ABSOLUTE SATURATION OF ONE-DIMENSIONAL LOCAL RINGS.

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Introduction. A closure operation, *absolute saturation*, is defined within certain commutative rings (Section 1). The definition is closely related to—and was inspired by—Pham and Teissier's definition of "Lipschitz-saturation" [4, p. 650]. The main result (Section 2) is that for a reduced one-dimensional noetherian local ring $\mathfrak{o}$ which is complete, with residue field of characteristic zero, if k is any field of representatives and x is any transversal parameter, then *the* $k((x))$*-saturation* $\tilde{\mathfrak{o}}_{k((x))}$ *of* $\mathfrak{o}$ (cf. [1, p. 968]) *is identical with the absolute saturation of* $\mathfrak{o}$ (*within the integral closure* A *of* $\mathfrak{o}$ *in its total ring of fractions* F).

Some basic results of Zariski's theory of saturation [GTS I, II, III] are direct consequences. First of all: $k((x))$*-saturation is an intrinsic operation on* $\mathfrak{o}$, i.e. it does not depend on the choice of k and x (simply because k and x do not enter into the definition of "absolute saturation"). Furthermore: $\tilde{\mathfrak{o}}_{k((x))}$ can also be characterized intrinsically as being the *smallest* among those rings between $\mathfrak{o}$ and A which are saturated in Zariski's sense, i.e. with respect to some (variable) subfield of F. (This is because any Zariski-saturated subring of A is absolutely saturated (remark (c), Section 1).)

As will be apparent from numerous references, this paper has many points of high-order contact with Zariski's work. I do feel, however, that the above-mentioned main result merits an independent treatment. Enough preliminary material from Zariski's theory is included in Section 0 to make the proof self-contained (modulo standard commutative algebra).

After this paper was completed, I received a preprint of E. Böger [7], in which he obtains still another proof of the intrinsic nature of $k((x))$-saturation. Bögers work makes use of the notion of Lipschitz-saturation, but his results do not contain the theorem given here in Section 2. On the other hand his proof, like Zariski's, shows that saturated rings have many automorphisms.

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Throughout, A will be a noetherian semi-local ring† which is equidimensional of dimension one, complete, reduced (i.e. without nonzero nilpotents), integrally closed in its total ring of fractions F, and which contains a field of characteristic zero. K will be a subfield of F—necessarily of characteristic zero —such that F is finite-dimensional as a K-vector space.

0. Preliminaries. For the most part, the material in this section is taken—with minor alterations—from the aforementioned papers of Zariski.

1. F has just finitely many prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_h$, all of them maximal, and F is canonically isomorphic with $\prod_{i=1}^h F/\mathfrak{p}_i$. For each $i=1, 2, \ldots, h$ the field $F/\mathfrak{p}_i$ is finite over K; hence if Ω is a fixed algebraic closure of K, then the set

$$H_i = \{\text{all } K\text{-algebra homomorphisms } \psi: F \to \Omega \text{ with kernel } \mathfrak{p}_i\}$$

is non-empty and finite. We choose *once and for all* one member of H_i and call it ψ_i. $F_i = \psi_i(F)$ $(\cong F/\mathfrak{p}_i)$ is a subfield of Ω containing K, and *the map*

$$\Psi: F \to \prod_{i=1}^{h} F_i$$

given by

$$\Psi(\eta) = (\eta_1, \eta_2, \ldots, \eta_h)$$

$$(\eta_i = \psi_i(\eta) \text{ for } \eta \in F \text{ and } i = 1, 2, \ldots, h)$$

is a K-algebra isomorphism.

For each $i = 1, 2, \ldots, h$, F_i is the field of fractions of its subring $\psi_i(A)$. Let A_i be the integral closure of $\psi_i(A)$ in F_i. *Then*

$$\Psi(A) = \prod_{i=1}^{h} A_i$$

(*so that actually* $A_i = \psi_i(A)$). [Indeed, if $\eta \in F$ and $\Psi(\eta) \in \prod_{i=1}^h A_i$, then for each i there exists a *monic* polynomial $f_i(X) \in A[X]$ such that $\psi_i(f_i(\eta)) = 0$; if $f = f_1 f_2 \cdots f_h$, then $\psi_i(f(\eta)) = 0$ for *all* i, whence $f(\eta) = 0$, and so $\eta \in A$ (since A is integrally closed in F); from this the assertion is immediate.] For convenience, *we shall identify—via* Ψ*—F with* $\prod_{i=1}^h F_i$ and A with $\prod_{i=1}^h A_i$.

The following proposition is somewhat peripheral—though not entirely

†It is understood that all rings are commutative, with identity; that homomorphisms of rings always preserve the identity; and that subrings of a ring always contain the identity of that ring.

irrelevant—to our objective, which is to prove the theorem in Section 2. The proposition could be omitted, with little subsequent effect, if we simply took its conclusion as an additional condition to be imposed throughout on A and K †.

Proposition (0.1) (*cf.* [*GTS I*, *p.* 578, *Prop.* 1.4]). *If A, K are as above, then A is a finite $A\cap K$-module, and $A\cap K$ is a complete discrete valuation ring with field of fractions K (so that* [3, *p.* 307, *Corollary*] *for some field k of characteristic zero and some element x, $A\cap K$ is the one-dimensional power series ring $k[[x]]$ and $K=k((x))$).*

The proof of (0.1) is based on:

Lemma (0.2) [6, *p.* 60]. *Let R be a* **complete** (*or, more generally,* **henselian** [5, *Section* (30.3)]) *local domain of dimension >0, and let v be a discrete rank one valuation of the field of fractions of R, with valuation ring R_v. Then $R\subseteq R_v$.*

Corollary (0.3) (*F. K. Schmidt*). *If in* (0.2) *R is a complete discrete valuation ring, then $R_v=R$.*

Proof of (0.2). Let $x\in\mathfrak{m}$, the maximal ideal of R. R being henselian, $1+x$ has an Nth root in R for every integer N prime to the characteristic of $R/\mathfrak{m}$; hence $v(1+x)$ is divisible by every such N, so $v(1+x)=0$, and $1+x\in R_v$, i.e. $x\in R_v$; thus $\mathfrak{m}\subseteq R_v$. Furthermore we have for any $y\in R$ and any $n\geqslant 0$ that $y^n x\in\mathfrak{m}\subseteq R_v$, so that $nv(y)+v(x)\geqslant 0$; taking $x\neq 0$ and $n>v(x)$, we see that $v(y)\geqslant 0$, i.e. $y\in R_v$. Q.E.D.

Proof of (0.1). Let $\psi_1, \psi_2,\dots,\psi_h$ be as above. We have already seen that for $i=1,2,\dots,h$, $A_i=\psi_i(A)$ is an integrally closed domain with field of fractions $F_i=\psi_i(F)$; moreover A_i is complete and semi-local (since A is), so A is local [CA p. 283, Cor. 2]; and finally, since A is equidimensional of dimension one, therefore A_i is also of dimension one; thus *A_i is a complete discrete valuation ring*. Since $[F_i:K]<\infty$, therefore *$A_i\cap K$ is a discrete valuation ring with field of fractions K*; and furthermore Corollary (0.3) (with $R=A_i$) implies that *A_i is the integral closure of $A_i\cap K$ in F_i*. Since F_i is *separable* over K (K being of characteristic zero) it follows that *A_i is a finite $A_i\cap K$-module.*

We show next that *the discrete valuation ring $A_i\cap K$ is complete*. The completion B_i of $A_i\cap K$ is a subring of A_i (since $A_i\cap K$ is clearly a subspace of A_i); since A_i is a finite $A_i\cap K$-module, so therefore is B_i. But the maximal ideal of $A_i\cap K$ generates that of B_i, and the residue fields of $A_i\cap K$ and B_i are

†To be precise, this would make the definition of "Zariski-saturated subset of A" (cf. 3 below) more restrictive, thereby weakening Corollaries 2 and 3 in Section 2.

canonically isomorphic; therefore Nákayama's Lemma shows that $A_i \cap K = B_i$, i.e. $A_i \cap K$ is complete.

From Corollary (0.3) we see now that $A_i \cap K = A_j \cap K$ for all $i, j = 1, 2, \ldots, h$.

To complete the proof, we show that $A \cap K = A_i \cap K$ $(i = 1, 2, \ldots, h)$. (Since, as before, $A \cong \prod_{i=1}^{h} A_i$ and, as above, A_i is a finite $A_i \cap K$-module, it will follow that A is a finite $A \cap K$-module.) If $\eta \in A \cap K$, then $\eta = \psi_i(\eta) \in A_i \cap K$, so $A \cap K \subseteq A_i \cap K$. Conversely, if $\eta \in A_i \cap K$, then there exists $\xi \in A$ such that $\psi_j(\xi) = \eta$ $(j = 1, 2, \ldots, h)$ (because $\eta \in A_j \cap K \subseteq A_j$, and $A \cong \prod_{j=1}^{h} A_j$ via the ψ_j). But then $\psi_j(\eta - \xi) = 0$ for all j, so $\eta - \xi = 0$, whence $\eta \in A \cap K$. Q.E.D.

Before proceeding, the reader may find it helpful to refer to Example (0.16) below.

2. Let $K = k((x))$ be as in (0.1), with algebraic closure Ω. Let $L \subseteq \Omega$ be a finite algebraic field extension, necessarily separable, of K. The integral closure R of $k[[x]]$ in L is a finite $k[[x]]$-module, hence a complete semilocal domain [CA p. 276, Theorem 15], i.e. a complete local domain [CA p. 283, Cor. 2], and consequently R is a complete discrete valuation ring. The residue field of R is finite over that of $k[[x]]$; it follows easily that the algebraic closure k_L of k in L is a maximal subfield of R, i.e. a field of representatives [CA pp. 280–281]. So R is the power series ring $k_L[[\tau]]$ for some τ.

For suitable $n > 0$ we have then

$$x = \tau^n \sum_{p=0}^{\infty} a_p \tau^p \qquad (a_p \in k_L,\ a_0 \neq 0).$$

Let k^* be a finite normal field extension of k containing k_L and such that the polynomial $X^n - a_0$ splits into linear factors over k^* (whence k^* contains n distinct nth roots of unity). Setting

$$\begin{aligned} u &= (x / a_0 \tau^n) - 1 \\ &= (a_1/a_0)\tau + (a_2/a_0)\tau^2 + \cdots \end{aligned}$$

we have, in $k^*[[\tau]]$, that $x = t^n$, where

$$\begin{aligned} t &= \tau a_0^{1/n} (1+u)^{1/n} \\ &= \tau a_0^{1/n} \left(1 + \frac{1}{n} u + \frac{1}{2!} \frac{1}{n} \left(\frac{1}{n} - 1 \right) u^2 + \cdots \right) \end{aligned}$$

($a_0^{1/n}$ being some root of $X^n - a_0$ in k^*). Moreover, $k^*[[\tau]] = k^*[[t]]$.

Now take for L the compositum of all the fields $\psi(F)$ as ψ runs through the set H of *all* K-algebra homomorphisms from F into Ω. (Note that H is finite, $H=\cup_{i=1}^{h} H_i$ with H_i as in 1 above, and that for each $\psi \in H$, $\psi(F)$ is a field containing, and finite over, K.) For each $\psi \in H$, $\psi(A)$ is a subring of L which is integral over $k[[x]]$. We conclude:

(0.4) *There exists a power series ring* $R^* = k^*[[t]]$ *and an integer* $n > 0$ *such that*

(i) $x = t^n$;

(ii) k^* *is a finite normal field extension of* k *containing* n *distinct* nth *roots of unity;*

and such that for all K*-homomorphisms* $\psi: F \to \Omega$, *we have*

(iii) $\psi(F) \subseteq F^* = k^*((t))$; *and*

(iv) R^* *is the integral closure of* $\psi(A)$ *in* F^*.

Remarks. (a). F^* is a Galois extension of K: indeed, the degree $[F^*:K]$ is easily seen to be $n[k^*:k]$, and we have *at least that many* automorphisms of F^*/K; namely, for each of the n nth roots of unity α in k^* and for each of the $[k^*:k]$ k-automorphisms σ_0 of k^*, there is a unique K-automorphism σ of F^* such that

$$\text{(0.5)} \qquad \sigma\left(\sum_{p=0}^{\infty} b_p t^p\right) = \sum_{p=0}^{\infty} \sigma_0(b_p)\alpha^p t^p \qquad (b_p \in k^*).$$

The Galois group G of F^*/K consists therefore *precisely* of all σ as in (0.5).

(b) Since F^* admits L-isomorphisms into Ω, *we may as well assume that* $F^* \subseteq \Omega$. Then

$$\text{(0.6)} \qquad H = \{\sigma \circ \psi_i \mid \sigma \in G, i = 1, 2, \ldots, h\}.$$

(H as above; ψ_i as in 1).

(c) For each $i = 1, 2, \ldots, h$, $A_i = \psi_i(A) \subseteq k^*[[t]]$, and since $A = \prod_{i=1}^{h} A_i$ (cf. 1), each element η of A is of the form

$$\text{(0.7)} \qquad \eta = (\eta_1, \eta_2, \ldots, \eta_h)$$

$$= \left(\sum_{p=0}^{\infty} a_{1p} t^p, \sum_{p=0}^{\infty} a_{2p} t^p, \ldots, \sum_{p=0}^{\infty} a_{hp} t^p\right)$$

$$(a_{ip} \in k^* \text{ for } 1 \leqslant i \leqslant h, 0 \leqslant p < \infty).$$

We fix a ring $R^* = k^*[[t]]$ as in (0.4), *with $R^* \subseteq \Omega$, and denote by* "ord" *the valuation of $F^* = k^*((t))$ associated with R^*:*

$$\operatorname{ord}(\xi) = \sup\{n \in \mathbf{Z} \mid \xi \in t^n R^*\} \qquad (\xi \in F^*)$$

3. For elements η, ζ of A, we say that η *dominates* ζ (*with respect to* $K = k((x))$) if for any $\psi, \psi' \in H$: the quotient

$$\frac{\psi(\eta) - \psi'(\eta)}{\psi(\zeta) - \psi'(\zeta)} \in \Omega$$

is *integral* over $A \cap K\ (= k[[x]])$ if $\psi(\zeta) \neq \psi'(\zeta)$, and $\psi(\eta) = \psi'(\eta)$ if $\psi(\zeta) = \psi'(\zeta)$.

The relationship of domination does not depend on the choice of Ω. Furthermore if $\phi: A \to A'$ is a ring-isomorphism, extending to an isomorphism $\Phi: F \to F'$ of total rings of fractions, then η dominates ζ (w.r.t. K)$\Leftrightarrow \phi(\eta)$ dominates $\phi(\zeta)$ (w.r.t. $\Phi(K)$).

If ord is the valuation defined above, then clearly: *η dominates ζ if and only if, for any $\psi, \psi' \in H$*

$$\operatorname{ord}(\psi(\eta) - \psi'(\eta)) \geq \operatorname{ord}(\psi(\zeta) - \psi'(\zeta)).$$

A subset T of A is said to be *K-saturated* (*in* A) if T contains every element of A which dominates an element of T. For any $S \subseteq A$, the set

$$\tilde{S} = \tilde{S}_K = \{\eta \in A \mid \eta \text{ dominates some member of } S \text{ (w.r.t. } K)\}$$

is evidently the *smallest K-saturated subset of A containing S*; $\tilde{S}_K$ is called the *K-saturation of S* (*in* A). The usual "closure" properties hold:

$$\text{(i)}\quad S \subseteq \tilde{S}; \qquad \text{(ii)}\quad S \subseteq T \subseteq A \Rightarrow \tilde{S} \subseteq \tilde{T}; \qquad \text{(iii)}\quad \tilde{\tilde{S}} = \tilde{S}.$$

A being fixed, we say that a subset S of A is *Zariski-saturated in A* if there *exists* a field K as in (0.1) such that $S = \tilde{S}_K$.

Lemma. *If $\zeta_1, \zeta_2, \ldots, \zeta_N$ are in A, then there exist positive integers $r_1, r_2, \ldots, r_N$ such that each one of $\zeta_1, \zeta_2, \ldots, \zeta_N$ dominates $r_1\zeta_1 + r_2\zeta_2 + \ldots + r_N\zeta_N$.*

Proof. (cf. [GTS II, p. 878, Prop. 1.3]). By assumption A is a $\mathbf{Q}$-algebra ($\mathbf{Q}$ = field of rational numbers). For $\psi, \psi' \in H$, the set

$$V_{\psi,\psi'} = \left\{(\rho_1, \rho_2, \ldots, \rho_N) \in \mathbf{Q}^N \,\middle|\, \operatorname{ord}\left[\psi\left(\sum_{i=1}^{N} \rho_i\zeta_i\right) - \psi'\left(\sum_{i=1}^{N} \rho_i\zeta_i\right)\right] > \min_{1 \leq i \leq N}\{\operatorname{ord}[\psi(\zeta_i) - \psi'(\zeta_i)]\}\right\}$$

is easily seen to be a *proper* $\mathbf{Q}$-vector-subspace of $\mathbf{Q}^N$. Hence there exists a point $(r_1, r_2, \ldots, r_N)$ with positive integer coordinates lying outside the finite union $\cup V_{\psi,\psi'}(\psi, \psi' \in H)$, and this is as required. Q.E.D.

COROLLARY (0.8). *Let A, K be as before and let S be an additively closed subset of $A (\eta_1, \eta_2 \in S \Rightarrow \eta_1 + \eta_2 \in S)$. Then $\eta \in \tilde{S}_K$ if and only if, for any ψ, ψ' in H there is an element $\zeta (= \zeta_{\psi,\psi'})$ in S such that*

$$\text{(0.9)} \qquad \operatorname{ord}(\psi(\eta) - \psi'(\eta)) \geqslant \operatorname{ord}(\psi(\zeta) - \psi'(\zeta)).$$

Proof. If η dominates $\xi \in S$ then (0.9) holds with $\zeta = \xi$ (for any ψ, ψ'). Conversely, if $\zeta_{\psi,\psi'}$ is as in (0.9), then, S being additively closed, the Lemma shows that there is an element ξ in S dominated by each one of the $\zeta_{\psi,\psi'} (\psi, \psi' \in H)$; clearly η dominates this ξ.

COROLLARY (0.10). *Let A, K, S be as in (0.8), with S non-empty. Then $\tilde{S}$ is a subring of A containing $A \cap K$.*

Proof. If η_1, $\eta_2 \in \tilde{S}$, then so do $\eta_1 - \eta_2$ and $\eta_1\eta_2$, as follows easily from (0.8) and the identities

$$\psi(\eta_1 - \eta_2) - \psi'(\eta_1 - \eta_2) = [\psi(\eta_1) - \psi'(\eta_1)] - [\psi(\eta_2) - \psi'(\eta_2)]$$

$$\psi(\eta_1\eta_2) - \psi'(\eta_1\eta_2) = [\psi(\eta_1) - \psi'(\eta_1)][\psi(\eta_2)] + [\psi'(\eta_1)][\psi(\eta_2) - \psi'(\eta_2)].$$

If $\eta \in A \cap K$, then $\psi(\eta) - \psi'(\eta) = 0$ for all ψ, $\psi' \in H$; hence η dominates every element of A, and so (since S is non-empty) $\eta \in \tilde{S}$; thus $A \cap K \subseteq \tilde{S}$. Q.E.D.

4. PROPOSITION (0.11). *Let A and $K = k((x))$ be as in (0.1), and let $\mathfrak{o}$ be a local subring of A containing $k[[x]] = A \cap K$. Then the saturation $\tilde{\mathfrak{o}} = \tilde{\mathfrak{o}}_K$ is a local ring, and if $\mathfrak{m}$, $\mathfrak{n}$ are the maximal ideals of $\mathfrak{o}$, $\tilde{\mathfrak{o}}$ respectively, then the canonical map $\mathfrak{o}/\mathfrak{m} \to \tilde{\mathfrak{o}}/\mathfrak{n}$ is bijective, and $\mathfrak{m}A = \mathfrak{n}A$.*†

Proof. Let $\mathfrak{o}_*$ be the ring $\mathfrak{o} + \mathfrak{m}A$. Then $\mathfrak{o}/\mathfrak{m} \to \mathfrak{o}_*/\mathfrak{m}A$ is *bijective*, so $\mathfrak{m}A$ is a maximal ideal of $\mathfrak{o}_*$; on the other hand, every maximal ideal of A contains $\mathfrak{m}A$ (since A is integral over $\mathfrak{o}$, cf. (0.1)), whence $\mathfrak{m}A$ is the *only* maximal ideal of $\mathfrak{o}_*$ (since A is integral over $\mathfrak{o}_*$). We shall show that *if $\eta \in A$ dominates $\zeta \in \mathfrak{o}$*

† $\mathfrak{o}$ and $\tilde{\mathfrak{o}}$ are readily seen to have the same total ring of fractions, say $F^{\#}$, so we may assume that A is the integral closure of $\mathfrak{o}$ in $F^{\#}$. In this case we find, from the "projection formula" [CA p. 299, Cor. 1], that the multiplicity of an $\mathfrak{m}$-primary ideal I in $\mathfrak{o}$ is equal to the length of the $\mathfrak{o}$-module A/IA. Consequently, what Proposition (0.11) says is that $\tilde{\mathfrak{o}}$ *is a local ring having the same multiplicity as* $\mathfrak{o}$ (cf. [GTS II, p. 900, Prop. 3.7.]).

(*w.r.t.* $K=k((x))$) *then* $\eta\in\mathfrak{o}_*$. (This means that $\tilde{\mathfrak{o}}\subseteq\mathfrak{o}_*$, whence $\tilde{\mathfrak{o}}$ is a local ring with maximal ideal $\mathfrak{n}=\mathfrak{m}A\cap\tilde{\mathfrak{o}}$, and $\tilde{\mathfrak{o}}/\mathfrak{n}\subseteq\mathfrak{o}_*/\mathfrak{m}A$; (0.11) follows at once.)

Since $\mathfrak{o}$ is a finite $k[[x]]$-module (cf. (0.1)) we see that $\mathfrak{o}$ is a complete local ring and that the integral closure k_1 of k in $\mathfrak{o}$ is a maximal subfield of $\mathfrak{o}$, hence is a field of representatives. We have then: $k[[x]]\subseteq k_1[[x]]\subseteq\mathfrak{o}$, and if K_1 is the fraction field of $k_1[[x]]$, then $K\subseteq K_1\subseteq F$ and $A\cap K_1=k_1[[x]]$. Since any $k_1((x))$-homomorphism of F into an algebraic closure of $k_1((x))$ is also a $k((x))$-homomorphism, it follows that η dominates ζ w.r.t. $k_1((x))$. We may therefore assume that $k=k_1$. Then $\zeta=b+\zeta'$ with $b\in k$ and $\zeta'\in\mathfrak{m}$, and η dominates ζ'; so *we may assume that* $\zeta\in\mathfrak{m}$. This being so, we shall show that

$$\text{(0.12)}\qquad \eta\in k+(x,\zeta)A \qquad (\subseteq\mathfrak{o}_*).$$

Let $\eta_i=\sum_{p=0}^{\infty}a_{ip}t^p$ be as in (0.7) $(1\leqslant i\leqslant h)$, and let $q_i=\inf\{p\mid a_{ip}t^p\notin k[[x]]\}$. If $q_i=\infty$ (i.e., $\eta_i\in k[[x]]$) let σ be the identity automorphism of F^*; if $q_i<\infty$, let σ be an automorphism of F^*/K such that $\sigma(a_{iq_i}t^{q_i})\neq a_{iq_i}t^{q_i}$; in either case, $\operatorname{ord}(\sigma(\eta_i)-\eta_i)=q_i$ (cf. (0.5)). With

$$\eta_i'=\Big(\eta_i-\sum_{p<q_i}a_{ip}t^p\Big)\in F_i=\psi_i(F)$$

we have (setting $\zeta_i=\psi_i(\zeta)$)

$$\begin{aligned}\operatorname{ord}(\zeta_i)&\leqslant\operatorname{ord}(\sigma(\zeta_i)-\zeta_i) &&[\text{since } \operatorname{ord}(\sigma(\zeta_i))=\operatorname{ord}(\zeta_i)]\\ &\leqslant\operatorname{ord}(\sigma(\eta_i)-\eta_i) &&[\text{since } \eta \text{ dominates } \zeta]\\ &=q_i=\operatorname{ord}(\eta_i').\end{aligned}$$

Hence $\eta_i'\in\zeta_iA_i$, and (since $(\sum_{1\leqslant p<q_i}a_{ip}t^p)\in xk[[x]]$)

$$\text{(0.13)}\qquad (\eta_i-a_{i0})\in xk[[x]]+\zeta_iA_i\subseteq\psi_i(x,\zeta)A.$$

Furthermore, $0<\operatorname{ord}(\zeta_i)$ (since $\zeta\in\mathfrak{m}$), so $0<q_i$, whence $a_{i0}\in k$; and since η dominates ζ,

$$\operatorname{ord}(\eta_i-\eta_j)\geqslant\operatorname{ord}(\zeta_i-\zeta_j)>0\qquad(i,j=1,2,\dots,h)$$

so that $a_{i0}=a_{j0}$ for all i, j. Thus

$$a_{10}=a_{20}=\dots=a_{h0}=a\quad(\text{say}),\qquad a\in k,$$

and hence (since $A=\prod_{i=1}^{h}A_i$, so that $(x,\zeta)A=\prod_{i=1}^{h}\psi_i(x,\zeta)A$) (0.13) gives (0.12). Q.E.D.

5. The following simple lemma helps to elucidate the structure of saturated rings (although we will not use it *explicitly* for that purpose).

LEMMA (0.14). *Let $\eta \in A$ be as in (0.7). Then for each $m \geq 0$ we have that*

$$\eta^{(m)} = (a_{1m}t^m, a_{2m}t^m, \ldots, a_{hm}t^m) \text{ dominates } \eta$$

(*so that any saturated set containing η also contains all the $\eta^{(m)}$, $m = 0, 1, 2, \ldots$*).

Proof. For any i, $j = 1, 2, \ldots, h$, and $\sigma, \tau \in G$ (the Galois group of F^*/K), we have (in view of (0.5))

$$(0.15) \qquad \operatorname{ord}[\sigma(\eta_i) - \tau(\eta_j)]$$

$$= \operatorname{ord}\left[\sigma\left(\sum_{p=0}^{\infty} a_{ip}t^p\right) - \tau\left(\sum_{p=0}^{\infty} a_{jp}t^p\right)\right]$$

$$= \operatorname{ord}\left[\sum_{p=0}^{\infty} (\sigma_0(a_{ip})\alpha^p - \tau_0(a_{jp})\,\beta^p)\,t^p\right] \qquad (\alpha^n = \beta^n = 1)$$

$$\leq \operatorname{ord}[(\sigma_0(a_{im})\alpha^m - \tau_0(a_{jm})\,\beta^m)\,t^m]$$

$$= \operatorname{ord}[\sigma(a_{im}t^m) - \tau(a_{jm}t^m)].$$

Taking $j = i$, $\tau =$ identity, we see that

$$[\sigma(\eta_i) = \eta_i] \Rightarrow [\sigma(a_{im}t^m) = a_{im}t^m]$$

and therefore $a_{im}t^m \in K(\eta_i) \subseteq F_i$, so that

$$a_{im}t^m \in F_i \cap R^* = A_i \qquad (i = 1, 2, \ldots, h).$$

Thus $\eta^{(m)} \in \prod_{i=1}^{h} A_i = A$. Because of (0.6) (and since $\eta_i = \psi_i(\eta)$, $a_{im}t^m = \psi_i(\eta^{(m)})$, etc.), (0.15) tells us that $\eta^{(m)}$ dominates η. Q.E.D.

Example (0.16). We drop, for just a moment, all the notational conventions previously made. Let $\mathfrak{o}$ be a reduced complete one-dimensional noetherian local ring with maximal ideal $\mathfrak{m}$, and assume that the residue field $\mathfrak{o}/\mathfrak{m}$ has characteristic zero. Let k be a subfield of $\mathfrak{o}$ such that $[\mathfrak{o}/\mathfrak{m} : \bar{k}] < \infty$, $\bar{k}$ being the canonical image of k in $\mathfrak{o}/\mathfrak{m}$. (For example, k could be a field of representatives [CA pp. 280–281]). k is obviously of characteristic zero. Let x be a parameter of

$\mathfrak{o}$, i.e. a non-unit regular element of $\mathfrak{o}$ (equivalently: the ideal $x\mathfrak{o}$ is primary for $\mathfrak{m}$). $\mathfrak{o}$ is then a finite module over its power series subring $k[[x]]$ (cf. [CA, p. 293, Remark]).

Let K be the field of fractions of $k[[x]]$ and let F be the total ring of fractions of $\mathfrak{o}$:

$$F = \mathfrak{o}[1/x] = \mathfrak{o} \otimes_{k[[x]]} K.$$

F is a finite-dimensional K-algebra. Taking A to be the integral closure of $\mathfrak{o}$ in F, we return to the situation described in (0.11),† but from a somewhat different path. This is actually the approach which will be taken in Section 2.

Note that in this example it is clear that $A \cap K = k[[x]]$, and furthermore A is a finite $A \cap K$-module (because, as in 1, $A = \prod_{i=1}^{h} A_i$ and for each i, F_i is a finite separable field extension of K, whence A_i is a finite $k[[x]]$-module.) In other words, *Proposition* (0.1) *is* obviously *satisfied here.*

1. Absolute Saturation. We shall say that a homomorphism of commutative semi-local rings $\phi: B \to C$ is *continuous* if $\phi(\text{rad.} B) \subseteq (\text{rad.} C)$, where "rad" means Jacobson radical (= intersection of all maximal ideals).

Definition (1.1). *Let S be a subset of A (where A is as in Section* 0). *The* ***absolute saturation*** *of S in A, S' in symbol, is the set consisting of all elements η in A such that: for each valuation ring R_v (with valuation v) and each pair of* ***continuous*** *homomorphisms $\phi_1, \phi_2: A \to R_v$, there exists an element ζ in S such that*

$$v(\phi_1(\eta) - \phi_2(\eta)) \geqslant v(\phi_1(\zeta) - \phi_2(\zeta)).$$

S is ***absolutely saturated*** *(in A) if $S = S'$.*

Remarks (a). It is immediate from the definition that the usual closure properties hold for absolute saturation:

$$\text{(i)}\quad S \subseteq S'; \qquad \text{(ii)}\quad S \subseteq T \subseteq A \Rightarrow S' \subseteq T'; \qquad \text{(iii)}\quad S'' = S'.$$

Hence *S' is the smallest absolutely saturated subset of A containing S.*

(b). Arguing as in (0.10) (replacing ψ, ψ' by ϕ_1, ϕ_2) we find that *if S is non-empty then S' is a subring of A.* (It is not necessary here that S be

†In (0.11) we did not assume that $\mathfrak{o}$ and A have the same total ring of fractions, but for our purpose—which is to study $\tilde{\mathfrak{o}}_K$—this makes no difference (cf. first sentence in footnote following (0.11)).

additively closed. We need to know that S is non-empty in order to show that $1 \in S'$.)

(c). If S is additively closed, then it follows from Corollary (0.8) that

$$S' \subseteq \tilde{S}_K.$$

(Indeed, any ψ, ψ' in H induce *continuous homomorphisms* from A into the valuation ring R^* of "ord", cf. (0.4).) In particular, *if* S *is additively closed and Zariski-saturated* (i.e. $S = \tilde{S}_K$ *for some* K), *then* S *is absolutely saturated.* (In this case $S = \tilde{S}_K$ is actually a *subring* of A, cf. (0.10).)

The following remark, which allows us to impose certain restrictions on ϕ_1 and ϕ_2 in Definition (1.1), will be useful. Notation is as in 1 of Section 0.

(d). Let $R \doteq R_v$ be a valuation ring and let $\phi_1, \phi_2 : A \to R$ be continuous homomorphisms. The kernel of ϕ_1 contains one of the minimal prime ideals $\mathfrak{p}_i \cap A$, and so $\phi_1 = \theta_1 \circ \psi_i$ for some i with $1 \leqslant i \leqslant h$, where $\theta_1 : A_i \to R$ is a continuous homomorphism ($A_i = \psi_i(A)$). θ_1 can be extended to $\phi_1^* : R^* \to \bar{R}$, where R^* is as in (0.4), and $\bar{R}$ is the integral closure of R in some algebraic closure of the field of fractions of R. So we have a commutative diagram

$$\begin{array}{ccc} A & & \\ \psi_i \downarrow & \searrow^{\phi_1} & \\ A_i & \xrightarrow[\theta_1]{} & R \\ \downarrow x & & x \downarrow \\ R^* & \xrightarrow[\phi_1^*]{} & \bar{R} \end{array}$$

If $\mathfrak{n}$ is a maximal ideal of $\bar{R}$ then

$$(\phi_1^*)^{-1}(\mathfrak{n}) \cap A_i = \theta_1^{-1}(\mathfrak{n} \cap R)$$

which is the maximal ideal of A_i since $\mathfrak{n} \cap R$ is maximal in R, and θ_1 is continuous; we conclude that $(\phi_1^*)^{-1}(\mathfrak{n})$ is the maximal ideal of R^*; thus ϕ_1^* is also a continuous homomorphism. In a similar way, we have $\phi_2 = \theta_2 \circ \psi_j$ for some j with $1 \leqslant j \leqslant h$, and θ_2 extends to a continuous homomorphism $\phi_2^* : R^* \to \bar{R}$.

Let $\bar{R}_{\mathfrak{n}}$ be the localization of $\bar{R}$ at one of its maximal ideals $\mathfrak{n}$. Then $\bar{R}_{\mathfrak{n}}$ is the valuation ring of a valuation $\bar{v}$ which is an extension of v [CA, p. 27, Cor. 2].

If $\eta, \zeta \in A$, then clearly

$$v(\phi_1(\eta)-\phi_2(\eta)) \geqslant v(\phi_1(\zeta)-\phi_2(\zeta))$$

if and only if

$$\bar{v}(\phi_1^*(\psi_i(\eta))-\phi_2^*(\psi_j(\eta))) \geqslant \bar{v}(\phi_1^*(\psi_i(\zeta))-\phi_2^*(\psi_j(\zeta))).$$

These considerations lead to the following conclusion:

$\eta \in S'$ if and only if: for each valuation ring R_v, each pair of continuous homomorphisms $\phi_1^*, \phi_2^* : R^* \to R_v$, and each pair i, j with $1 \leqslant i, j \leqslant h$, there exists an element ζ in S such that

$$v(\phi_1^*(\eta_i)-\phi_2^*(\eta_j)) \geqslant v(\phi_1^*(\zeta_i)-\phi_2^*(\zeta_j)) \qquad (\eta_i=\psi_i(\eta), \text{ etc.})$$

2. Equality with Zariski's Saturation. Notation remains as in Sections 0 and 1. Let $\mathfrak{o}$, $\mathfrak{m}$, $K=k((x))$, F, A be as in Example (0.16). We say that the parameter x of $\mathfrak{o}$ is *transversal* if $\mathfrak{m}A=xA$. This means that for any non-unit ξ of $\mathfrak{o}$ we have $\xi \in xA$, i.e. (with ψ_i as in **1** of Section 0)

$$\psi_i(\xi) \in \psi_i(x)A_i \qquad (i=1,2,\ldots,h)$$

or, equivalently,

$$\operatorname{ord}(\psi_i(\xi)) \geqslant \operatorname{ord}(x) \qquad (i=1,2,\ldots,h)$$

(cf. **2** of Section 0).

Since k is infinite, transversal parameters exist.†

THEOREM. *If x is a transversal parameter of $\mathfrak{o}$, then the $k((x))$-saturation of $\mathfrak{o}$ in A is identical with the absolute saturation of $\mathfrak{o}$ in A:*

$$\tilde{\mathfrak{o}}_{k((x))}=\mathfrak{o}'.$$

COROLLARY 1. *The $k((x))$-saturation of $\mathfrak{o}$ in A does not depend on the choice either of the field k or of the transversal parameter x.*

†Let $(\xi_1, \xi_2, \ldots, \xi_N)$ generate the ideal $\mathfrak{m}$; for $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant N$, let

$$V_{ij}=\left\{(a_1,a_2,\ldots,a_N) \in k^N \,\middle|\, \psi_i(\xi_j) \not\in \psi_i\left(\sum_{s=1}^{N} a_s\xi_s\right)A_i\right\}.$$

Then V_{ij} is a *proper* sub-k-vector-space of k^N, and for any $(b_1,b_2,\ldots,b_N) \in k^N - \cup_{i,j} V_{ij}$, $\Sigma b_s\xi_s$ is a transversal parameter.

The adjective "transversal" comes from the fact that $xA=\mathfrak{m}A \Leftrightarrow$ the ideals $x\mathfrak{o}$ and $\mathfrak{m}$ of $\mathfrak{o}$ have the same multiplicity (cf. footnote following (0.11)).

In fact, $\tilde{\mathfrak{o}}_{k((x))}$ can also be characterized as being *the smallest Zariski-saturated ring between* $\mathfrak{o}$ *and* A:

Corollary 2. [GTS III, *Appendix* A, *Lemma* A9] *If* $\mathfrak{o} \subseteq S \subseteq A$, *where* S *is additively closed and Zariski-saturated in* A, *then*

$$\tilde{\mathfrak{o}}_{k((x))} \subseteq S.$$

Proof.

$$\begin{aligned} S &= S' && \text{(Section 1, Remark (c))} \\ &\supseteq \mathfrak{o}' && \text{(Section 1, Remark (a))} \\ &= \tilde{\mathfrak{o}}_{k((x))} && \text{(by the Theorem).} \end{aligned}$$

Corollary 3. *If* $\mathfrak{o}$ *is Zariski-saturated in* A *then* $\mathfrak{o}$ *is* $k((x))$*-saturated in* A.

Proof. Take $S = \mathfrak{o}$ in Corollary 2.

Remark. The cited Lemma A9 is one of the main facts proved in [GTS]. Actually, since our definition of "Zariski-saturated" refers to *any* field K such that $[F:K] < \infty$, Corollary 2 is marginally stronger than Lemma A9.

Proof of Theorem. We already know that $\mathfrak{o}' \subseteq \tilde{\mathfrak{o}}$ $(= \tilde{\mathfrak{o}}_{k((x))})$ (Section 1, remark (c)), so we need only show:

If $\eta \in \tilde{\mathfrak{o}}$ *then* $\eta \in \mathfrak{o}'$; *in other words if* ϕ_1, ϕ_2 *are continuous homomorphisms of* A *into a valuation ring* R_v (*with valuation* v), *then there exists* $\zeta \in \mathfrak{o}$ *such that*

$$v(\phi_1(\eta) - \phi_2(\eta)) \geqslant v(\phi_1(\zeta) - \phi_2(\zeta)). \tag{2.1}$$

By Proposition (0.11), $\tilde{\mathfrak{o}} \subseteq \mathfrak{o} + \mathfrak{m}A$, so if $\eta \in \tilde{\mathfrak{o}}$ then there exists $\xi \in \mathfrak{o}$ such that

$$\eta - \xi \in \mathfrak{m}A = xA$$

(x being a transversal parameter). If $\eta - \xi \in \mathfrak{o}'$ then $\eta \in \mathfrak{o}'$, so we may replace η in (2.1) by $\eta - \xi$, i.e. we may assume that $\eta \in xA$, whence (cf. (0.4))

$$\operatorname{ord}(\eta_i) \geqslant \operatorname{ord}(x) = \operatorname{ord}(t^n) = n$$

$$(\eta_i = \psi_i(\eta),\ i = 1, 2, \ldots, h). \tag{2.2}$$

Furthermore, as in (0.14), $\eta^{(m)} \in \tilde{\mathfrak{o}}$ for all $m \geqslant 0$. Now in the finite $k[[x]]$-module

$\prod_{i=1}^{h} R_i^*$ ($R_i^* = k^*[[t]]$ for $i = 1, 2, \ldots, h$), x-adically topologized, we clearly have $\eta = \sum_{m=0}^{\infty} \eta^{(m)}$; moreover $\mathfrak{o}'$ is a *closed subspace* of $\prod_{i=1}^{h} R_i^*$ [CA, p. 262, Theorem 9]; so if we can show that $\eta^{(m)} \in \mathfrak{o}'$ for all m, then $\eta \in \mathfrak{o}'$, as required. By (2.2), $\eta^{(m)} = 0$ if $m < n$. Thus, *we may assume in proving* (2.1) *that* η *is of the form*

$$\boxed{\eta = (a_1 t^m, a_2 t^m, \ldots, a_h t^m)} \qquad (m \geqslant n; a_1, a_2, \ldots, a_h \in k^*) \tag{2.3}$$

By remark (d) of Section 1, we may also assume that

$$\boxed{\phi_1 = \phi_1^* \circ \psi_i, \qquad \phi_2 = \phi_2^* \circ \psi_j}$$

(for some i, j with $1 \leqslant i, j \leqslant h$, ϕ_1^*, ϕ_2^* being continuous homomorphisms of R^* into R_v). Then

$$\phi_1(\eta) - \phi_2(\eta) = \phi_1^*(\eta_i) - \phi_2^*(\eta_j) = \phi_1^*(a_i t^m) - \phi_2^*(a_j t^m).$$

Let

$$\nu = \min\{v(\phi_1^*(t)), v(\phi_2^*(t))\}.$$

Then

$$n\nu \leqslant v(\phi_1^*(t^n) - \phi_2^*(t^n)) = v(\phi_1(x) - \phi_2(x)) \tag{2.4}$$

and

$$m\nu \leqslant v(\phi_1^*(a_i t^m) - \phi_2^*(a_j t^m)) = v(\phi_1(\eta) - \phi_2(\eta)). \tag{2.5}$$

If *equality* holds in (2.4), then, since $m \geqslant n$, (2.1) holds with $\zeta = x$, and we are done.

So we assume henceforth that

$$n\nu < v(\phi_1^*(t^n) - \phi_2^*(t^n)). \tag{2.6}$$

It then follows that

$$\boxed{\nu = v(\phi_1^*(t)) = v(\phi_2^*(t)) < \infty.} \tag{2.7}$$

We shall now prove (2.1), *with* η *as in* (2.3), *by induction on* m. η dominates some element y in $\mathfrak{o}$ (by definition of $\tilde{\mathfrak{o}}$, cf. 3 in Section 0). Arguing as in the second paragraph of the proof of (0.11), *we may assume that* $y \in \mathfrak{m} \subseteq xA$, *so that* $y^{(p)} = 0$ *for* $p < n$. By (0.14), $y^{(p)} \in \tilde{\mathfrak{o}}$ for all integers $p \geqslant 0$, and so (since we are proceeding by induction) *we may assume that* $y^{(p)} \in \mathfrak{o}'$ *for* $n \leqslant p < m$. As in (2.5), we have, for all p,

$$v\big(\phi_1(y^{(p)}) - \phi_2(y^{(p)})\big) \geqslant p\nu.$$

Suppose that for some $p \leqslant m$

$$v\big(\phi_1(y^{(p)}) - \phi_2(y^{(p)})\big) = p\nu. \tag{2.8}$$

Set

$$z = y - \sum_{q<p} y^{(q)}.$$

Then $z \in \mathfrak{o}'$, so there exists $\zeta \in \mathfrak{o}$ such that

$$v(\phi_1(\zeta) - \phi_2(\zeta)) \leqslant v(\phi_1(z) - \phi_2(z)).$$

But

$$\psi_i(z - y^{(p)}) \in t^{p+1}R^*$$

so that

$$v\big(\phi_1(z - y^{(p)})\big) = v\big(\phi_1^* \circ \psi_i(z - y^{(p)})\big) \geqslant (p+1)\nu > p\nu$$

($\nu > 0$ because ϕ_1^*, ϕ_2^* are *continuous* homomorphisms); similarly

$$v\big(\phi_2(z - y^{(p)})\big) > p\nu.$$

In view of (2.8), we conclude then that

$$v(\phi_1(z) - \phi_2(z)) = v\big(\phi_1(z - y^{(p)}) - \phi_2(z - y^{(p)}) + \phi_1(y^{(p)}) - \phi_2(y^{(p)})\big) = p\nu.$$

Since (cf. (2.5))

$$p\nu \leqslant m\nu \leqslant v(\phi_1(\eta) - \phi_2(\eta))$$

we have finally that

$$v(\phi_1(\zeta) - \phi_2(\zeta)) \leqslant v(\phi_1(z) - \phi_2(z)) = p\nu \leqslant v(\phi_1(\eta) - \phi_2(\eta))$$

and so (2.1) holds.

So we assume henceforth that

$$v\big(\phi_1(y^{(p)})-\phi_2(y^{(p)})\big)>pv \quad \text{for } p\leqslant m. \tag{2.9}$$

We may *further assume that*, if E is the residue field of R_v, and $\bar{\phi}_1, \bar{\phi}_2 : k^* \to E$ are the homomorphisms induced by ϕ_1^*, ϕ_2^*, then $\bar{\phi}_1$ *and* $\bar{\phi}_2$ *have the same restriction to* k. (Otherwise, $v(\phi_1(c)-\phi_2(c))=0$ for some c in k, and (2.1) holds with $\zeta=c$.) Since k^* is normal over k (cf. (0.4)), we have then

$$\bar{\phi}_2=\bar{\phi}_1\circ\rho \quad \textit{for some } k\textit{-automorphism } \rho \textit{ of } k^*. \tag{2.10}$$

LEMMA (2.11). *Under the preceding assumptions* (2.6), (2.9), (2.10), *there exists a* $k((x))$*-automorphism* σ *of* F^* *such that, if* $\theta_1=\phi_1^*\circ\sigma$ *and* $\theta_2=\phi_2^*$, *then:*

(i) $\phi_1(\eta)-\phi_2(\eta)[=\phi_1^*(\eta_i)-\phi_2^*(\eta_j)]=\theta_1(\eta_j)-\theta_2(\eta_j)$

(ii) $v\left(\dfrac{\theta_1(t)}{\theta_2(t)}-1\right)>0 \qquad (\theta_2(t)\neq 0 \text{ by } (2.7)).$

(iii) $v(\theta_2(a)-\theta_1(a))>0 \quad \text{for any } a\in k^*.$

Before proving (2.11), let us see how it enables us to complete the proof of the Theorem. Let T be the set consisting of all elements ξ in R^* such that there exists ζ in $k[[x]]$ with

$$v(\theta_1(\xi)-\theta_2(\xi))\geqslant v(\theta_1(\zeta)-\theta_2(\zeta)).$$

Using the identities in (0.10) (with ψ, ψ' replaced by θ_1, θ_2, and η_1, η_2 by ξ_1, ξ_2), we see that T *is a subring of* R^*. Since $\theta_1(\zeta)=\phi_1(\zeta)$ and $\theta_2(\zeta)=\phi_2(\zeta)$ for all $\zeta\in k[[x]]$, (2.11) (i) shows that *for* (2.1) *to hold* (*with* η *as in* (2.3)), *it suffices that* $a_j t^m\in T$. Using (2.11), we shall now prove that:

(a) $t^q\in T \quad \text{for } q\geqslant n.$

(b) $k^*\subseteq T.$

(Since T is a ring and $m\geqslant n$, this implies that $a_j t^m\in T$, as desired).

Proof of (a). (cf. [GTS I, p. 623, Lemma 7.1]). By (2.11) (ii) we have

$$\frac{\theta_1(t)}{\theta_2(t)}=1+u, \qquad v(u)>0.$$

Hence for any integer $q > 0$

$$\frac{\theta_1(t^q)}{\theta_2(t^q)} = (1+u)^q \equiv 1 + qu \qquad (\text{mod}.\, u^2),$$

so that (since $v(u^2) > v(qu)$, q being a unit in R_v)

$$v(\theta_1(t^q) - \theta_2(t^q)) = v(\theta_2(t^q)\cdot qu) = q\nu + v(u) \qquad (\text{cf. } (2.7)).$$

This implies, for $q \geqslant n$, that

$$v(\theta_1(t^q) - \theta_2(t^q)) \geqslant v(\theta_1(t^n) - \theta_2(t^n)) = v(\theta_1(x) - \theta_2(x))$$

and so $t^q \in T$.

Proof of (b). (cf. [GTS I, p. 642, proof of Lemma 9.2]). Let $a \in k^*$ and let

$$f(X) = \sum_{s=0}^{N} a_s X^s \in k[X]$$

be the minimum polynomial of a over k. For any element ξ in R^*, we set

$$\xi' = \theta_1(\xi), \qquad \xi'' = \theta_2(\xi).$$

We shall show that

$$v(a' - a'') \geqslant \min_{0 < s < N} \{v(a_s' - a_s'')\} \tag{2.12}$$

(*whence* $a \in T$, *as required*). Let

$$f'(X) = \sum_{s=0}^{N} a_s' X^s \in R_v[X]$$

$$f''(X) = \sum_{s=0}^{N} a_s'' X^s \in R_v[X].$$

Then since $f(a) = 0$, therefore

$$f'(a')[= \theta_1(f(a))] = 0$$

$$f''(a'')[= \theta_2(f(a))] = 0.$$

Furthermore, if f_X (resp. f_X') is the derivative of f (resp. f') then $f_X(a)$ is a unit in k^*, whence $f_X'(a')$ $[= \theta_1(f_X(a)]$ *is a unit in* R_v, i.e. $v(f_X'(a')) = 0$.

We have therefore

$$f'(a'') = f'(a'') - f''(a'') = \sum_{s=0}^{N} (a'_s - a''_s)(a'')^s$$

so that

$$v(f'(a'')) \geqslant \min_{0 \leqslant s \leqslant N} \{v(a'_s - a''_s)\}. \tag{2.13}$$

Furthermore

$$f'(a'') = f'(a'') - f'(a')$$

$$\equiv (a'' - a')f'_X(a') \qquad (\text{mod.}\,(a'' - a')^2).$$

But (2.11) (iii) says that $v(a'' - a') > 0$, and we noted above that $v(f'_X(a')) = 0$; consequently

$$v(f'(a'')) = v(a'' - a'). \tag{2.14}$$

(2.14) and (2.13) give (2.12).

It remains to prove (2.11).

Let β be the canonical image in E (the residue field of R_v) of $\phi_2^*(t)/\phi_1^*(t)$ (recall that $\phi_1^*(t) \neq 0$, cf. (2.7)). (2.6) gives $\beta^n = 1$, and hence (cf. (0.4) (ii)) there is an α in k^* such that

$$\alpha^n = 1 \quad \text{and} \quad \bar{\phi}_1(\alpha) = \beta.$$

Let

$$y_i = \sum_{p=0}^{\infty} b_p t^p \qquad (b_p \in k^*)$$

$$y_j = \sum_{p=0}^{\infty} c_p t^p \qquad (c_p \in k^*).$$

(2.9) says that

$$v(\phi_1^*(b_p t^p) - \phi_2^*(c_p t^p)) > p\nu \qquad (p \leqslant m)$$

and hence

$$\bar{\phi}_1(b_p) = \bar{\phi}_2(c_p)\beta^p \qquad (p \leqslant m)$$

i.e. (cf. (2.10))

$$\bar{\phi}_1(b_p) = \bar{\phi}_1(\rho(c_p)\alpha^p)$$

whence

$$b_p = \rho(c_p)\alpha^p \qquad (p \leqslant m). \tag{2.15}$$

Let σ be the automorphism of $F^* = k^*((t))$ for which

$$\sigma\left(\sum_{p=0}^{\infty} d_p t^p\right) = \sum_{p=0}^{\infty} \rho(d_p)(\alpha t)^p \qquad (d_p \in k^*).$$

Since ρ restricts to the identity map on k, and since $\alpha^n = 1$, therefore σ restricts to the identity on $k[[t^n]] = k[[x]]$, hence on $k((x))$.

In view of (2.15), we have

$$\operatorname{ord}(y_i - \sigma(y_j)) = \operatorname{ord}\left(\sum_{p=0}^{\infty} (b_p - \rho(c_p)\alpha^p)t^p\right) > m.$$

Hence $m < \operatorname{ord}(y_i - \sigma(y_j)) \leqslant \operatorname{ord}(\eta_i - \sigma(n_j))$ (because η *dominates* y w.r.t. $k((x))$). Since $\eta_i = a_i t^m$, and $\eta_j = a_j t^m$, this implies that $\eta_i = \sigma(\eta_j)$, and (2.11) (i) follows.

Next, we have

$$\frac{\theta_1(t)}{\theta_2(t)} = \frac{\phi_1^*(\sigma t)}{\phi_2^*(t)} = \frac{\phi_1^*(\alpha t)}{\phi_2^*(t)} = \phi_1^*(\alpha)\frac{\phi_1^*(t)}{\phi_2^*(t)}.$$

Hence, the image of $\theta_1(t)/\theta_2(t)$ in E is $\bar{\phi}_1(\alpha)/\beta = 1$. This proves (2.11) (ii).

Finally, for $a \in k^*$, the image in E of

$$\theta_2(a) - \theta_1(a) = \phi_2^*(a) - \phi_1^*(\sigma a) = \phi_2^*(a) - \phi_1^*(\rho a)$$

is

$$\bar{\phi}_2(a) - \bar{\phi}_1(\rho(a)) = 0 \qquad \text{(cf. 2.10)},$$

so (2.11) (iii) holds.

This completes the proof of (2.11), and of the Theorem.

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