RELATIVE LIPSCHITZ-SATURATION.

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Introduction. We discuss algebraically the notion (due to Pham and Teissier [6]) of *relative Lipschitz-saturation of rings*. Among other things, we examine the radicial nature of this operation (Proposition (1.4)), and its compatibility with étale localization (Section 3) and with certain base changes (in particular, with completion (Corollary (2.6), Example (5.2)).

Then (Section 4) we consider the relation between relative Lipschitz-saturation and the relative saturation studied by Zariski in [8] and [9]. Under mild assumptions, we find that the Zariski-saturation of a ring is contained in the Lipschitz-saturation, with *equality* in the hypersurface case (for the precise statement cf. Corollary (4.2)). We have consequently an alternative approach to most of the "soft" properties of saturation which are given by Zariski in [GTS] (cf. remarks following Corollary (4.2)).

Actually our results are presented in a more general setting than the corresponding ones in [GTS]. This indicates that the Pham-Teissier definition of relative saturation is more amenable to algebraic manipulation than that of Zariski. It must be emphasized, however, that we do not obtain here any new insight into the central "hard" results of [GTS], namely those pertaining to the structure, automorphisms, and properties of transversal parameters of one-dimensional saturated local rings, and to the connections with the theory of equisingularity. Thus, our contribution is to the *technique* rather than to the core of saturation theory.

This paper could not have been written without the guidance of Zariski's theory; and the paper owes much to his interest during its preparation.

1. Miscellaneous Properties. *All rings will be commutative, with identity; "subring" means "subring containing the identity"; all homomorphisms of rings are understood to preserve the identity.*

The definition of relative Lipschitz-saturation is based on the concept of

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integral dependence on an ideal (a concept introduced by Prüfer [7]). Given an element x and an ideal I in a ring R, we say that x *is integral over* I if x satisfies a relation of the form

$$F(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0$$

for some integer $n>0$, with $a_i\in I^i$ for $i=1,2,\ldots,n$. An equivalent condition is that in the polynomial ring $R[X]$ (X an indeterminate) the element xX is integral over the subring $R[IX]=R\oplus IX\oplus I^2X^2\oplus\cdots$. (Thus statements about integral dependence over *ideals* can often be conveniently reformulated in terms of integral dependence over *rings*.) The set $\bar{I}$ consisting of those elements of R which are integral over I is an ideal, *the integral closure of I in R*. The usual closure properties hold: $I\subseteq\bar{I}=\bar{\bar{I}}$, and $I\subseteq J\Rightarrow\bar{I}\subseteq\bar{J}$.

We will make use of the following observation:

LEMMA (1.1). *Let $J_1,J_2,\ldots,J_n$ be ideals in R whose product $J_1J_2\ldots J_n$ is a* ***nilideal.*** *If x is integral over I (modulo J_i) for each $i=1,2,\ldots,n$, then x is integral over I.*

Proof. Let $F_i(x)\equiv 0$ (mod. J_i) be an equation (as above) of integral dependence for x over I (modulo J_i). Then

$$F(x)=\prod_{i=1}^{n}F_i(x)\in J_1J_2\cdots J_n,$$

so that for a suitably large integer r we have $F(x)^r=0$, and this is an equation of integral dependence for x over I.

Now we come to the basic definition.

Definition (1.2). *Let R be a ring and let $g:A\rightarrow B$ be a homomorphism of R-algebras. (In other words, we are given a sequence of ring-homomorphisms $R\rightarrow A\xrightarrow{g}B$.) The* ***Lipschitz-saturation*** $A^*=A^*_{B,R}$ *of A in B,* ***relative*** *to $R\rightarrow A\rightarrow B$, is the set*

$$A^*=\{x\in B\,|\,x\otimes_R 1-1\otimes_R x \text{ is integral over the kernel of}$$

$$\text{the canonical map } B\otimes_R B\rightarrow B\otimes_A B\}.$$

(This kernel is generated by all the elements $g(a)\otimes 1-1\otimes g(a)$, $a\in A$).

A is ***saturated*** *in B (relative to $R\rightarrow A\rightarrow B$) if $A^*=g(A)$.*

(Usually A will be a subring of B, and $g:A\rightarrow B$ will be the inclusion map.)

Some elementary properties of this operation on algebras are listed below. Straightforward proofs are omitted.

1. *A^* is a subring of B, containing $g(A)$.* (The proof uses the identity

$$xy\otimes 1 - 1\otimes xy = (x\otimes 1)(y\otimes 1 - 1\otimes y) + (1\otimes y)(x\otimes 1 - 1\otimes x)$$

and the fact that the integral closure of an ideal in $B\otimes_R B$ is again an ideal.)

2. (Closure properties). *Assume that A is an R-subalgebra of B, and for any ring C with $A\subseteq C\subseteq B$ let $C^* = C^*_{B,R}$ (the sequence $R\to C\to B$ being the obvious one). Then*

(i) $A\subseteq A^*$

(ii) $A\subseteq C\subseteq B \Rightarrow A^*\subseteq C^*$

(iii) $(A^*)^* = A^*$.

(Hence A^* is the *smallest* ring between A and B which is saturated in B relative to R.)

3. (Functoriality). *Given a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & A & \xrightarrow{g} & B \\ \downarrow & & \downarrow & & \downarrow f \\ R' & \longrightarrow & A' & \xrightarrow{g'} & B' \end{array}$$

we have that

$$f(A^*_{B,R})\subseteq (A')^*_{B',R'}.$$

4. (Inductive limits). *Given a (filtered) inductive system of sequences*

$$R_\alpha \longrightarrow A_\alpha \longrightarrow B_\alpha$$

with inductive limit

$$R\longrightarrow A\longrightarrow B$$

there is a natural isomorphism

$$\varinjlim_{\alpha} (A_\alpha)^*_{B_\alpha,R_\alpha} \xrightarrow{\approx} A^*_{B,R}.$$

5. (Direct products). *Let $g_i: A_i\to B_i$ $(i=1,2,\dots,n)$ be homomorphisms of R-algebras, and let $g: A\to B$ be the direct product of these maps:*

$$A = \left(\prod_{i=1}^{n} A_i\right) \xrightarrow{g=\prod g_i} \left(\prod_{i=1}^{n} B_i\right) = B$$

Then

$$A^*_{B,R} = \prod_{i=1}^{n} (A_i)^*_{B_i,R}$$

[Here the proof is a bit lengthy, though still straightforward. (Cf. also remark (ii) following Proposition (3.1).)]

6. (Faithfully flat descent). *Let* $R \to A \xrightarrow{g} B$ *be as in* (1.2). *Tensor this sequence with a faithfully flat R-algebra* R', *and denote the resulting sequence by* $R' \to A' \xrightarrow{g'} B'$ ($A' = A \otimes_R R'$, *etc.*) *If* A' *is* R'*-saturated in* B' (*i.e.* $(A')^*_{B',R'} = g'(A')$) *then* A *is* R*-saturated in* B (*i.e.* $A^*_{B,R} = g(A)$).

Proof. Let $f: B \to B'$ be the canonical map. By (1) and (3) we have

$$g(A) \subseteq A^*_{B,R} \subseteq f^{-1}((A')^*_{B',R'}) = f^{-1}(g'(A')).$$

But there is a natural commutative diagram of *R-module* homomorphisms

$$\begin{array}{ccc} & B/g(A) & \\ & \alpha \uparrow & \bar{f} \uparrow \\ (B/g(A)) \otimes_R R' & \underset{\beta}{\overset{\approx}{\longrightarrow}} & B'/g'(A') \end{array}$$

where β is an *isomorphism*, and the canonical map α is *injective* (since R' is faithfully flat over R [1, Ch. 1, p. 50, Prop. 8]); hence the map $\bar{f}$ induced by f is *injective*, i.e. $g(A) = f^{-1}(g'(A'))$, and the conclusion follows.

7. (Contraction). *Given a sequence of ring homomorphisms*

$$R \longrightarrow A \xrightarrow{g} B \xrightarrow{f} B'$$

assume that

either: (a) *the kernel of* f *is a nilideal and* B' *is integral over* $f(B)$

or: (b) B' *is faithfully flat over* B (*via* f).

Then

$$A^*_{B,R} = f^{-1}(A^*_{B',R}).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} B\otimes_R B & \xrightarrow{\phi} & B\otimes_A B \\ {\scriptstyle f\otimes_R f}\downarrow & & \downarrow{\scriptstyle f\otimes_A f} \\ B'\otimes_R B' & \xrightarrow{\phi'} & B'\otimes_A B' \end{array}$$

ϕ and ϕ' being the canonical maps. It is clear (by considering generators) that

$$\ker(\phi') = \ker(\phi)\cdot(B'\otimes_R B') \qquad (\text{“ker”} = \text{“kernel of”})$$

Now, *if* (a) *holds, then the kernel N of $f\otimes_R f$ is a nilideal.* (To see this it suffices to show that every prime ideal p in $B\otimes_R B$ is of the form $(f\otimes_R f)^{-1}(q)$ for some prime ideal q in $B'\otimes_R B'$ (so that p contains N). But p is the kernel of a map α of $B\otimes_R B$ into an algebraically closed field F; to this map there corresponds a pair of maps $B \rightrightarrows F$ with the same “restriction” to R; because of the assumption (a) these two maps can be “extended” to maps of B' into F; there results a map $\beta: B'\otimes_R B' \to F$ whose composition with $f\otimes_R f$ is α; the kernel of β is the desired q.)

So, assuming (a), (7) results from the following lemma (with $h: S \to T$ replaced by $(f\otimes_R f): B\otimes_R B \to B'\otimes_R B'$, and I by $\ker(\phi)$):

LEMMA. *Let $h: S\to T$ be a ring homomorphism whose kernel is a nilideal, and such that T is integral over $h(S)$. Then for any ideal I in S (with integral closure $\bar{I}$) we have*

$$\bar{I} = h^{-1}(\overline{IT}).$$

Proof. It is clear that $h(\bar{I}) \subseteq \overline{IT}$. Conversely if $x\in S$ and $y = h(x)$ is integral over IT, then, in the polynomial ring $T[X]$, yX is integral over the subring $T[(IT)X]$, and this ring is integral over its subring $h(S)[h(I)X]$ (being generated over the subring by the elements of T). Hence there is an equation

$$y^n + b_1 y^{n-1} + \ldots + b_n = 0$$

with $b_i \in h(I)^i$, say $b_i = h(a_i)$, $a_i \in I^i$ $(i = 1, 2, \ldots, n)$. So

$$F(x) = x^n + a_1 x^{n-1} + \ldots + a_n \in \ker(f),$$

and for suitably large r, $F(x)^r = 0$. Thus x is integral over I, and the lemma is proved.

Similarly, if (b) holds, it is enough to prove the preceding lemma assuming only that T is *faithfully flat* over S. In this case, if $h(x)$ is integral over IT, then an equation of integral dependence shows that, for some $n>0$,

$$h(x)^n \in I(I,x)^{n-1}T$$

whence

$$x^n \in h^{-1}\left(I(I,x)^{n-1}T\right) = I(I,x)^{n-1}.$$

(By faithful flatness we have $h^{-1}(JT)=J$ for all ideals J in S.) We conclude that x is integral over I. Q.E.D.

Remark. More generally, it can be shown that the conclusion of (7) is valid if we assume only the following condition on f:

"For every B-algebra V which is a valuation ring, there exist two prime ideals $q \subseteq p$ in $V \otimes_B B'$ whose inverse images in V are respectively (0) and the maximal ideal of V."

For a sequence of homomorphisms $R \to R' \to A \to B$, (3) shows that $A^*_{B,R} \subseteq A^*_{B,R'}$. The following proposition gives conditions under which these two saturations are *equal*.

Recall first that, given a ring homomorphism $h: T \to T'$, we say that T' is a *radicial* T-algebra (via h) if for every prime ideal p in T there is *at most* one prime ideal q in T' such that $h^{-1}(q)=p$, and furthermore for any such pair p, q the map $T/p \to T'/q$ induced by h makes the field of fractions of T'/q purely inseparable over that of T/p. *Equivalently*: any two *distinct* homomorphisms of T' into a field have *distinct* compositions with h; *or again*: $t' \otimes 1 - 1 \otimes t'$ is nilpotent in $T' \otimes_T T'$ for all $t' \in T'$, i.e. the kernel K of the canonical map $T' \otimes_T T' \to T'$ is a *nilideal* (cf. [2, p. 246, Prop. (3.7.1)]).

We shall say that T' is an *unramified* T-algebra if the above kernel K is finitely generated (which is the case, e.g., if T' is a ring of fractions of a finitely generated T-algebra) and furthermore satisfies $K^2=K$ (i.e. the module of differentials $\Omega^1_{T'/T} = K/K^2$ vanishes). It is *equivalent* to say that K is generated by a single element e such that $e^2=e$. (For if K is finitely generated and satisfies $K^2=K$, then a standard argument shows that $K(1-e)=0$ for some $e \in K$, and this e is as required.)

Proposition (1.3). *Given a sequence of ring homomorphisms $R \to R' \to A \to B$, if R' is either radicial or unramified over R, then*

$$A^*_{B,R} = A^*_{B,R'}.$$

Proof. In either case, the kernel K of the canonical map $R'\otimes_R R'\to R'$ contains an element e' such that $(e')^2=e'$ and $K\subseteq\sqrt{e'}$.† Consider the commutative diagram of canonical maps

$$\begin{array}{ccccc} R'\otimes_R R' & \xrightarrow{\theta} & B\otimes_R B & \xrightarrow{\phi} & B\otimes_A B \\ & & \psi\big\downarrow & \nearrow\phi' & \\ & & B\otimes_{R'} B & & \end{array}$$

The kernel J_1 of ψ is generated by $\theta(K)$, and so if $e=\theta(e')$ we have

$$e\in J_1\subseteq\sqrt{e}\quad\text{and}\quad e^2=e.$$

ψ, being surjective, maps the kernel I of ϕ *onto* the kernel of ϕ'. So if $x\in A^*_{B,R'}$, then, in $B\otimes_R B$, $\bar{x}=x\otimes 1-1\otimes x$ *is integral over* I $(\text{mod}\, J_1)$. But, setting

$$J_2=(B\otimes_R B)(1-e)$$

we have that $\bar{x}$ *is integral over* I $(\text{mod}\, J_2)$: indeed

$$\bar{x}-a\equiv 0\ (\text{mod}\, J_2)\qquad (a=\bar{x}e\in J_1\subseteq I).$$

Since $J_1\subseteq\sqrt{e}$ and $e(1-e)=0$, therefore J_1J_2 is a nilideal. Hence Lemma (1.1) shows that $\bar{x}$ is integral over I, i.e. $x\in A^*_{B,R}$. Thus $A^*_{B,R'}\subseteq A^*_{B,R}$, and, as remarked above, the opposite inclusion is given by (3). Q.E.D.

The next proposition expresses the "radicial" nature of saturation.

PROPOSITION (1.4). *Given* $R\to A\xrightarrow{g} B$, *if* B *is integral over* $g(A)$ *then* $A^*_{B,R}$ *is a radicial* A*-algebra (via* g*).*

Proof. Let $C=A^*_{B,R}$. We must show, for $c\in C$, that $c\otimes 1-1\otimes c$ is *nilpotent* in $C\otimes_A C$. Applying the canonical map $\phi: B\otimes_R B\to B\otimes_A B$ to an equation of integral dependence for $c\otimes_R 1-1\otimes_R c$ over the kernel of ϕ, we find that $c\otimes 1-1\otimes c$ is nilpotent in $B\otimes_A B$. Since B is integral over $C(\supseteq g(A))$, we can argue as in the proof of (7a) above to see that the kernel of $C\otimes_A C\to B\otimes_A B$ is a nilideal, and the conclusion follows.

†This means that the diagonal map takes $\text{Spec}(R')$ homeomorphically onto an open subspace of $\text{Spec}(R')\times_{\text{Spec}(R)}\text{Spec}(R')$.

2. Quasi-Normal Base Change. Let $R \to A \xrightarrow{g} B$ be as before. Tensoring this sequence with an R-algebra R', we obtain a sequence $R' \to A' \xrightarrow{g'} B'$ ($A' = A \otimes_R R'$, etc.). Let I (resp. I') be the kernel of the canonical map $B \otimes_R B \to B \otimes_A B$ (resp. $B' \otimes_{R'} B' \to B' \otimes_{A'} B'$). We have

$$g'(A') = R'[h(A)] \subseteq B'$$

where h is the composition of the canonical map $B \to B'$ with $g: A \to B$; hence the ideal I' in $B' \otimes_{R'} B'$ is generated by all the elements $h(a) \otimes 1 - 1 \otimes h(a)$ ($a \in A$), or in other words: $I' = I(B' \otimes_{R'} B')$. ($B' \otimes_{R'} B'$ is a $B \otimes_R B$-algebra via the canonical map $B \otimes_R B \to B' \otimes_{R'} B'$.) Consequently, if $\bar{I}$, $\bar{I}'$ are the integral closures of I, I' respectively, then we have

$$\bar{I}(B' \otimes_{R'} B') \subseteq \bar{I}' \tag{2.1}$$

In a variety of interesting cases, *equality* holds in (2.1) (cf. discussion following the next proposition).

PROPOSITION (2.2). *If R' is flat over R, and if equality holds in* (2.1), *then, with notation as above,*

$$(A')^*_{B',R'} = (A^*_{B,R}) \otimes_R R'.$$

Proof. Let τ_1, τ_2 be the R-module homomorphisms from B into $(B \otimes_R B)/\bar{I}$ given by

$$\tau_1(b) = b \otimes 1 \ (\text{mod.}\, \bar{I}), \qquad \tau_2(b) = 1 \otimes b \ (\text{mod.}\, \bar{I}).$$

Let τ_1', $\tau_2' : B' \to (B' \otimes_{R'} B')/\bar{I}'$ be similarly defined. We have, by definition, an *exact sequence of R-module homomorphisms*

$$0 \longrightarrow A^*_{B,R} \longrightarrow B \xrightarrow{\tau_1 - \tau_2} (B \otimes_R B)/\bar{I} \tag{2.3}$$

and similarly, an *exact sequence of R'-module homomorphisms*

$$0 \longrightarrow (A')^*_{B',R'} \longrightarrow B' \xrightarrow{\tau_1' - \tau_2'} (B' \otimes_{R'} B')/\bar{I}'. \ (2.3)'$$

Using the functoriality of saturation (Section 1, (3)) we get a natural commutative diagram

$$\begin{array}{ccccccc}
0\longrightarrow & (A^*_{B,R})\otimes_R R' & \longrightarrow B\otimes_R R'\longrightarrow & B'\otimes_{R'}B'/\bar{I}(B'\otimes_{R'}B') \\
& \alpha\downarrow & \Vert & \downarrow\beta \\
0\longrightarrow & (A')^*_{B',R'} & \longrightarrow\ B'\ \longrightarrow & B'\otimes_{R'}B'/\bar{I}'
\end{array}$$

where the first row is obtained from (2.3) by tensoring with R' and making the canonical identification

$$(B\otimes_R B)\otimes_R R' = B'\otimes_{R'}B',$$

and the second row is (2.3)′. (The verification of commutativity is left to the reader.) The first row is exact because (2.3) is exact and R' is flat over R; the second row (2.3)′ is also exact. Equality in (2.1) means that the map β is bijective, whence α is bijective. Q.E.D.

We shall show that the following condition on the R-algebra R' guarantees equality in (2.1), and then give examples of algebras for which the condition holds.

($N_{R,R'}$): *If C is any R-algebra, and D is a C-algebra in which C is integrally closed,*† *then also $C\otimes_R R'$ is integrally closed in $D\otimes_R R'$.*

If ($N_{R,R'}$) holds, and T is any R-algebra, then clearly ($N_{T,T'}$) holds, where T' is the T-algebra $T\otimes_R R'$. In particular, if B is an R-algebra, and $B' = B\otimes_R R'$, then, since

$$B'\otimes_{R'}B' = (B\otimes_R B)\otimes_R R',$$

we see that ($N_{T,T'}$) holds for the pair

$$T = B\otimes_R B, \qquad T' = B'\otimes_{R'}B'.$$

Hence the fact that ($N_{R,R'}$) implies equality in (2.1) is contained in the following lemma:

LEMMA (2.4). *Let T be a ring and let T' be a T-algebra such that ($N_{T,T'}$) holds. If I is an ideal in T, with integral closure $\bar{I}$ in T, then $\bar{I}T'$ is the integral closure in T' of the ideal $I' = IT'$.*

†If $h: C\to D$ is the structural map, then "C is integrally closed in D" means "$h(C)$ is integrally closed in D."

Proof. Let X be an indeterminate, and let

$$C_I = T[IX] \subseteq D = T[X].$$

The integral closure of C_I in D is a graded ring $C = T \oplus \bar{I}X \oplus \cdots$ (cf. beginning of Section 1, and [AC, Ch. 5, p. 30, Prop. 20]). We have a natural *surjective* map of graded rings

$$C_I \otimes_T T' \longrightarrow T'[I'X]$$

and hence the integral closure of $T'[I'X]$ in $T'[X]$ is the same as that of $C_I \otimes_T T'$. Because of $(N_{T,T'})$, we see that this integral closure is the canonical image in $T'[X]$ of $C \otimes_T T'$, viz. $T' \oplus \bar{I}T'X \oplus \cdots$; the conclusion follows.

Q.E.D.

We say that an R-algebra R' is a ***quasi-normal*** R-algebra if R' is *flat* over R and $(N_{R,R'})$ holds. So *the hypotheses of Proposition* (2.2) *are satisfied whenever* R' *is a quasi-normal* R*-algebra.*

Examples of quasi-normal R-algebras:

(i) $R' = R_M$, where M is a multiplicatively closed subset of R [AC, Ch. 5, p. 22, Prop. 16].

(ii) $R' = R[X_1, X_2, \ldots, X_n]$, with independent indeterminates $X_1, \ldots, X_n$. [AC, Ch. 5, p. 19, Prop. 13]

(iii) If R' is *étale* over R (i.e. finitely presented, flat, and unramified) then R' is a quasi-normal R-algebra [EGA IV, (18.12.15)].

We say that a ring S is *locally normal* if for every prime ideal p in S the local ring S_p is an *integrally closed domain*. If S is *noetherian*, then S is locally normal if and only if S is a *finite direct product* of integrally closed domains (cf. [EGA 01, p. 145, Section (6.5.1)].

We say that an S-algebra S' is *normal over* S, if S' is *flat* over S and for any prime ideal p in S and any finite purely inseparable field extension K of the field of fractions of S/p, the ring $S' \otimes_S K$ is locally normal.

(iv) If R and R' are *noetherian*, and R' is *normal over* R, then R' is a quasi-normal R-algebra [EGA IV, (6.14.5)].

(v) Let R be an *excellent* ring [EGA IV, (7.8.2)], let I be an ideal of R, and let $\hat{R}$ be the I-adic completion of R. Then $\hat{R}$ is normal over R [EGA IV, (7.4.6)].

(vi) If k is a field of characteristic zero and K is any field extension of k, then the power series ring $K[[X_1, X_2, \ldots, X_n]]$ is normal over $k[[X_1, X_2, \ldots, X_n]]$.

(This follows from [EGA IV, (18.11.10)]. More general examples, involving *formally smooth* algebras, are indicated in [EGA IV, (7.5.4) (i) and the sentence preceding it].)

(vii) If R' is a quasi-normal R-algebra and R'' is a quasi-normal R'-algebra, then R'' is a quasi-normal R-algebra.

(viii) If R' is a quasi-normal R-algebra and T is any R-algebra, then $T' = T \otimes_R R'$ is a quasi-normal T-algebra.

(ix) A (filtered) inductive limit of quasi-normal R-algebras is a quasi-normal R-algebra.

Consider once again a sequence $R \to A \to B$. In case B is the *integral closure* of A in its total ring of fractions, we may set

$$A_R^* = A_{B,R}^*$$

and refer to it as *the* R-saturation of A (omitting any explicit reference to B).

Proposition (2.5). *Let R be a noetherian ring and let A be a reduced R-algebra such that every prime ideal in A consisting entirely of zero-divisors is a minimal prime ideal. (In other words, the total ring of fractions F of A has Krull dimension zero.) Let R' be a noetherian R-algebra,* **normal over** *R, and set $A' = A \otimes_R R'$. Then*

$$(A')_{R'}^* = (A_R^*) \otimes_R R'.$$

Proof. Let B be the integral closure of A in F. Since R' is normal over R we can apply Proposition (2.2) to conclude that

$$(A')_{B',R'}^* = (A_R^*) \otimes_R R' \qquad (B' = B \otimes_R R').$$

Thus it will suffice to show that B' *is the integral closure of A' in the total ring of fractions F' of A'.*

Since R' is *flat* over R, it follows that

$$A' \subseteq B' \subseteq F \otimes_R R' = F \otimes_A A' \subseteq F'.$$

(The last inclusion results from the fact that every regular element in A remains regular in A'). B' is integral over A', and since R' is quasi-normal over R, B' is integrally closed in $F \otimes_R R'$; hence we need only show that *$F \otimes_R R'$ is integrally closed in its total ring of fractions F'.*

For each prime ideal q in F, F_q is a reduced local ring of Krull dimension zero, i.e. F_q is a field; thus F is a locally normal ring, and hence $F \otimes_R R'$ is also locally normal [EGA IV, (6.14.1)]. So our conclusion follows from the simple fact that *every locally normal ring S is integrally closed in its total ring of fractions T*.

(*Proof*. Let $x \in T$ be integral over S, and let $I = \{s \in S \mid sx \in S\}$. If q is any prime ideal in S, then $T_q = T \otimes_S S_q$ is canonically contained in the total ring of fractions of S_q, and since S_q is an integrally closed domain, the canonical image of x in T_q lies in S_q. It follows easily that $IS_q = S_q$, i.e. $I \not\subseteq q$, and since this holds for every q, therefore I is the unit ideal in S, i.e. $x \in S$.)

Corollary (2.6). (***Permutability of saturation and completion***). *Let R, I, $\hat{R}$ be as in example* (v) *above, and let A be a reduced R-algebra which is a finite R-module. Then*

$$(\hat{A})^*_{\hat{R}} = (A^*_R)^\wedge .$$

Here "^" denotes "I-adic completion," and otherwise notation is as in the remarks immediately preceding Proposition (2.5). (2.6) is obtained from (2.5) by setting $R' = \hat{R}$; for $\hat{R}$ is normal over R, and furthermore, R is pseudogeometric [EGA IV, (7.8.3) (vi)] so that the integral closure B of A in the total ring of fractions of A is a finite R-module, as is $A^*_R \subseteq B$; hence [AC, Ch. 3, p. 68, Theorem 3]

$$\hat{A} = A \otimes_R \hat{R}, \qquad (A^*_R)^\wedge = A^*_R \otimes_R \hat{R}.$$

3. Compatibility of Saturation with Étale Localization. Let T be a ring. A T-algebra T' is said to be *étale* if T' is finitely presented, flat, and unramified over T (cf. definition preceding Proposition (1.3)). As mentioned before (example (iii), Section 2) an étale T-algebra is quasi-normal over T.

Proposition (3.1). *Let $R \to A \to B$ be as usual, let A' be a (filtered) inductive limit of étale A-algebras, and let $B' = B \otimes_A A'$. Then the canonical map*

$$(A^*_{B,R}) \otimes_A A' \to (A')^*_{B',R} \tag{3.2}$$

is bijective.

Remarks. (i). In the special case when $A' = A \otimes_R R'$ where R' is an étale R-algebra, Proposition (1.3) gives

$$(A')^*_{B',R} = (A')^*_{B',R'}.$$

Thus, *in this case*, Proposition (3.1) follows from Proposition (2.2).

(ii) Let M be a multiplicatively closed subset of A. Then the ring of fractions A_M is, in an obvious way, the inductive limit of the étale A-algebras $A[1/f]$ $(f \in M)$, and so (3.1) gives the *compatibility of saturation with rings of fractions*:

$$(A^*_{B,R})_M = (A_M)^*_{B_M,R}.$$

Further, if N is a multiplicatively closed subset of R consisting of elements whose image in A_M is invertible, then $R \to A_M$ factors canonically as $R \to R_N \to A_M$, and from (1.3) we get

$$(A_M)^*_{B_M,R} = (A_M)^*_{B_M,R_N}.$$

It follows directly that the notion of saturation *globalizes*: if $f: X \to Y$ is a morphism of schemes, $\mathcal{A} = \mathcal{O}_X$, and $\mathcal{B}$ is a quasi-coherent $\mathcal{A}$-algebra, then there is a (unique) *quasi-coherent* $\mathcal{A}$-algebra $\mathcal{A}^* \subseteq \mathcal{B}$ such that, for each $x \in X$,

$$\mathcal{A}^*_x = (\mathcal{A}_x)^*_{\mathcal{B}_x, \mathcal{O}_{f(x)}}.$$

(It is illuminating, for example, to think about the compatibility of saturation and products (Section 1, (5)) from such a scheme-theoretic point of view.)

Similar remarks apply to algebraic spaces.

Proof of Proposition (3.1). (I). In the first place, if $A' = \varinjlim_{\alpha} A_\alpha$ (A_α étale over A) then the canonical map (3.2) is easily seen (in view of (4) of Section 1) to be the inductive limit of the canonical maps

$$(A^*_{B,R}) \otimes_A A_\alpha \to (A_\alpha)^*_{B_\alpha,R} \qquad (B_\alpha = B \otimes_A A_\alpha).$$

We may therefore assume that A' is actually étale over A.

(II). Let I (resp. J), be the kernel of the canonical map

$$B \otimes_R B \longrightarrow B \otimes_A B$$

$$\text{(resp.)} \qquad B' \otimes_R B' \longrightarrow B' \otimes_{A'} B'$$

and let $\bar{I}$, $\bar{J}$ be the respective integral closures of I, J.

Define the maps $\tau_1, \tau_2 : B \to (B \otimes_R B)/\bar{I}$ by

$$\tau_1(b) = b \otimes 1 \ (\text{mod.}\, \bar{I}) \qquad \tau_2(b) = 1 \otimes b \ (\text{mod.}\, \bar{I}).$$

Note that $(B \otimes_R B)/\bar{I}$, as a homomorphic image of $B \otimes_A B$, has a natural A-module structure, and so τ_1, τ_2 can be regarded as *A-module* *homomorphisms*. We have similar *A'-module homomorphisms* $\tau_1', \tau_2' : B' \to (B' \otimes_{R'} B')/\bar{J}$, and (cf. Section 1, (3)) a natural commutative diagram whose rows are *exact* (since A' is *flat* over A):

$$\begin{array}{ccccccc}
0 \longrightarrow & A^*_{B,R} \otimes_A A' & \longrightarrow & B \otimes_A A' & \overset{(\tau_1 - \tau_2) \otimes_A A'}{\longrightarrow} & \left((B \otimes_R B)/\bar{I}\right) \otimes_A A' \\
 & \downarrow & & \| & & \downarrow \gamma \\
0 \longrightarrow & (A')^*_{B',R} & \longrightarrow & B' & \overset{\tau_1' - \tau_2'}{\longrightarrow} & (B' \otimes_R B')/\bar{J}
\end{array}$$

We have, moreover, the natural identifications

$$\left((B \otimes_R B)/\bar{I}\right) \otimes_A A' = ((B \otimes_A B)/I_1)$$

$$\left((B \otimes_R B)/\bar{I}\right) \otimes_A A' = ((cb \otimes dcacb)/I_1) \otimes_A A' \qquad \left(I_1 = \bar{I}/I\right)$$

$$= (B' \otimes_{A'} B')/I'B(B' \otimes_A a')$$

and

$$(B' \otimes_R B')/\bar{J} = (B' \otimes_{A'} B')/J_1 \qquad \left(J_1 = \bar{J}/J\right)$$

and after these identifications are made, γ is given by

$$\gamma[x \otimes y \; (\text{mod.}\, I_1(B' \otimes_{A'} B'))] = x \otimes y \; (\text{mod.}\, J_1) \qquad (x, y \in B').$$

Thus *it will suffice to show that $J_1 = I_1(B' \otimes_{A'} B')$, or in other words, that*

$$\bar{J} = \bar{I}(B' \otimes_R B') + J. \tag{3.3}$$

(III). Let I' be the kernel of the canonical map

$$B' \otimes_R B' \longrightarrow B' \otimes_A B'.$$

I' is generated by all the elements $h(a) \otimes 1 - 1 \otimes h(a)$ ($a \in A$, $h: A \to B'$ the natural map), so that

$$I' = I(B' \otimes_R B').$$

Since A' is étale over A, and since

$$B' \otimes_R B' = A' \otimes_A (B \otimes_R B) \otimes_A A'$$

therefore $B' \otimes_R B'$ is étale, and hence *quasi-normal*, over $B \otimes_R B$. So, by

Lemma (2.4),

$$\bar{I}(B'\otimes_R B')=\bar{I}' \qquad \text{(the integral closure of } I'). \tag{3.4}$$

So (3.3) becomes

$$\bar{J}=\bar{I}'+J. \tag{3.5}$$

(IV). As in the proof of (1.3), we see that the kernel of $B'\otimes_A B' \longrightarrow B'\otimes_{A'} B'$ is generated by an idempotent element; hence

$$J=(I',e) \qquad (e^2\equiv e(\text{mod}.\, I'))$$

and so

$$(1-e)J\subseteq I'. \tag{3.6}$$

To prove (3.5)—and hence Proposition (3.1)—let $z\in\bar{J}$, so that z satisfies an equation

$$z^n+r_1z^{n-1}+r_2z^{n-2}+\dots+r_n=0 \qquad (r_i\in J^i).$$

Multiplying by $(1-e)^n$, and using (3.6), we deduce that

$$(1-e)z\in\bar{I}'$$

i.e.

$$z-ez\in\bar{I}',$$

and since $ez\in J$, therefore $z\in\bar{I}'+J$. Thus $\bar{J}\subseteq\bar{I}'+J$. The opposite inclusion is obvious, and so the proof is complete.

4. Comparison with Zariski's Saturation. The comparison between the relative saturation concepts of Pham-Teissier and of Zariski is based on Lemma (1.1). To apply that lemma, we need some preliminary remarks concerning geometrically unibranch domains and "universal going-down" homomorphisms.

An integral domain R is *geometrically unibranch* if its integral closure $\bar{R}$ (in its field of fractions) is a *radicial* R-algebra. In particular, if $R=\bar{R}$, i.e. R is integrally closed, then R is geometrically unibranch.

Let R be a geometrically unibranch domain, and let $f\colon R\to S$ be a ring homomorphism making S integral over R and torsion-free as an R-module. Then "*going-down*" *holds for* f: if q is any prime ideal in S, and $p'\subseteq f^{-1}(q)$ is a prime ideal in R, then $p'=f^{-1}(q')$ for some prime ideal $q'\subseteq q$. (This can be deduced easily from the usual going-down theorem of Cohen-Seidenberg [AC, Ch. 5, p. 56, Theorem 3], applied to $\bar{f}:\bar{R}\to\bar{f}(\bar{R})[S]\subseteq L$, where L is the total ring of fractions of S, and $\bar{f}$ is the canonical extension of f.) Any polynomial ring $R[X]$ (X, an indeterminate) is also geometrically unibranch (its integral closure being $\bar{R}[X]$ (cf. [AC, Ch. 5, p. 19, Cor. 1]), which is then radicial over $R[X]$ (cf. [EGA 01, p. 249, Cor. (3.7.6)(iii)])); and $S[X]$ is an integral $R[X]$-algebra, torsion-free over $R[X]$. Similar conditions hold for the pair $R[(X_\alpha)]$, $S[(X_\alpha)]$ for *any* family (X_α) of independent indeterminates. (Reduce to the case where the family is finite, then use induction.) Thus, the class of R-algebras T such that going-down holds for the canonical map $f_T:T\to S\otimes_R T$ contains any polynomial ring over R. Since this class evidently contains with any T all its homomorphic images, it must contain *all* R-algebras.†

In particular, going down holds for $f_S:S\to S\otimes_R S$, and hence also for $f_S\circ f:R\to S\otimes_R S$. Consequently—and this is the only fact which we will need below—*if $f:R\to S$ is as above, then every minimal prime ideal in $S\otimes_R S$ contracts to (0) in R.*

Now consider an integral domain R and an R-algebra B, B being integral over R and torsion-free over R (so that R may be identified with a subring of B). Let Ω be an algebraic closure of the field of fractions K of R. Say that $\eta\in B$ *R-dominates* $\zeta\in B$ if, for any two R-homomorphisms $\psi_1,\psi_2:B\to\Omega$, the quotient

$$\frac{\psi_1(\eta)-\psi_2(\eta)}{\psi_1(\zeta)-\psi_2(\zeta)}$$

is *integral* over R. (If $\psi_1(\zeta)=\psi_2(\zeta)$ this is taken to mean that $\psi_1(\eta)=\psi_2(\eta)$.)

LEMMA (4.1). *In the preceding situation assume furthermore that R is geometrically unibranch. Then $\eta\in B$ R-dominates $\zeta\in B$ if and only if, in $B\otimes_R B$, $\eta\otimes 1-1\otimes\eta$ is integral over the ideal generated by $\zeta\otimes 1-1\otimes\zeta$.*

Proof. If $\eta\otimes 1-1\otimes\eta$ is integral over $(\zeta\otimes 1-1\otimes\zeta)(B\otimes_R B)$ then there exists a *homogeneous* polynomial of degree, say, n,

$$H(X,Y)\in(B\otimes_R B)[X,Y]$$

such that $H(X,0)=X^n$, and $H(\eta\otimes 1-1\otimes\eta,\ \zeta\otimes 1-1\otimes\zeta)=0$. For ψ_1, ψ_2 as above, we obtain a map

$$\psi_1\otimes\psi_2:B\otimes_R B\to\Omega,$$

†(Added in proof). A stronger result, viz. that $\mathrm{Spec}(S)\to\mathrm{Spec}(R)$ is *universally open*, can be deduced from [EGA IV, (14.4.4)], or, more simply, cf. H. Seydi, *C. R. Acad. Sc. Paris, Série A*, **271** (1970), pp. 1107–1108.

and hence a relation

$$\overline{H}(\psi_1(\eta)-\psi_2(\eta),\psi_1(\zeta)-\psi_2(\zeta))=0,$$

where $\overline{H}(X,Y)=(\psi_1\otimes\psi_2)H(X,Y)$ is a form of degree n with coefficients which are *integral over* R, and $\overline{H}(X,0)=X^n$. It follows easily that η dominates ζ.

Suppose conversely that η dominates ζ. Let $S=R[\eta,\zeta]\subseteq B$, and let p be any minimal prime ideal in $S\otimes_R S$. By the preceding remarks, p contracts to (0) in R. Thus p is the kernel of an R-homomorphism of $S\otimes_R S$ into Ω, and (cf. proof of (7a), Section 1) such a homomorphism is necessarily of the form $\psi_1\otimes\psi_2$, ψ_1, ψ_2 being R-homomorphisms of B into Ω (more precisely, the restrictions to S of such R-homomorphisms). Since η dominates ζ, there exists a homogeneous polynomial $G(X,Y)\in R[X,Y]$, of degree, say, m, such that

$$G(X,0)=X^m$$

and

$$G(\psi_1(\eta)-\psi_2(\eta),\quad \psi_1(\zeta)-\psi_2(\zeta))=0$$

i.e.

$$G(\eta\otimes 1-1\otimes\eta,\zeta\otimes 1-1\otimes\zeta)\in p.$$

Thus, in $S\otimes_R S$, $\eta\otimes 1-1\otimes\eta$ is integral over $(\zeta\otimes 1-1\otimes\zeta)(S\otimes_R S)$ modulo each minimal prime p; and there are only finitely many such p since $S\otimes_R S$ is finite over the integral domain R. So by Lemma (1.1) (with $J_1,J_2,\dots,J_n$ the various minimal primes in $S\otimes_R S$), $\eta\otimes 1-1\otimes\eta$ is integral over $(\zeta\otimes 1-1\otimes\zeta)(S\otimes_R S)$. Applying the canonical map $S\otimes_R S\to B\otimes_R B$, we obtain the desired conclusion. Q.E.D.

Let $R\subseteq B$ be as above and let A be a ring with $R\subseteq A\subseteq B$. Say that A *is R-saturated in B—in Zariski's sense*—if every element of B which R-dominates an element of A is itself in A. The *R-saturation of A in B*, $A_{B,R}^{\sim}$ in symbol, is the intersection of all such R-saturated rings between A and B; i.e. $A_{B,R}^{\sim}$ is the smallest such R-saturated ring.

Corollary (4.2). *Under the preceding circumstances (viz.* $R\subseteq A\subseteq B$, R *a geometrically unibranch domain, B integral over R, no nonzero element of R a zero-divisor in B) we have that* $A^*_{B,R}$ *is (Zariski-) R-saturated in B, so that*

$$A_{B,R}^{\sim}\subseteq A^*_{B,R}.$$

If there exists in A an element y such that $A_{B,R}^{\sim}=R[y]_{B,R}^{\sim}$ *(i.e. y is an "R-saturator of A in B"), then*

$$A_{B,R}^{\sim}=A^*_{B,R}.$$

Proof. The first assertion follows easily from Lemma (4.1). As for the second, Lemma (4.1) shows that every element of $R[y]^*$ dominates y, so $R[y]^* \subseteq R[y]^{\sim}$; hence

$$A^{\sim} \subseteq A^* \subseteq (A^{\sim})^* = (R[y]^{\sim})^* \subseteq (R[y]^*)^*$$

$$= R[y]^* \subseteq R[y]^{\sim} = A^{\sim}. \qquad \text{Q.E.D.}$$

Remarks. (i) An example where $A^{\sim} \neq A^*$ is described in the introduction to [GTS III].

(ii) If in (4.2) R is a *one-dimensional noetherian local domain with* infinite *residue field*, and A is a *finite* R-module, then there *does* exist an R-saturator of A in B (cf. [GTS II, p. 878, Prop. 1.3] and also [4, p. 681, Lemma 5.2]). In this case, therefore, $A^{\sim} = A^*$.

(iii) With the hypotheses of Corollary (4.2), we see now, from Proposition (1.4), that $A^{\sim}_{B,R}$ *is a radicial A-algebra*. (This is Theorem 4.1 of [8, p. 997]).

(iv) In the special situation where *equality* holds in Corollary (4.2), we can deduce a number of results of [GTS III] from those in this paper. Most notable is the *compatibility of saturation and completion* (Corollary (2.6) and also example (5.2) below; [GTS III, Theorem 2.7]). Some other—more elementary—pairs of related results are listed below.

[This paper]	[GTS III]
Section 1, (3)	Lemma A.8.(1)
Prop. (1.3)	Lemma A.8.(2)
Section 1, (5)	Prop. 1.9
Section 1, (6)	Prop. 2.5
Remark (ii) following Prop. (3.1)	Prop. 1.2
or Section 2, Prop. (2.2) and Example (i).	

(v) Theorem 4.1 of [GTS III]—which asserts, with suitable hypotheses, the preservation of multiplicity under saturation—has no analogue in this paper. But *the proof* (loc. cit) *applies almost verbatim* to the case where relative Zariski-saturation is replaced by relative Lipschitz-saturation. (Since $A^{\sim} \subseteq A^*$, the resulting statement is actually stronger.)

(vi) Which definition of relative saturation is to be preferred? The deeper results of Zariski's theory, as it now stands, refer either to the one-dimensional or to the hypersurface case, situations in which the two definitions agree. As far as I know, for *every* established theorem involving Zariski-saturation there is a corresponding theorem involving Lipschitz-saturation. So the Pham-Teissier definition enjoys, at present, a certain advantage, at least of generality (not to mention its nice analytic interpretation [6]). But only future developments will show which definition—if either—leads to more significant results in the non-hypersurface case. (Added in proof: cf. [10].)

5. Saturation and Base Change in the Hypersurface Case. This rather technical section is an appendix to Section 2. We show that under certain circumstances, Proposition (2.5) holds with weaker hypotheses on the R-algebra R'. The basic idea, due to Zariski, is to use techniques similar to those of [GTS III, Section 1] to reduce the question to the case where R and R' are discrete valuation rings.

Let T be a ring and let T' be a T-algebra. We shall say that a prime ideal q of T is *geometrically unramified* in T' if, for every prime ideal p of T' which is *minimal* among those containing qT', we have that qT'_p is the maximal ideal of T'_p and that T'_p/qT'_p is a *separable* field extension of T_q/qT_q.

PROPOSITION (5.1). *Let R be an integrally closed noetherian domain, and let A be an R-algebra of the form*

$$A = R[x] = R[X]/f(X)$$

where X is an indeterminate and $f(X) \in R[X]$ is a monic polynomial with non-vanishing discriminant. Let R' be an R-algebra such that (i) R' is an integrally closed noetherian domain, (ii) R' is flat over R, and (iii) every prime ideal of height $\leqslant 1$ in R is geometrically unramified in R'. Then, setting $A' = A \otimes_R R'$, we have

$$(A')^*_{R'} = (A^*_R) \otimes_R R'.$$

(Here A^*_R is *the* R-saturation of A (cf. remarks immediately preceding Proposition (2.5)); by Corollary (4.2) A^*_R is also the Zariski-R-saturation of A in its integral closure.)

Example (5.2). Let R be an *analytically normal, pseudogeometric* noetherian local ring (cf. [5]). Then conditions (i), (ii), and (iii) of (5.1) are satisfied by $R' = \hat{R}$, the usual completion of R. (For (iii) cf. [EGA IV, (7.6.4)]). So, *with A as in* (5.1), we have (cf. Corollary (2.6)):

$$(\hat{A})^*_{\hat{R}} = (A^*_R)^\wedge .$$

Proof of (5.1). (I) We begin with some preliminary remarks on *reflexive R-modules*. Let M be an R-module, let $M^{\cdot}$ be the R-module $\mathrm{Hom}_R(M,R)$, and let $M^{\cdot\cdot}$ be the R-module $\mathrm{Hom}_R(M^{\cdot},R)$. M is *reflexive* if the canonical map $c:M\to M^{\cdot\cdot}$ is *bijective*, where c is given by

$$[c(m)](f)=f(m) \qquad (m\in M, f\in M^{\cdot}).$$

Any torsion element of M is clearly in the kernel of c, so if M is reflexive, then M is *torsion-free*.

Let K be the field of fractions of R, and let P be the set of height one prime ideals in R. If N is a finitely generated torsion-free R-module, then for each $p\in P$, we have

$$N\subseteq N_p\subseteq N\otimes_R K \qquad (N_p=N\otimes_R R_p),$$

and N *is reflexive if and only if*

$$N=\bigcap_{p\in P} N_p.$$

[AC, Ch. 7, p. 50, Theorem 2].

Finally, if M is a finitely-generated reflexive R-module, and R' is *any* flat R-algebra, then $M\otimes_R R'$ *is a reflexive R'-module*. [AC, Ch. 7, p. 53, Prop. 8].

(II) Now let A be as in (5.1), let F be the total ring of fractions of A, let B be the integral closure of A in F, and let $B'=B\otimes_R R'$. We first observe that B *is a finitely generated reflexive R-module* (*whence B' is a finitely generated reflexive R'-module*). Indeed it is immediate that

$$F=A\otimes_R K=K[X]/f(X)$$

($K=$fraction field of R), and since f has nonzero discriminant, F is a finite product of *separable field extensions* of K; hence B is a finitely generated torsion free R-module. Arguing as in Lemma 1.7 of [GTS III], we find that $B=\cap B_p$ ($p\in P$), i.e. B is reflexive.

Let P' be the set of height one prime ideals in R'. We shall show below that:

(5.3) *B'_p is locally normal for each prime ideal $p\in P'$.*

It will then follow that:

(5.4) *B' is the integral closure of A' in its total ring of fractions F'.*

For, if K' is the field of fractions of R', then we find easily that

$$F' = A' \otimes_{R'} K' = A \otimes_R K' = F \otimes_K K' = B \otimes_R K' = B' \otimes_{R'} K'.$$

Furthermore, we have just seen that B' is a finitely generated reflexive R'-module, so that, in F', $B' = \cap B'_p$ ($p \in P'$). If we know that each B'_p is locally normal, then each B'_p is integrally closed in its total ring of fractions F' (cf. end of proof of Proposition (2.5)), and consequently B' is integrally closed in F', whence the assertion.

(III) Next, using the functoriality of saturation Section 1, (3) and the flatness of R' over R, we get

$$(A^*_{B,R}) \otimes_R R' \subseteq (A')^*_{B',R'} \subseteq B'.$$

For convenience, put

$$M = (A^*_{B,R}) \otimes_R R' = A^*_R \otimes_R R'$$

$$N = (A')^*_{B',R'} = (A')^*_{R'} \qquad \text{(by (5.4))}.$$

Proposition (5.1) asserts then that $M = N$. Since B' is torsion-free and finitely generated over R', so also are M and N, and for each $p \in P'$, we have

$$M_p \subseteq N_p \subseteq F'.$$

So to prove (5.1) it will suffice to show—in addition to (5.3)—that

(5.5) *A^*_R is a finitely generated reflexive R-module*

(whence M is a finitely generated reflexive R'-module, i.e. $M = \cap M_p$ ($p \in P'$)), and that

(5.6) *for each $p \in P'$, $M_p = N_p$*

(whence $N \subseteq \cap_{p \in P'} N_p = \cap_{p \in P'} M_p = M$).

(IV) (5.5) is contained in Theorem 1.8 of [GTS III]; for, as remarked before, A^*_R is identical with the Zariski-R-saturation of A. [Actually what we need here is that part of the proof of the cited Theorem 1.8 which follows (17) (loc. cit), where it should be noted that the left-hand side of (17) is contained in B because, as above, $B = \cap B_p$ ($p \in P$)].

(V) *It remains to prove* (5.3) *and* (5.6). Let $p \in P'$, and let q be the contraction of p in R. Then, setting $A^* = A^*_{B,R}$, $A^*_q = A^* \otimes_R R_q$, we have

$$M_p = (A^* \otimes_R R') \otimes_{R'} R'_p = (A^* \otimes_R R_q) \otimes_{R_q} R'_p = A^*_q \otimes_{R_q} R'_p.$$

Since saturation commutes with localization (cf. Section 2 or Section 3) we have

$$A_q^* = (A_q)^*_{B_q, R_q}$$

Similarly,

$$N_p = (A')^*_{B', R'} \otimes_{R'} R'_p = (A'_p)^*_{B'_p, R'_p}$$

where

$$A'_p = A' \otimes_{R'} R'_p = A \otimes_R R'_p = A_q \otimes_{R_q} R'_p$$

$$B'_p = B' \otimes_{R'} R'_p = B \otimes_R R'_p = B_q \otimes_{R_q} R'_p.$$

If $q=(0)$, we have $A_q = B_q = F$, whence

$$M_p = N_p = B'_p.$$

If $q \neq (0)$, then q has height one, (since $\mathrm{Spec}(R'_p) \to \mathrm{Spec}(R_q)$ is surjective, R'_p being *faithfully flat* over R_q). Since p has height one, p is a minimal prime divisor of q. By assumption, q is geometrically unramified in R', and so is the prime ideal (0) of R. Hence, if k is the residue field of R_q, we have

—$k \otimes_{R_q} R'_p$ is a separable field extension of k

—$K \otimes_{R_q} R'_p$ is a separable field extension of K.

It follows that R'_p is *normal over* R_q (Section 2, example (iv)). Hence by results of Section 2,

$$(A_q)^*_{B_q, R_q} \otimes_{R_q} R'_p = (A'_p)^*_{B'_p, R'_p}$$

i.e.

$$M_p = N_p.$$

This proves (5.6).

To prove (5.3), suppose first that $q \neq (0)$. Then, as above, R'_p is a normal R_q-algebra, and since B_q is locally normal, therefore so is B'_p [EGA IV, (6.14.1)]. If $q=(0)$, then

$$B'_p = B_q \otimes_{R_q} R'_p = F \otimes_K R'_p.$$

But F is a normal K-algebra (since F is a finite product of *separable* field extensions of K), and R'_p is a discrete valuation ring, so again by [EGA IV, (6.14.1)] B'_p is locally normal.

This completes the proof.

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1. The diagram in the middle of page 794 should be:

$$\begin{array}{ccc} & B/g(A) & \\ {}^{\alpha}\swarrow & & \searrow^{\bar{f}} \\ (B/g(A))\otimes_R R' & \underset{\beta}{\overset{\approx}{\longrightarrow}} & B'/g'(A') \end{array}$$

2. On page 804, starting on line 6, read:

We have, moreover, the natural identifications

$$\begin{aligned} ((B\otimes_R B)/\bar{I})\otimes_A A' &= ((B\otimes_A B)/I_1)\otimes_A A' \qquad (I_1=\bar{I}/I) \\ &= (B'\otimes_{A'} B')/I_1(B'\otimes_{A'} B') \end{aligned}$$

and

$$(B'\otimes_R B')/\bar{J}=(B'\otimes_{A'} B')/J_1 \qquad (J_1=\bar{J}/J)$$