INTRODUCTION TO RESOLUTION OF SINGULARITIES

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ABSTRACT

These lectures will be strictly introductory in nature. The object is to communicate some feeling for the resolution problem by focusing on a selected few of the many methods available for dealing with singularities of surfaces.

INTRODUCTION

The problem of resolution of singularities, simply put, is to prove (or disprove):

(RES) For any algebraic variety X over an algebraically closed field k there exists a proper map $f:Y \to X$, with Y non-singular (smooth), such that f is an isomorphism over some open dense subset U of X (i.e. f maps $f^{-1}(U)$ isomorphically onto U).

Of course we can generalize (RES) by allowing k to be, say, an excellent local ring[1], or by throwing k out altogether and letting X be an arbitrary reduced locally noetherian excellent scheme (in which case Y should be regular, i.e. the local rings of points on Y should be regular). We can also require that U be the set of all regular points of X, and that f be obtained by successive blowing up of nice subvarieties.[2]

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[1] In Hironaka's work this is found to be necessary, even for proving (RES) when k is an algebraically closed field.

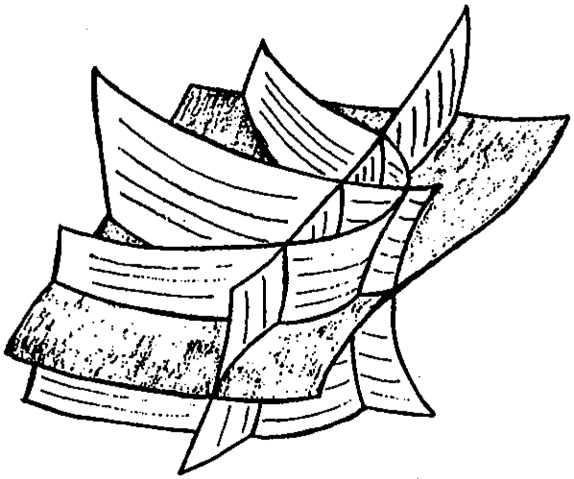
[2] For a definition and discussion of "blowing up" cf. [H1, chapter 0, §2].

But even as it stands (RES) poses a fundamental question, which is unanswered when k is of characteristic > 0 and X has dimension ≥ 4.[3]

In those cases where (RES) has been proved (cf. historical remarks below) the proof has usually been by induction on the dimension d of X. As part of the inductive step from $d - 1$ to d, it has been necessary to establish something akin to the following theorem ("Embedded Resolution" or "Simplification of the Boundary"), with $\dim . X' = d$:

(EMB) Let X' be a non-singular algebraic variety over k, and let W be a closed subvariety of X', with dense complement. Then there exists a proper map $f: Y \to X'$, with Y non-singular, such that f induces an isomorphism of $Y - f^{-1}(W)$ onto $X' - W$ and $f^{-1}(W)$ is a divisor on Y having only normal crossings.

"Normal crossings" means that for each $y \in Y$, the defining ideal of $f^{-1}(W)$ is generated locally around y by an element of the form $\xi_1^{a_1}\xi_2^{a_2}\ldots\xi_d^{a_d}$ where $\{\xi_1\xi_2,\ldots,\xi_d\}$ is a regular system of parameters (= local coordinate system) at y, and each $a_i \geq 0$.



Divisor with normal crossings

[3] Nevertheless, at present - as in the past - not many people are working on the problem. At this Institute, the only report on recent progress in the area was Hironaka's lecture on Giraud's theory of "maximal contact".

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Here are some brief historical indications.

(RES) was proved for curves over the complex numbers $\mathbb{C}$ in the last century by Kronecker, Max Noether, and others (cf. chapter VI in [N-B]). I will say a few things about the one-dimensional situation at the beginning of Lecture 1 below.

Several approaches to proving (RES) for X a surface over $\mathbb{C}$ were proposed by Italian geometers (cf. chapter 1 in [Z1]). One such approach, due to Albanese, has turned out to be quite practicable (Lecture 1, §5). The first completely rigorous proof was given by Walker in 1936 [W1]. What he did, basically, was to show how to patch together "local" resolutions constructed by Jung in 1908 [J]. His methods (and Jung's) were in part complex analytic; but the whole line of argument can be carried out algebraically, and this is what will be done in Lecture 2. Another proof based on Jung's work was given, in the context of analytic geometry, by Hirzebruch [Hz]. Jung's ideas have, in addition, influenced a number of other works on resolution (cf. end of Lecture 2).

Purely algebraic proofs for surfaces over fields of characteristic zero were first given by Zariski [Z2], [Z4][4] (Lecture 1, §§2, 3). Zariski's work on resolution in the late 1930's and early 1940's culminated in the memoir [Z6], in which he proved (RES) for surfaces, with f obtained by successively blowing up points and non-singular curves of maximal multiplicity ("theorem of Beppo Levi"), and then deduced (RES) for three-dimensional X (always with char.k = 0). More recently, Zariski found a simpler proof of the possibility of desingularizing surfaces by such a procedure [Z7]. A still simpler proof, due to Abhyankar [unpublished] will be presented in Lecture 3.

In his great 1964 paper [H1], Hironaka proved (RES) and (EMB) for any equicharacteristic zero excellent scheme. At the same time he solved the corresponding problems for real analytic spaces and locally for complex analytic spaces. In the last few years he has solved (affirmatively) the global resolution problem for complex spaces. The main ideas of this solution are outlined in the preprint [H6], and details have begun to appear [H7].

For k of characteristic > 0, most of the published results are due essentially to Abhyankar. (For an introduction to his work cf. [A4].) In his Harvard thesis (1954) Abhyankar proved (RES) for dim.X = 2 and k of any characteristic; some years later [A1] he disposed of the more general case where k is an excellent Dedekind domain with perfect residue fields. (In fact he has announced a proof for dim.X = 2 and k any excellent

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[4] All of Zariski's papers on resolution are in volume 1 of his collected works [Z8].

domain [A5, §5].) In his book [A3], Abhyankar deduced (EMB) for dim.X' = 3 from a previously developed algorithm of his on monic polynomials [A2]; and he also obtained (RES) for dim.X = 3 and char.k ≠ 2,3,5.

At the same time (January, 1967) that Abhyankar announced his most general results on surfaces [A5, §1 and §5], Hironaka announced a proof of (EMB) for X' an arbitrary excellent scheme and dim.W = 2.[5] Hironaka's results include Abhyankar's; but full details of the proof have not yet been published (cf. however [H2], [H3], [H4], [H5]).

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<u>Apology</u>. For lack of time, and because of its technical complexity, Hironaka's fundamental work will hardly be touched on. I can only express the hope that these lectures will help the reader to acquire the right frame of mind for exploring Hironaka's work on his own.

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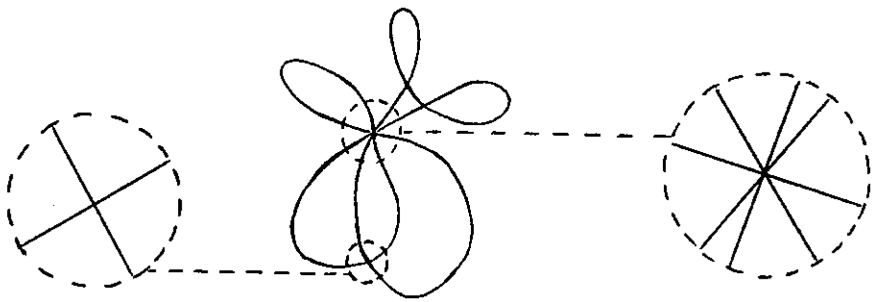
[5]Abhyankar and Hironaka were both at Purdue University during the Fall semester of 1966.

LECTURE 1: GENERALITIES ON CURVES AND SURFACES

§1. Curves.

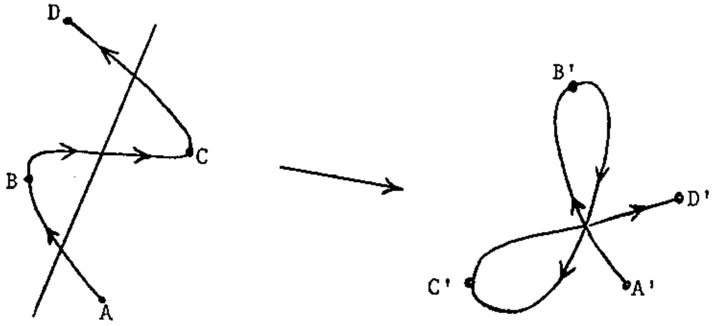
Max Noether dealt with singularities of projective plane curves, to which he applied a succession of well-chosen quadratic transformations of $\mathbb{P}^2$. [If the curve C has a singular point at Q, choose coordinates so that (i) $Q = (1,0,0)$; (ii) C has no other singular point on any of the coordinate axes $X_0 = 0$, $X_1 = 0$, $X_2 = 0$; and (iii) these axes are not tangent to C at any point. Now apply the transformation $(x_0,x_1,x_2) \to (x_1x_2,x_2x_0,x_0x_1)$. For details cf. e.g. [W2, chapter III, §7].]

A careful analysis of the effect of such transformations on the singular points shows that eventually the curve is transformed into one having only ordinary multiple points (i.e. points at which C looks locally like several distinct lines through the origin).



Ordinary multiple points

Such points are unavoidable because of the fact that global transformations are being used which blow the coordinate axes down to points; for each intersection of C with a coordinate axis, a branch of the transformed curve will pass through the corresponding point.



To avoid such irrelevancies, one can isolate the effect of the transformations on the points in which one is really interested (viz. the original singular points); this leads to the intrinsic (coordinate-free) notion of <u>locally quadratic transformation (blowing-up points)</u>. Quite generally:

> <u>successive blowing up of singular points transforms any projective curve</u> X <u>into a non-singular projective curve</u> Y;

so we have (RES) in this case.

Conceptually, one can achieve the same thing by one operation, namely <u>normalization</u> (cf. [Z-M]).[6] But the method of successive blowing up gives us a step-by-step analysis of the original singularities, leading to the notions of equivalent singularities, equisingularity, etc., (notions which cannot be pursued here).

Incidentally, the desingularization Y may not be embeddable in $\mathbb{P}^2$, even if X is. However any sufficiently general projection of a non-singular curve from $\mathbb{P}^n$ into $\mathbb{P}^3$ will still be non-singular, and into $\mathbb{P}^2$ will have at worst finitely many ordinary double points (nodes), cf. [Z-M], [S, p. 46, Theorem 1]. Thus <u>any curve is birationally equivalent to a non-singular curve in</u> $\mathbb{P}^3$, <u>and to a projective plane curve having only nodes</u>.

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[6] Both methods work equally well for any reduced excellent one-dimensional scheme X. The idea is that blowing up a <u>singular point</u> of X gets you a scheme lying "<u>strictly closer</u>" than X to the normalization $\bar{X}$; since $\bar{X}$ is finite over X, <u>after finitely</u> many blowing-ups you must get $\bar{X}$ itself, and $\bar{X}$ is regular.

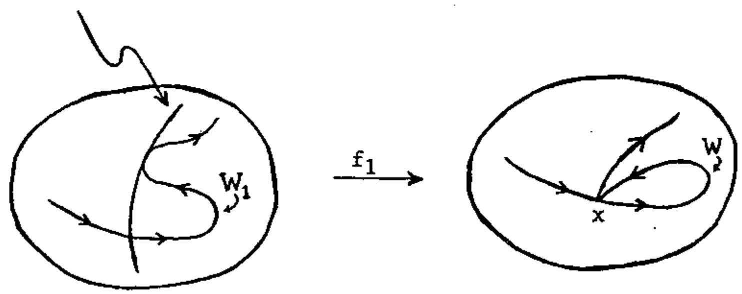
We also have (EMB) *for a curve* W *in a non-singular surface* X'. (Here again the method will work for one-dimensional subschemes of arbitrary two-dimensional regular excellent schemes.). We proceed as follows. Let $f_1 : X_1 \to X'$ be obtained by blowing up a point $x \in X'$ at which W has a singularity. f_1 induces an isomorphism over $X' - \{x\}$. Let W_1 be the closure in X_1 of $f_1^{-1}(W - \{x\})$. Then the map $W_1 \to W$ induced by f_1 is identical with the one obtained by blowing up x, considered as a point of W. Hence by repeating the process sufficiently often (blow up a singular point x_1 of W_1 to get $f_2 : X_2 \to X_1$, let W_2 be the closure in X_2 of $f_2^{-1}(W_1 - \{x_1\})$, etc. etc.) we *resolve the singularities of W*, so that we have

$$\begin{array}{ccccccc} W_n & \to & \cdots & \to & W_1 & \to & W \\ \downarrow & & & & \downarrow & & \downarrow \\ X_n & \to & \cdots & \to & X_1 & \xrightarrow{f_1} & X' \end{array}$$

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n$$

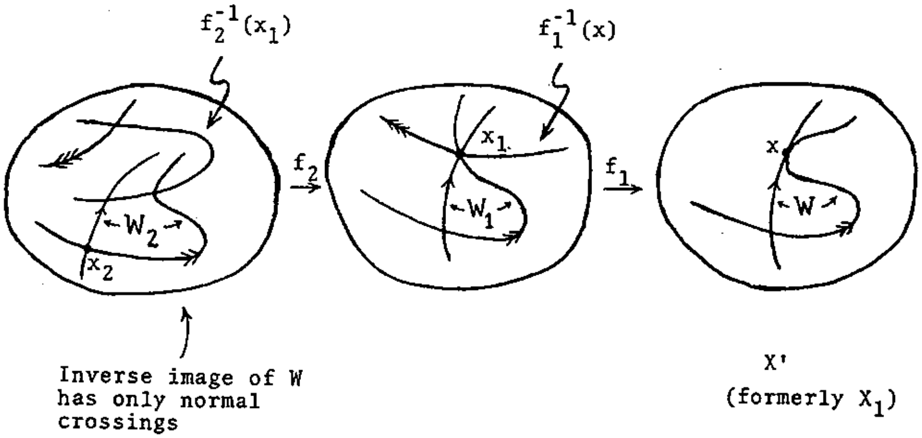
with W_n *non-singular*. (In the following picture $n = 1$).

$f_1^{-1}(x)$, a non-singular rational curve



X_1 X'

The irreducible curves in X_n whose image under F_n is a single point are all non-singular. So, after replacing X' by X_n and W by $F_n^{-1}(W)$, *we may assume that each irreducible component of* W *is non-singular*. But with this assumption on W it is easily seen that if $m \geq 1$ is such that W_m is non-singular (in other words, the irreducible components of W have become completely detached from each other in X_m), then $F_m^{-1}(W)$ has only normal crossings.



(Here, strictly speaking, we should blow up x_2 to detach the components of W_2 and obtain a non-singular W_3; but we needn't bother, since W_2 already has a normal crossing at x_2.)

§2. Local Uniformization.

We turn now to surfaces, and begin with Zariski's approach to the problem of desingularization. Until otherwise indicated, "surface" means "irreducible two-dimensional algebraic variety over a field k of characteristic zero".

Zariski's first step was to prove a much weakened version of (RES) called local uniformization:

(LU) Let X be a surface, and let v be a valuation of the function field k(X), with v centered at $x \in X$. Then there exists a birational projective map $f: Y \to X$ such that v is centered on Y at a point y where Y is smooth.

(Subsequently Zariski proved (LU) for varieties of any dimension over a field of characteristic zero [Z3].)

For surfaces Zariski gave two ways of deducing (RES) from (LU) [Z2], [Z4]. The second is much shorter, but the first gives us a nice canonical desingularization process (blowing up and normalizing) which we discuss in §3 below.

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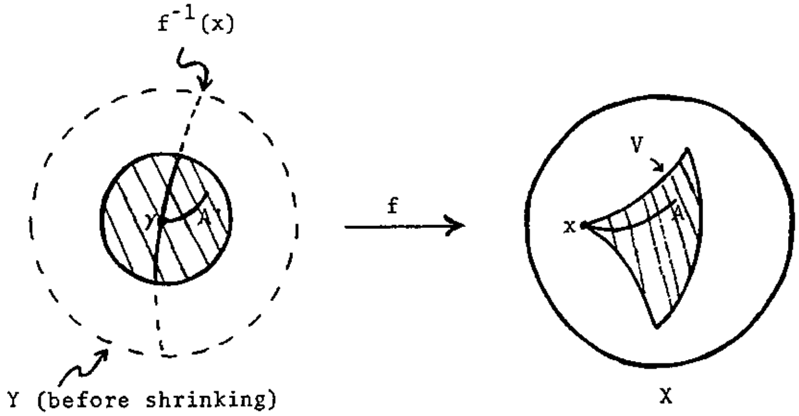
To get some sort of picture of what (LU) means when $k = \mathbb{C}$, think of a valuation centered at x as being a little arc A on X emanating from x, and (after replacing Y by some complex neighborhood of y) think of Y as being a bounded open disc around $y \in \mathbb{C}^2$, containing a little arc A' emanating from y, with $f(A') = A$. Here f is no longer birational, but rather a <u>holomorphic</u> map inducing an isomorphism from an open dense subset of Y onto a (complex) open set V of X, f^{-1} being given on V by <u>rational</u> functions.



(LU) says that X can be covered - not only pointwise, but, so to speak, arcwise - by such "local parametrizations"[7] (in fact by <u>finitely many</u> of them, if X is compact, as one sees using the compactness of the "Riemann manifold" cf. [Z5] and also the introduction to [Z3]).

<u>Example</u> (Jung [J]). Let x be the origin on the affine surface

$$X = \{(\xi,\eta,\zeta) \in \mathbb{C}^3 \mid \zeta^5 = \xi^3\eta\}$$

Consider the following three parametrizations of X, (u,v) being coordinates in the unit disc $D \subset \mathbb{C}^2$.

I.	$\xi = u^5v^3$	$\eta = v$	$\zeta = u^3v^2$
II.	$\xi = u^3v$	$\eta = uv^2$	$\zeta = u^2v$
III.	$\xi = u$	$\eta = u^2v^5$	$\zeta = uv$

Together these three parametrizations cover a neighborhood of x. For any valuation v centered at x, one of them can serve as the desired f. (And they are all required: for example to lift the arc $\xi = t^5$, $\eta = t^5$, $\zeta = t^4$ to an arc centered at the origin of the unit disc take $u = t$, $v = t^2$ in II; no such lifting is possible via I or III.) Of course the unit disc is

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[7] In [W1], these are called "wedges".

not an algebraic variety; but in I, for example, we can solve for u and v in terms of ξ, η, ζ:

$$u = \xi^2/\zeta^3, \qquad v = \eta \qquad (\zeta \neq 0)$$

so that I can be factored as

$$\begin{array}{ccccc} & \overset{\lambda}{\nearrow} & Y & \hookrightarrow & \mathbb{C}^4 \\ & & \downarrow & & \downarrow \text{ projection} \\ D & \xrightarrow[I]{} & X & \hookrightarrow & \mathbb{C}^3 \end{array}$$

where Y is the variety whose general point is $(\xi,\eta,\zeta,\ \xi^2/\zeta^3)$ [(ξ,η,ζ) being, by abuse of notation, the general point of X] and λ is the local analytic isomorphism given by

$$\lambda(u,v) = (u^5v^3,\ v,\ u^3v^2,\ u).$$

II and III can be treated similarly.

This example is too special in two respects. First of all, λ extends to an <u>algebraic</u> isomorphism from all of $\mathbb{C}^2$ onto Y; in general (for instance if we start with a non-rational surface X) this can't happen; our parametrizations will be given by <u>convergent power series</u> in u, v, rather than by polynomials.

Secondly, I, II and III happen to patch together into a single map $Z \to X$ which desingularizes an entire neighborhood of x. In general, if we cover a neighborhood of a point by local parametrizations, they will <u>not</u> patch together, nor will it be clear how to modify them so that they <u>will</u> patch. This is precisely the difficulty in deducing (RES) from (LU).

§3. Desingularization of Surfaces by Blowing up and Normalizing.

Let X be a surface, and let X_0 be the normalization of X. The set Σ_0 of singular points of X_0 is finite. Blow up Σ_0, and let X_1 be the normalization of the resulting surface. We have than a projective map $f_1: X_1 \to X_0$ inducing an isomorphism $X_1 - f_1^{-1}(\Sigma_0) \to X_0 - \Sigma_0$, and such that the inverse image of any point in Σ_0 is one-dimensional. Repeating the same process (blowing up the singular set and then normalizing) we get a sequence

$$X_n \xrightarrow{f_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{f_1} X_0.$$

Using (LU), Zariski showed in [Z2] that <u>this process must terminate, i.e. X_n is non-singular for some n</u>.

<u>Example.</u> Let S be the normal surface in $\mathbb{C}^3$ given by the equation $Z^2 + X^3 + Y^7 = 0$. The origin is the only singular point (in other words the only point where the first partial derivatives of $Z^2 + X^3 + Y^7$ all vanish). Blowing up this point, we obtain a surface S_1 which is again normal; in fact, if R is the coordinate ring of S:

$$R = \mathbb{C}[X,Y,Z]/(Z^2 + X^3 + Y^7) \cong \mathbb{C}[x,y,z]$$

then S_1 has an open covering by two affine surfaces S_1', S_1'', whose coordinate rings are respectively

$$R_1' = R[y/x,\ z/x] = \mathbb{C}[x,\ y/x,\ z/x] \cong \mathbb{C}[X,Y,Z]/(Z^2 + X + Y^7X^5)$$

$$R_1'' = R[x/y,\ z/y] = \mathbb{C}[x/y,\ y,\ z/y] \cong \mathbb{C}[X,Y,Z]/(Z^2 + X^3Y + Y^5);$$

S_1' is smooth, while S_1'' has just one singular point, at the origin.

Blowing up the singular point of S_1'', we get a surface S_2 covered by two affine surfaces S_2', S_2'' whose coordinate rings are respectively

$$R_2' = R_1''[y/(x/y),\ (z/y)/(x/y)] = \mathbb{C}[x/y,\ y^2/x,\ z/x]$$

$$\cong \mathbb{C}[X,Y,Z]/(Z^2 + X^2Y + X^3Y^5)$$

$$R_2'' = R_1''[(x/y)/y,\ (z/y)/y] = \mathbb{C}[x/y^2,\ y,\ z/y^2]$$

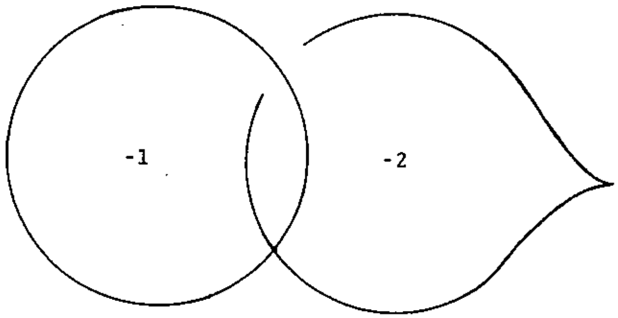
$$\cong \mathbb{C}[X,Y,Z]/(Z^2 + X^3Y^2 + Y^3).$$

The element $(z/x)/(x/y)$ is integral over R_2', and

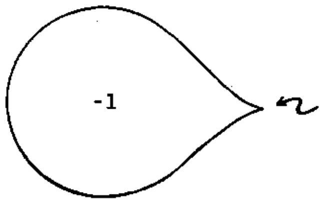
$$R_2'[(z/x)/(x/y)] = \mathbb{C}[x/y,\ y^2/x,\ zy/x^2] = \mathbb{C}[X,Y,Z]/(Z^2 + Y + XY^5).$$

Since the surface $Z^2 + Y + XY^5 = 0$ is smooth, S_2' has a smooth normalization. Similarly by adjoining $(z/y^2)/y$ to R_2'', we see that S_2'' has a smooth normalization. Thus the normalization $\bar{S}_2$ of S_2 is smooth, and so we have a desingularization $T \to S$ with $T = S_1' \cup \bar{S}_2$.

In this example, one finds that the inverse image on T of the singular point of S consists of a pair of rational curves E_1, E_2, with intersection numbers $(E_1 \cdot E_2) = 1$, $(E_1 \cdot E_1) = -1$, $(E_2 \cdot E_2) = -2$. E_1 is non-singular, while E_2 has a simple cusp (with completed local ring isomorphic to $\mathbb{C}[[U,V]]/(U^2 - V^3)$).



The curve E_1 can be blown down, so that $T \to S$ factors as $T \to T_0 \to S$, where T_0 is a smooth surface on which the inverse image of the singular point of S looks like this:



(T_0 can be obtained more directly from S by blowing up the ideal (x, y^2, z) and then normalizing.)

§4. Minimal Desingularization; Rational Singularities; Factorization Theorem.

It is a fact that any surface S over $\mathbb{C}$ has a unique minimal desingularization $S^* \to S$, i.e. a desingularization through which every other desingularization $f: S' \to S$ factors (uniquely):

$$\begin{array}{ccc} & \overset{f^*}{\nearrow} & S^* \\ & & \downarrow \\ S' & \xrightarrow{f} & S \end{array} \qquad \text{(diagram commutes)}$$

{This is mentioned in [H1], p. 151; a proof can be found in [B; Lemma 1.6]. For the same result in the case of arbitrary reduced two-dimensional excellent schemes cf. [L, p. 277, Cor. (27.3)]}.

The above example (§3) shows that successive blowing up and normalization does not always produce the minimal desingularization.

There is one class of normal singularities for which Zariski's process does give the minimal desingularization - the class of rational singularities. This fact, along with other applications of rational singularities in resolution questions, comes out of the following properties, [L, parts I and II].

Let X be a normal surface having only rational singularities[8] (X may be smooth) and let $f:Y \to X$ be a proper birational map. Then:

(A) If Y is normal, then Y has only rational singularities.

(B) If f is obtained by blowing up a point of X, then Y is normal.

(C) (Factorization Theorem) If all the local rings of points on Y are factorial, then f is obtainable by successively blowing up points.

[(A) and (B) enter into the proof of (C).]

(C), with Y the minimal desingularization of X, shows that X can be minimally desingularized by blowing up points (normalization being unnecessary).

Furthermore, using suitably formulated local versions of (A) and (C), one can make Zariski's proof that (LU) ⇒ (desingularization by blowing up and normalization) work for any reduced excellent two-dimensional scheme [L, §2]. In this context (LU) can be taken to mean:

(LU)(bis) If R is an excellent two-dimensional local domain and v is a valuation of the quotient field of R such that the valuation ring R_v contains R, then there exist elements $x_1, x_2, \ldots, x_n$ in R_v such that the local ring

$$R[x_1, x_2, \ldots, x_n]_q \quad (q = \{x \in R[x_1, \ldots, x_n] \mid v(x) > 0\})$$

is regular.

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[8] In case X is proper over a field, what this means is tnat $\chi(\mathcal{O}_X) \le \chi(\mathcal{O}_{X'})$ for any proper normal surface X' birationally equivalent to X (where $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$ = arithmetic genus of X.)

To prove (LU)(bis), even when R is complete (a special case to which the general one reduces without much trouble), is quite difficult. As indicated before, proofs were announced by Abhyankar and Hironaka in 1967, but the complete details have yet to appear in print.

§5. Albanese's Method.

A pretty method of Albanese [Alb] was revived in 1963 by M. Artin, who showed [unpublished] that it could be used to simplify greatly the resolution problem for surfaces over fields of any characteristic. The method extends in a straightforward way to irreducible projective varieties V of any dimension d over an algebraically closed field k. We sketch the basic idea[9].

For convenience we shall say that "$\pi: W \to W_1$ is a permissible projection" if W is a positive-dimensional irreducible closed subvariety of a projective space $\mathbb{P}^M$ over k, π is the projection of $\mathbb{P}^M$ into $\mathbb{P}^{M-1}$ from a point w on W of multiplicity $\mu < \deg.W$ ("deg" = "degree of"), and W_1 is the closure in $\mathbb{P}^{M-1}$ of $\pi(W-w)$. The condition $\mu < \deg.W$ guarantees that $\dim.W_1 = \dim.W$, so that we have a finite algebraic field extension $k(W)/k(W_1)$, whose degree we denote by $[W:W_1]$ (the generic covering degree of W over W_1). It is easily seen that

$$[W:W_1](\deg.W_1) = \deg.W - \mu \tag{5.1}$$

(Cut W_1 by a generic linear space L of complementary dimension in $\mathbb{P}^{M-1}$; then consider $\pi^{-1}(L)\cap W\ldots$).

Assume that $V = V_0 \subseteq \mathbb{P}^N$ and that V is not contained in any hyperplane of $\mathbb{P}^N$. Let r be a fixed integer. We are going to try to find a succession of permissible projections

$$V_0 \xrightarrow{\pi_0} V_1 \xrightarrow{\pi_1} V_2 \longrightarrow \cdots \xrightarrow{\pi_{n-1}} V_n \tag{5.2}$$

such that

$$\text{every point on } V_n \text{ has multiplicity} \le r/[V_0:V_n]. \tag{5.3}$$

By the "projection formula", it will then follow that on the normalization of V_n in $k(V_0)$ - a variety birationally equivalent to V - every point has multiplicity $\le r$.

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[9] For the raw technical details - which are quite elementary - cf. [A3, §12]. Abhyankar uses the result in his proof of (LU) for 3-dimensional varieties over fields of characteristic > (3!).

Of course we want r to be as small as possible, and in fact, as we will now see, if we begin with a suitable projective embedding of V, then we can succeed with $r = d!$ $(d = \dim.V)$. In particular, for curves $(d = 1)$, this will give another proof of (RES).

We begin by projecting into $\mathbb{P}^{N-1}$ from a point v_0 on V_0 of multiplicity $> r$ and $< \deg.V$ (if there is such a point); call this projection π_0, and let V_1 be the closure of $\pi_0(V_0 - v_0)$. Next, choose a point v_1 on V_1 of multiplicity $> (r/[V_0:V_1])$ and $< \deg.V_1$, and project into $\mathbb{P}^{N-2}$ from v_1; call this projection π_1, and let V_2 be the closure of $\pi_1(V_1 - v_1)$. Now choose a point v_2 on V_2 of multiplicity $> (r/[V_0:V_2])$ and $< \deg.V_2$, etc.etc.etc. Continue in this way as long as possible; obviously the process must stop after a finite number of steps: in other words we obtain a sequence (5.2) such that every point v on V_n of multiplicity $>(r/[V_0:V_n])$ must have multiplicity $= \deg.V_n$ (i.e. V_n is a cone with vertex v). So it will suffice to show that V_n cannot be a cone.

First of all, (5.1) gives

$$[V_0:V_1]\deg.V_1 = \deg.V_0 - (\text{mult. of } v_0) \le \deg.V_0 - r - 1.$$

Similarly,

$$[V_1:V_2]\deg.V_2 \le \deg.V_1 - (r/[V_0:V_1]) - 1$$

whence

$$[V_0:V_2]\deg.V_2 \le [V_0:V_1]\deg.V_1 - r - [V_0:V_1] \le \deg V_0 - 2r - 2.$$

Continuing in this way, we find that

$$[V_0:V_n]\deg.V_n \le \deg.V_0 - n(r + 1).$$

So we have the crude inequality

$$(5.4) \qquad n(r + 1) \le \deg.V_0.$$

Next, if V_n is a cone with vertex v, then a generic linear space L through v of dimension $N - n - d + 1$ will intersect V_n in $\deg.V_n$ distinct lines. Hence the inverse image in $\mathbb{P}^N$ of L (under the composed projection $\pi_{n-1}\circ\pi_{n-2}\circ\cdots\circ\pi_0$) will cut V_0 in a curve having at least $\deg.V_n$ irreducible components. Setting

$$c = c(V_0) = \min\{\text{degree of irreducible curves on } V_0\}$$

we conclude that

$$(5.5) \qquad (\deg.V_n)c \le \deg.V_0$$

But V_n is not contained in any hyperplane of $\mathbb{P}^{N-n}$ (otherwise, taking inverse images, we would find that V_0 is contained in a hyperplane of $\mathbb{P}^N$, contrary to assumption). An easy argument shows that consequently

$$\deg . V_n > N - n - d.$$

So from (5.5) we get

$$(N - n - d)c < \deg . V_0$$

i.e.

(5.6) $$N - d - c^{-1}\deg . V_0 < n.$$

Combining (5.6) and (5.4), we have

$$N - d - c^{-1}\deg . V_0 < (r + 1)^{-1}\deg . V_0 .$$

We conclude, therefore, that <u>if</u>

(5.7) $$r + 1 \geq \deg . V_0/(N - d - c^{-1}\deg . V_0)$$

<u>then</u> V_n <u>is not a cone</u> (and hence every point on V_n has multiplicity $\leq r/[V_0:V_n]$).

Now finally consider the embedding

$$V \xrightarrow{\approx} V_0^\delta \subseteq \mathbb{P}^{N(\delta)}$$

of V via the linear system of sections of V by hypersurfaces in $\mathbb{P}^N$ of degree δ (or equivalently, for large δ, via the very ample sheaf $\mathcal{O}_V(\delta)$). V_0^δ is not contained in any hyperplane of $\mathbb{P}^{N(\delta)}$. Any curve on V_0^δ has degree $\geq \delta$, since cutting by hyperplanes in $\mathbb{P}^{N(\delta)}$ corresponds to cutting by hypersurfaces of degree δ in $\mathbb{P}^N$; thus $c_\delta = c(V_0^\delta) \geq \delta$. For large δ, $N(\delta) + 1$ is given by $\chi(\mathcal{O}_V(\delta))$, while $(1/d!)\deg . V_0^\delta$ is the coefficient of n^d in the polynomial $\chi(\mathcal{O}_V(n\delta))$. From these remarks, it follows that

$$\lim_{\delta \to \infty} \deg . V_0^\delta/(N(\delta) - d - c_\delta^{-1}\deg . V_0^\delta) = d!$$

So for large δ, if $V_0 = V_0^\delta$, then (5.7) holds with $r = (d!)$.

In summary, the argument has given us the <u>existence of a normal variety birationally equivalent to</u> V, <u>and on which each point has multiplicity</u> $\leq (d!)$ $(d = \dim . V)$.

*　　　*　　　*

If V is a surface $(d = 2)$, then we have a surface birational to V, and on which every point has multiplicity ≤ 2. To prove resolution for surfaces as in §3 above, all that is really needed is the weaker form of (LU) in which Y is required only to be birationally equivalent to X (the map f need not exist.) So we are reduced to "uniformizing" valuations centered at double points.

Taking advantage of the good behavior of blowing up and normalization vis-à-vis completion, we come down to proving (LU)(bis) for complete local rings R of multiplicity 2. In such an R there exist elements u, v, such that the field of fractions of R is the same as that of a ring

$$R' = k[[u,v]][w] \qquad w^2 = f(u,v)w + g(u,v)$$

$(f,g \in k[[u,v]],\ g \neq 0;$ and $f = 0$ when char. $k \neq 2)$. There is no harm in assuming that $R = R'$. An application of (EMB) to the curve $g = 0$ (resp. $fg = 0$ if $f \neq 0$) in the surface $\mathrm{Spec}(k[[u,v]])$ (cf. §1) gives us a further reduction to the situation where

$$g(u,v) = \mu(u,v)u^a v^b$$

and, in case $f \neq 0$,

$$f(u,v) = \nu(u,v)u^c v^d$$

where μ and ν are units in $k[[u,v]]$. At this point, if the characteristic of k is $\neq 2$, then after normalizing, at most one blowing up will be needed to resolve the singularity of R.

On the other hand, when k has characteristic 2, the analysis of the situation is much harder. It is recommended to the interested reader to play around a little with this case, as a gentle initiation into the intricacies of Abhyankar's algorithms.

LECTURE 2: JUNG'S METHOD FOR LOCAL DESINGULARIZATION

We noted in Lecture 1 that for surfaces resolution of singularities — for example by a succession of blowing-ups (of points) and normalizations — can be deduced from local uniformization. A fruitful method for achieving local uniformization on a surface X over an algebraically closed field k of characteristic zero was given by Jung in 1908 [J]. Jung's ideas have had a great influence on subsequent work on the resolution problem: last time the work of Walker and Hirzebruch was mentioned, and further instances will be indicated at the end of this lecture.

Actually, as observed by Walker [W1], a slight elaboration of Jung's method gives us a "local resolution" theorem which says considerably more than local uniformization:

THEOREM: *There exists a family of desingularizations* $(f_x : Y_x \to U_x)_{x \in X}$,[1] *where* U_x *is an open neighborhood of* x, *the restriction of* f_x *to* $Y_x - f_x^{-1}(\{x\})$ *is the normalization of* $U_x - \{x\}$, *and furthermore for all* x *outside some zero-dimensional subset* S *of* X, f_x *is just the normalization of* U_x *itself*.

Resolution for X follows at once from the Theorem: assuming, without loss of generality, that $U_x \cap S = \{x\}$ for each x in S, cover X by the open sets U_x $(x \in S)$ together with the open set $U_0 = X - S$; let $f_0 : Y_0 \to U_0$ be the normalization of U_0; then because of the uniqueness properties of normalization it is evident that the local desingularizations f_x $(x \in S)$ and f_0 will patch together to give a global desingularization $f : Y \to X$, where the restriction of f to $Y - f^{-1}(S)$ is the normalization of $X - S$.[2]

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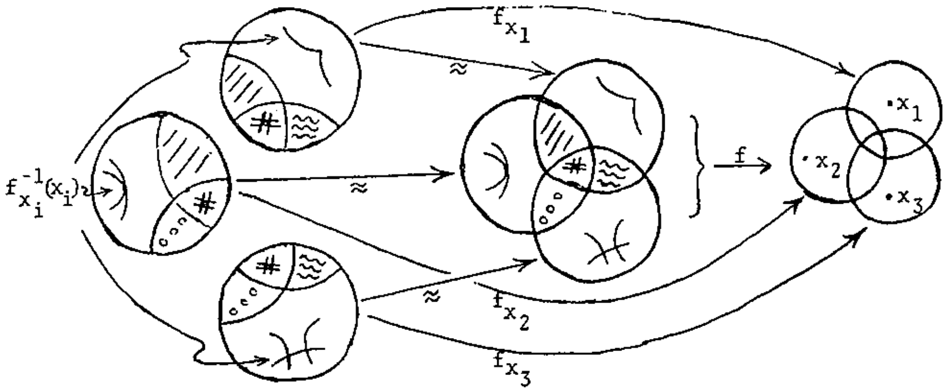
[1] In other words Y_x is smooth, f_x is proper, and for some open dense subset V_x of U_x, f_x maps $f_x^{-1}(V_x)$ isomorphically onto V_x.

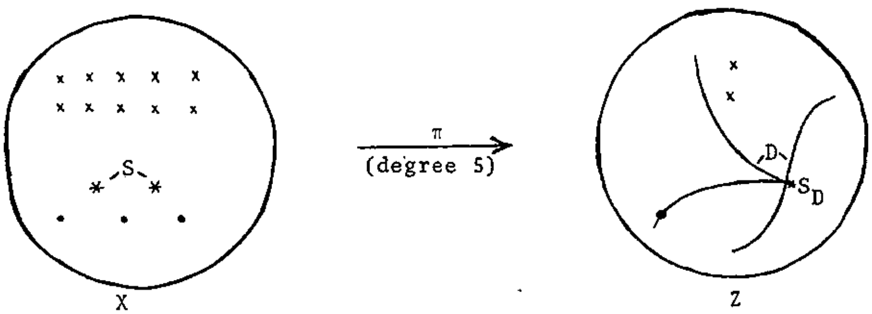
[2] cf. also [W1, p.343, Theorem 4].



f_{x_1}, f_{x_2}, f_{x_3} induce same map (normalization) over overlaps; so they patch.

* * *

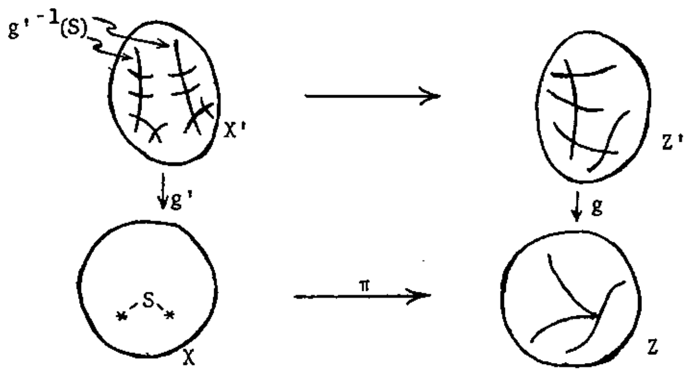
Now let us outline a <u>proof of the Theorem.</u> The Theorem is local on X, so we may assume that there exists a finite map (= branched covering) $\pi: X \to Z$ of X onto an open subset Z of affine 2-space over k. Let $D \subseteq Z$ be the <u>critical</u> (or <u>discriminant</u>) <u>variety</u> for π, i.e. the smallest closed subset of Z over whose complement π is an etale covering. [Equivalently: D consists of those z in Z such that the number of (geometric) points in $\pi^{-1}(z)$ is less than the covering degree of π.] D has dimension ≤ 1.



The first step is to reduce to the case where D has only normal crossings. This is done by applying embedded resolution to $D \subset Z$, as follows. Let S_D be the (zero-dimensional) set of singular points of D, and let $S = \pi^{-1}(S_D)$. As we saw in Lecture 1, there exists a map $g: Z' \to Z$, obtained as a succession of blowing-ups (of points), such that g induces an isomorphism $Z' - g^{-1}(S_D) \to Z - S_D$ and such that $g^{-1}(D)$ is a curve having only normal crossings, together with some isolated points (corresponding to the isolated points - i.e. the zero-dimensional components - of D). Let X' be the fibre product $X \times_Z Z'$ and let $g': X' \to X$ be the projection. Then g' induces an isomorphism

$$X' - g'^{-1}(S) \xrightarrow{\sim} X - S.$$

It will clearly be enough, therefore, to find a desingularization $f':Y' \to X'$ which induces the normalization map over $X' - g'^{-1}(S)$.



We will show that:

(*) <u>Every singular point of</u> $\tilde{X}'$, <u>the normalization of</u> X', <u>can be resolved by blowing up a zero-dimensional ideal in its local ring</u>.

("Zero-dimensional" means "containing some power of the maximal ideal".)

The existence of the desired $f':Y' \to X'$ follows immediately, since all the singularities of the normal surface $\tilde{X}'$ are isolated. (In fact, as will drop out of the following proof, the image of every singular point of $\tilde{X}'$ under the composed map $\tilde{X}' \to X' \to Z'$ is a double point of $g^{-1}(D)$.)

* * *

To prove (*), we begin by formulating it in more algebraic terms. Consider a two-dimension regular local ring Q with residue field k (algebraically closed, characteristic zero) and field of fractions K; let $\tilde{Q}$ be the integral closure of Q in a finite field extension L of K, and let R be the localization of $\tilde{Q}$ at one of its maximal ideals. Let u, v generate the maximal ideal of Q. The problem is then to show that <u>if</u> $\tilde{Q}[1/uv]$ <u>is étale over</u> $Q[1/uv]$ (i.e. $\mathrm{Spec}(\tilde{Q})$ is étale over $\mathrm{Spec}(Q)$ outside the two "lines" $u = 0$, $v = 0$), <u>then</u> $\mathrm{Spec}(R)$ <u>can be desingularized by blowing up a zero-dimensional ideal</u>.

For this, we may pass to completions[3], i.e. <u>we may assume that</u> Q <u>is a power series ring</u> $k[[u, v]]$ <u>and</u> $R = \tilde{Q}$.

[3] This would not be necessary if we were in the analytic category.

Suppose then that $R[1/uv]$ is étale over $Q[1/uv]$. A basic observation is that for some integer $\alpha > 0$

$$K = k((u,v)) \subseteq L \subsetneq k((u^{1/\alpha}, v^{1/\alpha})).$$

(This was proved classically by topological methods, the point being that the complex plane minus two intersecting lines is topologically the product of two punctured discs, so its fundamental group is $\mathbb{Z} \times \mathbb{Z}$. A purely algebraic proof was given by Abhyankar [Amer. J. Math., 77 (1955) p. 585].) By Galois theory we find that L is generated over $k((u, v))$ by a collection of "monomials" $u^{\beta/\alpha}v^{\gamma/\alpha}$ where the pair (β,γ) runs through some subgroup of $\mathbb{Z} \times \mathbb{Z}$. Elementary considerations then show that

$$L = k((u^{1/c}, v^{1/d}))(u^{1/cn}v^{p/dn})$$

where c, d, n, p are non-negative integers (with nc and d both dividing α), $p < n$, and $(n, p) = 1$. Setting $\bar{u} = u^{1/c}$, $\bar{v} = v^{1/d}$, we have

$$L = k((\bar{u}, \bar{v}))(w) \qquad\qquad (w = \bar{u}^{1/n}\bar{v}^{p/n}).$$

From now on we write u for $\bar{u}$ and v for $\bar{v}$. Thus:
R <u>is the normalization of the ring</u>

$$R_0 = k[[u, v, w]] = k[[U, V, W]]/(W^n - UV^p);$$

<u>more explicitly</u>:

(**) $\quad R = k[[u^{1/n}, v^{1/n}]] \cap L =$ free $k[[u, v]]$-module with basis

$$(u^{i/n}v^{p(i)/n})_{\,0 \le i < n}$$

where $\quad 0 \le p(i) < n$

and

$$p(i) \equiv pi \pmod{n}^{4}.$$

<u>Example</u> $(n = 5, p = 2)$.

$$R = k[[u,v,u^{1/5}v^{2/5},u^{2/5}v^{4/5},u^{3/5}v^{1/5},u^{4/5}v^{3/5}]]$$

$$= k[[u,v,u^{1/5}v^{2/5},u^{3/5}v^{1/5}]].$$

---

[4] R is the ring of invariants of the cyclic group of k-automorphisms of $k[[u^{1/n},v^{1/n}]]$ generated by ϕ, where $\phi(u^{1/n}) = \varepsilon^{n-p}u^{1/n}$, $\phi(v^{1/n}) = \varepsilon v^{1/n}$ (ε = primitive n-th root of unity). A detailed study of such "quotient singularities" has been carried out recently by Riemenschneider [Math. Ann. 209 (1974), pp. 211-248].

(Remark. We note in passing that if $R[1/v]$ is étale over $Q[1/v]$, then $n = 1$ and R is regular. Hence every point of $\bar{X}'$ which does not lie over a double point of $g^{-1}(D)$ is already smooth.)

It is convenient to work with R_0 instead of R. As we will soon see, it is quite simple to desingularize $\mathrm{Spec}(R_0)$; and of course every desingularization of $\mathrm{Spec}(R_0)$ factors through $\mathrm{Spec}(R)$, so we will have a desingularization of $\mathrm{Spec}(R)$ too. In fact the procedure to be used will indicate how to compute explicitly a zero-dimensional ideal in R whose blowing up is a desingularization, thereby proving (*).

One can avoid the explicit computation - at some cost, perhaps, in understanding - by using the fact that <u>any desingularization of a two-dimensional normal local ring is obtainable by blowing up a zero-dimensional ideal</u>. (Because of the negative-definiteness of the intersection matrix of the components of the closed fibre, there exists a relatively ample invertible sheaf supported on the closed fibre...)

Alternatively, since R is a quotient singularity we know - once some desingularization has been shown to exist - that R has a <u>rational singularity</u> (cf. E. Brieskorn, Inventiones math. <u>4</u> (1968), p. 340]). Hence (Lecture 1, §4) R <u>can be minimally desingularized by successively blowing up points</u>, and from this (*) follows at once.

* * *

<u>To desingularize</u> $\mathrm{Spec}(R_0)$, let

$$\cdots \to \Sigma_{j+1} \xrightarrow{\phi_j} \Sigma_j \to \cdots \xrightarrow{\phi_1} \Sigma_1 \xrightarrow{\phi_0} \Sigma_0 = \mathrm{Spec}(R_0)$$

be the sequence such that ϕ_j is obtained by blowing up the unique reduced irreducible subscheme L_j of Σ_j whose image in Σ_0 is the "line" $v = w = 0$. It is routine to verify (cf. following example) that there is, for each j, just one closed point σ_j on L_j, that all other closed points on Σ_j are regular, and that the maximal ideal in the local ring of σ_j on Σ_j is generated by three elements u_j, v_j, w_j satisfying a relation

$$w_j^{n_j} = u_j v_j^{p_j} \qquad (0 < n_j,\ 0 \le p_j).$$

Moreover, if $p_j \neq 0$, then

$$n_{j+1}p_{j+1} < n_j p_j.$$

Hence for some $J >> 0$, we must have $p_J = 0$; so σ_J is regular, and $\Sigma_J \to \Sigma_0$ is a desingularization.

The following example should make clear what's going on.

Example (n = 10, p = 7).

$R_0 = k[[u,v,w]]$ $\qquad\qquad$ $w^{10} = uv^7$

L_0 is given by $v = w = 0$; blowing this up, we get

(1) $$\Sigma_1 = \mathrm{Spec}(R_1) \cup \mathrm{Spec}(R_1')$$

$$R_1 = R_0[v/w] \qquad\qquad R_1' = R_0[w/v].$$

Setting $u_1 = u$, $v_1 = v/w$, $w_1 = w$, we have

$$w_1^3 = u_1 v_1^7$$

$$u = u_1, \quad v = v_1 w_1, \quad w = w_1.$$

The inverse image of L_0 in $\mathrm{Spec}(R_1)$ has two components, given by $v_1 = w_1 = 0$, $u_1 = w_1 = 0$; the second maps onto $u = v = w = 0$, so $L_1 \cap \mathrm{Spec}(R_1)$ is $v_1 = w_1 = 0$. In a similar way, we see that $L_1 \cap \mathrm{Spec}(R_1')$ is empty.

Next, $(u,v,w)R_1' = vR_1'$, and the ideal $(u,v,w)R_1$ is invertible wherever $v_1 \neq 0$ (at such points u_1 is a multiple of w_1); thus (u,v,w) is <u>invertible</u> on $\Sigma_1 - L_1$. It follows that any point outside L_1 is regular (note that the quotient of the corresponding local ring by the principal ideal (u,v,w) is regular...). Finally, there is just one closed point σ_1 on L_1, defined by the maximal ideal $(u_1,v_1,w_1)R_1$.

Now blow up L_1 to get Σ_2. Since w_1/v_1 is integral over R_1, we find that

(2) $$\Sigma_2 = \mathrm{Spec}(R_2) \cup \mathrm{Spec}(R_1')$$

$$R_2 = R_1[w_1/v_1] = R_0[v/w,\ w^2/v].$$

Setting $u_2 = u_1 = u$, $v_2 = v_1 = v/w$, $w_2 = w_1/v_1 = w^2/v$, we have

$$w_2^3 = u_2 v_2^4.$$

Here L_2 is the subscheme of $\mathrm{Spec}(R_2)$ defined by $v_2 = w_2 = 0$, and σ_2 is the point $u_2 = v_2 = w_2 = 0$.

Blowing up L_2, we get

(3) $$\Sigma_3 = \mathrm{Spec}(R_3) \cup \mathrm{Spec}(R_1')$$

$$R_3 = R_2[w_2/v_2] = R_0[v/w,\ w^3/v^2]$$

and $$w_3^3 = u_3v_3 \qquad (u_3 = u,\ v_3 = v/w,\ w_3 = w^3/v^2).$$

$L_2 \subseteq \mathrm{Spec}(R_3)$ is defined by $v_3 = w_3 = 0$.

(4)
$$\textstyle\sum_4 = \mathrm{Spec}(R_4) \cup \mathrm{Spec}(R_4') \cup \mathrm{Spec}(R_1')$$
$$R_4 = R_3[v_3/w_3] = R_0[w^3/v^2,\ v^3/w^4]$$
$$R_4' = R_3[w_3/v_3] = R_0[v/w,\ w^4/v^3]$$
$$w_4^2 = u_4v_4 \qquad (u_4 = u,\ v_4 = v^3/w^4,\ w_4 = w^3/v^2)$$

$\mathrm{Spec}(R_4')$ is regular, and $L_4 \subseteq \mathrm{Spec}(R_4)$ is given by $v_4 = w_4 = 0$.

(5)
$$\textstyle\sum_5 = \mathrm{Spec}(R_5) \cup \mathrm{Spec}(R_5') \cup \mathrm{Spec}(R_4') \cup \mathrm{Spec}(R_1')$$
$$R_5 = R_4[v_4/w_4] = R_0[v^5/w^7]$$
$$R_5' = R_4[w_4/v_4] = R_0[v^3/w^4,\ w^7/v^5]$$
$$w_5 = u_5v_5 \qquad (u_5 = u,\ v_5 = v^5/w^7,\ w_5 = w^3/v^2).$$

The point $\sigma_5 \in \mathrm{Spec}(R_5)$ defined by the ideal $(u_5,v_5,w_5) = (u_5,v_5)$ is regular, and $\sum_5$ is a desingularization of $\mathrm{Spec}(R_0)$, as desired.

REMARKS 1. In the above desingularization process the map ϕ_j is <u>finite</u> precisely when $n_j < p_j$. When $n_j > p_j$, then $\phi_j^{-1}(\sigma_j) \cong \mathbb{P}^1$.

2. In the preceding example, the rings R_5, R_5', R_4', R_1' appearing in the expression for $\sum_5$ each give rise to a "wedge" of $\mathrm{Spec}(R_0)$ (cf. Lecture 1, §2). From R_5', for example, we get, with $v_* = v^3/w^4$, $w_* = w^7/v^5$,

$$u = v_*w_*^2 \qquad v = v_*^7w_*^4 \qquad w = v_*^5w_*^3.$$

This is precisely the kind of wedge which Jung constructed in his proof of local uniformization. (The example in Lecture 1, §2, corresponds to $n = 5$, $p = 3$.)

3. Walker observed that Jung's wedges (cf. remark 2) could always be pasted together to give a local desingularization. (This is illustrated by the preceding example.) A more explicit description of what happens (for any n, p) was given by Hirzebruch [Hz]: let

$$\frac{n}{n-p} = b_1 - \frac{1}{b_2 -}\,\frac{1}{b_3 -}\cdots\frac{1}{b_s} \qquad \text{(continued fraction; each } b_j \geq 2)$$

Then the above described desingularization $\sum$ of R_0 is covered by $\mathrm{Spec}(T_i)$, $0 \le i \le s$,

$$T_i = R_0[v^{\nu_i}/w^{\mu_i}, w^{\mu_{i+1}}/v^{\nu_{i+1}}] = R[v^{\nu_i}/w^{\mu_i}, w^{\mu_{i+1}}/v^{\nu_{i+1}}]$$

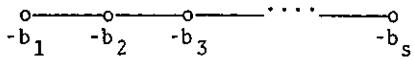
where the integers μ_j, ν_j $(0 \le j \le s+1)$ are defined inductively by

$$\mu_0 = 0,\ \mu_1 = 1, \quad \mu_{j+1} = b_j\mu_j - \mu_{j-1}$$

$$\nu_0 = 1,\ \nu_1 = 1, \quad \nu_{j+1} = b_j\nu_j - \nu_{j-1}.$$

(It can be shown that $\mu_{s+1} = n$, $\nu_{s+1} = p$.)

Furthermore, the desingularization $\sum$ is <u>minimal</u> (cf. Lecture 1, §4). The reduced closed fibre, as given by Hirzebruch, is a chain of non-singular rational curves intersecting transversally, and with self-intersections $-b_1, -b_2, \ldots, -b_s$.



<u>Example</u> (n = 10, p = 7, as in the above example.)

$$\frac{10}{3} = 4 - \frac{1}{2-}\,\frac{1}{2} \qquad (b_1 = 4,\ b_2 = 2,\ b_3 = 2)$$

μ: 0, 1, 4, 7, 10.

ν: 1, 1, 3, 5, 7 .

$T_0 = R_1'$ $\quad T_1 = R_4'$ $\quad T_2 = R_5'$ $\quad T_3 = R_5$.

<u>4</u>. (This was shown to me by E. Viehweg.) We can get $\sum$ from R by blowing up the ideal I generated by the elements $c_0, c_1, \ldots, c_s$ defined by

$$c_0 = v^{\nu_1}v^{\nu_2}\ldots v^{\nu_s}$$

$$c_i = c_{i-1}(w^{\mu_i}/v^{\nu_i}) \qquad (0 < i \le s).$$

Indeed,

$$T_i = R[Ic_i^{-1}] = R[\frac{c_0}{c_i}, \frac{c_1}{c_i}, \ldots, \frac{c_s}{c_i}].$$

I is not zero-dimensional, but consider the following elements a_i, b_i $(0 \le i \le s)$:

let $\mu = \mu_1 + \mu_2 + \ldots + \mu_s$, choose an integer $r > \mu/n$, and set

$$a_i = c_i v^{rn-\mu-rp}$$

$$b_i = c_i(u^r/w^\mu) = a_i(w/v)^{rn-\mu}$$

Using the fact that $n\nu_j > p\mu_j$ $(0 \le j \le s)$, one checks that all the a_i and b_i are in R; and the ideal $\bar{I}$ which they generate is <u>zero-dimensional</u> (since $b_s = u^r$ and $a_0 = v^\lambda$, $\lambda > 0$). Moreover, since

$$w/v \in T_0, \quad v/w \in T_i \qquad (0 < i \le s)$$

one sees that

$$\mathrm{Spec}(T_0) = \mathrm{Spec}(R[\bar{I}a_0^{-1}]) \supseteq \mathrm{Spec}(R[\bar{I}b_0^{-1}])$$
$$\mathrm{Spec}(T_i) = \mathrm{Spec}(R[\bar{I}b_i^{-1}]) \supseteq \mathrm{Spec}(R[\bar{I}a_i^{-1}]) \qquad (0 < i \le s).$$

Thus $\sum$ <u>is also gotten by blowing up</u> $\bar{I}$, and (*) is proved.

(This proof of (*) can be made somewhat shorter: just pull the ideal $\bar{I}$ out of a hat, and show directly that blowing it up gives a regular scheme!)

<u>5</u>. It is a curious fact, given without proof by Zariski in 1954 [Z8, p.521], that if $\sum' \to \mathrm{Spec}(R)$ is the blowing up of the closed point of Spec(R), then <u>every singular point on</u> $\sum'$ <u>is rational of type</u> A_m, i.e. the completion of its local ring is isomorphic to

$$k[[U,V,W]]/(W^{m+1} - UV) \qquad (m \ge 1).$$

More precisely, Zariski's result is that if we set (as we may, uniquely)

$$\frac{n}{n-p} = h_1 + \frac{1}{h_2+}\,\frac{1}{h_3+}\cdots\frac{1}{h_{2t+1}}$$

then the singularities on $\sum'$ are in one-one correspondence with the integers i such that $h_{2i} > 1$, the singularity corresponding to i being of type $A_{h_{2i}-1}$.

One way to see this is to begin by noting that if $E_1, E_2, \ldots, E_s$ are the components of the closed fibre on $\sum$ (remark 3), and if E is the cycle $E_1 + E_2 + \ldots + E_s$, then $E.E_i \le 0$ for all i $(1 \le i \le s)$, with equality if and only if $i \ne 1$, $i \ne s$, and $b_i = -2$. Thus E is the "fundamental cycle", and the theory of rational singularities tells us that

$$\sum' = \mathrm{Proj}(\bigoplus_{n\ge 0} H^0(\mathcal{O}(-nE)))\ .$$

From this, and from the relation between the b's and h's, we can reach the desired conclusion.

Presumably however there is a more direct and elementary proof, starting from the explicit description (**) of R. Since - as is easily shown blowing up a singularity of type A_m leads to a scheme with just one singularity of type A_{m-2} (and no singularities at all if $m = 1$ or 2), this will give us another way to show that R can be desingularized by blowing up only points. This type of desingularization descends through completion, so we will have another proof of (*). [Cf. also Remark at end of this Lecture.]

6. There is an illuminating approach to the problem of desingularizing R, due to D. Lieberman:

Any unramified covering of $C' = \mathbb{C}^2 - \{xy = 0\}$ can be realized by a map $\theta: C' \to C'$ of the form

$$(v,w) \to (v^d, v^{-pc}w^{nc}) \qquad (0 \le p < n;\ (n,p) = 1).$$

From a complex analytic point of view, the problem is to find a manifold Y' containing C' as a dense open subset such that θ extends to a proper map of $Y' \to \mathbb{C}^2$, with $\theta(Y' - C') \subseteq \{xy = 0\}$. It is well-known that this can be accomplished by starting with $\mathbb{C}^2 \supseteq C'$ and successively blowing up points. Making things explicit, one comes up with essentially the same formal calculations as before, but with a different motivation. (Roughly, what we did before was, instead of successively blowing up points of indeterminacy of θ, to blow up their inverse image on the graph of θ.)

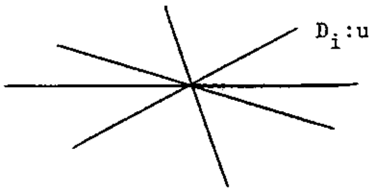
Algebraically speaking, let $\mathbb{C}\langle v,w\rangle$ be the ring of convergent power series, let $u = v^{-p}w^n$ and let $G = \mathbb{C}\langle v,w\rangle(u)$; then the completion of G at the maximal ideal generated by u, v, w is our old friend R_0 (with $k = \mathbb{C}$). Lieberman's approach produces a desingularization of G, and hence of R_0.

* * *

Examples. We illustrate Jung's method by describing local resolutions for some singularities of multiplicity two. Consider the origin (0,0,0) on the surface defined over k by $w^2 = f(u,v)$ $(f(0,0) = 0)$. If π is the projection to the (u,v)-plane, then the critical variety D is given by $f(u,v) = 0$.

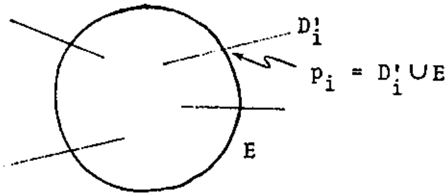
A. $w^2 = (u - a_1v)(u - a_2v)\dots(u - a_nv)$

($a_1, a_2, \dots, a_n$, distinct elements of k).



$D = \bigcup D_i$

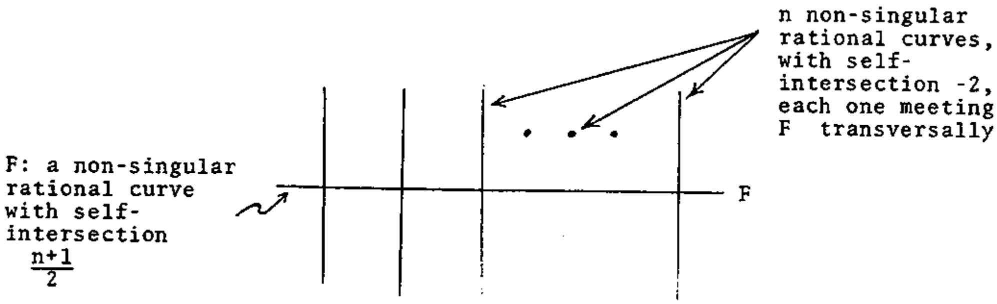
D is resolved by one blowing up:



(Here E is the curve coming out of the point $u = v = 0$, and D_i' is the proper transform of D_i).

When n is even, say $n = 2g + 2$, $\bar{X}'$ is already non-singular (notation as at the beginning of the lecture.) The inverse image on $\bar{X}'$ of the original singularity (0,0,0) is a non-singular double covering of E, ramified at the points $p_1, p_2, \ldots, p_{2g+2}$, hence hyperelliptic of genus $g = (n-2)/2$. The self intersection of this curve is -2.

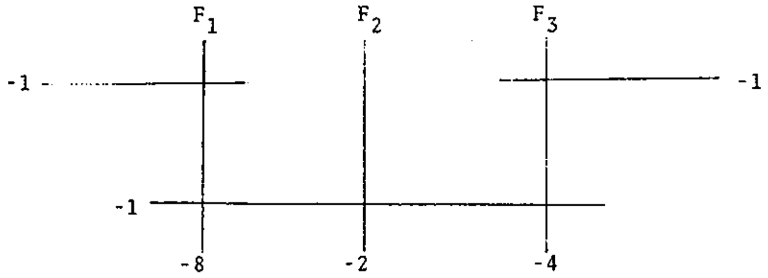
When n is odd, the inverse image of E on $\bar{X}'$ is a non-singular rational curve mapping isomorphically onto E. The singularities of $\bar{X}'$ occur at the points lying over the p_i; and they are rational, of type A_1, so they are each resolved by one blowing-up. On the resulting desingularization, the inverse image of (0,0,0) looks like



In either case, the resolution obtained is minimal.

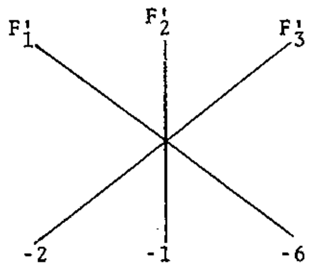
B. $$w^2 = -(u^3 + v^7)$$ (cf. Lecture 1, §3).

One finds that on $\bar{X}'$ there is just one singularity, which is rational of type A_1. Blowing this up, we get a desingularization of (0,0,0), on which the inverse image of (0,0,0) looks like

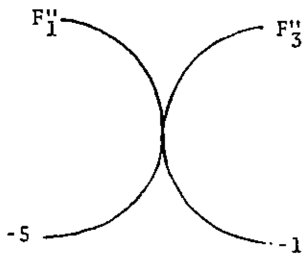


(All curves non-singular rational; all intersections transversal; self-intersections as shown.)

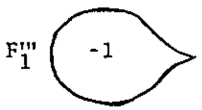
This is certainly not a minimal resolution. Blowing down the -1's, we get three non-singular rational curves meeting transversally at a point:



Blowing down F_2', we get two non-singular rational curves meeting tangentially at one point (intersection multiplicity 2):



Finally, blowing down F_3'', we get the minimal desingularization on which (as in Lecture 1) the inverse image of the original singularity (0,0,0) is an irreducible rational curve with a cusp, having self-intersection -1.



*   *   *

Here are three examples of the <u>influence of Jung's idea</u> of simplifying singularities by resolving critical varieties (for suitable projections).

I. As a graduate student, Abhyankar tried to adapt Jung's method to surfaces over fields of positive characteristic. He found (with notation as above) that even when $g^{-1}(D)$ has normal crossings, the structure of R may be quite horrible (cf. [Amer. J. Math. 77, (1955), 575-592]).

As an example of what may happen, consider the point $P = (0,0,0)$ on the surface

$$w^2 + wv^3 + u^5 = 0$$

over a field of characteristic 2. The critical variety for the projection into the (u,v)-plane is the line $v = 0$. P is an isolated singularity, hence normal, but one checks easily that on the blowing up of P, the inverse image of P is a singular curve; so P is not even a rational singularity.

But then about ten years ago, Abhyankar discovered that in certain cases (projections giving cyclic Galois coverings of degree = characteristic of the ground field) if one keeps on blowing up certain (possibly smooth) points of the critical variety, even when the critical variety has only normal crossings, then eventually a stage is reached where the structure of R becomes essentially as simple as in the characteristic zero case. (R is a "Jungian domain", in Abhyankar's terminology.) This discovery was a key point in his solution of the resolution problem for surfaces over excellent Dedekind domains.

(In the above example, for instance, blowing up the point $u = v = 0$ and normalizing in the function field of the given surface produces a surface covered by two affine surfaces whose equations are respectively

$$w^2 + wuv^3 + u = 0$$
$$w^2 + wv + vu^5 = 0.$$

The first of these is non-singular, while the second has just one singularity (at $(0,0,0)$), the singularity being rational of type A_9.)

II. What Jung's method suggests for higher dimensions is to apply embedded resolution for divisors in a d-dimensional non-singular variety to a certain critical variety in order to prove local uniformization on d-dimensional varieties. This idea works out well in characteristic zero, though the details are elaborate (cf. the main result (10.25) of chapter 10 in Abhyankar's book [A3]). In fact, using his generalization of Albanese's method (Lecture 1, §5), Abhyankar is able to make the idea work over fields of characteristic $> (d!)$. This is his approach to local uniformization on three dimensional varieties over fields of characteristic $> (3!)$. (The main difficulty is to prove embedded resolution for surfaces in non-singular threefolds.)

III. In 1954 Zariski proposed a global version of the local inductive procedure mentioned in II, as a method for resolution of singularities in characteristic zero [Z8, pp. 512-521]. In other words, apply embedded resolution to a certain (d-1)-dimensional critical variety in a non-singular d-dimensional variety, and then deduce embedded resolution for divisors in a non-singular variety of dimension $d + 1$. So far this idea has not met with much success, but it remains of interest as a possible source of a simpler alternative to Hironaka's proof.

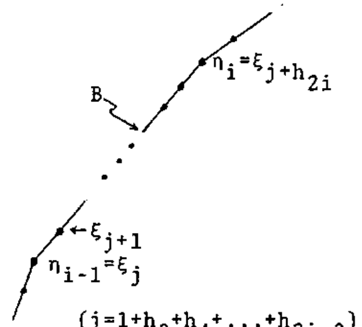
*   *   *

What we have done in this lecture is to use embedded resolution of curves in non-singular surfaces to prove resolution for surfaces. But our resolution procedure is not an "embedded" one, in that it is not entirely made up of transformations of some non-singular ambient variety containing the surface X to be resolved. An embedded resolution procedure, in which Jung's method plays an important role, was given by Zariski in some Lincei notes in the 1960's (cf. [Z7]). But Zariski considered only the <u>proper</u> transform, and not the <u>total</u> transform of X, so that his result is still not as strong as (EMB) (Lecture 1) for surfaces in threefolds over fields of characteristic zero. However Zariski's work did stimulate Abhyankar to find a very simple proof of the stronger result. In Abhyankar's proof, embedded resolution is applied not to the <u>discriminant</u>, but rather to the <u>coefficients</u> of some Weierstrass polynomial. We will discuss these matters in detail in Lecture 3.

*   *   *

<u>Remark</u> ("Added in proof"). The quotient singularity R can be studied by the methods of Chapter I of Kempf, Knudsen, Mumford and Saint-Donat's <u>Toroidal Embeddings I</u> (Lecture Notes in Math., no. 339, Springer-Verlag), where higher-dimensional analogues of R are also treated. In <u>loc. cit.</u>, R is represented by the <u>plane sector</u> $\sigma = \langle \xi_0, \xi_{s+1} \rangle \subset \mathbb{R}^2$, where $\xi_0 = (1,0)$, $\xi_{s+1} = (p,n)$, and $\langle \xi_0, \xi_{s+1} \rangle$ consists of all points $c\xi_0 + d\xi_{s+1}$ with c, d real and ≥ 0. The desingularization
$\sum = \bigcup_{i=0}^{s} \operatorname{Spec}(R[v^{\nu_i}/w^{\mu_i}, w^{\mu_{i+1}}/v^{\nu_{i+1}}])$ (cf. remark $\underline{3}$ above) corresponds to a "subdivision" $\sigma = \bigcup_{i=0}^{s} \langle \xi_i, \xi_{i+1} \rangle$, with $\xi_i = (\nu_i, \mu_i)$.

Blowing up the maximal ideal of R corresponds to a coarser subdivision $\sigma = \langle \xi_0, \eta_0 \rangle \cup \langle \eta_0, \eta_1 \rangle \cup \langle \eta_1, \eta_2 \rangle \cup \ldots \cup \langle \eta_{t-1}, \eta_t \rangle \cup \langle \eta_t, \xi_{s+1} \rangle$ where $\eta_0 = \xi_1$, $\eta_t = \xi_s$, and $\eta_1, \eta_2, \ldots, \eta_{t-1}$ are the "corners" (= vertices) lying strictly between ξ_1 and ξ_s on the boundary (B) of the convex hull of $(\sigma \cap \mathbb{Z}^2) - (0,0)$. [The "division points" η_i $(0 \leq i \leq t)$ can be characterized as being the points $(p,q) \in \mathbb{Z}^2$ for the which the line $pX + qY = 1$ has a segment (of positive length) in common with the boundary of the convex hull of $(\hat{\sigma} \cap \mathbb{Z}^2) - (0,0)$, where $\hat{\sigma} = \{(x,y) \mid ax + by \geq 0 \text{ for all } (a,b) \in \sigma\}$.]



To establish the result of remark $\underline{5}$, one shows that precisely h_{2i-1} of the ξ's are in the interior of the line segment joining η_{i-1} to η_i $(1 \leq i \leq t)$.

LECTURE 3: EMBEDDED RESOLUTION OF SURFACES (CHAR. 0)

The main topic of this lecture is the following weak form of (EMB) (cf. Lecture 1), for surfaces in non-singular threefolds:

(EMB*) Let X be a smooth irreducible three-dimensional variety over an algebraically closed field k of characteristic zero, and let S be a surface in X (i.e. S is a reduced pure two-dimensional closed subvariety of X). Then there exists a proper map $f:Y \to X$, with Y smooth, inducing an isomorphism

$$Y - f^{-1}(\mathrm{Sing}(S)) \xrightarrow{\approx} X - \mathrm{Sing}(S)$$

$$[\mathrm{Sing}(S) = \text{Singular locus of } S]$$

and such that S_Y, the closure in Y of $f^{-1}(S - \mathrm{Sing}(S))$, is smooth (whence the induced map $f:S_Y \to S$ is a desingularization).

We will see that f can in fact be obtained by successively blowing up points and "nice" curves.

As indicated earlier, (EMB*) was proved by Zariski, first in [Z6] and then again in [Z7]. (The latter proof is discussed briefly near the end of this lecture.) We shall present here a fairly detailed version of a previously unpublished proof due to Abhyankar (1966). This proof is in some vague ways influenced by Zariski's, but it is simpler, and accomplishes more (see end of lecture).

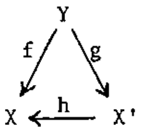
Of course Hironaka [H1] has proved far more general results than those to be mentioned in this lecture. But still Abhyankar's proof is worth being acquainted with, because it brings out very nicely - and quickly - the spirit and flavor of the subject.

*   *   *

(EMB*) has many applications. In the first place, it gives us resolution of singularities for surfaces (say, for simplicity, irreducible) over k; for, since any irreducible surface S' is birational to a surface S in $\mathbb{P}^3$ (= X), (EMB*) gives us a smooth surface (viz. S_Y) with function field $k(S')$. This is the weakest form of resolution; but it shows that every valuation of $k(S')/k$ is centered at a smooth point on some variety with function field $k(S')$; and even this weak form of local uniformization is enough for Zariski's proof that any surface over k can be desingularized by blowing up points and normalizing (cf. Lecture 1).

Secondly, (EMB*) is a central point in Zariski's proof of resolution of singularities of three-dimensional varieties [Z6], and even more so in Abhyankar's treatment [A3]. (Abhyankar works also over fields of positive characteristic, in which case the proof of (EMB*) is far more involved.) In order to deduce resolution from local uniformization (which he had already proved for varieties of any dimension [Z3]) Zariski needed the following theorem ("Dominance") on the elimination of the fundamental locus of a birational transformation:

(DOM) *Let* $h: X' \to X$ *be a proper birational map* (X *as above*). *Then there exists a smooth variety* Y *and a commutative diagram of birational maps*



f *being obtained as a succession of nice blowing-ups*.

Zariski's approach to this theorem is first to reformulate it as a statement about the elimination of the base points of a linear system Λ of divisors on X. He then resolves the singularities of a generic member S of Λ, *according to* (EMB*). Because of a theorem of Bertini, this creates a situation where each base point of Λ is smooth on almost all members of Λ, and under this condition, the matter becomes relatively straightforward.

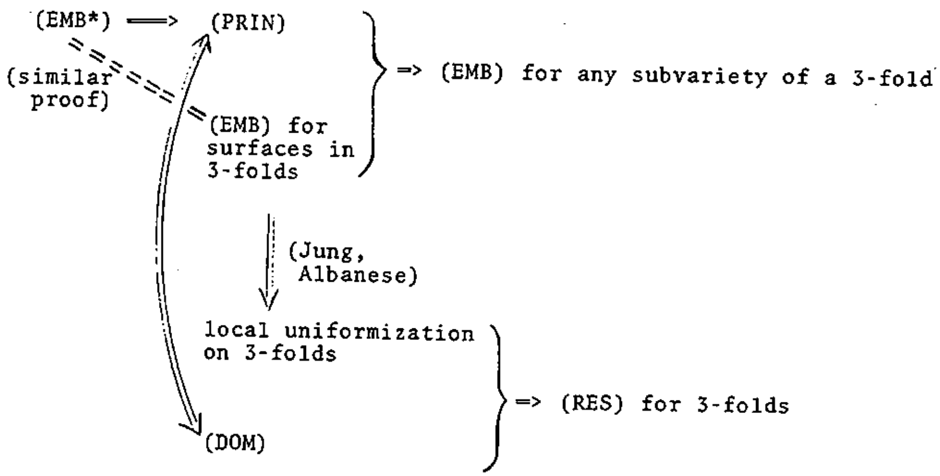
Abhyankar has extracted from Zariski's procedure its essential local algebraic content, which turns out to be quite elementary, so that he can give a much simpler deduction of (DOM) from (EMB*) (cf. [A3, p. 52 and p. 229]). Abhyankar's argument is valid also in characteristic $p > 0$, where Bertini's theorem may fail to hold.

Now (DOM) is (almost trivially) equivalent with the following statement ("Principalization"):

(PRIN) *Let* X *be as before, and let* $\mathcal{J} \neq 0$ *be a coherent sheaf of* $\mathcal{O}_X$*-ideals. Then there exists a proper birational map* $f: Y \to X$, *with* Y *smooth, such that* $\mathcal{J}\mathcal{O}_Y$ *is locally principal; here* f *can be obtained as a succession of nice blowing-ups, and in such a way that* f *induces an isomorphism over the dense open subset of* X *consisting of those points* x *at which the stalk* $\mathcal{J}_x$ *is already principal*.

(PRIN) reduces the strong embedded resolution theorem (EMB)(Lecture 1) for a subvariety W of X to the case where W is of pure codimension one, i.e. W is a surface (possibly reducible) in X. We will see at the end of this lecture that a simple modification of the following proof of (EMB*) allows us to add to the conclusion of (EMB*) the further condition that "$f^{-1}(S)$ has only normal crossings". In this way, we see that (EMB) holds for subvarieties of X.

Finally, recall that (EMB) was the basic point in Abhyankar's proof (via Jung and Albanese, cf. end of Lecture 2) of local uniformization on threefolds.



* * *

PROOF OF (EMB*). 1. First of all, recall that the multiplicity of a point s on S can be described as follows: we can represent the completed local ring of s on S in the form

$$\hat{\mathcal{O}}_{S,s} = k[[T,U,V]]/f(T,U,V)$$

for some power series f (T,U,V being local parameters at s on the smooth threefold X); then the multiplicity ν_s is the order of f, i.e. the unique integer ν such that

$$f = f_\nu + f_{\nu+1} + f_{\nu+2} + \dots$$

where $f_i = f_i(T,U,V)$ is a form of degree i, with $f_\nu \neq 0$. We say then that s is a ν-fold point of S. A point s is 1-fold if and only if S is smooth at s. For each ν, the set $S^{(\nu)}$ of points on S of multiplicity $\geq \nu$ is closed in S [N, §40]. An irreducible curve C on S is said to be ν-fold if $C \subseteq S^{(\nu)}$, $C \not\subseteq S^{(\nu+1)}$ (equivalently: the generic point of C is ν-fold).

Since $S^{(\nu)}$ is empty for $\nu >> 0$ (S being compact), we can set

$$\nu_S = \max_{s\in S} (\nu_s).$$

To prove (EMB*), it is clearly enough to prove the weaker assertion obtained by replacing "S_Y is smooth" by the condition

$$\nu_{S_Y} < \nu_S \quad (\text{if } \nu_S > 1).$$

* * *

2. We are going to construct Y from X by a sequence of permissible transformations, a permissible transformation being one obtained by blowing up either a point, or a smooth curve. One good thing about permissible transformations is that they preserve smoothness, i.e. if $X' \to X$ is a permissible transformation, with X smooth (as above), then also X' is smooth (cf. [Z8; p. 241]).

Some other good properties of permissible transformations are contained in the following preliminary Lemma (cf. [Z6; §§3,6]):

LEMMA 1. Let P be a ν-fold point of S.

(a) Let $f:S' \to S$ be obtained by blowing up P. Then any point $P' \in f^{-1}(P)$ has multiplicity $\leq \nu$ on S'; and any irreducible ν-fold curve $C' \subseteq f^{-1}(P)$ is smooth.

(b) Let $g:S'' \to S$ be obtained by blowing up an irreducible ν-fold curve C on which P is a smooth point. Then $g^{-1}(P)$ is finite; any point $P'' \in g^{-1}(P)$ has multiplicity $\leq \nu$ on S''; and any irreducible ν-fold curve $C'' \subseteq g^{-1}(C)$ passing through P'' is smooth at P''.

Proof.[1] Let R (resp. R', resp. R'') be the completed local ring of P on S (resp. P' on S', resp. P'' on S'').

Using the Weierstrass preparation theorem we can write

$$R \cong k[[X,Y,Z]]/f(X,Y,Z) = k[[x,y,z]]$$

$$f(X,Y,Z) = Z^{\nu} + a_2(X,Y)Z^{\nu-2} + a_3(X,Y)Z^{\nu-3} + \ldots + a_{\nu}(X,Y)$$

where, for $2 \leq i \leq \nu$, $a_i(X,Y)$ is a power series of order $\geq i$. (To get rid of terms involving $Z^{\nu-1}$, we need that $\nu.1 \neq 0$ in k.) Along any ν-fold subscheme of Spec(R), we have that

[1]Zariski has somewhat stronger results in loc. cit.; his proof is valid also in char. $p > 0$. We give the following char. 0 argument because most of it is used again later on anyway. (By oversight "X" and "Y" are used in the sequel to denote indeterminates, and "f" for power series; no confusion should result.)

$$\nu! Z = \partial^{\nu-1} f / \partial Z^{\nu-1} = 0$$

so that

$$Z = 0 = a_\nu(X,Y).$$

It follows that if C is a ν-fold curve smooth at P, then we may assume that the "germ" of C at P is given by $Z = X = 0$, with $a_i(X,Y)$ divisible by X^i for $2 \le i \le \nu$. Hence z/x is integral over R, and we conclude that $g^{-1}(P)$ is finite, as asserted in (b).

Now it is easily seen that

$$R' \cong k[[X,Y,Z]]/f'(X,Y,Z)$$

where, for suitable elements $c, d \in k$, <u>either</u>

(i): $$f'(X,Y,Z) = \frac{1}{X^\nu} f(X, X(Y+c), X(Z+d))$$

<u>or</u>

(ii): $$f'(X,Y,Z) = \frac{1}{Y^\nu} f(YX, Y, Y(Z+d)).$$

Suppose, for example, that (i) holds. Then

$$f'(X,Y,Z) = Z^\nu + \nu d Z^{\nu-1} + \ldots$$

so that P' has multiplicity $\le \nu$. Moreover, if some ν-fold curve $C' \subseteq f^{-1}(P)$ passes through P', then first of all $X = 0$ on C' (in fact $X = 0$ on $f^{-1}(P)$); secondly, $d = 0$ (otherwise $\nu d \neq 0$ and P' has multiplicity $< \nu$); and finally, as above, Z vanishes on C', so that C' is given at P' by $Z = X = 0$. This proves (a).

The rest of (b) is proved similarly, starting with the power series

$$f''(X,Y,Z) = \frac{1}{X^\nu} f(X, Y, X(Z+d)).$$

Q.E.D.

* * *

<u>3</u>. The underlying <u>idea of the proof</u> of (EMB*) is that by blowing up certain ν_S-fold points often enough, one achieves a situation where the maximal multiplicity ν_S can be lowered by blowing up smooth ν_S-fold curves.

To make this more precise, we introduce some convenient <u>terminology</u>. Let $\nu = \nu_S$. We say that a ν-fold point s of S is <u>good</u> if the following two conditions hold:

(i) Locally around s, $S^{(\nu)}$ is a <u>curve</u> having at s either a smooth point or an ordinary double point (node).

(ii) Let

(*) $$\cdots \to S_{i+1} \to S_i \to \cdots \to S_1 \to S_0 = S$$

be any sequence in which, for each $i \geq 0$, $S_{i+1} \to S_i$ is obtained by blowing up an irreducible ν-fold curve on S_i; and let $f_i : S_i \to S$ be the composed map $S_i \to S_{i-1} \to \cdots \to S$. Then, for any $i \geq 0$, every point of $f_i^{-1}(s)$ which is ν-fold for S_i lies on an irreducible ν-fold curve of S_i.

The usefulness of this notion is brought out by the following Proposition:

PROPOSITION 1. (a) The number of bad (= not good) ν_S-fold points on S is finite.

(b) If all ν_S-fold points on S are good, and if all the irreducible ν_S-fold curves on S are smooth, then for any sequence (*) as above there is an i such that $\nu_{S_i} < \nu_S$.

The Proposition results easily from the next Lemma. (Details are left to the reader; for (b) of Proposition 1 use also Lemma 1(b)).

LEMMA 2: For any S with $\nu_S > 1$, there exist only finitely many distinct sequences

$$S_n \to S_{n-1} \to \cdots \to S_0 = S$$

in which, for $0 \leq i < n$, S_{i+1} is obtained from S_i by blowing up an irreducible ν_S-fold curve whose image on S is a ν_S-fold curve of S.

(Proof: Consider the generic points of the irreducible ν_S-fold curves of S ...)

We can now reduce to proving the following:

THEOREM 1. Let $\nu = \nu_S$, and let $\cdots \to S'_{i+1} \xrightarrow{\psi_i} S'_i \to \cdots \to S'_0 = S$ be the sequence such that ψ_i is obtained by blowing up all the (finitely many) bad ν-fold points on S'_i $(i \geq 0)$. [Note that $\nu_{S'_i} \leq \nu$, (Lemma 1(a))]. Then the sequence terminates, i.e. for some n all the ν-fold points on S'_n are good.

Indeed, if S'_n is as in Theorem 1, then any irreducible ν-fold curve on S'_n has as its singularities at worst nodes; blowing up all such nodes, we get a surface S^* on which all irreducible ν-fold curves are smooth

(Lemma 1(a)); applying the Theorem to S^* instead of S, we obtain, again by Lemma 1(a), a surface on which all ν-fold points are good and on which all irreducible ν-fold curves are smooth; and then Proposition 1(b) completes the proof of (EMB*).

*   *   *

4. We sketch the proof of Theorem 1.

Theorem 1 is essentially local: since any surface S_i' has only finitely many bad ν-fold points, and since an inverse limit of non-empty finite sets is non-empty, we see that if Theorem 1 fails, then there exists an infinite sequence

$$(\sigma) \qquad R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq \cdots$$

where R_i is the local ring of a bad ν-fold point on S_i'; so we must show that such a sequence cannot exist.

Since multiplicities and blowing up behave well with respect to completion, we can consider instead of (σ) the corresponding sequence of complete local rings

$$(\hat{\sigma}) \qquad \hat{R}_0 \to \hat{R}_1 \to \hat{R}_2 \to \cdots$$

Arguing as in the proof of Lemma 1, we find that

$$\hat{R}_n \cong k[[X,Y,Z]]/f_n(X,Y,Z)$$

$$f_n(X,Y,Z) = Z^\nu + a_{n,2}(X,Y)Z^{\nu-2} + a_{n,3}(X,Y)Z^{\nu-3} + \ldots + a_{n,\nu}(X,Y)$$

where $a_{n,i}$ is a power series of order $\geq i$ $(2 \leq i \leq \nu)$; and furthermore, for each n, either

$$(\#) \qquad a_{n+1,i}(X,Y) = \frac{1}{X^i}\, a_{n,i}(X, X(Y+c_n)) \qquad (c_n \in k;\ 2 \leq i \leq \nu)$$

or

$$(\#\#) \qquad a_{n+1,i}(X,Y) = \frac{1}{Y^i}\, a_{n,i}(YX, Y) \qquad (2 \leq i \leq \nu).$$

Now let C_n be the algebroid curve $\mathrm{Spec}(k[[X,Y]]/\prod_j a_{n,j})$ where the product is taken over all j such that $a_{n,j} \neq 0$. Then the preceding remarks show that, modulo completion, C_{n+1} is a closed subscheme of the inverse image of C_n under the map obtained by blowing up the closed point of $\mathrm{Spec}(k[[X,Y]])$. By the embedded resolution theorem for curves in surfaces (Lecture 1), it follows that for large n, C_n has a normal crossing at its closed point, so that

$$a_{n,i} = \mu_{n,i}(X,Y)X^{\alpha_{n,i}}Y^{\beta_{n,i}} \qquad (2 \le i \le \nu)$$

where each $\mu_{n,i}$ is either a unit in $k[[X,Y]]$, or identically zero, and

$$\alpha_{n,i} + \beta_{n,i} \ge i.$$

It is straightforward to check that since R_n is the local ring of a bad ν-fold point, we cannot have $\beta_{n,i} = 0$ for all i.

Similarly, since R_{n+1} is the local ring of a bad point, we cannot have $\beta_{n+1,i} = 0$ for all i; hence if the above relation (#) holds, we must have $c_n = 0$.

In summary, we have, for large n,

$$f_n(X,Y,Z) = Z^\nu + \sum_{i=2}^{\nu} \mu_{n,i}(X,Y)X^{\alpha_{n,i}}Y^{\beta_{n,i}}Z^{\nu-i}$$

$$f_{n+1}(X,Y,Z) = Z^\nu + \sum_{i=2}^{\nu} \mu_{n+1,i}(X,Y)X^{\alpha_{n+1,i}}Y^{\beta_{n+1,i}}Z^{\nu-i}$$

where

either $(\alpha_{n+1,i}, \beta_{n+1,i}) = (\alpha_{n,i} + \beta_{n,i} - i, \beta_{n,i}) \qquad (2 \le i \le \nu)$

or $(\alpha_{n+1,i}, \beta_{n+1,i}) = (\alpha_{n,i}, \alpha_{n,i} + \beta_{n,i} - i) \qquad (2 \le i \le \nu)$

* * *

5. The whole matter can now be reduced to a game played with the $(\nu-1)$-tuples of integer pairs $(\alpha_{n,i}, \beta_{n,i})$ $2 \le i \le \nu$.

For any real number ρ, $\{\rho\}$ denotes the fractional part of ρ, i.e. the number in $[0,1)$ which differs from ρ by an integer.

We write "$(\alpha,\beta) \le (\gamma,\delta)$" to signify "$\alpha \le \gamma$ and $\beta \le \delta$".

LEMMA 3. If for some i, we have $\mu_{n,i} \ne 0$ and

$$\left(\frac{\alpha_{n,i}}{i}, \frac{\beta_{n,i}}{i}\right) \le \left(\frac{\alpha_{n,j}}{j}, \frac{\beta_{n,j}}{j}\right) \quad \text{for all } j \text{ with } \mu_{n,j} \ne 0$$

and if furthermore

$$\left\{\frac{\alpha_{n,i}}{i}\right\} + \left\{\frac{\beta_{n,i}}{i}\right\} < 1$$

then R_n is the local ring of a good point.

<u>Proof:</u> Left to reader.

All that we need now is the following two numerical Lemmas, whose (simple) proofs are omitted.

LEMMA 4. Let

$$\delta_{n,i,j} = \left(\frac{\alpha_{n,i}}{i} - \frac{\alpha_{n,j}}{j}\right)\left(\frac{\beta_{n,i}}{i} - \frac{\beta_{n,j}}{j}\right)$$

Then $\delta_{n+1,i,j} \geq \delta_{n,i,j}$;

and if $\delta_{n,i,j} < 0$, then

$$\delta_{n+1,i,j} - \delta_{n,i,j} \geq 1/\nu^4.$$

COROLLARY. There exists n_0 such that for each $n \geq n_0$, and for each i, j, we have $\delta_{n,i,j} \geq 0$, so that the sequence of pairs

$$\left(\frac{\alpha_{n,j}}{j}, \frac{\beta_{n,j}}{j}\right)_{2 \leq j \leq \nu}$$

is <u>totally ordered</u>.

<u>Remark.</u> If n_0 is as in the Corollary, and i is such that $\mu_{n_0,i} \neq 0$

and $$\left(\frac{\alpha_{n_0,i}}{i}, \frac{\beta_{n_0,i}}{i}\right) \leq \left(\frac{\alpha_{n_0,j}}{j}, \frac{\beta_{n_0,j}}{j}\right)$$ for all j with $\mu_{n_0,j} \neq 0$

then clearly for each $n \geq n_0$, we have $\mu_{n,i} \neq 0$

and $$\left(\frac{\alpha_{n,i}}{i}, \frac{\beta_{n,i}}{i}\right) \leq \left(\frac{\alpha_{n,j}}{j}, \frac{\beta_{n,j}}{j}\right)$$ for all j with $\mu_{n,j} \neq 0$.

LEMMA 5. If n_0, i are as in the preceding Remark, then for <u>some</u> $n \geq n_0$, we have

$$\left\{\frac{\alpha_{n,i}}{i}\right\} + \left\{\frac{\beta_{n,i}}{i}\right\} < 1.$$

Hence (Lemma 3), R_n is the local ring of a good point.

This contradiction completes the proof.

* * *

<u>Concluding Remarks.</u>

<u>A</u>. We have previously mentioned Zariski's proof of (EMB*) [Z7]. This proof, though not as elementary as Abhyankar's, has some attractive features.

In Zariski's approach, the "good" singularities are those of "dimensionality type 1", i.e. those s at which S is <u>equisingular</u> along the singular locus (which is supposed to be a curve locally around s). If <u>all</u> the singularities are good (in this sense), and if

$$\cdots \to S_{i+1} \xrightarrow{\phi_i} S_i \to \cdots \to S_0 = S$$

is the sequence such that ϕ_i is obtained from S_i by blowing up the singular locus ($i \geq 0$), then for all i, the singular locus of S_i is a smooth curve, so that ϕ_i is induced by a <u>permissible transformation</u> of ambient threefolds; S_i <u>is finite over</u> S; and for some n, S_n <u>is smooth</u>. Thus the problem is to get rid of the (finitely many) "bad" singularities, those of dimensionality type > 1, which Zariski calls "exceptional singularities".

Let ν be the largest among the multiplicities of the exceptional singularities. Zariski uses a Jungian type argument to show that after the exceptional ν-fold singularities are blown up often enough, those whose multiplicity is not decreased become of a simple type, called "quasi-ordinary". (This means that some neighborhood can be projected into the affine plane in such a way that the branch locus has only normal crossings.) A direct analysis of quasi-ordinary singularities shows first of all that by blowing up the ν-fold locus of S sufficiently often, one reaches a situation where every ν-fold exceptional singularity is "isolated", in the sense that it does not lie on any ν-fold curve of S; and secondly, that the multiplicity of these isolated ν-fold exceptional singularities can be lowered by blowing them up often enough.

Repeating the process, one eventually eliminates the exceptional singularities.

For further insight into (EMB*), part III of [Z6] is recommended.

<u>B</u>. Finally, we indicate how Abhyankar modifies his proof of (EMB*) to obtain the stronger version in which $f^{-1}(S)$ has only normal crossings[2]. Actually it is convenient to show a little more; the precise - and somewhat lengthy - formulation of the result is as follows:

A <u>resolution datum</u> $\mathcal{R} = (E_0, E_1, S, X)$ consists of three surfaces E_0, E_1, S in a smooth threefold X, where each irreducible component of E_0 or E_1 is smooth, and $E = E_0 \cup E_1$ has only normal crossings in X. (Either or both of E_0, E_1 are allowed to be empty.) $\mathcal{R}$ is <u>resolved</u> at a point $s \in S$ if S is smooth at s, and if $E \cup S$ has only normal crossings at s.

Let $\nu = \nu(\mathcal{R})$ be the greatest among the multiplicities on S of points at which $\mathcal{R}$ is not resolved ($\nu = 0$ if $\mathcal{R}$ is resolved everywhere on S); and let Δ be the set of all such ν-fold points. Δ is a closed subset of S, of dimension ≤ 1. ($\Delta = S^{(\nu)}$ if $\nu > 1$).

[2] Zariski has recently found a proof of this stronger result, along the lines of [Z7]. (Oral communication).

An irreducible curve $C \subset \Delta$ is permissible at a point $s \in C$ if C has normal crossings with E at s [this means that there exist local coordinates x, y, z at s (in X) such that E is defined near s by $x^{\varepsilon_1}y^{\varepsilon_2}z^{\varepsilon_3} = 0$ ($\varepsilon_i = 0$ or 1) and C by $x = y = 0$], and if there is no irreducible curve $C' \subset \Delta$ which has normal crossings with E at s and which lies in more components of E_1 than does C. (Any curve lies on at most two components of E_1). C is permissible if it is so at all its points.

A permissible transformation of $\mathcal{R}$ is a map $g: X' \to X$ obtained by blowing up a subvariety B of Δ, where B is either a point or a permissible curve. The g-transform $\mathcal{R}' = (E_0', E_1', S', X')$ of $\mathcal{R}$ is defined by:

S' = proper transform of S [= closure in X' of $g^{-1}(S-B)$]

E_1' = proper transform of E_1

$E_0' = g^{-1}(E_0 \cup B)_{\text{reduced}}$

unless $\nu_{S'} < \nu_S$, in which case we set

$E_1' = g^{-1}(E_0 \cup E_1 \cup B)_{\text{reduced}}$

E_0' = empty set.

$\mathcal{R}'$ is a resolution datum, and $\nu(\mathcal{R}') \leq \nu(\mathcal{R})$. It is clear what is meant by the transform of $\mathcal{R}$ under a succession of permissible transformations[3].

The theorem to be proved is that a resolution datum (E_0, E_1, S, X) with E_0 empty can be resolved, i.e. there exists a succession of permissible transformations $f = g_1 \circ g_2 \circ \ldots$ under which $\mathcal{R}$ is transformed into a resolved datum (E_0^*, E_1^*, S^*, Y). (Then S^* is smooth, and $f^{-1}(S)$ has only normal crossings in Y.) It is clearly enough to show that if $\nu(\mathcal{R}) > 0$, then $\mathcal{R}$ can be transformed into a datum $\mathcal{R}^*$ with $\nu(\mathcal{R}^*) < \nu(\mathcal{R})$.

For the proof, one says that a point $s \in \Delta$ is pregood if locally around s, Δ is a curve having either a simple point or a node at s, each (formal) branch* of Δ having a normal crossing with E at s. s is a good point if, roughly speaking, any ν-fold point t obtained from s by successively blowing up "locally" permissible curves is still pregood. (In other words there is a sequence $s = s_0, s_1, \ldots, s_n = t$ in which each s_{i+1} is a point lying over s_i on the surface obtained by blowing up a curve which is permissible at s_i ...). The number of bad points is finite. If all points of Δ are good, and if all components of Δ are smooth curves then we can lower $\nu(\mathcal{R})$, as desired, by successively blowing up components of Δ, subject to the following restriction: never blow up a component B of Δ if there is another component B' of Δ which lies in more components of E_1 than B does.

In this way (cf. §3 above) one reduces to proving the central local fact, viz. there cannot exist an infinite "quadratic sequence" of bad ν-fold points. The technique for showing this is similar to that of §§4,5 above except that one must work simultaneously with the local equations of E_0, E_1, S. Details are left to the interested reader.

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[3] To understand fully the motivation behind all the foregoing definitions, one must work through the details of the proof indicated below.

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