RINGS WITH DISCRETE DIVISOR CLASS GROUP: THEOREM OF DANILOV-SAMUEL

By JOSEPH LIPMAN†

Introduction. In [3], Danilov has studied normal noetherian domains A for which the canonical map of divisor class groups i^*: $\mathcal{C}\ell(A) \to \mathcal{C}\ell(A[[T]])$ is bijective; such A are said to have *discrete divisor class group* (abbreviated DCG).[1] One of Danilov's main results [3, p. 374, Theorem 1] is

(D-S). *If the normal noetherian domain A is such that the localization A_p has DCG at every prime ideal p for which* $\text{depth}(A_p) = 2$, *then A itself has DCG.*[2]

This result generalizes a theorem of Samuel [10, p. 5, Theorem 3.2]. (Danilov states [3, p. 368] that Samuel's proof is incomplete.) Danilov's proof, in which certain additional mild restrictions are placed on A, uses difficult cohomological results of Grothendieck [5]. The *purpose of this note* is to give a proof of (D-S) which uses nothing deeper than a well-known theorem of Rees on the connection between depth and Ext, and moreover needs no additional hypotheses on A.

Actually we obtain somewhat more than (D-S). Danilov defines a map j^*: $\mathcal{C}\ell(A[[T]]) \to \mathcal{C}\ell(A)$ such that $j^* \circ i^* =$ identity (cf. [4, p. 109, Theorem 18.8], or else just use the equivalent definition given in Section 1 below); so *A has DCG $\Leftrightarrow$ j^* is injective.* But in fact Danilov's definition works more generally to give a map $j_{B,t}$: $\mathcal{C}\ell(B) \to \mathcal{C}\ell(B/tB)$ for any noetherian normal ring B and nonunit $t \in B$ such that B/tB is also normal. [Thus the DCG property begins to look like one of "Lefschetz type," i.e., it has to do with the comparison of something on Spec(B) to the corresponding

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[1] All rings are understood to be commutative. The definition of i^* can be found, e.g., in [4, Sect. 6 (cf. especially pp. 29-31, 35)]. A survey of the theory of rings with DCG is given in [8, Sects. 1-3].

[2] The converse "A has DCG $\Rightarrow$ A_M has DCG for every multiplicative set M in A" is an open question. (For an affirmative answer in case A is an excellent $\mathbf{Q}$-algebra, cf. [3, p. 371, Prop. 4].)

thing on the "hypersurface section" Spec(B/tB). This indicates why Danilov finds [5] so useful.] This being so, what we show is

(D–S)′. *Let B be a normal noetherian domain and t a regular element* (= nonzerodivisor) *in the Jacobson radical of B such that the ring $A = B/tB$ is a normal domain. Suppose that for every prime ideal p in A such that* depth$(A_p) = 2$, *there exists a normal noetherian flat B-algebra C such that*

$$C/tC \cong A_p \qquad (\textit{isomorphic as } A\textit{-algebras})$$

and such that $j_{C,t}\colon \mathcal{C}\ell(C) \to \mathcal{C}\ell(A_p)$ *is injective. Then* $j_{B,t}\colon \mathcal{C}\ell(B) \to \mathcal{C}\ell(A)$ *is injective.*[3]

(Taking $B = A[[T]]$, $t = T$, $C = A_p[[T]]$ in (D–S)′, we get (D–S).)

Using a familiar interpretation of $\mathcal{C}\ell(A)$ as being the group of isomorphism classes of reflexive rank one A-modules—an interpretation which, for lack of convenient references, we review briefly in Section 0—we reformulate the injectivity of $\mathcal{C}\ell(B) \to \mathcal{C}\ell(B/tB)$ as a simple statement about free modules. This gives us a straightforward reduction of (D–S)′ to Theorem 1 (Section 2), which is the main result. (This reduction is carried out in Section 1; Theorem 1 and its proof (Section 2) are independent—except for motivation—of anything which precedes them.)

As a corollary of Theorem 1, we obtain a generalization of a result of [5], including the following statement:

If B is a noetherian local ring of depth ≥ 4 and if t is a regular nonunit in B, then B/tB parafactorial $\Rightarrow$ B parafactorial.

0. Reflexive Modules and the Divisor Class Monoid. Let A be a *noetherian integral domain.* For any A-module E, let E^* be the A-module $\mathrm{Hom}_A(E, A)$. Write E^{**} for $(E^*)^*$. Recall that E is said to be *reflexive* if the canonical map $\alpha\colon E \to E^{**}$ is *bijective.* $\{[\alpha(x)](f) = f(x)$ for all $x \in E$, $f \in E^*.\}$

Let K be the field of fractions of A. We shall say that an A-module E has *rank one* if E is *finitely generated* over A and the K-vector space $E \otimes_A K$ is *one-dimensional.* If E has rank one, then so does E^*; and if

[3] In (D–S)′, if $j_{C,t}$ is injective then so is $j_{B_q,t}$, where q is the inverse image of p in B. So the hypothesis in (D–S)′ could be changed to: "*Suppose that $j_{B_q,t}$ is injective for all prime ideals q containing t and such that* depth$(B_q) = 3$."

two A-modules E_1 and E_2 both have rank one, then so does $E_1 \otimes_A E_2$. For any rank one reflexive A-module E, the canonical map $E \to E \otimes_A K$ is injective, and so E is isomorphic to an A-submodule of K.

If E_1 and E_2 are rank one A-submodules of K, then any A-homomorphism of E_1 into E_2 extends uniquely to a K-homomorphism of $E_1 \otimes_A K$ $(=K)$ into $E_2 \otimes_A K$ $(=K)$; it follows that we can identify $\mathrm{Hom}_A(E_1, E_2)$ with the A-module

$$E_2 : E_1 = \{x \in K \mid xE_1 \subseteq E_2\}.$$

The canonical map $E_1 \to E_1^{**}$ is then identified with the inclusion of E_1 into $A:(A:E_1)$. So E_1 is *reflexive* if and only if $E_1 = A:(A:E_1)$, i.e. if and only if E_1 is a *divisorial fractionary ideal* of A (cf. [2, Section 1.1, Definition 2 and Proposition 1]). And E_1 is *A-isomorphic* to E_2 if and only if there is an $x \neq 0$ in K such that $xE_1 = E_2$, i.e. E_1 and E_2 determine *equivalent divisors* [2, Section 1.2, Proposition 4].

With this in mind, it is now straightforward to see that the elements of the *divisor class monoid*—denoted $\mathcal{C}\ell(A)$—as defined in [2, Section 1.2], are in one-to-one correspondence with the *isomorphism classes of reflexive rank one A-modules.*

The *monoid structure* on $\mathcal{C}\ell(A)$ can be described as follows (verification left to the reader): For any rank one A-module E let

$$[E]_A = (\text{isomorphism class of } E^{**}) \in \mathcal{C}\ell(A).$$

(We may write $[E]$ for $[E]_A$ if no confusion results.) Multiplication in $\mathcal{C}\ell(A)$ is such that

$$[E_1][E_2] = [E_1 \otimes_A E_2]$$

for any two rank one A-modules E_1, E_2. So *multiplication is determined by* $\otimes$ (even though the tensor product of two reflexive A-modules need not itself be reflexive!). The *identity element* of $\mathcal{C}\ell(A)$ is $[A]$. One checks that $[E_1] = [E_2]$ if and only if E_1^* and E_2^* are isomorphic; in particular, *$[E]$ is the identity element of $\mathcal{C}\ell(A)$ $\Leftrightarrow$ E^* is free.*

1. Reformulation of (D–S)′. Let B, t, A be as in the statement of (D–S)′. It is easily seen that $j_{B,t}\colon \mathcal{C}\ell(B) \to \mathcal{C}\ell(A)$ is such that for each rank one reflexive B-module F,

$$j_{B,t}([F]_B) = [F \otimes_B A]_A = [F/tF]_A.$$

[Since the localization B_q of B at the prime ideal $q = tB$ is a U.F.D., therefore the reflexive rank one B_q-module $F \otimes_B B_q$ is *free*, and so the *dimension* of

$$(F/tF) \otimes_A (B_q/qB_q) = (F \otimes_B B_q) \otimes_{B_q} (B_q/qB_q)$$

over the field B_q/qB_q is *one*, i.e. F/tF is a rank one A-module (not necessarily reflexive!).] In view of the characterization of the identity element of $\mathcal{C}\ell(A)$ given at the end of Section 0, we conclude that $j_{B,t}$ *is injective* $\Leftrightarrow$ *the following condition holds*:

$(*)_{B,t}$. With $A = B/tB$, if F is any reflexive rank one B-module such that $\mathrm{Hom}_A\,(F/tF, A)$ is a free A-module, then $\mathrm{Hom}_B\,(F, B)$ is a free B-module (i.e. F itself is free).

Next, we need a simple observation:

LEMMA. *Let B, t, A, p, C be as in* (D–S)′. *Let F be a reflexive rank one B-module such that* $\mathrm{Hom}_A\,(F/tF, A)$ *is a free A-module. Then $(F/tF) \otimes_A A_p$ is a free A_p-module.*

Proof. Since C is *flat* over B, therefore $F_C = F \otimes_B C$ is a reflexive rank one C-module [2, Section 4.2, Proposition 8]. Furthermore

$$F_C/tF_C = (F \otimes_B C) \otimes_C (C/tC) = (F \otimes_B A) \otimes_A (C/tC) = (F/tF) \otimes_A A_p$$

whence

$$\mathrm{Hom}_{A_p}\,(F_C/tF_C, A_p) = \mathrm{Hom}_A\,(F/tF, A) \otimes_A A_p,$$

and so $\mathrm{Hom}_{A_p}\,(F_C/tF_C, A_p)$ is a free A_p-module. Since by assumption $j_{C,t}$ is injective, therefore $(*)_{C,t}$ holds, and we conclude that F_C is free over C, whence $(F/tF) \otimes_A A_p$ $(= F_C/tF_C)$ is free over A_p $(=C/tC)$. Q.E.D.

One final remark: B, t, A being as above, if F is any finitely generated reflexive B-module then F is *torsion-free* (in other words, if $0 \neq f \in F$ and b is a regular element of B, then $bf \neq 0$); and furthermore *F/tF is a torsion-free A-module.* (Proof left to reader; the only hypothesis really needed is that $F = \mathrm{Hom}_B(G, B)$ for some B-module G.) It follows that if p is a prime ideal of depth ≤ 1 in the normal noetherian domain A, then $(F/tF) \otimes_A A_p$ is a free A_p-module.

It should now be evident—and up to this point nothing very substantial has been done—that (D-S)′ follows from Theorem 1 below.

2. The Basic Result.

THEOREM 1. *Let B be a noetherian ring and let t be a regular element in the Jacobson radical of B. Let F be a finitely generated B-module such that t is not a zero-divisor in F. Set $A = B/tB$, $E = F/tF$, and suppose that*

(i) $\mathrm{Hom}_A(E, A)$ *is a free A-module, and*
(ii) $E_p = E \otimes_A A_p$ *is a free A_p-module for every prime ideal p in A such that* $\mathrm{depth}(A_p) \leq 2$.

Then $\mathrm{Hom}_B(F, B)$ *is a free B-module.*[4]

Proof. Consider the exact sequence

$$0 \to \mathrm{Hom}_B(F, B) \xrightarrow{t} \mathrm{Hom}_B(F, B) \to \mathrm{Hom}_B(F, A)$$
$$\to \mathrm{Ext}^1_B(F, B) \xrightarrow{t} \mathrm{Ext}^1_B(F, B) \to \mathrm{Ext}^1_B(F, A)$$

(obtained from the short exact sequence $0 \to B \xrightarrow{t} B \to A \to 0$). If we can show that $\mathrm{Ext}^1_B(F, A) = 0$, then $\mathrm{Ext}^1_B(F, B) = t(\mathrm{Ext}^1_B(F, B))$, so by Nakayama's lemma [1, Chapter 2, Section 3.2, Proposition 4], $\mathrm{Ext}^1_B(F, B) = 0$, whence

$$\mathrm{Hom}_B(F, B)/t(\mathrm{Hom}_B(F, B)) \cong \mathrm{Hom}_B(F, A) = \mathrm{Hom}_A(E, A);$$

since $\mathrm{Hom}_A(E, A)$ is a free A-module, it follows from [1, Chapter 2, Section 3.2, Proposition 5] (applied to the ideal tB) that $\mathrm{Hom}_B(F, B)$ is free over B, as required.

Now $\mathrm{Ext}^1_B(F, A)$ *is canonically isomorphic to* $\mathrm{Ext}^1_A(E, A)$. [This can be seen—for example—as follows: Let $G_{.} = \cdots \to G_2 \to G_1 \to G_0 \to 0$

[4] One can check, using [7, p. 879, Proposition 2.2], that (ii) can be replaced by the following two weaker conditions:

(ii)′ E_p is a torsion-free A_p-module whenever $\mathrm{depth}(A_p) = 1$;
(ii)″ E_p is a reflexive A_p-module whenever $\mathrm{depth}(A_p) = 2$.

[(ii)′ is automatic if F is reflexive.]

be a projective resolution of the B-module F (so that $F = H_0(G.)$). A brief examination of the long exact homology sequence of the exact sequence of complexes

$$0 \to G. \xrightarrow{t} G. \to G./tG. \to 0$$

shows that $G./tG.$ is an A-projective resolution of the A-module E. (It should be kept in mind here that, by assumption, multiplication by t in $H_0(G.)$ is an injective map.) Hence, for all $i \geq 0$, and any A-module M

$$\begin{aligned} \operatorname{Ext}^i{}_A(E, M) &= H^i(\operatorname{Hom}_A(G./tG., M)) \\ &= H^i(\operatorname{Hom}_B(G., M)) = \operatorname{Ext}^i{}_B(F, M).] \end{aligned}$$

Let us show then that $\operatorname{Ext}^1{}_A(E, A) = 0$.

Let $\alpha: E \to E^{**}$ be the canonical map of E into its "bidual" (over A) (cf. Sect. 0), and let K, I, C be the kernel, image, and cokernel (respectively) of α. From the exact sequences

$$0 \to K \to E \to I \to 0, \qquad 0 \to I \to E^{**} \to C \to 0$$

we obtain exact seqences

$$\operatorname{Ext}^1{}_A(I, A) \to \operatorname{Ext}^1{}_A(E, A) \to \operatorname{Ext}^1{}_A(K, A),$$

$$\operatorname{Ext}^1{}_A(E^{**}, A) \to \operatorname{Ext}^1{}_A(I, A) \to \operatorname{Ext}^2{}_A(C, A).$$

Since E^* is free (by assumption), also E^{**} is free, and $\operatorname{Ext}^1{}_A(E^{**}, A) = 0$. So it will be enough to show that

$$\operatorname{Ext}^2{}_A(C, A) = \operatorname{Ext}^1{}_A(K, A) = 0.$$

Now for any prime p in A, there is a *natural identification of* $\alpha \otimes_A A_p$ *with the canonical map of* $E \otimes_A A_p$ *into its bidual* (*over* A_p). [Indeed, for any A-algebra R, we have a natural commutative diagram

$$\begin{array}{ccc} E \otimes_A R & \xrightarrow{\beta} & \operatorname{Hom}_R(\operatorname{Hom}_R(E \otimes_A R, R), R) \\ {\scriptstyle \alpha \otimes_A R}\downarrow & & \downarrow{\scriptstyle \gamma} \\ E^{**} \otimes_A R & \xrightarrow[\delta]{} & \operatorname{Hom}_R(E^* \otimes_A R, R) \end{array}$$

where β is the natural map of $E \otimes_A R$ into its bidual (over R); γ is obtained by applying $\mathrm{Hom}_R(\cdot, R)$ to the natural map

$$\omega_E\colon \mathrm{Hom}_A(E, A) \otimes_A R \to \mathrm{Hom}_R(E \otimes_A R, R);$$

and δ is ω_{E^*}. And if R is *flat* over A, then γ and δ are isomorphisms [1, Chapter 1, Section 2.10, Proposition 11].]

Thus if $\mathrm{depth}(A_p) \le 2$, so that $E \otimes_A A_p$ is free, then $\alpha \otimes_A A_p$ is bijective, i.e. $K_p = C_p = 0$. By a theorem of Rees ([9, p. 31, Theorem 1.3] or also [5, p. 33, Proposition 2.9]), this means that

$$\mathrm{Ext}^i(K, A) = \mathrm{Ext}^i(C, A) = 0 \qquad (i \le 2). \qquad \text{Q.E.D.}$$

We note, in closing, the following generalization of [5, p. 133, Lemma 3.16]:

COROLLARY. *Let B, t, and A be as in Theorem* 1 [*so that* Spec (A) *can be identified with the closed subscheme $t = 0$ of Spec (B)*]. *Let X be an open subset of* Spec (B), *let $Y \subset X$ be an open subset of* Spec (A) *such that* depth $(A_p) \ge 3$ *for all prime ideals p in* Spec $(A) - Y$,[5] *and let $i\colon Y \to X$ be the inclusion map. If $\mathfrak{F}$ is any locally free coherent sheaf on X with $i^*\mathfrak{F} \cong \mathcal{O}_Y{}^n$, then $\mathfrak{F} \cong \mathcal{O}_X{}^n$. In particular, the canonical map* Pic $(X) \to$ Pic (Y) *is injective.*

Proof. There exists a finitely generated B-module F such that, if $F^\sim$ is the corresponding coherent sheaf on Spec (B), then $F^\sim|X \cong \mathfrak{F}$ [6, p. 318, Cor. (6.9.5)]. We may replace F by its bidual, i.e. we may assume that F is reflexive. (In particular, t is not a zero-divisor in F.) We shall show that $E = F/tF$ satisfies conditions (i) and (ii) of Theorem 1; then $\mathrm{Hom}_B(F, B)$ is free, so F itself, being reflexive, is free, and the Corollary results.

For (i), let $j\colon Y \to$ Spec (A) be the inclusion map. Since Y contains all primes p such that depth $(A_p) \le 1$, we have that: for any finitely generated *free* A-module G, with associated sheaf $G^\sim$ on Spec (A), the canonical map $G^\sim \to j_* j^* G^\sim$ is an isomorphism (cf. for example [5, p. 37, Cor. 3.5]). Applying $\mathrm{Hom}_A(\cdot, A)$ to a finite presentation of E, we obtained an exact sequence

$$0 \to E^* \to G_1 \to G_2$$

[5] This means precisely that $H^0(Y, \mathcal{O}_Y) = A$ and $H^1(Y, \mathcal{O}_Y) = 0$.

with G_1 and G_2 finitely generated free A-modules; in the resulting commutative diagram (with exact rows)

$$\begin{array}{ccccccc}
0 & \longrightarrow & E^{*\sim} & \longrightarrow & G_1{}^\sim & \longrightarrow & G_2{}^\sim \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\
0 & \longrightarrow & j_*j^*E^{*\sim} & \longrightarrow & j_*j^*G_1{}^\sim & \longrightarrow & j_*j^*G_2{}^\sim
\end{array}$$

we see then that ϕ is an isomorphism. But

$$j^*E^\sim = i^*\mathfrak{F} \cong \mathcal{O}_{Y}{}^n$$

and hence

$$j^*E^{*\sim} \cong \mathcal{O}_{Y}{}^n 0 \qquad [=j^*(A^n)^\sim].$$

Applying j_*, we conclude that $E^{*\sim} \cong (A^n)^\sim$, so E^* is free.

As for (ii), if $\operatorname{depth}(A_p) \leq 2$, then $p \in Y$; but then E_p is just the stalk at p of $i^*\mathfrak{F}$, so E_p is free. Q.E.D.

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