally algebraic group is discrete (i.e. zero-dimensional) if and only if \( i \) is bijective.

A survey of results about rings with DCG is given in [F, ch. V].
Recall that $A$ is factorial if and only if $C(A) = (0)$ [AC, ch. 7, §3]. Also, $A$ local $\Rightarrow A[[T]]$ local, with the same residue field as $A$; and $A$ complete $\Rightarrow A[[T]]$ complete [AC, ch. 3, §2.6]. Hence (by induction):

**COROLLARY 1.** If $A$ (as in Theorem 1) is factorial, then so is any formal power series ring $A[[T_1, T_2, \ldots, T_n]]$.

When the singularities of $A$ are resolvable, more can be said:

**THEOREM 1'.** Let $A$ be as in Theorem 1, with $C(A)$ finitely generated, and suppose that there exists a proper birational map $X \to \text{Spec}(A)$ with $X$ a regular scheme (i.e. all the local rings of points on $X$ are regular). Let $B$ be a noetherian local ring and let $f:A \to B$ be a local homomorphism making $B$ into a formally smooth $A$-algebra (for the usual maximal ideal topologies on $A$ and $B$). Then $B$ is normal, and the canonical map $C(A) \to C(B)$ is bijective.

Some brief historical remarks are in order here. Corollary 1 was conjectured by Samuel [S2, p. 171]; however Samuel did not

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(3) "Formal smoothness" means that the completion $\hat{B}$ is $A$-isomorphic to a formal power series ring $\hat{A}[[T_1, T_2, \ldots, T_n]]$, where $\hat{A}$ is a complete local noetherian flat $A$-algebra with maximal ideal generated by that of $A$ (cf. [EGA 0IV, §§19.3, 19.6, 19.7]). In particular, $B$ is flat over $A$.

(4) For some earlier work on unique factorization in power series rings cf. [S1] and [K].
assume that the residue field of $A$ was algebraically closed, and without this assumption, the conjecture was found by Salmon to be false [SMN]. Later, a whole series of counterexamples was constructed by Danilov [DI] and Grothendieck [unpublished]. (5)

Danilov's work led him to the following modification of Samuel's conjecture [DI, p. 131]:

If $A$ is a local ring which is "geometrically factorial" (i.e. the strict henselization of $A$ is factorial) then also $A[[T]]$ is geometrically factorial.

In this general form, the conjecture remains open, though some progress has been made by Boutot [unpublished].

The study of Samuel's conjecture evolved into the study of rings with DCG. A complete normal noetherian local ring $A$ has been shown to have DCG in the following cases (6):

1. (Scheja [SH]). $A$ is factorial and depth $A \geq 3$.
2. (Storch [ST2]) $A$ contains a field, and the residue field of $A$ is algebraically closed and uncountable, with cardinality greater than that of $\mathbb{C}(A)$.

[Actually, for such $A$, Storch essentially proves Theorem 1 without needing any desingularization $X \to \text{Spec}(A)$. Storch's proof uses a theorem of Ramanujam-Samuel (cf. proof of Theorem 1' in §1) and an elementary counting argument.]

(5) In these counterexamples the locally algebraic group of footnote (2) above has dimension > 0, but has just one point - namely zero - rational over the residue field of $A$.

(6) For some investigations in the context of analytic geometry, cf. [ST1] and [P].
(iii) (Danilov [D3]) If either
(a) $A$ contains a field of characteristic zero
or
(b) $A$ contains a field, the residue field of $A$ is separably closed, and there exists a projective map $g: X \to \text{Spec}(A)$ with $X$ a regular scheme, such that $g$ induces an isomorphism

$$X - g^{-1}(\mathfrak{m}) \to \text{Spec}(A) - \{\mathfrak{m}\}$$

($\mathfrak{m} = \text{maximal ideal of } A$)

then $C(A)$ finitely generated $\Rightarrow A$ has DCG.

[Danilov uses a number of results from algebraic geometry, among them the theory of the Picard scheme of schemes proper over a field, and the resolution of singularities (by Hironaka in case (a), and by assumption in case (b)).]

Significant simplifications have been brought about by Boutot. His lemma (§1) enabled him to eliminate all assumptions about resolution of singularities in the above-quoted result of Danilov, and also to modify the proof of Theorem 1' to obtain the proof of Theorem 1 which appears in §1 below.

Our proof of Theorem 1' is basically a combination of ideas of Danilov and Storch, except that in order to treat the case when $A$ does not contain a field, we need a theory of Picard schemes for schemes proper over a complete local ring of mixed characteristic. This theory - which is the main underlying novelty in the paper - is given in §§2-3.
§1. Proofs of Theorems 1 and 1'.

The two theorems have much in common, and we will prove them together. Let $A, B$ be as in Theorem 1'; for Theorem 1 we will simply take $B = A[[T]]$. Since $A$ is local and $B$ is faithfully flat over $A$, the canonical map $C(A) \to C(B)$ is injective [F, Prop. 6.10]; so we need only show that $C(A) \to C(B)$ is surjective.

Both $B$ and its completion $\hat{B}$ are normal: when $B = A[[T]]$ this is clear; and under the assumption of Theorem 1', since $B$ and $\hat{B}$ are formally smooth over $A$, it follows from the existence of the "desingularization" $X \to \text{Spec}(A)$ [L, Lemma 16.1]. As above, since $\hat{B}$ is faithfully flat over $B$, $C(B) \to C(\hat{B})$ is injective, and consequently we may assume that $B = \hat{B}$ (* $\hat{A}[[T_1, T_2, \ldots, T_n]]$, cf. footnote (3) in the Introduction).

[Note here that if $R \subseteq S \subseteq T$ are normal noetherian rings with $S$ flat over $R$ and $T$ flat over $S$ (and hence over $R$), then the composition of the canonical maps

$$C(R) \to C(S) \to C(T)$$

is the canonical map $C(S) \to C(T)$.)

Let $M$ be the maximal ideal of $A$. Then $\hat{A}$ is the maximal ideal of $\hat{A}$, and by the theorem of Ramanujam-Samuel [F, Prop. 19.14],

$$C(B) \to C(B_{MB})$$

is bijective. Furthermore [EGA 01, p. 170, Cor. (6.8.3)], there
exists a complete local noetherian flat $B_{MB}$-algebra $B^*$ such that $B^*/MB^*$ is an algebraically closed field. $B^*$ is formally smooth over $A$ (footnote (3) above) so under the hypotheses of Theorem 1', $B^*$ is normal; furthermore $B^*$ is faithfully flat over $B_{MB}$, so that, as before

$$C(B_{MB}) \to C(B^*)$$

is injective. Thus for Theorem 1' it suffices to show that $C(A) \to C(B^*)$ is surjective.

To continue the proof of Theorem 1', let $U_A$ be the domain of definition of the rational map inverse to $X \to \text{Spec}(A)$. Then $U_A$ is isomorphic to an open subscheme of $X$, so we have a surjective map $\text{Pic}(X) \to \text{Pic}(U_A)$ [EGA IV, (21.6.11)]; furthermore the codimension of $\text{Spec}(A) \smallsetminus U_A$ in Spec($A$) is $\geq 2$, so there is a natural isomorphism $\text{Pic}(U_A) \cong C(A)$ [ibid, (21.6.12)]. Similar considerations hold with $B^*$ in place of $A$, and $X^* = X \otimes_A B^*$ in place of $X$. (The projection $X^* \to \text{Spec}(B)$ is proper and birational, and $X^*$ is a regular scheme [LI, Lemma 16.1].) There results a commutative diagram

$$\begin{array}{ccc}
\text{Pic}(X) & \longrightarrow & \text{Pic}(U_A) \\
\downarrow & & \downarrow \\
\text{Pic}(X^*) & \longrightarrow & \text{Pic}(U_{B^*})
\end{array}$$

Since $\text{Pic}(X^*) \to \text{Pic}(U_{B^*})$ is surjective, it will be more than enough to show that $\text{Pic}(X) \to \text{Pic}(X^*)$ is bijective.
The corresponding step in the proof of Theorem 1 is more involved, and goes as follows. Let $B = A[[T]]$, let $B^*$ be as above, and let $I$ be a divisorial ideal in $B$. We will show below that there exists an open subset $U_A$ of $\text{Spec}(A)$ whose complement has codimension $\geq 2$, and such that, with

$$U_B = (U_A) \otimes_A B \quad (\subseteq \text{Spec}(B)), \quad U^* = (U_A) \otimes_A B^* \quad (\subseteq \text{Spec}(B^*))$$

we have that

(i) $IB_q$ is a principal ideal in $B_q$ for all prime ideals $q \in U_B$, and

(ii) the canonical map $\nu: \text{Pic}(U_B) \to \text{Pic}(U^*)$ is injective.

Now there is a natural commutative diagram

$$
\begin{array}{ccc}
\text{Pic}(U_A) & \longrightarrow & \text{Pic}(U_B) \\
\downarrow \lambda & & \downarrow \mu_B \\
C(A) & \longrightarrow & C(B)
\end{array}
$$

cf. [EGA IV, (21.6.10)]. Since $B$ is flat over $A$, it is immediate (from the corresponding property for $U_A$) that the complement of $U_B$ in $\text{Spec}(B)$ has codimension $\geq 2$; hence (i) signifies that the element of $C(B)$ determined by $I$ is of the form $\mu_B(\zeta)$ for some $\zeta \in \text{Pic}(U_B)$. So if we could show that $\zeta$ lies in the image of $\lambda$, then we would have the desired surjectivity of $C(A) \to C(B)$.

At this point we need:
LEMMA (J.-F. Boutot)\(^{(1)}\). There exists a projective birational map \(\phi: X \to \text{Spec}(A)\) such that \(\phi\) induces an isomorphism \(\phi^{-1}(U_A) \to U_A\), and such that \(\xi\) lies in the image of the canonical map \(\text{Pic}(X \otimes_A B) \to \text{Pic}(U_B)\).

(Here \(X\) may be taken to be normal, but not necessarily regular.) Setting \(X^* = X \otimes_A B^*\), we have a natural commutative diagram

\[
\begin{array}{ccc}
\text{Pic}(X) & \longrightarrow & \text{Pic}(X \otimes_A B) \\
\downarrow & & \downarrow \\
\text{Pic}(U_A) & \xrightarrow{\lambda} & \text{Pic}(U_B) \\
& & \downarrow^\nu \\
& & \text{Pic}(U^*)
\end{array}
\]

with \(\nu\) injective (cf. (ii) above). A simple diagram chase shows then that for \(\xi\) to lie in the image of \(\lambda\), it more than suffices that \(\text{Pic}(X) \to \text{Pic}(X^*)\) be bijective.

Let us finish off this part of the argument by constructing \(U_A\) satisfying (i) and (ii). [It will then remain - for proving both Theorems 1 and 1' - to examine the map \(\text{Pic}(X) \to \text{Pic}(X^*)\).]

Let

\[U_A = \{p \in \text{Spec}(A) | A_p \text{ is a regular local ring}\}.
\]

By a theorem of Nagata [EGA IV (6.12.7)], \(U_A\) is open in \(\text{Spec}(A)\); and certainly, \(A\) being normal, the codimension of \(\text{Spec}(A) - U_A\) in \(\text{Spec}(A)\) is \(\geq 2\). Since the fibres of \(\text{Spec}(B) \to \text{Spec}(A)\) are regular [EGA IV, (7.5.1)], therefore \(B_q\) is regular for all \(q \in U_B\) [EGA IV, (17.3.3)], and (i) follows.

\(^{(1)}\)The proof, which will appear in Boutot's thèse, was presented at a seminar at Harvard University in January, 1972.
As for (ii), setting $U' = U \otimes_A B_{MB}$ ($M$ = maximal ideal of A) we have the commutative diagram

$$
\begin{array}{ccc}
\text{Pic}(U_B) & \longrightarrow & \text{Pic}(U') \\
\downarrow & & \downarrow \\
\text{C}(B) & \longrightarrow & \text{C}(B_{MB})
\end{array}
$$

in which the vertical arrows are isomorphisms [EGA IV, (21.6.12)], and also $\text{C}(B) \rightarrow \text{C}(B_{MB})$ is an isomorphism (cf. above); so we have to show that $\text{Pic}(U') \rightarrow \text{Pic}(U^*)$ is injective. Since $\text{Pic}(U')$ is isomorphic to $\text{C}(B_{MB})$, this injectivity amounts to the following statement:

(\#) Let $I$ be a divisorial ideal of $B_{MB}$, and let $\mathcal{F}^*$ be the coherent ideal sheaf on $\text{Spec}(B^*)$ determined by the ideal $IB^*$. If $\mathcal{F}^*|U^* \cong \mathcal{O}_{U^*}$ then $I$ is a principal ideal.

Since $B_{MB}$ is local, and $B^*$ is faithfully flat over $B_{MB}$, we have

$I$ principal $\Rightarrow$ $I$ invertible $\Rightarrow$ $IB^*$ invertible.

Now $I$ is a reflexive $B_{MB}$-module [CA, p. 519, Ex. (2)], and therefore $IB^*$ is a reflexive $B^*$-module [ibid, p. 520, Prop. 8]. Since $B^*$ is flat over $B_{MB}$, it follows (from the corresponding property of $U'$) that for every prime ideal $P$ in $B^*$ such that $P \not\in U^*$, the local ring $B_P^*$ has depth $\geq 2$. This being so, if $i:U^* \rightarrow \text{Spec}(B^*)$ is the inclusion map, then the natural map

$$\mathcal{O}_{\text{Spec}(B^*)} \rightarrow i_*(\mathcal{O}_{U^*})$$
is an isomorphism \([\text{EGA IV, (5.10.5)}]\). Since \(IB^*\) is reflexive, application of \(\text{Hom}_{B^*}(\cdot, B^*)\) to a "finite presentation"

\[
(B^*)^n \to (B^*)^m \to \text{Hom}_{B^*}(IB^*, B^*) \to 0,
\]
gives an exact sequence

\[
0 \to IB^* \to (B^*)^m \to (B^*)^n,
\]
whence a commutative diagram, with exact rows,

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{I}^* & \to & \mathcal{O}^m & \to & \mathcal{O}^n & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & [\mathcal{O} = \mathcal{O}_{\text{Spec}(B^*)}] \\
0 & \to & i_*((\mathcal{I}^*|U^*)) & \to & i_*(\mathcal{O}^m_{U^*}) & \to & i_*(\mathcal{O}^n_{U^*}) & & \\
\end{array}
\]

from which we conclude that the canonical map

\[
\mathcal{I}^* \to i_*((\mathcal{I}^*|U^*)) \cong i_*(\mathcal{O}_{U^*})
\]
is an isomorphism. Thus \(\mathcal{I}^*\) is isomorphic to \(\mathcal{O}_{\text{Spec}(B^*)}\), and (ii) is proved.

The rest of the discussion applies to both Theorems (1 and i'). We must now examine the map \(\text{Pic}(X) \to \text{Pic}(X^*)\).

The kernel of the surjective map \(\text{Pic}(X) \to \text{Pic}(U_A^*)\) consists of the linear equivalence classes of those divisors on \(X\) which are supported on \(X - U_A\); hence (\(X\) being assumed to be normal) this kernel is isomorphic to a subgroup of the free
abelian group generated by those irreducible components of $X - U_A$ having codimension one in $X$; since $\text{Pic}(U_A) \subseteq \text{C}(A)$, and $\text{C}(A)$ is finitely generated, therefore $\text{Pic}(X)$ is finitely generated.

Let $k$ (resp. $k^*$) be the residue field of $A$ (resp. $B^*$). There is an obvious map $k \to k^*$. In §2 we will show that

(1.1) There exists a $k$-group-scheme $P$ and a commutative diagram

$$
\begin{array}{ccc}
P(k) & \longrightarrow & P(k^*) \\
\downarrow \mathcal{N} & & \downarrow \mathcal{N} \\
\text{Pic}(X) & \longrightarrow & \text{Pic}(X^*)
\end{array}
$$

Here $P(k) \to P(k^*)$ is the map from $k$-valued points of $P$ to $k^*$-valued points corresponding to the map $k \to k^*$; and the vertical maps are isomorphisms.

Furthermore, in §3 it will be shown that

(1.2) There exists a closed irreducible $k$-subgroup $P^0$ of $P$, whose underlying subspace is the connected component of the zero point of $P$, and such that:

(i) $P^0$ is the inverse limit of its algebraic (= finite type over $k$) quotients; moreover if $\hat{P}$ is such a quotient, then $P(k) \to \hat{P}(k)$ is surjective.
(ii) \( P/P^O = \lim_{n>0} Q_n \), where \( Q_n \) is a discrete (= reduced and zero-dimensional) locally algebraic \( k \)-group; moreover \( P(K) \to (P/P^O)(K) \) is surjective for any algebraically closed field \( K \supseteq k \).

To show that \( \text{Pic}(X) \to \text{Pic}(X^*) \) is bijective, it will then suffice to show that \( P^O \) is infinitesimal [in other words, every algebraic quotient of \( P^O \) is zero-dimensional, so that \( P^O(k) = P^O(k^*) = 0 \), whence \( \text{Pic}(X) \to \text{Pic}(X^*) \) can be identified with the map \( \lim_{n} (Q_n(k) \to Q_n(k^*)) \) which is obviously bijective].

But since \( P^O(k) \subseteq P(k) \) is finitely generated, so is \( P(k) \) for any algebraic quotient \( P \) of \( P^O \). By the structure theorem for connected reduced commutative algebraic groups over an algebraically closed field, we know that \( P_{\text{red}} \) has a composition series whose factors are multiplicative groups, additive groups, and abelian varieties. It follows easily that if \( P(k) = P_{\text{red}}(k) \) is finitely generated, then \( P(k) = 0 \), i.e. \( P \) is zero-dimensional.

§2. The Picard Scheme of a Formal Scheme.

In this section we establish the existence of a natural group-scheme structure on \( \text{Pic}(\mathcal{X}) \) for certain formal schemes \( \mathcal{X} \). (If
p\mathcal{O}_X = (0) (cf. (2.2)) there will be nothing new here. For the case
p\mathcal{O}_X \neq (0), most of the work is carried out in [L2], whose results
will be quoted and used.) From this we will obtain (1.1). However,
for completeness, we prove more general results than are required
in the proof of Theorems 1 and 1'.

DEFINITION (2.1). A formal scheme \((X, \mathcal{O}_X)\) is weakly
noetherian if \(X\) has a fundamental system of ideals of definition
\(I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots\) such that for each \(n \geq 0\) the scheme
\((X, \mathcal{O}_X/I_n)\) is noetherian.

It amounts to the same thing to say: in the category of
formal schemes,

\[ X = \lim_{\longrightarrow} X_n \]

where \(X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots\) is a sequence of immersions of noetherian
schemes \(X_n\), the underlying topological maps being homeomorphisms
(cf. [EGA 01, §10.6, pp. 411-413]).

Any noetherian formal scheme is weakly noetherian [ibid,
middle of p. 414].

If \(X\) is weakly noetherian and \(I\) is any ideal of definition,
then \((X, \mathcal{O}_X/I)\) is a noetherian scheme; indeed, \(I \supseteq I_n\) for
some \(n\) (since \(X\) is quasi-compact) so that \((X, \mathcal{O}_X/I_n)\) is a
closed subscheme of the noetherian scheme \((X, \mathcal{O}_X/I_n)\). In
particular, taking \(I\) to be the largest ideal of definition of
we see that we may - and, for convenience, we always will - assume that the scheme $\mathfrak{X}_{\text{red}} = (\mathfrak{X}, \mathcal{O}_\mathfrak{X}/\mathfrak{f}_0)$ is reduced. (Cf. [EGA 01, p. 172 (7.1.6)].)

Next, let $k$ be a perfect field of characteristic $p \geq 0$. For $p > 0$ let $W(k)$ be the ring of (infinite) Witt vectors with coefficients in $k$; and for $p = 0$ let $W(k)$ be the field $k$ itself. $W(k)$ is complete for the topology defined by the ideal $pW(k)$; the corresponding formal scheme $\text{Spf}(W(k))$ will be denoted by $W_k$.

(2.2) In what follows we consider a triple $(\mathfrak{X}, k, f)$ with:

(i) $\mathfrak{X}$ a weakly noetherian formal scheme.

(ii) $k$ a perfect field of characteristic $p \geq 0$.

(iii) $f: \mathfrak{X} \to W_k$ a morphism of formal schemes such that for every ideal of definition $\mathfrak{f}$ of $\mathfrak{X}$, the induced map of schemes

$$f_\mathfrak{f}:(\mathfrak{X}, \mathcal{O}_\mathfrak{X}/\mathfrak{f}) \to \text{Spec}(W(k))$$

is proper. (1)

Remarks. Morphisms $f: \mathfrak{X} \to W_k$ are in one-one correspondence with continuous homomorphisms $i:W(k) \to H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ [EGA 01, p. 407, (10.4.6)] (2). The above map $f_\mathfrak{f}$ corresponds to the composed $i$.

(1) For (iii) to hold it suffices that $f_\mathfrak{f}$ be proper for one $\mathfrak{f}$ (cf. (2.6) below).

(2) The existence of such an $i$ implies that $p$ is topologically nilpotent in $H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ (since the image of a topologically nilpotent element under a continuous homomorphism is again topologically nilpotent). On the other hand, if $p$ is topologically nilpotent in $H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$, then clearly every ring homomorphism $W(k) \to H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is continuous.
homomorphism

\[ W(k) \xrightarrow{1} H^0(X, \mathcal{O}_X) \xrightarrow{\text{canonical}} H^0(X, \mathcal{O}_X/f). \]

It is practically immediate that \( f_/(1) \) is supported in the closed point of \( \text{Spec}(W(k)) \).

**Example.** Let \( R \) be a complete noetherian local ring with maximal ideal \( M \) and residue field \( k \) (perfect, of characteristic \( p \neq 0 \)); let \( g:X \to \text{Spec}(R) \) be a proper map; and let \( \mathcal{X} \) be the formal completion of \( X \) along the closed fibre \( g^{-1}(\{M\}) \). The structure theory of complete local rings gives the existence of a (continuous) homomorphism \( W(k) \to R \); composing with the map

\[ R \to H^0(X, \mathcal{O}_X) \]

determined by \( g \), we obtain \( i:W(k) \to H^0(X, \mathcal{O}_X) \), whence a triple \((\mathcal{X}, k, f)\) as above.

**2.3** For any \( k \)-algebra \( A \) let \( W_n(A) \) (resp. \( W(A) \)) be the ring of Witt vectors of length \( n \) (resp. of infinite length) with coefficients in \( A \). (\( W_n(A) = W(A) = A \) if \( p = 0 \).) We consider \( W_n(A) \) to be a discrete topological ring, and give \( W(A) \) the topology for which \( K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots \) is a fundamental system of neighborhoods of \( 0 \), \( K_n \) being the kernel of the canonical map \( W(A) \to W_n(A) \) (\( n \geq 1 \)); then, in the category of topological rings,

\[ W(A) = \lim_{\text{proj}} W_n(A). \]
It is not hard to see that $K^n_{21} = p K^n_{1}$, whence

$$K^n_{1} = p^n K^n_{1} \subseteq K^n_1;$$

so $W(A)$ is an "admissible" ring, and we may let $B_A$ be the affine formal scheme

$$B_A = \text{Spf}(W(A)).$$

In particular, for $A = k$, we get the same $B_k$ as in (2.1). If $B$ is an $A$-algebra, then $W(B)$ is in an obvious way a topological $W(A)$-algebra, so that $B_A$ varies functorially with $A$.

With $f: \mathfrak{X} \to B_k$ as in (2.2), we set

$$\mathfrak{X}_A = \mathfrak{X} \times_{B_k} B_A = \mathfrak{X} \otimes_{W(k)} W(A)$$

(product in the category of formal schemes). We have then the covariant functor of $k$-algebras

$$A \to \text{Pic}(\mathfrak{X}_A).$$

What we show below is that the fpqc sheaf $\mathcal{P}$ associated to this functor is a $k$-group scheme, and that furthermore the canonical map $\text{Pic}(\mathfrak{X}_A) \to \mathcal{P}(A)$ is bijective if $A$ is an algebraically closed field.

Example (continued from (2.2)). Suppose that $\mathfrak{X}$ is obtained from a proper map $g: X \to \text{Spec}(R)$ as in the example of (2.2). For
any k-algebra A, setting $R_A = R \hat{\otimes}_{W(k)} W(A)$ (completed tensor product, R being topologized as usual by its maximal ideal M), we have

$$\mathfrak{f}_A = \mathfrak{f} \hat{\otimes}_{W(k)} W(A) = \mathfrak{f} \hat{\otimes}_R R_A.$$ 

Now if A is a perfect field, then $R_A$ has the following properties, which characterize $R_A$ as an R-algebra (up to isomorphism): $R_A$ is a complete local noetherian flat R-algebra such that

$$R_A/M_{R_A} \cong A \quad (\text{cf. [EGA 01, p. 190, (7.7.10)] and [EGA 0IV, (19.7.2)]}).$$

Furthermore, $\mathfrak{f}_A$ is then the completion of the scheme $X_A = X \hat{\otimes}_R R_A$ along the closed fibre of the projection $g_A: X_A \to \text{Spec}(R_A)$. Hence Grothendieck's algebrization theorem [EGA III, (5.1.6)] gives that "completion" is an equivalence from the category of coherent $\mathfrak{f}_{X_A}$-modules to the category of coherent $\mathfrak{f}_A$-modules. Since an $\mathfrak{f}_{X_A}$-module is invertible if and only if so is its completion\(^{(3)}\), we deduce a natural isomorphism

$$\text{Pic}(X_A) \cong \text{Pic}(\mathfrak{f}_A).$$

Hence, restricting our attention to those A which are algebraically closed fields, we will have an A-functorial isomorphism

$$\text{Pic}(X_A) \cong P(A).$$

\(^{(3)}\)This follows easily from the fact that the completion $\hat{B}_I$ of a noetherian ring B w.r.t. an ideal I is faithfully flat over the ring of fractions $B_{1+I}$, so that if J is a B-ideal with $J\hat{B}_I$ a projective $\hat{B}_I$-module, then $J\hat{B}_{1+I}$ is a projective $B_{1+I}$-module.
This gives us the diagram (1.1) which is needed in the last step of the proof of Theorems 1 and 1'.

(2.4) We fix a fundamental system $\mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \ldots$ of defining ideals of $\mathfrak{x}$, and for $n \geq 0$ let $X_n$ be the scheme $(\mathfrak{x}, \mathfrak{g}_n/\mathfrak{g}_n)$. For any $k$-algebra $A$, let $X_n, A$ be the scheme

$$X_n, A = X_n \otimes W(k) W(A).$$

The ringed spaces $X_0, A, X_1, A, \ldots, X_n, A, \ldots$ and $X_A$ all have the same underlying topological space, say $X$, and on this space $X$ we have $\mathfrak{g}_A = \lim_{n} \mathfrak{g}_n, A$. Hence there is a natural map

$$\text{Pic}(X_A) \rightarrow \lim_{n} \text{Pic}(X_n, A).$$

(*)

LEMMA. Let $A$ be a $k$-algebra, and if $p > 0$ assume that $A^p = A$ (i.e. the Frobenius endomorphism $x \rightarrow x^p$ of $A$ is surjective). Then the above map (*) is bijective.

Remark. When $p > 0$ and $A^p = A$, or when $p = 0$, then $X_n, A = X \otimes W(k) W(A)$.

Proof of Lemma. Say that an open subset $U$ of $X$ is affine if $(U, \mathfrak{g}_A | U)$ is an affine formal scheme. The affine open sets form a base for the topology of $X$.

For each $n$, let $\mathfrak{m}_n$ be the sheaf of multiplicative units in the sheaf of rings $\mathfrak{g}_n, A$ (on the topological space $X$) and let
\[ \mathcal{F} = \lim_{\rightarrow n} \mathcal{F}_n = \text{sheaf of units in } \Omega_{X_A} \]

For \( m \geq n \), the kernel of \( \Omega_{X_m, A} \rightarrow \Omega_{X_n, A} \) is nilpotent; so a simple argument ([L2, Lemma (7.2)], with the Zariski topology in place of the étale topology) shows that for affine \( U \) the canonical maps

\[ H^i(U, \mathcal{F}_m) \rightarrow H^i(U, \mathcal{F}_n) \]

are bijective if \( i > 0 \), and surjective if \( i = 0 \). Applying [EGA 0III, (13.3.1)], we deduce that for all \( i > 0 \), the maps

\[ H^i(X, \mathcal{F}) \rightarrow \lim_{\rightarrow n} H^i(X, \mathcal{F}_n) \]

are surjective. Furthermore, in order that

\[ H^1(X, \mathcal{F}) \rightarrow \lim_{\rightarrow n} H^1(X, \mathcal{F}_n) \]

be bijective, it is sufficient that the inverse system \( H^0(X, \mathcal{F}_n) \) satisfies the Mittag-Leffler condition (ML); and for this it is enough that the inverse system \( H^0(X, \Omega_{X_n, A}) \) should satisfy (ML); that is, for each fixed \( n \), if \( I_{mn} \) (\( m \geq n \)) is the image of \( H^0(X, \Omega_{X_m, A}) \rightarrow H^0(X, \Omega_{X_n, A}) \), then the sequence

\[ I_{n,n} \supseteq I_{n+1,n} \supseteq I_{n+2,n} \supseteq \ldots \]
should **stabilize** (i.e. \( I_{N,n} = I_{N+1,n} = I_{N+2,n} = \ldots \) for some \( N \)).

For \( p > 0 \) it is shown in [L2, Corollary (0.2) and Theorem (2.4)] that the fpqc sheaf \( H_n \) associated to the functor

\[
A \mapsto H^0(X, \mathcal{O}_X^n, A)
\]

(of \( k \)-algebras \( A \)) is an **affine algebraic \( k \)-group**; furthermore [ibid, Corollary (4.4)] the canonical map

\[
H^0(X, \mathcal{O}_X^n, A) \rightarrow H_n(A)
\]

is **bijective** whenever \( A^p = A \); and finally, for \( m \geq n \), if \( I_{mn} \) is the image (in the category of algebraic \( k \)-groups) of the natural map \( H_m \rightarrow H_n \), and if \( A^p = A \), then the canonical map

\[
H_m(A) \rightarrow I_{mn}(A)
\]

is **surjective**, so that \( I_{mn} = I_{mn}(A) \) [cf. ibid, last part of proof of (6.3)]. Similar facts when \( p = 0 \) are well-known (and more elementary).

Now the sequence

\[
I_{n,n} \supseteq I_{n+1,n} \supseteq I_{n+2,n} \supseteq \ldots
\]

of closed subgroups of \( H_n \) must stabilize, whence so must the sequence (**). Q.E.D.
(2.5) Before stating the basic existence theorem we need some more notation. For any scheme $Y$, $\text{Br}(Y)$ will be the cohomological Brauer group of $Y$:

$$\text{Br}(Y) = H^2_{\text{étale}}(Y, \text{multiplicative group}).$$

For any ring $R$ we set:

$$\text{Br}(R) = \text{Br}(	ext{Spec}(R))$$

$$\text{Pic}(R) = \text{Pic}(	ext{Spec}(R))$$

$$R_{\text{red}} = R/\text{nilradical of } R.$$

For any defining ideal $J$ of $X$ and any $k$-algebra $A$:

$$X_J = \text{the scheme } (X, \mathcal{O}_X/J)$$

$$X_J, A = X_J \otimes_{W(k)} W(A).$$

Finally, we set

$$k_0 = H^0(X_{\text{red}}, \mathcal{O}_{X_{\text{red}}}).$$

Since $X_{\text{red}}$ is proper over $k$ (cf (2.2)), therefore $k_0$ is a finite product of finite field extensions of $k$.

Now for any $J$, we have (cf (2.2)) a proper map

$$f_J : X_J \to \text{Spec}(W(k))$$

whose image is supported in the closed point of $\text{Spec}(W(k))$. 
Hence, when \( p > 0 \), \([L2, \text{Theorem (7.5)}]\) gives us a \( k \)-group-scheme \( P_j \) and, for all \( k \)-algebras \( A \) with \( A^p = A \), an exact \( A \)-functorial sequence

\[
0 \to \text{Pic}(k_0 \otimes_k A_{\text{red}}) \to \text{Pic}(\mathcal{X}_j A) \to P_j(A) \to \text{Br}(k_0 \otimes_k A_{\text{red}}) \to \text{Br}(\mathcal{X}_j A).
\]

A similar result is well-known for \( p = 0 \), or more generally when \( p\mathcal{X}_j = (0) \), with no condition on \( A \), since then \( \mathcal{X}_j \) is proper over the field \( k \) (cf \([GR, \text{Cor. 5.3}]\)).

Also, if \( j \subseteq j' \), then the canonical map

\[
P_j \to P_{j'}
\]

is affine (\([SGA 6, \text{Exposé XII, Prop. (3.5)}]\) when \( p = 0 \), and \([L2, \text{Prop. (2.5)}]\) when \( p > 0 \)). Thus \( P = \lim P_j \) exists as a \( k \)-group-scheme (cf. \([EGA IV, \S 8.2]\)).

Now, in view of Lemma (2.4), a simple passage to inverse limits gives the desired result:

**THEOREM.** There exists a \( k \)-group scheme \( P \), and for \( k \)-algebras \( A \) such that \( A^p = A \) (the condition \( A^p = A \) is vacuous when \( p = 0 \)) an exact sequence, varying functorially with \( A \),

\[
0 \to \text{Pic}(k_0 \otimes_k A_{\text{red}}) \to \text{Pic}(\mathcal{X}_A) \to P(A) \to \bigcap_j \ker[\text{Br}(k_0 \otimes_k A_{\text{red}}) \to \text{Br}(\mathcal{X}_j A)].
\]
COROLLARY. If \( A \) is an algebraically closed field, then the above map \( \text{Pic}(\mathcal{I}_A) \to P(A) \) is bijective.

For, then \( \text{Pic}(k_0 \otimes_k A_{\text{red}}) = \text{Br}(k_0 \otimes_k A_{\text{red}}) = (0) \). (4)

Remarks. 1. The \( k \)-group-scheme \( P \) is uniquely determined by the requirements of the Theorem. Indeed, since for every \( k \)-algebra \( A \) there exists a faithfully flat \( A \)-algebra \( \tilde{A} \) with \( \tilde{A}^P = \tilde{A} \) [L2, Lemma (0.1)], and since every element in \( \text{Pic}(k_0 \otimes_k A_{\text{red}}) \) or in \( \text{Br}(k_0 \otimes_k A_{\text{red}}) \) is locally trivial for the étale topology on \( A \), it follows easily that \( P \) is the fpqc sheaf associated to the functor \( A \to \text{Pic}(\mathcal{I}_A) \) of \( k \)-algebras \( A \).

2. \( P^0 \), the connected component of zero in \( P \), is described in (3.2) below. The remarks following (1.2) suggest that the following conjecture - or some variant - should hold:

**Conjecture:** \( P^0 \) is infinitesimal if and only if the natural (split injective) map

\[
\text{Pic}(\mathcal{I}) \to \text{Pic}(\mathcal{I} \otimes_W W[[T]]) \quad (W = W(k))
\]

is bijective.

(4) The Corollary, which is what we need for Theorems 1 and 1', could be proved more directly, using [L2, §1, comments on part II]; then we could do without our Lemma (2.4), and without introducing "\( \text{Br} \)". In a similar vein it can be deduced from the Theorem - or shown more directly - that if \( K \) is a normal algebraic field extension of \( k \) such that every connected component of \( \mathcal{I}_{\text{red}} \) has a \( K \)-rational point, and if \( A \) is any perfect field containing \( K \), then \( \text{Pic}(\mathcal{I}_A) \to P(A) \) is bijective.
(2.6) (Appendix to §2). The following proposition is meant to give a more complete picture of how our basic data \((I, k, f)\) can be defined. It will not be used elsewhere in this paper.

To begin with, observe that if \((I, k, f)\) is as in (2.2), then \(f\) induces a proper map

\[ f_0: (I, \theta_I/f_0) = I_{\text{red}} \to \text{Spec}(k) \]

(cf. (2.2)). Hence \(H^0(I, \theta_I)\) is a finite \(k\)-module (equivalently: a finite \(W(k)\)-module) and - a fortiori - a finite \(H^0(I, \theta_I)\) module. Conversely:

**PROPOSITION.** Let \(I\) be a weakly noetherian formal scheme, and assume that the \(H^0(I, \theta_I)\)-module \(H^0(I, \theta_I)\) is finitely generated. Let \(k\) be a perfect field of characteristic \(p \geq 0\), and let

\[ f_0: I_{\text{red}} \to \text{Spec}(k) \]

be a proper map of schemes. Then \(f_0\) extends (uniquely, if \(p > 0\)) to a map of formal schemes \(f: I \to \mathbb{B}_k\). Furthermore, all the maps \(f_\gamma\) (cf. (2.2)) are proper.

**Proof.** (Sketch) \(f_0\) corresponds to a homomorphism \(i_0: k \to H^0(I, \theta_I)\); the problem is to lift \(i_0\) to a continuous homomorphism

\[ i: W(k) \to H^0(I, \theta_I). \]
Let $\mathcal{J}_0 \supseteq \mathcal{J}_1 \supseteq \mathcal{J}_2 \supseteq \ldots$ be a fundamental system of defining ideals of $\mathcal{X}$ (cf. (2.1)), and let $H_0 = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/H^0(\mathcal{X}, \mathcal{J}_0)$. We will show below that:

\[(*) \text{ the canonical map } H_0 \xrightarrow{n} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}_{\text{red}}}) \text{ is bijective.} \]

Then the existence of the lifting $i$ follows (since $W(k)$ is formally smooth over its subring $\mathbb{Z}_p$) from [EGA 0 IV, (19.3.10)] (with $\mathcal{J} = H^0(\mathcal{X}, \mathcal{J}_0)$). For the uniqueness when $p > 0$, cf. [loc. cit. (20.7.5) or (21.5.3)(ii)]. (Or else note that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}_{\text{red}}})$, being reduced and finite over $k$, is perfect, and argue as in [SR, p. 48, Prop. 10], using the following easily proved fact in place of [ibid., p. 44, Lemme 1]:

If $a, b \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ satisfy $a \equiv b \pmod{H^0(\mathcal{X}, \mathcal{J}_n)}$, then for some $N$ depending only on $n$ we have

$$a^p \equiv b^p \pmod{H^0(\mathcal{X}, \mathcal{J}_{n+1})}.$$ 

Now $(*)$ simply says that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}_{\text{red}}})$ is surjective, and to prove this we may assume that $\mathcal{X}$ is connected; then $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}_{\text{red}}})$, being finite over $k$, is a perfect field, as is its subring $H_0$ (since $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}_{\text{red}}})$ is finite over $H_0$, by assumption), say $H_0 = K$. As above, the identity map $K \to K$ lifts to a homomorphism $W(K) \to H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and thereby, for every ideal of definition $\mathcal{J}$, the scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}/\mathcal{J}})$ is a $W(K)$-scheme. For $\mathcal{J} = \mathcal{J}_0$, the structural map $(\mathcal{X}, \mathcal{O}_{\mathcal{X}/\mathcal{J}_0}) \to \text{Spec}(W(K))$ factors as
\((\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{I}_0) = \mathfrak{X}_{\text{red}} \to \text{Spec}(\mathbb{H}^0(\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}_{\text{red}}})) \xrightarrow{\text{finite}} \text{Spec}(K) \subseteq \text{Spec}(W(K))\).

Note that \(\mathfrak{X}_{\text{red}}\), being proper over \(k\), is proper over \(\mathbb{H}^0(\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}_{\text{red}}})\), and hence also over \(K\). Arguing as below, we see that 

\((\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{I}_n)\) is proper over \(W(K)\), whence the kernel of 

\(\pi_n : \mathbb{H}^0(\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{I}_n) \to \mathbb{H}^0(\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{I}_0)\) is a \(W(K)\)-module of \(\text{finite length}\).

So by [EGA 0III, (13.2.2)], \(\pi = \lim \pi_n\) will be surjective if \(\pi_n\) is surjective for all \(n\). Let us show more generally for any scheme map \(\phi : X \to \text{Spec}(W(K))\) that if \(\phi\) induces a \(\text{proper}\) map 

\[Y = X_{\text{red}} \to \text{Spec}(K) \subseteq \text{Spec}(W(K))\]

then \(\mathbb{H}^0(X, \mathfrak{O}_X) \to \mathbb{H}^0(Y, \mathfrak{O}_Y)\) is surjective.

Let \(\bar{K}\) be an algebraic closure of \(K\). Then \(W(\bar{K})\) is a \(\text{faithfully flat}\) \(W(K)\)-algebra. In view of [EGA III, (1.4.15)].

(K"unneth formula for flat base change) and the fact that

\[Y \otimes_{\text{W}(K)} \text{W}(\bar{K}) = Y \otimes_{\text{K}} \bar{K}\]

is reduced (\(K\) being perfect), we may replace \(X\) by \(X \otimes_{\text{W}(K)} \text{W}(\bar{K})\), i.e. we may assume that \(K\) is algebraically closed. But then 
\(\mathbb{H}^0(Y, \mathfrak{O}_Y)\) is a product of copies of \(X\), one for each connected component of \(Y\), so the assertion is obvious.

It remains to be shown that the maps \(f_{\mathfrak{X}}\) are all proper. 
\((\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{f})\) is noetherian, and \(\mathfrak{X}_{\text{red}} = (\mathfrak{X}, \mathfrak{O}_{\mathfrak{X}}/\mathfrak{f})_{\text{red}}\). By [EGA II (5.4.6) and EGA 01, p. 279, (5.3.1)(vi)] it suffices to show
that \( f \) is locally of finite type; so what we need is that if \( A \) is a noetherian \( W(k) \)-algebra with a nilpotent ideal \( N \) such that \( A/N \) is finitely generated over \( W(k) \), then also \( A \) is finitely generated over \( W(k) \). But if \( a_1, a_2, \ldots, a_r \) in \( A \) are such that their images in \( A/N \) are \( W(k) \)-algebra generators of \( A/N \), and if \( b_1, b_2, \ldots, b_s \) are \( A \)-module generators of \( N \), then it is easily seen that

\[
A = W(k)[a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s].
\]

Q.E.D.

§3. Structure of inverse limits of locally algebraic k-groups.

In this section, we establish (1.2)- and a little more- for any group-scheme \( P \) of the form \( \lim \rightarrow P_n \), where \((P_n, f_{mn})\)
\((n, m, \text{non-negative integers, } n \geq m)\) is an inverse system of locally algebraic k-groups (\( k \) a field), the maps \( f_{mn}: P_n \rightarrow P_m \)
\((n \geq m)\) being affine (cf. [EGA IV, §8.2]). (Note that the group-scheme \( P \) of §(2.5) is of this form.) This is more or less an exercise, and the results are presumably known, but I could not find them recorded anywhere.

(3.1) By [SGA 3, p. 315], \( f_{mn}: P_n \rightarrow P_m \) \((n \geq m)\) factors uniquely as

\[
P_n \xrightarrow{u} P_{mn} \xrightarrow{v} P_n
\]

where \( v \) is a closed immersion and \( u \) is affine, faithfully
flat, and finitely presented. \((P_{mn} \) is the image, or coimage, of \(f_{mn}\).) For \(n_1 \geq n_2\), \(P_{mn_1}\) is a closed subgroup of \(P_{mn_2}\), and we can set

\[
\hat{P}_m = \bigcap_{n \geq m} P_{mn} = \lim_{\leftarrow n \geq m} P_{mn}.
\]

\(\hat{P}_m\) is a closed subgroup of \(P_m\), its defining ideal in \(\mathfrak{O} P_m\) being the union of the defining ideals of the \(P_{mn}\). Clearly \(f_{mn}\) induces a map \(\hat{f}_{mn}: \hat{P}_n \to \hat{P}_m\), so we have an inverse system \((\hat{P}_n, \hat{f}_{mn})\).

**PROPOSITION.** (i) \(P\) (together with the natural maps \(f_{mn}: P_n \to P_m\)) is equal to \(\lim_{\leftarrow n} P_n\).

(ii) The maps \(\hat{f}_{mn}: \hat{P}_n \to \hat{P}_m\) and \(\hat{f}_m: P \to \hat{P}_m\) are affine, faithfully flat, and universally open.

(iii) If \(K\) is any algebraically closed field containing \(k\), then

\[
\hat{f}_m(K): P(K) \to \hat{P}_m(K)
\]

is surjective.

(iv) \(\ker(\hat{f}_{mn})\) is a closed subgroup of \(\ker(f_{mn})\).

**Proof.** (i) and (iv) are left to the reader. It is clear that all the maps \(\hat{f}_{mn}\) and \(\hat{f}_m\) are affine. We show below that \(\hat{f}_m\) is faithfully flat for all \(m\). Since \(\hat{f}_m = \hat{f}_{mn} \circ \hat{f}_n\) for \(n \geq m\), it will follow that \(\hat{f}_{mn}\) is faithfully flat [EGA IV, (2.2.13)]. This implies that \(\hat{f}_{mn}\) is universally open [EGA IV,
(2.4.6)] and hence so is \( f_m \) [EGA IV, (8.3.8)], proving (ii).

As for (iii), since \( f_{st} \) is locally of finite type and surjective, it follows that \( f_{st}(K) \) is surjective for all \( t \geq s \); in particular, \( f_{n,n+1}(K) \) is surjective for all \( n \geq m \), so any element of \( \hat{P}_n(K) \) can be lifted back to \( P(K) = \lim_{n \geq m} \hat{P}_n(K) \), i.e. \( \hat{f}_m(K) \) is surjective.

So let us show that \( \hat{f}_m \) is faithfully flat. Let \( y \in \hat{P}_m \), and let \( U \) be an affine open neighborhood of \( y \) in \( P_m \). Since \( U \) is noetherian, we see that for some \( n_0 \)

\[
\hat{P}_m \cap U = P_{mn} \cap U \quad \text{for all } n \geq n_0.
\]

But \( f_{mn} \) induces a faithfully flat map

\[
P_n \times_{P_m} U \rightarrow P_{mn} \times_{P_m} U = \hat{P}_m \cap U \quad \text{for } n \geq n_0.
\]

Since \( P_n \times_{P_m} U \) and \( \hat{P}_m \cap U \) are affine, and since for any ring \( R \) an inductive limit of faithfully flat \( R \)-algebras is still a faithfully flat \( R \)-algebra, we conclude that

\[
P \times_{\hat{P}_m} (\hat{P}_m \cap U) = P \times_{P_m} U = \lim_{n \geq n_0} (P_n \times_{P_m} U)
\]

is faithfully flat over \( \hat{P}_m \cap U \). Thus \( \hat{f}_m \) is faithfully flat.

(3.2) Because of Proposition (3.1), we can assume from now on that \( P_m = \hat{P}_m \) (so that all the maps \( f_{mn} (= \hat{f}_{mn}) \) are
faithfully flat etc. etc.). Furthermore, certain additional conditions which may be imposed on the original $f_{mn}$ (for example the condition that $\ker(f_{mn})$ be unipotent) will not be destroyed by this replacement of $P_m$ by $P_m$ (because of (iv) in Prop. (3.1)).

We examine now the connected component of the zero-point of $P$. Let $P_n^0$ be the open and closed subgroup of $P_n$ supported by the connected component of zero in $P_n$ (cf. [DG, ch. II, §5, no. 1]). Then $f_{mn}:P_n \rightarrow P_m (n \geq m)$ induces a map $f_{mn}^0:P_n^0 \rightarrow P_m^0$, so we have an inverse system $(P_n^0, f_{mn}^0)$. Set $P^0 = \lim_{\leftarrow} P_n^0$.

**PROPOSITION.** (i) The maps $f_{mn}^0$ are affine, faithfully flat and finitely presented; and $\ker(f_{mn}^0)$ is a closed subgroup of $\ker(f_{mn})$.

(ii) $P^0$ is a closed irreducible subgroup of $P$, and the underlying subspace of $P^0$ is the connected component of zero in $P$. Furthermore, if $x \in P^0$, then the canonical map of local rings $\mathcal{O}_P,x \rightarrow \mathcal{O}_{P^0,x}$ is bijective.

**Proof.** (i) is immediate except perhaps for the surjectivity of $f_{mn}^0$, which follows from the fact that the (topological) image of $f_{mn}^0$ is open [EGA IV, (2.4.6)] and closed [DG, p. 249, (5.1)].

As for (ii), it is clear that $P^0$ is a closed subgroup of $P$; and if $Q$ is any connected subspace of $P$ containing zero, then $f_n(Q) \subseteq P_n^0$ for all $n$ ($f_n:P \rightarrow P_n$ being the natural map) whence $Q \subseteq P^0$ (since $P^0 = \lim_{\leftarrow} P_n^0$ in the category of
topological spaces [EGA IV, 8.2.9]). So for the first assertion of (ii), it remains to be shown that \( P^0 \) is irreducible (hence connected). For this it suffices to show that \( P^0 \) is covered by open irreducible subsets, any two of which have a non-empty intersection. \( P^0 \), being irreducible, has such a covering by irreducible affine subsets, and we can cover \( P^0 \) by their inverse images. Since all the maps \( f^0_{\alpha n} \) are affine and each \( P^0_n \) is irreducible, we need only check that a direct limit of rings with irreducible spectrum has irreducible spectrum. But this is easily seen, since "A has irreducible spectrum" means that "for \( a, b \in A \), \( ab \) is nilpotent ⇔ either \( a \) or \( b \) is nilpotent".

Finally, for \( x \in P^0 \), we have

\[
\Theta_{P^0,x} = \lim_{\to} \Theta_{P^0_n,f_n(x)} = \lim_{\to} \Theta_{P^0_n,f_n(x)} = \Theta_{P^0,x}
\]

Q.E.D.

Remark. Though \( P^0 \) is not algebraic over \( k \) in general, it may nevertheless have certain finite-dimensional structural features. For example, when \( k \) is perfect, if \( A_n \) is the abelian variety which is a quotient of \( (P^0_n)_{\text{red}} \) by its maximal linear subgroup \( L_n \) (structure theorem of Chevalley) then \( f_{mn} \) \((n \geq m)\) induces an epimorphism \( A_n \to A_m \), with infinitesimal kernel. If furthermore the kernel of \( f_{mn} \) is unipotent (as would be the case, e.g. in (2.5) [L2; Cor. (2.11)]), then, writing

\[
L_n = M_n \times U_n \quad (M_n \text{ multiplicative}, \; U_n \text{ unipotent})
\]

we find that \( f_{mn} \) induces an isomorphism \( M_n \to M_m \).
For each $n$, let $\pi_0(P_n)$ be the étale $k$-group $P_n/P_n^0$ (cf. [DG, p. 237, Prop. (1.8)]). The natural map $q_n : P_n \to \pi_0(P_n)$ is faithfully flat and finitely presented (loc. cit). $f_{mn}$ induces a map $\pi_0(f_{mn}) : \pi_0(P_n) \to \pi_0(P_m)$, so we have an inverse system $(\pi_0(P_n), \pi_0(f_{mn}))$. We set $\pi_0(P) = \lim_{\leftarrow} \pi_0(P_n)$.

**Proposition.** (i) The maps $\pi_0(f_{mn})$ are finite, étale, surjective; and $\ker(\pi_0(f_{mn}))$ is a quotient of $\ker(f_{mn})$.

(ii) The canonical map $q : P \to \pi_0(P)$ is faithfully flat and quasi-compact, with kernel $P^0$ (so that the sequence

$$0 \to P^0 \to P \to \pi_0(P) \to 0$$

is exact in the category of fpqc sheaves). The (topological) fibres of $P \to \pi_0(P)$ are irreducible, and they are the connected components of $P$. For any $x \in P$, the canonical map of local rings $\theta_{P,x} \to \theta_{\pi_0(P),x}$ is bijective. If $K$ is an algebraically closed field containing $k$, then $P(K) \to \pi_0(P)(K)$ is surjective.

**Proof.** (i) Consider the commutative diagram (with $n \geq m$):

$$
\begin{array}{ccc}
0 & \longrightarrow & P_n^0 \\
\downarrow f_{mn}^0 & & \downarrow f_{mn} \\
0 & \longrightarrow & P_m^0 \\
\end{array}
\begin{array}{ccc}
& & \\
\downarrow & & \downarrow \\
0 & \longrightarrow & P_m \\
& & \\
\end{array}
\begin{array}{ccc}
q_n & \longrightarrow & \pi_0(P_n) \\
\downarrow & & \downarrow \\
q_m & \longrightarrow & \pi_0(P_m) \\
\end{array}
\begin{array}{ccc}
& & \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0 \\
\end{array}
$$

The maps in the rows are the natural ones, and the rows are exact.
in the category of fppf sheaves (when we identify k-groups with functors of k-algebras...). Since $f_{mn}^\circ$ is an epimorphism of fppf sheaves (Prop. (3.2)), so therefore is the natural map $\ker(f_{mn}) \to \ker(\pi_0(f_{mn}))$, and we have the second assertion of (i).

$f_{mn}$, $q_m$, and $q_n$ are all faithfully flat - hence surjective and quasi-compact, and then so is $\pi_0(f_{mn})$. Since $\pi_0(P_n)$ and $\pi_0(P_m)$ are étale over $k$, therefore the map $\pi_0(f_{mn})$ is étale. Thus the kernel of $\pi_0(f_{mn})$-being quasi-compact and étale over $k$ - is finite over $k$, and it follows that the map $\pi_0(f_{mn})$ is finite.

(ii) For the last assertion, note that we have an inverse system of exact sequences

$$0 \to P_0^O(k) \to P_n(k) \to \pi_0(P_n)(k) \to 0$$

and that $P_{n+1}(k) \to P_n^O(k)$ is surjective for all $n$ (Prop. (3.2)); so on passing to the inverse limit we obtain an exact sequence

$$0 \to P_0^O(k) \to P(k) \to \pi_0(P)(k) \to 0.$$  

The exactness of $0 \to P_0^O \to P \xrightarrow{q} \pi_0(P)$ is straightforward. To show that $q$ is flat let $x \in P$, $y = q(x)$, and let $x_n$, $y_n$ be their images in $P_n$, $\pi_0(P_n)$ respectively. Then $\vartheta_\pi_{P_n}x_n$ is flat over $\vartheta_\pi_{\pi_0(P_n)}y_n$, and passing to inductive limits, we see that $\vartheta_\pi_{P,x}$ is flat over $\vartheta_\pi_{\pi_0(P),y}$. Next let $z \in \pi_0(P)$, let
Let \( z \) be the image of \( z \) in \( \pi_0(P_n) \), and let \( Q = q^{-1}(z) \).

\( Q_n = q_n^{-1}(z) \). Note that \( Q_n \) is irreducible, and is a connected component of \( P_n \). The \( Q_n \) form an inverse system of schemes, in which the transition maps are affine, and

\[
Q = \lim_{\leftarrow} Q_n.
\]

We show next that \( Q_n \to Q_m \) is surjective; then it follows that \( Q \) is non-empty (so that \( q \) is surjective - hence faithfully flat) and the proof of Prop. (3.2) (ii) can be imitated to give all the assertions about the fibres of \( q \).

Let \( \bar{k} \) be the algebraic closure of \( k \). By a simple translation argument, we deduce from the surjectivity of \( p_0^0 \to p_0^m \) that every component of \( Q_n \otimes_k \bar{k} \) maps surjectively onto a component of \( Q_m \otimes_k \bar{k} \); since every component of \( Q_m \otimes_k \bar{k} \) projects surjectively onto \( Q_m \), we find that \( Q_n \to Q_m \) is indeed surjective.

It remains to be seen that \( q \) is quasi-compact. The fibres of the maps \( \pi_0(f_n):\pi_0(P) \to \pi_0(P_n) \) (\( n \geq 0 \)) form a basis of open sets on \( \pi_0(P) \) (since \( \pi_0(P_n) \) is discrete as a topological space); furthermore these fibres are quasi-compact (since \( \pi_0(f_n) \) is an affine map), and their inverse images in \( P \) are quasi-compact (the affine map \( P \to P_n \) and the finitely presented map \( P_n \to \pi_0(P_n) \) are both quasi-compact, so the composed map \( P \to \pi_0(P_n) \) is quasi-compact); it follows that \( q \) is quasi-compact.

Q.E.D.
Remarks.

1. Say that a $k$-group $Q$ is \textit{pro-étale} if it is of the form $\lim_\leftarrow Q_n$, where $(Q_n, g_{mn})$ is an inverse system of the type we have been considering, with all the $Q_n$ \textit{étale} over $k$. For example $\pi_0(P)$ is pro-étale. It is immediate that if $Q$ is pro-étale and $f: G \to Q$ is a map of $k$-groups, with $G$ connected, then $f$ is the zero-map. From this we see that, with $P$ as above, \textit{every map of $P$ into a pro-étale $k$-group factors uniquely through $P \to \pi_0(P)$}.

2. Let $(P_n, f_{mn})$ be as above, and assume that the kernel of $f_{mn}$ is \textit{unipotent} for all $m,n$. Set $Q_n = \pi_0(P_n)$, $g_{mn} = \pi_0(f_{mn})$; by (i) of Proposition (3.3), the kernel of $g_{mn}$ is \textit{étale} and also unipotent (i.e. annihilated by $p^t$ for some $t$, with $p = \text{char. of } k$). Assume also that the abelian group $Q_n(\bar{k})$ ($\bar{k} = \text{algebraic closure of } k$) is \textit{finitely generated} (for each $n$). (These assumptions hold in the situation described in (2.5), cf. [L2; Prop. (2.7), Cor. (2.11)].)

Let $Q_n^t$ be the kernel of multiplication by $p^t$ in $Q_n$. Then $Q_n^0 \subseteq Q_n^1 \subseteq Q_n^2 \subseteq \ldots$, and since $Q_n(\bar{k})$ is finitely generated, we have, for large $t$, $Q_n^t = Q_n^{t+1} = \ldots$; so we can set

$$Q_n^{(p)} = \bigcup_t Q_n^t = Q_n^t \text{ for large } t.$$ 

Clearly $Q_n^{(p)}$ is finite \textit{étale} over $k$, and unipotent; and the quotient $R_n = Q_n/Q_n^{(p)}$ is \textit{étale} over $k$. Consider the commutative diagram (n $\geq$ m):
Straightforward arguments give that:

(i) Multiplication by \( p \) in \( R_n \) is a monomorphism.

(ii) \( R_n \to R_m \) is an isomorphism.

(iii) \( Q_n^{(p)} \to Q_m^{(p)} \) is an epimorphism.

Then, passing to the inverse limit, we obtain:

There exists an exact sequence

\[
0 \to Q^{(p)} \to \pi_0(P) \to R \to 0
\]

\( Q^{(p)} \) = inverse limit of unipotent finite étale \( k \)-groups.

\( R = \) étale \( k \)-group such that the abelian group \( R(\bar{k}) \) (\( \bar{k} \) = algebraic closure of \( k \)) is finitely generated and without \( p \)-torsion.

Here \( R \) is already determined by \( P_1 \).
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