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# RESIDUES AND TRACES OF DIFFERENTIAL FORMS VIA HOCHSCHILD HOMOLOGY

by

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## Contents



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## §0. INTRODUCTION

The *residue symbol* introduced by Grothendieck [**RD**, pp. 195-199] has been found useful in various contexts: duality theory of algebraic varieties, Gysin homomorphisms of manifolds with vector fields having isolated zeros, integral representations in several complex variables, just to mention a few (cf. for example [**L**], [**AC**], [**AY**], and their bibliographies).

However, in spite of its broad interest, the theory of the residue symbol does not seem to have been written down in a really satisfactory manner. One difficulty is that Grothendieck's approach depends on the global duality machinery developed in [**RD**]; and furthermore proofs are not given there. (A more detailed version is presented in [**Bv**]; and for a complete treatment of the case of algebraic varieties, with a somewhat different slant, cf. [**L**].) Grothendieck considers a smooth map $f: X \to Y$ of locally noetherian schemes, with q-dimensional fibres, and a closed subscheme Z of X defined by an ideal $I$ which is locally generated by q elements, and such that Z is finite over Y. With $i: Z \to X$ the inclusion, and $g = f \circ i: Z \to Y$, there is a *residue isomorphism* $i^! f^! \xrightarrow{\sim} g^!$, or, more concretely, a sheaf isomorphism:

$$g_*(\mathcal{H}om_{O_Z}(\Lambda^q(I/I^2), i^*\Omega^q_{X/Y})) \xrightarrow{\sim} \mathcal{H}om_{O_Y}(g_* O_Z, O_Y)$$

($\Omega^q_{X/Y}$ = relative differential q-forms) upon which the theory of the residue symbol is built.

But in fact the residue symbol can be viewed as a formal algebraic construct, which can be defined and studied directly with only the elements of ring theory and homological algebra. Indeed, while duality theory may provide the primary motivation for residues,(1) eliminating it from their theoretical foundation results not only in greater simplicity, but also in greater generality, and ultimately, one hopes, in more

(1) and that is why [L] appeared before this paper. (The relation of this paper to [L] is made explicit in Appendix A of §3 below.) My own interest in the subject was inspired by p. 81 of [S], and by §§10 and 15 of [Z].

interconnections with other areas (see the end of this Introduction). In any case, the purpose of this paper is *to provide an elementary development of the theory of residues*.

The possibility of carrying out such a development of residues was known long ago to Cartier. He proposed a local definition, which could, in principle, be used to establish the properties listed in [**RD**], just as an exercise. It turned out to be quite a long exercise [**L**, p. 137]. In print, a beginning along these lines was made by Hopkins in [**H**]. The definition in [**H**], somewhat simpler than Cartier's, uses Koszul complexes, Ext functors, etc. I personally was uncomfortable with this definition, because Koszul complexes seem somehow too specialized; but I knew of no alternative. Then, around 1980, in an attempted proof of the "exterior differentiation" formula (R9) of [**RD**, p. 199] (given here in Appendix B of §3), the formalism of Hochschild homology began to extrude itself. It quickly became clear that this formalism provided a very convenient and surprisingly natural framework for the whole theory. Such, in brief, is the background of this paper.

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The basic situation considered is the following: A is a commutative ring, R is an A-algebra (not necessarily commutative), and there is given a *representation* of R, i.e. an A-algebra homomorphism $R \to \mathrm{Hom}_A(P, P)$, where P is a finitely generated projective A-module. For each $q \geq 0$, there is then a natural $R^c$-linear pairing ($R^c$ = center of R):

$$H^q(R, \mathrm{Hom}_A(P, P)) \otimes_{R^e} H_q(R, R) \to H_0(R, \mathrm{Hom}_A(P, P))$$

where $H^q$ and $H_q$ denote Hochschild cohomology and homology (reviewed at the beginning of §1). The usual trace map $\mathrm{Hom}_A(P, P) \to A$ factors through

$$H_0(R, \mathrm{Hom}_A(P, P)) = \mathrm{Hom}_A(P, P)/\{\text{commutators}\},$$

and composing with the preceding pairing we obtain the *residue homomorphism* (cf. (1.5)):

$$\mathrm{Res}^q\colon H^q(R, \mathrm{Hom}_A(P, P)) \otimes_{R^e} H_q(R, R) \to A$$

which is our basic object of study.

To get the *residue symbol*, we need to relate $H^q$ and $H_q$ to more concrete objects. Suppose for simplicity that R is commutative. Assume further that $P = R/I$ for some ideal I in R, and set

$$(I/I^2)^* = \mathrm{Hom}_P(I/I^2, P).$$

There are then natural homomorphisms of graded R-algebras

(1.8.3) $$\bigoplus_{n\geq 0} \otimes^n[(I/I^2)^*] \to \bigoplus_{n\geq 0} H^n(R, \mathrm{Hom}_A(P, P))$$

(1.10.2) $$\bigoplus_{n\geq 0} \Omega^n{}_{R/A} \to \bigoplus_{n\geq 0} H_n(R, R)$$

so that, via $\mathrm{Res}^q$, we get a natural map

$$t^q: \otimes^q[(I/I^2)^*] \otimes_R \Omega^q{}_{R/A} \to A$$

(equal, when $q = 0$, to the trace map $P \to A$). For $\nu \in \Omega^q$, and $\alpha_1, \ldots, \alpha_q \in (I/I^2)^*$, we set

$$\mathrm{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = t^q(\alpha_1 \otimes \cdots \otimes \alpha_q \otimes \nu).$$

Finally, if $I/I^2$ is free over $P$, with basis, say, $(f_i + I^2)_{1\leq i\leq q}$ $(f_i \in I)$, and if $(\alpha_1, \ldots, \alpha_q)$ is the dual basis of $(I/I^2)^*$, then we set

$$\mathrm{Res}\begin{bmatrix} \nu \\ f_1, \ldots, f_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}.$$

Details are worked out in §1, which culminates with the "determinant formula" (1.10.5) and its corollaries.

Sections 2, 3, and 4 are more or less independent of each other.

In section 2, we study the behavior of $\mathrm{Res}^q$ when the data $(A, R, P)$ vary. In particular we prove a "base-change" formula relative to a ring homomorphism $\psi: A \to A'$:

$$\mathrm{Res}'\begin{bmatrix} \nu' \\ \alpha_1', \ldots, \alpha_q' \end{bmatrix} = \psi\left(\mathrm{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}\right)$$

where "$'$" means "apply the functor $\otimes_A A'$ to everything in sight". (Cf. (2.4) for an exact formulation.) We also show how the residues in this paper lead to the residues in [H]; and then deduce the "transition formula" (2.8):

$$\mathrm{Res}\begin{bmatrix} \nu \\ g_1, \ldots, g_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \det(r_{ij})\nu \\ f_1, \ldots, f_q \end{bmatrix}$$

for *regular* sequences $\mathbf{g} = (g_1, \ldots, g_q)$, $\mathbf{f} = (f_1, \ldots, f_q)$ in $R$, with

$$f_i = \sum_{j=1}^{q} r_{ij} g_j \qquad\qquad r_{ij} \in R, \quad 1 \le i \le q,$$

and such that $R/\mathbf{g}R$ and $R/\mathbf{f}R$ are finite and projective over $A$. (For this formula, at least, Koszul complexes remain unavoidable.)

At this point, we will have, among other things, reworked and extended most of the material in [H].

The first "hard" result appears in §3 (Corollary (3.7)): it is a formula for residues with respect to powers of the members of a quasi-regular sequence $\mathbf{f} = (f_1, \dots, f_q)$ in the A-algebra R, with $R/\mathbf{f}R$ finite and projective over A. Such a formula in the case of power series rings is well-known; and we relate our "formally Cohen-Macaulay" situation to this case by embedding R into a power series ring in $(f_1, \dots, f_q)$, with coefficients in the (usually) non-commutative finite projective A-algebra $\mathrm{Hom}_A(R/\mathbf{f}R, R/\mathbf{f}R)$. As a corollary we obtain in (3.10) a relation between Jacobian determinants, traces, and residues, which enables us, in particular, to derive the residues defined in [L] from those in this paper (cf. Appendix A). We also use (3.7) in Appendix B, to obtain the "exterior differentiation" formula alluded to above.

The second "hard" result is the *trace formula* (4.7.1), expressing a kind of adjointness relation between certain "trace" and "cotrace" maps in the Hochschild formalism. In terms of residue symbols, one consequence is the following.

We consider as above a commutative A-algebra R, and an ideal $I \subset R$ such that $P = R/I$ is finite and projective over A. We consider further a finite projective commutative R-algebra $R'$, and set $I' = IR'$ (so that $P' = R'/I'$ is also finite and projective over A). Then, for any $\alpha \in \mathrm{Hom}_P(I/I^2, P)$ there exists a unique $\alpha' \in \mathrm{Hom}_{P'}(I'/I'^2, P')$ (the "cotrace" of $\alpha$) making the following diagram (with horizontal arrows representing obvious maps) commute:

$$\begin{array}{ccc} I/I^2 & \longrightarrow & I'/I'^2 \\ \alpha\big\downarrow & & \big\downarrow\alpha' \\ P & \longrightarrow & P' \end{array}$$

Furthermore, under suitable hypotheses (e.g. R smooth over A, or $R'$ étale over R) there is a "trace map"

$$\tau_q : \Omega^q_{R'/A} \to \Omega^q_{R/A} ;$$

and we have, for any $\nu \in \Omega^q_{R'/A}$:

$$\operatorname{Res}\begin{bmatrix} \nu \\ \alpha_1', \ldots, \alpha_q' \end{bmatrix} = \operatorname{Res}\begin{bmatrix} \tau_q(\nu) \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}.$$

The problem of defining a trace map $\tau_q$ for differential forms is indicated in [RD, p. 188]. Considerable work has been done on this problem, best documented in [K,§16]. A novel definition was discovered by Angéniol [A, pp. 108 ff]. His approach was computational; but it turned out that the definition could best be understood via Hochschild homology (cf. (4.6.5)). In fact, with R and R′ as above, and $H = \mathrm{Hom}_R(R', R')$, there is a trace map on homology, defined to be the composition

$$H_q(R', R') \overset{\text{natural}}{\longrightarrow} H_q(H, H) \to H_q(R, R), \tag{0.1}$$

where the second arrow comes from "Morita equivalence". (We give a different description in §4.5). D. Burghelea has informed me that this type of composition also arose independently in work on Chern classes in cyclic homology. Differential forms are brought into the picture through the natural map $\Omega^q{}_{R/A} \to H_q(R, R)$ (cf. (1.10.2)); but since this map is not fully understood, several hard questions concerning conditions for the existence of a trace map for differential forms remain (cf. §(4.6)). Anyway, once residues and traces are both defined via Hochschild homology, the road to the "trace formula" in (4.7) is open.

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The constructions in §4 suggest some tantalizing possibilities with respect to recent developments in other areas. One connection with cyclic homology has been indicated above (following (0.1)). Secondly, there is a natural homotopy class of maps, C, defined in (4.1), which underlies both the trace and the cotrace. A concrete – but highly non-canonical – representative of this class is described in (4.2). From this description, one can see that the "intermediate fundamental classes" recently defined by Angéniol and Lejeune-Jalabert [AL] could conveniently (i.e. with little or no computation) be formulated in terms of homotopy classes like C.

Further connections with cyclic homology might come out of arguments in Appendix B of §3; but I am unable to say more.

This Introduction began with the claim that there has not yet appeared a really satisfactory exposition of residues, a situation which this paper is meant to remedy, at least in part. The preceding remarks indicate that there might well be a more fundamental approach to the subject, encompassing a great deal more than we have dealt with here. If this paper helps someone toward such a discovery, it will have served its purpose.

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## §1. THE RESIDUE HOMOMORPHISM

The general definition of residues, due basically to Cartier, has numerous formulations in terms of homological products. In this section we give a concrete description, more or less self-contained, of one such formulation (Definition (1.5.1)) via Hochschild homology and cohomology of associative algebras. The reader may wish to begin with (1.11), where the main results of §1 are summarized.

We begin with a quick review of some basic notions in the Hochschild theory (as presented in [M, Chapter 10]).

Let A be a commutative ring, and let R be an A-algebra (associative, but not necessarily commutative), i.e. R is a ring together with a ring homomorphism $h: A \to R$ such that $h(A) \subset R^c$, the center of R. An R-R *bimodule* is by definition an A-module M equipped with compatible left and right R-module structures both of which induce (via h) the A-module structure; in other words there are given A-bilinear "scalar multiplication" maps $R \times M \to M$ (respectively $M \times R \to M$) satisfying the usual conditions for *left* (respectively *right*) R-modules; and "compatibility" means that (with self-explanatory notation) $(rm)r' = r(mr')$ for all $r, r' \in R$ and $m \in M$.

With $R^{op}$ the opposite algebra of R (that is, the A-module R together with the multiplication $\mu: R \times R \to R$ given by $\mu(x, y) = yx$), and $R^e$ the "enveloping algebra" $R \otimes_A R^{op}$, an R-R bimodule M is essentially the same thing as a left $R^e$-module, the scalar multiplications being related by

$$(r \otimes r')m = rmr' \qquad (r, r' \in R;\ m \in M);$$

and also the same as a right $R^e$-module, with scalar multiplication

$$m(r' \otimes r) = rmr'.$$

(Via the antiautomorphism of $R^e$ taking $r \otimes r'$ to $r' \otimes r$, every left $R^e$-module becomes a right $R^e$-module and vice versa.)

(1.0) The "bimodule bar resolution" $\epsilon: B.(h) \to R$:

$$\cdots \xrightarrow{\partial_3} B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0 = R^e \xrightarrow{\epsilon} R$$

is defined as follows [M, p.282]. For $n \geq 0$, $B_n = B_n(h)$ is the left $R^e$-module $R^e \otimes_A T_A^n(R/A)$ where "R/A" denotes the cokernel of h, and

$$T_A^n(R/A) = (R/A) \otimes (R/A) \otimes \cdots \otimes (R/A) \qquad (n \text{ factors}; \otimes = \otimes_A).$$

With $r^*$ the natural image of $r \in R$ in $R/A$, we denote the element

$$(r \otimes r') \otimes [r_1^* \otimes \cdots \otimes r_n^*] \in B_n$$

by

$$r[r_1 \mid r_2 \mid \dots \mid r_n]r'.$$

(The notation suggests that we think of $B_n$ as an R-R bimodule.) Here we may omit r if $r = 1$, and similarly for $r'$. In particular we set

$$r[\ ]r' = (r \otimes r') \otimes 1 \in R \otimes R \otimes A = R^e = B_0.$$

Then the $R^e$-linear maps $\epsilon: R^e \to R$ and $\partial_n: B_n \to B_{n-1}$ $(n \geq 1)$ are determined by

$$\epsilon(r[\ ]r') = rr',$$

$$\begin{aligned}\partial_n(r[r_1 \mid r_2 \mid \dots \mid r_n]r') &= rr_1[r_2 \mid \dots \mid r_n]r' \\ &+ \sum_{i=1}^{n-1} (-1)^i r[r_1 \mid \dots \mid r_i r_{i+1} \mid \dots \mid r_n]r' \\ &+ (-1)^n r[r_1 \mid \dots \mid r_{n-1}]r_n r'.\end{aligned}$$

**B**.(h) is a *positive complex* of left $R^e$-modules (i.e. $\partial_n\partial_{n+1} = 0$ for $n \geq 1$, and we take $B_m = (0)$ for $m < 0$), and $\epsilon: \mathbf{B}.(h) \to R$ is a *resolution* of the left $R^e$-module R, R being considered as a left $R^e$-module (= R-R bimodule) in the obvious way. In fact, with the right R-module homomorphisms

$$s_{-1}: R \to R^e = B_0$$

$$s_n: B_n \to B_{n+1} \qquad (n \geq 0)$$

determined by

$$s_{-1}(r') = 1 \otimes r' = [\ ]r'$$

$$s_n(r[r_1 \mid \dots \mid r_n]r') = [r \mid r_1 \mid \dots \mid r_n]r'$$

we have

$$\epsilon s_{-1} = \text{identity}$$

$$\partial_1 s_0 + s_{-1}\epsilon = \text{identity}$$

$$\partial_{n+1}s_n + s_{n-1}\partial_n = \text{identity} \qquad (n \geq 1);$$

in other words, the $s_i$ constitute a *right* R-*module splitting* (= contracting homotopy) of the bimodule resolution $\epsilon\colon \mathbf{B}.(h) \to R$; and furthermore

$$s_n s_{n-1} = 0 \qquad (n \geq 0).$$

(Our terminology is as in [M, pp. 41, 87].)

As indicated above, any R-R bimodule M can be considered as a left $R^e$-module and as a right $R^e$-module. The Hochschild homology and cohomology A-modules of the R-R bimodule M are defined then by

$$H_n(R, M) = H_n(M \otimes_{R^e} \mathbf{B}.(h))$$

$$H^n(R, M) = H^n(\mathrm{Hom}_{R^e}(\mathbf{B}.(h), M))^{(1)}.$$

[The notation $H_n(R, M)$, $H^n(R, M)$ is customary, though it would be more precise to write $H_n(h, M)$, $H^n(h, M)$]. In particular

(1.0.1) $$H_0(R, M) = M \otimes_{R^e} R = M/\{rm - mr\}$$

where $\{rm - mr\}$ is the A-submodule of M consisting of all sums of elements of the form $rm - mr$ $(r \in R, m \in M)$; and

(1.0.2) $$H^0(R, M) = \mathrm{Hom}_{R^e}(R, M) = \{m \in M \mid rm = mr \text{ for all } r \in R\}.$$

If $r \in H^0(R, R) = R^c$, the *center* of R, then multiplication by $r \otimes 1$ is an $R^e$-endomorphism of the complex $\mathbf{B}.(h)$ (or of the $R^e$-module M); and hence $H_n(R, M)$ and $H^n(R, M)$ are left $R^c$-modules. Similarly multiplication by $1 \otimes r$ gives rise to right $R^c$-module structures. These left and right $R^c$-module structures actually *coincide* (i.e. $rz = zr$ for all $r \in R^c$ and $z \in H_n(R, M)$ or $H^n(R, M)$): for given $r \in R^c$, if $t_n\colon B_n \to B_{n+1}$ is the unique $R^e$-homomorphism satisfying

$$t_n(r'[r_1 \mid \dots \mid r_n]r'') = \sum_{i=0}^{n} (-1)^i r'[r_1 \mid \dots \mid r_i \mid r \mid r_{i+1} \mid \dots \mid r_n]r''$$

then we have (for $n \geq 0$, with $t_{-1} = 0$):

(1) As in [M, p.42] we use the following *sign convention*: the coboundary of an n-cochain $f \in \mathrm{Hom}_{R^e}(B_n, M)$ is the n+1-cochain $(-1)^{n+1} f \circ \partial_{n+1}$.

$$\partial_{n+1}t_n + t_{n-1}\partial_n = \text{multiplication by } r\otimes 1 - 1\otimes r,$$

so that multiplication by $r\otimes 1$ in $\mathbf{B}.(h)$ is homotopic to multiplication by $1\otimes r$. Thus we can just think of $H^n(R,M)$ and $H_n(R,M)$ as being $R^c$-modules.

* * *

A basic component of our definition of residues will be a natural $R^c$-linear map

(1.1) $$\rho_M^q\colon H^q(R,M)\otimes_{R^e} H_q(R,R) \to H_0(R,M) \qquad (q\geq 0)$$

defined as follows. For any $x\in B_q$, let

$$\bar{x} = 1\otimes x \in R\otimes_{R^e} B_q;$$

and for any $R^e$-linear map $f\colon B_q\to M$, let $\bar{f}$ be the A-linear map

$$\bar{f} = 1\otimes f\colon R\otimes_{R^e} B_q \to R\otimes_{R^e} M = M\otimes_{R^e} R = H_0(R,M).$$

If $f$ is a q-cocycle representing $\xi\in H^q(R,M)$ and $\bar{x}$ is a q-cycle representing $\eta\in H_q(R,R)$, then $\bar{f}(\bar{x})\in H_0(R,M)$ depends only on $\xi$ and $\eta$, as we see at once from the relation $\overline{\delta g}(\bar{y}) = \pm\bar{g}(\bar{\partial}\bar{y})$ where $\delta$ (respectively $\bar{\partial}$) is the boundary map in the complex $\mathrm{Hom}_{R^e}(\mathbf{B}.(h),M)$ (respectively: in $R\otimes_{R^e}\mathbf{B}.(h)$); furthermore $\bar{f}(\bar{x})$ depends $R^c$-bilinearly on $\xi$ and $\eta$; so we can set

$$\rho_M^q(\xi\otimes\eta) = \bar{f}(\bar{x}).$$

*Remark* (1.1.1). The map (1.1) "varies functorially" with M. In particular when R is *commutative* ($R^c = R$), then setting $\overline{M} = H_0(R,M)$ we can put (1.1) into a commutative diagram

$$\begin{array}{ccc} H^q(R,M)\otimes_R H_q(R,R) & \longrightarrow & H_0(R,M) = \overline{M} \\ \downarrow & & \| \\ H^q(R,\overline{M})\otimes_R H_q(R,R) & \longrightarrow & H_0(R,\overline{M}) \end{array}$$

So when R is commutative, (1.1) is essentially determined by its restriction to the category of R-modules, any R-module $M^*$ being considered as an R-R bimodule with $rm = mr$ for all $r\in R$, $m\in M^*$.

*Example* (1.2) ($q=0$). As above, $H^0(R,M)\subseteq M$ and $H_0(R,M)$ is a homomorphic image of M. Denoting by $\bar{m}$ the natural image of $m\in M$ in

$H_0(R, M)$, and by $\bar{r}$ the natural image of $r \in R$ in $H_0(R, R)$, we find that

$$\rho_M^0(m \otimes \bar{r}) = \overline{mr} = \overline{rm}.$$

If $R$ is commutative then $H_0(R, R) = R = R^c$,

$$H^0(R, M) \otimes_{R^e} H_0(R, R) = H^0(R, M) \otimes_R R = H^0(R, M),$$

and (1.1) (with $q = 0$) is just the natural composition

$$H^0(R, M) \to M \to H_0(R, M).$$

*Example* (1.3) $(q = 1)$. Let $J = \partial_1(B_1)$ be the kernel of $\epsilon: R^e \to R$ $(\epsilon(r_1 \otimes r_2) = r_1 r_2)$; and let $J'$ be the kernel of $\epsilon': R^e \to R$ $(\epsilon'(r_1 \otimes r_2) = r_2 r_1)$. We have an obvious commutative diagram, with an exact column

$$\begin{array}{ccccc} & & R \otimes_{R^e} B_2 & & \\ & & \downarrow{\scriptstyle 1\otimes\partial_2} & & \\ R \otimes_A R/A & = & R \otimes_{R^e} B_1 & & \\ \delta\downarrow & & \downarrow & \searrow{\scriptstyle 1\otimes\partial_1} & \\ J/J'J & = & R \otimes_{R^e} J & \longrightarrow & R \otimes_{R^e} R^e = R^e/J' \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

from which we see that $\delta$ *maps*

$$H_1(R, R) = \ker(1 \otimes \partial_1)/\mathrm{im}(1 \otimes \partial_2) = \delta^{-1}(J' \cap J/J'J)/\ker(\delta)$$

$R^c$-*isomorphically onto* $J' \cap J/J'J$ (multiplication by $r \in R^c$ in $J' \cap J/J'J$ is induced by left multiplication by $r \otimes 1$ – or by $1 \otimes r$ – in $J' \cap J$).

In particular *if* $R$ *is commutative* then we have an $R$-isomorphism

$$H_1(R, R) \simeq J/J^2 = \Omega_{R/A}$$

where $\Omega_{R/A}$ is the $R$-module of Kähler $A$-differentials, and moreover if $\psi: R \to B_1$ is defined by $\psi(r) = 1[r]1$, then

$$\partial_1(\psi(r)) = r[\ ]1 - 1[\ ]r = r \otimes 1 - 1 \otimes r \in J,$$

and under the usual identification $R \otimes_{R^e} B_1 = R \otimes_A (R/A)$, $1 \otimes \psi(r)$ gets identified with $1 \otimes r^*$ ($r^* =$ image of $r$ in $R/A$); thus the *universal derivation* $d: R \to J/J^2$

given by

$$d(r) = r \otimes 1 - 1 \otimes r \quad (\text{mod. } J^2)$$

(cf. [B, Chap. III, § 10.11, Prop. 18]) can be identified with the R-linear map $d\colon R \to H_1(R, R)$ defined by

$$d(r) = \text{homology class of the 1-cycle } 1 \otimes r^* \in R \otimes_A (R/A).$$

Next, a 1-cocycle in $\mathrm{Hom}_{R^e}(B_1, M)$ can be viewed either as an $R^e$-linear map $L\colon J \to M$ or as an A-derivation $D\colon R \to M$ (i.e. an A-linear map D satisfying $D(r_1 r_2) = D(r_1)r_2 + r_1 D(r_2)$ for all $r_1, r_2 \in R$, and consequently $D(A) = AD(1) = 0$). More precisely, for any $L \in \mathrm{Hom}_{R^e}(J, M)$, $L \circ \partial_1$ is a 1-cocycle, and $L \circ \partial_1 \circ \psi$ (see above) is an A-derivation; and in this way we define A-isomorphisms

$$\mathrm{Hom}_{R^e}(J, M) \xrightarrow{\sim} \{\text{1-cocycles}\} \xrightarrow{\sim} \mathrm{Der}_A(R, M) \tag{1.3.1}$$

(where $\mathrm{Der}_A(R, M)$ is the A-module consisting of all A-derivations $R \to M$). The first of these isomorphisms identifies the 1-coboundaries with $R^e$-homomorphisms $J \to M$ which extend to $R^e \supset J$; and the second isomorphism identifies the 1-coboundaries with *inner derivations* , i.e. those of the form $D_m$, where $m \in M$ and, for all $r \in R$,

$$D_m(r) = (r \otimes 1 - 1 \otimes r)m = rm - mr. \tag{1.3.2}$$

Thus we have the (well known) identification

$$H^1(R, M) = \mathrm{Der}_A(R, M)/\{\text{inner derivations}\}. \tag{1.3.3}$$

Now the point is that *the following diagram commutes* (check!), and therefore determines (1.1) when $q = 1$:

$$\begin{array}{ccc}
\mathrm{Der}_A(R, M) \xrightarrow{\sim} \mathrm{Hom}_{R^e}(J, M) & \xrightarrow{\text{natural}} & \mathrm{Hom}_{R^e}(J/J'J, M/J'M) \\
\Big\downarrow \text{(surjective)} & & \Big\downarrow \text{restriction} \\
 & & \mathrm{Hom}_{R^e}(J' \cap J/J'J, M/J'M) \\
 & & \Big\downarrow \text{(isomorphism)} \\
H^1(R, M) & \xrightarrow{(1.1)'} & \mathrm{Hom}_{R^e}(H_1(R, R), H_0(R, M)).
\end{array} \tag{1.3.4}$$

Here $(1.1)'$ corresponds naturally to (1.1), and the other maps are as indicated in the preceding discussion.

If R is commutative ($R^c = R$) and M is an R-module (cf. remark (1.1.1)), then all the maps in the above diagram are bijective; and so (1.1)′ gets identified with the composition

$$\mathrm{Der}_A(R, M) \to \mathrm{Hom}_{R^e}(J/J'J, M/J'M) = \mathrm{Hom}_R(\Omega_{R/A}, M)$$

which is just the usual map given by the universal property of $\Omega_{R/A}$.

*Example* (1.4) (q = 1). Let I be a two-sided ideal in R, and set P = R/I. For any left P-module N, there is a unique A-linear map

$$\psi_N: \mathrm{Der}_A(R, \mathrm{Hom}_A(P, N)) \to \mathrm{Hom}_P(I/I^2, N)$$

such that, for each $D \in \mathrm{Der}_A(\ )$ and $x \in I$, with natural image $\bar{x}$ in $I/I^2$:

$$[\psi_N(D)](\bar{x}) = [D(x)](1).$$

*The kernel of* $\psi_N$ *consists of all inner derivations*, and hence (cf. (1.3.3)) we have an $R^c$-linear injective map, varying functorially with N,

$$\bar{\psi}_N: H^1(R, \mathrm{Hom}_A(P, N)) \to \mathrm{Hom}_P(I/I^2, N).$$

Indeed, it is easily checked that $\psi_N$ annihilates any inner derivation; and conversely, if $\psi_N(D) = 0$, then the A-linear map $\phi: R \to N$ given by

$$\phi(r) = [D(r)](1) \qquad\qquad (r \in R)$$

vanishes on I, hence gives an A-linear map $\bar{\phi}: P \to N$, and one checks that for all $r \in R$,

$$D(r) = \bar{\phi}r - r\bar{\phi}$$

i.e. D is the inner derivation $D_{(-\bar{\phi})}$ (cf. (1.3.2)).

*If the natural map* $\pi: R/I^2 \to R/I$ *has an A-linear section* (= *right inverse*) $\sigma$ (for example if R/I is projective as an A-module), *then* $\psi_N$ *is surjective, and so* $\bar{\psi}_N$ *is bijective.* For, if $\alpha: I/I^2 \to N$ is any P-linear map, then $\alpha = \psi_N(D_\alpha)$ where $D_\alpha$ is the derivation given by

(1.4.1) $$D_\alpha(r) = \alpha \circ (r\sigma - \sigma r).$$

Note here that for any $p \in P$,

$$\pi[(r\sigma - \sigma r)(p)] = r(\pi\sigma)(p) - \pi\sigma(rp) = rp - rp = 0$$

so that $r\sigma - \sigma r$ maps P into $I/I^2$, and $\alpha \circ (r\sigma - \sigma r)$ maps P into N.

[In other words, when $\sigma$ exists, the functor $H^1(R, \mathrm{Hom}_A(P, N))$ of left P-modules N is represented by $I/I^2$, together with the element of $H^1(R, \mathrm{Hom}_A(P, I/I^2))$

coming from the derivation D such that $D(r) = r\sigma - \sigma r$ $(r \in R)$, a derivation which is independent, modulo inner derivations, of the choice of $\sigma$.]

Observing that the above derivation $D_\alpha$ corresponds to the $R^e$-linear map $J \to \mathrm{Hom}_A(P, N)$ taking $j \in J$ to $\alpha \circ j\sigma$, we deduce from (1.3) that:

*The map* $\rho = \rho^1_M$ $(M = \mathrm{Hom}_A(P, N))$ *of* (1.1) *is uniquely determined by the following commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_P(I/I^2, N) \otimes_A (J' \cap J) & \xrightarrow{\quad \rho' \quad} & \mathrm{Hom}_A(P, N) \\ \downarrow & & \downarrow \\ H^1(R, \mathrm{Hom}_A(P, N)) \otimes_R H_1(R, R) & \xrightarrow{\quad \rho \quad} & H_0(R, \mathrm{Hom}_A(P, N)) \end{array}$$

*in which vertical arrows come from previously described surjective maps, and*

$$\rho'(\alpha \otimes j) = \alpha \circ j\sigma.$$

*In particular, if* R *is commutative, then* $\rho$ *can be identified with the map*

$$\rho'': \mathrm{Hom}_P(I/I^2, N) \otimes_R \Omega_{R/A} \to H_0(R, \mathrm{Hom}_A(P, N))$$

*such that*

$$\rho''(\alpha \otimes dr) = \text{natural image of } \alpha \circ (r\sigma - \sigma r) \in \mathrm{Hom}_A(P, N).$$

(1.5). We come now to the definition of the *residue homomorphism.*

As before, A is a commutative ring and R is an A-algebra with center $R^c$. Via the "structure map" $h: A \to R^c \subset R$, any left R-module P becomes an A-module, and $\mathrm{Hom}_A(P, P)$ is then an R-R bimodule, with

$$(r_1\phi r_2)(p) = r_1\phi(r_2p) \qquad (r_1, r_2 \in R,\ \phi \in \mathrm{Hom}_A(P, P),\ p \in P).$$

Suppose further that P is, *as an* A*-module*, finitely generated and projective. Then we have the canonical A-linear *trace map* $\mathrm{Tr}_{P/A}$, which is the composition

$$\mathrm{Tr}_{P/A}: \mathrm{Hom}_A(P, P) \xrightarrow{\nu^{-1}} \mathrm{Hom}_A(P, A) \otimes_A P \xrightarrow{ev} A$$

where $\nu$ is the isomorphism such that $[\nu(\phi \otimes p)](p') = \phi(p')p$, and $ev(\phi \otimes p) = \phi(p)$, cf. [B, Chap. II,§4.3]. Since $\mathrm{Tr}_{P/A}$ annihilates any element of the form $r\phi - \phi r$ $(r \in R,\ \phi \in \mathrm{Hom}_A(P,P))$ [*ibid*, Prop. 3], we get an induced map (which we continue to denote by $\mathrm{Tr}_{P/A}$):

$$H_0(R, \mathrm{Hom}_A(P,P)) = \mathrm{Hom}_A(P,P)/\{r\phi - \phi r\} \to A.^{(1)}$$

Definition (1.5.1). *For each* $q \geq 0$, *the* residue homomorphism $\mathrm{Res}^q = \mathrm{Res}^q{}_{A,R,P}$ *is the* A-*linear composition*

$$H^q(R, \mathrm{Hom}_A(P,P)) \otimes_{R^c} H_q(R,R) \longrightarrow H_0(R, \mathrm{Hom}_A(P,P)) \xrightarrow{\mathrm{Tr}_{P/A}} A$$

*where* $\rho = \rho^q{}_M$ $(M = \mathrm{Hom}_A(P,P))$, cf. (1.1).

*Remark* (1.5.2). By the definitions involved, if

$$f \in \mathrm{Hom}_{R^e}(B_q, \mathrm{Hom}_A(P,P)) = \mathrm{Hom}_A(T^q_A(R/A), \mathrm{Hom}_A(P,P))$$

is a q-cocycle representing $\xi \in H^q(R, \mathrm{Hom}_A(P,P))$, and $x \in B_q$ is such that

$$1 \otimes x \in R \otimes_{R^e} B_q$$

is a q-cycle representing $\eta \in H_q(R,R)$, then

$$\mathrm{Res}^q(\xi \otimes \eta) = \mathrm{Tr}_{P/A}(f(x)).$$

More generally, if $r \in R^c$ and $r_P \in \mathrm{Hom}_A(P,P)$ is the map "multiplication by r", then

$$\begin{aligned}\mathrm{Res}^q(r\xi \otimes \eta) &= \mathrm{Res}^q(\xi \otimes r\eta)\\ &= \mathrm{Tr}_{P/A}(r_P \circ f(x))\\ &= \mathrm{Tr}_{P/A}(f(x) \circ r_P).\end{aligned}$$

*Example* (1.6) $(q = 0)$. We have

$$H^0(R, \mathrm{Hom}_A(P,P)) = \{f \in \mathrm{Hom}_A(P,P) \mid rf = fr \text{ for all } r \in R\} = \mathrm{Hom}_R(P,P).$$

Thus (cf. example (1.2)) for $f \in \mathrm{Hom}_R(P,P)$, $r \in R$, we have

$$\mathrm{Res}^0(f \otimes \bar{r}) = \mathrm{Tr}_{P/A}(rf) = \mathrm{Tr}_{P/A}(fr).$$

In particular, if R is *commutative*, then $H_0(R,R) = R = R^c$, and

(1) One could consider, more generally, a *perfect complex* $P^\bullet$ of A-modules, together with an A-algebra homomorphism $R \to \mathrm{Ext}^0{}_A(P^\bullet, P^\bullet)$. Then one still has a trace map... (cf. [SGA 6, Exposé I, §8]). In particular, instead of assuming the R-module P to be finitely generated and projective over A, we could just assume that P is *perfect* over A, i.e. that P has a finite resolution by finitely generated projective A-modules.

$$\mathrm{Res}^0\colon \mathrm{Hom}_R(P,P) \to A$$

is just the restriction of $\mathrm{Tr}_{P/A}$ to $\mathrm{Hom}_R(P,P) \subset \mathrm{Hom}_A(P,P)$.

*Example* (1.7) ($q = 1$). As in (1.4), let $P = R/I$ ($I$ a two-sided R-ideal); and assume further that $P$ is finitely generated and projective over $A$, so that the natural map $R/I^2 \to R/I$ has an A-linear section $\sigma$. (It would suffice to assume only that $P$ is A-perfect, cf. (1.5), and that $\sigma$ exists.) Putting $N = P$ in (1.4), and setting

$$(I/I^2)^* = \mathrm{Hom}_P(I/I^2, P)$$

($P$ and $I/I^2$ being considered as *left* P-modules) we get an $R^e$-isomorphism

(1.7.1)$_I$ $$(I/I^2)^* \xrightarrow{\sim} H^1(R, \mathrm{Hom}_A(P,P)).$$

From (1.3) and (1.4) we see then that

$$\mathrm{Res}^1\colon (I/I^2)^* \otimes_{R^e} (J' \cap J/J'J) \to A$$

is given (for $\alpha \in (I/I^2)^*$, and $j \in J' \cap J$, with natural image $\bar{j}$ in $J' \cap J/J'J$) by

(1.7.2) $$\mathrm{Res}^1(\alpha \otimes \bar{j}) = \mathrm{Tr}_{P/A}(\alpha \circ j\sigma).$$

When $R$ is commutative,

$$\mathrm{Res}^1\colon (I/I^2)^* \otimes_R \Omega_{R/A} \to A$$

is given by

(1.7.3) $$\mathrm{Res}^1(\alpha \otimes dr) = \mathrm{Tr}_{P/A}(\alpha \circ (r\sigma - \sigma r)).$$

(1.7.4) To illustrate (1.7.3), take $R = A[X]$ ($X$ an indeterminate), and $I = FR$ where

$$F = X^n + a_1X^{n-1} + \cdots + a_n \qquad (a_i \in A).$$

Then $P = R/I$ is a free A-module, with basis $(1, x, \ldots, x^{n-1})$ where $x = X + I$, the coset of $X$; and $I/I^2$ is a free P-module with generator $f = F + I^2$. Let $[1/F] \in (I/I^2)^*$ be the P-linear map taking $f$ to $1$. Let $G \in A[X]$, and set

$$G + I = g = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} \in P.$$

Then, making the obvious choice for $\sigma$ (i.e. $\sigma(x^i) = X^i + I^2$, $0 \le i < n$), we calculate by (1.7.3):

(1.7.4.1) $$\mathrm{Res}^1([1/F] \otimes G\,dF) = \mathrm{Tr}_{P/A}(g)$$

(1.7.4.2) $$\mathrm{Res}^1([1/F]\otimes G\,dX) = b_{n-1}.$$

It follows, since $dF = F_X dX$, that if $\tau: P \to A$ is given by

$$\tau(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) = c_{n-1} \qquad (c_i \in A)$$

then, in the R-module $\mathrm{Hom}_A(P, A)$, we have the (well-known) relation

$$\mathrm{Tr}_{P/A} = F_X\tau.$$

*Exercise* (1.7.4.3). Replace A by A[Y] (Y an indeterminate), R by R[Y] in the above, but leave $F \in A[X]$ as it is (so that P gets replaced by P[Y]). Show, for any $G \in A[X]$, with remainder $G_0$ when divided by F, that

$$\mathrm{Res}^1\left([1/F]\otimes G(X)\,\frac{F(Y)-F(X)}{Y-X}\,dX\right) = G_0(Y).$$

Using (1.7.4.1), relate this formula to "Lagrange Interpolation".

*Example* (1.8). (The "residue symbol"). For any two R-R bimodules M, N, we consider $M\otimes_R N$ to be an R-R bimodule via the left R-module structure on M and the right R-module structure on N; in other words, scalar multiplication is specified by

$$r(m\otimes n)r' = rm\otimes nr' \qquad (r,r' \in R;\ m \in M,\ n \in N).$$

We are going first to define a *cohomology product*, i.e. an $R^c$-linear map

(1.8.1) $$H^p(R, M)\otimes_{R^e} H^q(R, N) \to H^{p+q}(R, M\otimes_R N).$$

Recall that $H^\cdot(R, M)$ is the cohomology of a complex in which the group of p-cochains is

$$\mathrm{Hom}_{R^e}(B_p, M) = \mathrm{Hom}_A(T^p, M)$$

where $T^p$ is the p-th tensor power $T^p_A(R/A)$. For any p-cochain $f: T^p \to M$ and any q-cochain $g: T^q \to N$, let

$$f\otimes g: T^{p+q} = T^p\otimes_A T^q \to M\otimes_R N$$

be the (p+q)-cochain such that

$$(f\otimes g)(\tau\otimes\tau') = f(\tau)\otimes g(\tau') \qquad (\tau \in T^p,\ \tau' \in T^q).$$

Let $\delta$ denote the coboundary map in the complex $\mathrm{Hom}_{R^e}(B.(h), M)$, or in $\mathrm{Hom}_{R^e}(B.(h), N)$, or in $\mathrm{Hom}_{R^e}(B.(h), M\otimes_R N)$, as the case may be. Then,

$$\delta(f\otimes g) = (-1)^q(\delta f)\otimes g + f\otimes(\delta g).$$

(The proof, which proceeds directly from definitions, is a straightforward computation, somewhat tedious and mildly surprising; we leave it to the reader.) Hence if $f$ and $g$ are cocyles then so is $f\otimes g$; and if furthermore either $f$ or $g$ is a coboundary, then so is $f\otimes g$. This leads at once to the product (1.8.1).

Now suppose that we are given a homomorphism of bimodules $\mu: M\otimes_R M \to M$ such that the corresponding multiplication $M\times M\to M$ is *associative*. For example, if $E$ is any left $R$-module, and $M = \mathrm{Hom}_A(E,E)$, then the usual composition of maps in $M$ gives rise to such a $\mu$. Then (1.8.1) combines with $\mu$ to make the direct sum $\oplus_{n\geq 0}H^n(R,M)$ into an *associative graded* $R^c$*-algebra.*

*Notation* (1.8.2). Assume that $\mu: M\otimes_R M\to M$ as above exists. For derivations $D_1, D_2, \ldots, D_q$ of $R$ into $M$, we let

$$[D_i]\in H^1(R,M) \qquad\qquad (1\leq i\leq q)$$

be the corresponding cohomology classes (cf. (1.3.3)), and set

$$[D_1D_2\cdots D_q] = [D_1][D_2]\cdots[D_q]\in H^q(R,M)$$

where the product is as described above.

A q-cocycle $f: T^q\to M$ representing $[D_1D_2\cdots D_q]$ is given then by

$$f[r_1\mid r_2\mid \ldots\mid r_q] = D_1(r_1)D_2(r_2)\cdots D_q(r_q)$$

where the product of the $D_i(r_i)\in M$ is defined via $\mu$.

(1.8.3) In particular, with $P = R/I$ as in (1.7) we obtain, via the isomorphism $(1.7.1)_I$ and the universal property of tensor algebras, a *homomorphism of graded* $R^c$-*algebras*

$$\oplus_{n\geq 0}\otimes^n_{R^c}[(I/I^2)^*]\to\oplus_{n\geq 0}H^n(R,\mathrm{Hom}_A(P,P)).$$

[For $n=0$, this map associates to $r\in R^c$ the map "multiplication by $r$", which is an element of $\mathrm{Hom}_R(P,P) = H^0(R,\mathrm{Hom}_A(P,P))$.]

In this case, for $\alpha_1,\alpha_2,\ldots,\alpha_q$ in $(I/I^2)^*$, we denote the image of $\alpha_1\otimes\alpha_2\otimes\cdots\otimes\alpha_q$ under the above map by $[\alpha_1\alpha_2\cdots\alpha_q]$, i.e.

$$[\alpha_1\alpha_2\cdots\alpha_q] = [D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_q}]\in H^q(R,\mathrm{Hom}_A(P,P))$$

(cf. (1.4.1)). If $\sigma_i: P\to R/I^2$ $(1\leq i\leq q)$ is an A-linear section of the natural map $R/I^2\to R/I = P$, then $[\alpha_1\alpha_2\cdots\alpha_q]$ is represented by the q-cocycle $f: T^q\to\mathrm{Hom}_A(P,P)$ given by

$$f[r_1 \mid r_2 \mid \dots \mid r_q] = \alpha_1 \circ (r_1\sigma_1 - \sigma_1 r_1) \circ \alpha_2 \circ (r_2\sigma_2 - \sigma_2 r_2) \circ \cdots \circ \alpha_q \circ (r_q\sigma_q - \sigma_q r_q).$$

(1.8.4) For $\omega \in H_q(R, R)$ we will sometimes follow custom and use the typographically inconvenient notation

$$\begin{bmatrix} \omega \\ \alpha_1, \dots, \alpha_q \end{bmatrix} = [\alpha_1\alpha_2 \cdots \alpha_q] \otimes \omega \in H^q(R, \mathrm{Hom}_A(P, P)) \otimes_{R^c} H_q(R, R).$$

Thus (cf. (1.5.1)) we have the **residue symbol**

$$\mathrm{Res} \begin{bmatrix} \omega \\ \alpha_1, \dots, \alpha_q \end{bmatrix} = \mathrm{Res}^q([\alpha_1\alpha_2 \cdots \alpha_q] \otimes \omega) \in A.$$

*Example* (1.9). With notation as in (1.8), suppose that the left P-module $I/I^2$ is *free*, with basis $\bar{f}_1, \dots, \bar{f}_q$, where $f_i \in I$ and $\bar{f}_i$ is the natural image of $f_i$ in $I/I^2$. [This is the case, for example, if $I = (f_1, \dots, f_q)R$, where $f_1, \dots, f_q$ are in the center $R^c$ and furthermore the sequence $\mathbf{f} = (f_1, \dots, f_q)$ is *regular*, i.e.

$$rf_i \in (f_1, \dots, f_{i-1})R \Rightarrow r \in (f_1, \dots, f_{i-1})R \quad (1 \le i \le q)$$

or, more generally, if $H_1(K_R(\mathbf{f})) = (0)$, where $K_R(\mathbf{f})$ is the Koszul complex over $R$ determined by $\mathbf{f}$ ]. Let $\alpha_1, \dots, \alpha_q$ be the basis of $(I/I^2)^*$ *dual to* the basis $(\bar{f}_1, \dots, \bar{f}_q)$ of $I/I^2$. Then we set

$$\mathrm{Res} \begin{bmatrix} \omega \\ f_1, \dots, f_q \end{bmatrix} = \mathrm{Res} \begin{bmatrix} \omega \\ \alpha_1, \dots, \alpha_q \end{bmatrix}.$$

*Example* (1.10). We begin by recalling a well-known connection between differential forms and Hochschild homology (cf. (1.10.2)). This will lead us to a "determinant formula" for residues, with several direct consequences.

Assume that $R$ is *commutative* (i.e. $R^c = R$). Then we have a "shuffle product",

$$S: \mathbf{B}.(h) \otimes_A \mathbf{B}.(h) \to \mathbf{B}.(h)$$

which is the A-linear map given by

$$S(r[r_1 \mid \dots \mid r_p]\bar{r} \otimes r'[r_{p+1} \mid \dots \mid r_{p+q}]\bar{r}') = \sum_{\mu}(-1)^{|\mu|} rr'[r_{\mu(1)} \mid r_{\mu(2)} \mid \dots \mid r_{\mu(p+q)}]\bar{r}\bar{r}'$$

where:

- $\mu$ runs through all permutations of $\{1,2,\dots, p+q\}$ such that $\mu^{-1}(i) < \mu^{-1}(j)$ whenever $i < j \leq p$ or $p < i < j$, and
- $|\mu| = 0$ or $1$ according as the permutation $\mu$ is even or odd.

The bimodule bar resolution $\mathbf{B}.(h)$ may be viewed as a complex of $(R^e \otimes_A R^e)$-modules (via the multiplication map $\epsilon^e: R^e \otimes_A R^e \to R^e$); and one checks that $S$ is the unique homomorphism of graded $(R^e \otimes_A R^e)$-modules reducing to $\epsilon^e$ in degree zero, and satisfying (for $p + q > 0$)

$$S([r_1 \mid \dots \mid r_p] \otimes [r_{p+1} \mid \dots \mid r_{p+q}])$$

$$= s_{p+q-1}S(\partial_p[r_1 \mid \dots \mid r_p] \otimes [r_{p+1} \mid \dots \mid r_{p+q}] + (-1)^p[r_1 \mid \dots \mid r_p] \otimes \partial_q\ [r_{p+1} \mid \dots \mid r_{p+q}])$$

(where $s$ and $\partial$ are as in the description of $\mathbf{B}.(h)$ at the beginning of this section). It follows at once that $S$ is a homomorphism of complexes. ($S$ is the "canonical comparison" of [M, p.267, Theorem 6.2].)

Now $S$ together with the multiplicaton $\epsilon: R \otimes_A R \to R$ induces a homomorphism of complexes

$$(R \otimes_{R^e} \mathbf{B}.(h)) \otimes_R (R \otimes_{R^e} \mathbf{B}.(h)) \to R \otimes_{R^e} \mathbf{B}.(h);$$

and passing to homology we get maps

$$H_p(R, R) \otimes_R H_q(R, R) \to H_{p+q}(R, R)$$

which, as is easily checked, make the direct sum $\oplus_{n \geq 0} H_n(R, R)$ into a *graded, anticommutative* R-*algebra*, with $\xi^2 = 0$ for every $\xi \in H_1(R, R)$.

Now, as in (1.3), we have an isomorphism

$$\Omega_{R/A} \xrightarrow{\sim} H_1(R, R).$$

By the preceding remarks, and the universal property of exterior algebras, this isomorphism extends to a unique homomorphism of graded R-algebras

(1.10.1) $$\Lambda_R \Omega_{R/A} \to \oplus_{n \geq 0} H_n(R, R)$$

(here $\Lambda_R$ denotes "exterior algebra"; and recall that $H_0(R, R) = R$).

For an arbitrary (not necessarily commutative) A-algebra $R$, with center $R^c$, and any integer $n \geq 0$, let $\Omega^n = \Omega^n{}_{R/A}$ be the exterior power

$$\Omega^n = \Lambda^n_{R^e}(\Omega_{R^e/A}).$$

There are obvious natural maps

$$H_n(R^e, R^e) \to H_n(R, R),$$

which combined with (1.10.1) (with $R^e$ in place of R) yield:

(1.10.2) *There are unique* $R^e$*-homomorphisms*

$$\theta_q : \Omega^q \to H_q(R, R) \qquad (q \geq 0)$$

*such that for any* $r_1, r_2, \ldots, r_q \in R^e$, $\theta_q(dr_1 \wedge dr_2 \wedge \cdots \wedge dr_q)$ *is the homology class of the* q*-cycle*

$$\sum_\tau (-1)^{|\tau|} \otimes [r_{\tau(1)} \mid r_{\tau(2)} \mid \ldots \mid r_{\tau(q)}] \in R \otimes_{R^e} B_q$$

*where* $\tau$ *runs through all permutations of* $\{1,2,\ldots, q\}$ *and* $|\tau| = 0$ *or* 1 *according as* $\tau$ *is an even or an odd permutation.*

Corollary (1.10.3). *With notation as in (1.8.2), and* $\rho^q_M$ *as in (1.1), we have, for* $r_1, r_2, \ldots, r_q \in R^e$:

$$\rho^q_M([D_1 D_2 \cdots D_q] \otimes \theta_q(dr_1 \wedge dr_2 \wedge \cdots \wedge dr_q)) = \det(D_i(r_j))^{\sim}$$

where, for any element $m \in M$,

$$m^{\sim} = m \otimes 1 \in M \otimes_{R^e} R = H_0(R, M),$$

and where the "determinant" $\det(D_i(r_j)) \in M$ is given by

$$\det(D_i(r_j)) = \sum_\tau (-1)^{|\tau|} D_1(r_{\tau(1)}) D_2(r_{\tau(2)}) \cdots D_q(r_{\tau(q)}).$$

*Remark.* To appreciate the explicit formula in (1.10.3), the reader might try to prove *ab ovo* the existence of a map

$$\otimes^q_{R^e}(\mathrm{Der}_A(R, M)/\{\text{inner derivations}\}) \otimes_{R^e} \Omega^q \to M/\{rm - mr\}$$

given by that formula.

*Notation* (1.10.4). Let I be a two-sided ideal in R such that $P = R/I$ is finitely generated and projective as an A-module.[1] For $\nu \in \Omega^q$, and

[1] It suffices that P be *perfect* (cf. (1.5)), and that $R/I^2 \to R/I$ have an A-linear section.

$$\alpha_1, \alpha_2, \ldots, \alpha_q \in (I/I^2)^* = \mathrm{Hom}_P(I/I^2, P)$$

(left P-module homomorphisms) we set

$$\mathrm{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \theta_q(\nu) \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}$$

From (1.10.3) and (1.5.2) we obtain:

Proposition (1.10.5). ("determinant formula for residues"). *With the notation of (1.10.4), let* $r, r_1, \ldots, r_q \in R^c$. *Let* $\pi{:}R/I^2 \to R/I = P$ *be the natural map, and let* $\sigma_i \in \mathrm{Hom}_A(P, R/I^2)$ $(1 \le i \le q)$ *be such that* $\pi\sigma_i = \text{identity}$. *Then*

$$\mathrm{Res}\begin{bmatrix} r\,dr_1 \cdots dr_q \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = \mathrm{Tr}_{P/A}(r_P \circ \det\{\alpha_i \circ (r_j\sigma_i - \sigma_i r_j)\})$$

$$= \mathrm{Tr}_{P/A}(\det\{\alpha_i \circ (r_j\sigma_i - \sigma_i r_j)\} \circ r_P)$$

*where* $\mathrm{Tr}_{P/A}{:}\mathrm{Hom}_A(P,P) \to P$ *is the trace map ;* $r_P \in \mathrm{Hom}_R(P,P)$ *is "multiplication by* r"; *and* $\det\{\ \} \in \mathrm{Hom}_A(P,P)$ *is as in (1.10.3).*

Corollary (1.10.6). *In (1.10.5), assume further that* $r_1, r_2, \ldots, r_q \in I$. *Let* $\overline{r}_j$ *be the natural image of* $r_j$ *in* $I/I^2$ $(j = 1,2,\ldots, q)$ ; *and let* $r'$ *be the natural image of* r *in* P. *Suppose that* $\alpha_i(\overline{r}_j)$ *is in the center* $P^c$ *for all* i, j. *Then*

$$\mathrm{Res}\begin{bmatrix} r\,dr_1 \cdots dr_q \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = \mathrm{Tr}_{P/A}(r'\det(\alpha_i(\overline{r}_j)))$$

*where* $\det(\alpha_i(\overline{r}_j)) \in P^c$, *and for any* $p \in P^c$, $\mathrm{Tr}_{P/A}(p)$ *is the trace of "multiplication by* p*". In particular, if* $\overline{r}_1, \ldots, \overline{r}_q$ *form a free basis of the left* P*-module* $I/I^2$, *then*

$$\mathrm{Res}\begin{bmatrix} r\,dr_1 \cdots dr_q \\ r_1, \ldots, r_q \end{bmatrix} = \mathrm{Tr}_{P/A}(r').$$

*Proof.* If $r_j \in I$, then for all $p \in P$, we have

$$\alpha_i \circ (r_j\sigma_i - \sigma_i r_j)(p) = \alpha_i(r_j\sigma_i(p)) = p\alpha_i(\bar{r}_j),$$

i.e. $\alpha_i \circ (r_j\sigma_i - \sigma_i r_j) \in \mathrm{Hom}_A(P, P)$ is multiplication by $\alpha_i(\bar{r}_j)$. The conclusion follows directly from (1.10.5). Q.E.D.

More generally, we have

COROLLARY (1.10.7). *With assumptions as in (1.10.6) let* $\beta_1, \ldots, \beta_t \in (I/I^2)^*$ *and suppose there is a two sided* R*-ideal* K *such that*

$$(r_1, \ldots, r_q)R \subset K \subset I$$

*and*

$$\beta_i((K + I^2)/I^2) = 0 \qquad (1 \leq i \leq t).$$

*Let*

$$R' = R/K, \qquad I' = I\,R'$$

*so that* $P = R'/I'$ *and* $\beta_i$ *induces a* P*-linear map*

$$\beta'_i : I'/(I')^2 \to P \qquad (1 \leq i \leq t).$$

*For any* $\omega \in \Omega^t_{R/A}$, *let* $\omega'$ *be its natural image in* $\Omega^t_{R'/A}$. *Let* $\Delta(\alpha, r)$ *be any element of* R *whose natural image in* P *is* $\det(\alpha_i(\bar{r}_j))$. *Then:*

$$\mathrm{Res}\begin{bmatrix} dr_1 \cdots dr_q \wedge \omega \\ \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_t \end{bmatrix} = \mathrm{Res}\begin{bmatrix} (\Delta(\alpha, r)\omega)' \\ \beta'_1, \ldots, \beta'_t \end{bmatrix}.$$

*Proof.* We may assume that $\omega = \rho_0 d\rho_1 d\rho_2 \cdots d\rho_t$, with $\rho_i \in R^c$ $(0 \leq i \leq t)$. Let $\sigma : P \to R/I^2$ be an A-linear section of $R/I^2 \to P$ and let $\sigma'$ be the composition

$$\sigma' : P \to R/I^2 \xrightarrow{\text{natural}} R'/(I')^2,$$

so that $\sigma'$ is a section of $R'/(I')^2 \to P$. Then, as in (1.10.5)

$$\alpha_i \circ (r_j\sigma - \sigma r_j) = \text{multiplication by } \alpha_i(\bar{r}_j)$$

$$\beta_i \circ (r_j\sigma - \sigma r_j) = \text{multiplication by } \beta_i(\bar{r}_j) = 0$$

and clearly, if $\rho'_j$ is the natural image of $\rho_j$ in $R'$ then

$$\beta_i \circ (\rho_j \sigma - \sigma \rho_j) = \beta'_i \circ (\rho'_j \sigma' - \sigma' \rho'_j).$$

The rest is a simple exercise in determinants.

COROLLARY (1.10.8). *With notations as in (1.10.4), let* $\delta_1, \ldots, \delta_q$ *be derivations of* $R/I^2$ *into* P, *and let* $\alpha_i = \delta_i \mid (I/I^2)$, *the restriction of* $\delta_i$ $(1 \le i \le q)$. *Then*

$$\mathrm{Res}\begin{bmatrix} r\,dr_1 \cdots dr_q \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = \mathrm{Tr}_{P/A}(r' \det(\delta_i(\bar{r}_j))$$

*where* $r'$ *(resp.* $\bar{r}_j$*) is the natural image of* r *in* P *(resp. of* $r_j$ *in* $R/I^2$*); and for any* $p \in P$, $\mathrm{Tr}_{P/A}(p)$ *is the trace of "left multiplication by* p*".*

*Proof.* The $R^e$-linear map $P \to \mathrm{Hom}_A(P, P)$ taking $p \in P$ to "left multiplication by p" induces a map

$$H^1(R/I^2, P) \to H^1(R/I^2, \mathrm{Hom}_A(P, P))$$

whose composition with the map $\psi_P$ of (1.4) takes the homology class of a derivation $\delta$ to $\delta \mid (I/I^2)$. Hence in (1.10.5) we may replace $\alpha_i \circ (r_j \sigma_i - \sigma_i r_j)$ by "left multiplication by $\delta_i(\bar{r}_j)$". Q.E.D.

*Summary* (1.11). Let R be an associative A-algebra (A a commutative ring), and let $R^c$ be the center of R. Let I be a two-sided R-ideal such that the A-module $P = R/I$ is finitely generated and projective (or just *perfect*, cf. (1.5), and such that $R/I^2 \to R/I$ has an A-linear section). Set

$$(I/I^2)^* = \mathrm{Hom}_P(I/I^2, P)$$

($I/I^2$ and P being considered as *left* P-modules) and for any $n \ge 0$,

$$\Omega^n = \Lambda^n{}_{R^c}(\Omega_{R^c/A}) \qquad \text{(Kähler differentials).}$$

We have defined in this section natural A-linear maps

(1.5.1) $$\mathrm{Res}^q : H^q(R, \mathrm{Hom}_A(P, P)) \otimes_{R^c} H_q(R, R) \to A \qquad (q \ge 0)$$

and also $R^c$-linear maps

(1.8.3) $$\otimes^q{}_{R^c}[(I/I^2)^*] \to H^q(R, \mathrm{Hom}_A(P, P))$$

(1.10.2) $$\Omega^q \to H_q(R, R)$$

which combine to give A-linear maps

$$t^q : \otimes^q_{R^c}[(I/I^2)^*] \otimes_{R^c} \Omega^q \to A \qquad (q \geq 0).$$

For $\nu \in \Omega^q$, and $\alpha_1, \ldots, \alpha_q \in (I/I^2)^*$, we set

$$\operatorname{Res}\begin{bmatrix} \nu \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = t^q(\alpha_1 \otimes \cdots \otimes \alpha_q \otimes \nu).$$

In particular, for $r, r_1, r_2, \ldots, r_q \in R^c$, we have the explicit determinant formula (cf. (1.10.5)):

$$\operatorname{Res}\begin{bmatrix} r\,dr_1 \cdots dr_q \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = \operatorname{Tr}_{P/A}(r_P \circ \det\{\alpha_i \circ (r_j\sigma_i - \sigma_i r_j)\}).$$

## §2. FUNCTORIAL PROPERTIES

We consider in this section the behavior of the residue homomorphism

$$\mathrm{Res}^q : H^q(R, \mathrm{Hom}_A(P, P)) \otimes_{R^e} H_q(R, R) \to A$$

(cf. (1.5)) when the data A, R, P vary. (Recall that these data constitute a *representation:* P is a finitely generated projective A-module, and there is given a homomorphism of A-algebras $R \to \mathrm{Hom}_A(P, P)$, i.e. an R-module structure on P...).

Principal results are given in (2.2) and its corollary (2.2.1), and in (2.3) and its corollary (2.4). Also, in (2.6) and (2.7) we describe the connection between residues as defined in this paper and as defined in [H]. In particular, we can then deduce the "transition formula" (2.8) (whose connection with the determinant formula (1.10.5) remains less direct than one would hope for). Finally, in (2.9), we state without proof some possibly amusing technical elaborations of preceding arguments.

(2.1) Suppose then that we have a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\ \phi\ } & R' \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h'} \\ A & \xrightarrow[\ \psi\ ]{} & A' \end{array}$$

where A and A′ are commutative, $h(A) \subset R^c$ (the center of R), and $h'(A') \subset (R')^c$. Suppose further that we have a left R-module P (respectively: left R′-module P′) which is finitely generated and projective as a module – via h – over A (respectively: – via h′ – over A′), and a homomorphism of R-modules $\chi : P \to P'$ (where P′ is an R-module via $\phi$).

Then $\phi$ and $\psi$ induce an obvious $(R \otimes_A R)$-linear map of bimodule bar resolutions (cf. (1.0))

$$\mathbf{B}.(h) \to \mathbf{B}.(h'),$$

and hence (via $\chi$) $R^c$-linear maps

(2.1.1) $H^q(R', \mathrm{Hom}_{A'}(P', P')) \xrightarrow{u} H^q(R, \mathrm{Hom}_A(P', P')) \xrightarrow{v} H^q(R, \mathrm{Hom}_A(P, P')) \xleftarrow{w} H^q(R, \mathrm{Hom}_A(P, P))$

and

(2.1.2) $$t: H_q(R, R) \to H_q(R', R') \qquad (q \geq 0).$$

If $\phi(R^c) \subset (R')^c$, then $t$ commutes with the map $\theta$ of (1.10.2) in an obvious sense.

Proposition (2.2). *With assumptions as in (2.1), suppose further that* $P = P'$, *that* $\chi: P \to P'$ *is the identity map (so that* $v$ *and* $w$ *in (2.1.1) are identity maps), and that* $\psi: A \to A'$ *makes* $A'$ *into a finitely generated projective* $A$*-module. Then for any* $\xi' \in H^q(R', \mathrm{Hom}_{A'}(P', P'))$ *and* $\omega \in H_q(R, R)$ *we have, with* $\mathrm{Tr}$ *and* $\mathrm{Res}^q$ *as in (1.5):*

$$\mathrm{Res}^q{}_{A,R,P}(u(\xi') \otimes \omega) = \mathrm{Tr}_{A'/A}(\mathrm{Res}^q{}_{A',R',P}(\xi' \otimes t(\omega)).$$

*Proof.* Using (1.5.2) and the definition of $u$ and $t$, we reduce easily to showing that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{A'}(P, P) & \xrightarrow{\mathrm{Tr}_{P/A'}} & A' \\ \Big\downarrow{\scriptstyle \text{inclusion}} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_{A'/A}} \\ \mathrm{Hom}_A(P, P) & \xrightarrow[\mathrm{Tr}_{P/A}]{} & A \end{array}$$

But commutativity clearly holds if $P = A'$; and it holds for $P = P_1 \oplus P_2$ if and only if it holds for $P = P_1$ and for $P = P_2$. Hence the diagram commutes for $P =$ any direct summand of a finitely generated free $A'$-module, i.e. for $P$ any finitely generated projective $A'$-module.

Corollary (2.2.1). *In the diagram (2.1), suppose that* $A'$ *is finitely generated and projective as an* $A$*-module (via* $\psi$*). Assume that* $K, I$ *are two-sided ideals in* $R$ *such that* $K \subset I^2$, $R' = R/K$ *(with* $\phi: R \to R/K$ *the natural map), and such that*

$$P = R/I = R'/I\,R'$$

*is finitely generated and projective over* $A'$ *(hence over* $A$*). Let*

$$\alpha_1, \alpha_2, \ldots, \alpha_q \in \mathrm{Hom_P}(I/I^2, P) = \mathrm{Hom_P}(I\,R'/(I\,R')^2, P)$$

*(left* P*-module homomorphisms). Then, with* $\omega$ *and* $\omega' = t(\omega)$ *as in (2.2), and with the notation of (1.8.4), we have*

$$\mathrm{Res}_{A,R,P}\begin{bmatrix} \omega \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = \mathrm{Tr}_{A'/A}\left(\mathrm{Res}_{A',R',P}\begin{bmatrix} \omega' \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}\right).$$

*Remark* (2.2.2). Note in particular the case $K = (0)$ $(\Rightarrow \omega' = \omega)$ of (2.2.1), and also the case $\{K = I^2, A = A', \psi = \text{identity}\}$. These two cases together are equivalent to (2.2.1).

*Proof of* (2.2.1). To deduce (2.2.1) from (2.2) we need to show that $u$ maps $[\alpha_1 \cdots \alpha_q] \in H^q(R', \mathrm{Hom}_{A'}(P,P))$ to $[\alpha_1 \cdots \alpha_q] \in H^q(R, \mathrm{Hom}_A(P,P))$. But it is easily checked that $u$ is compatible with the cohomology product described in (1.8), so we are reduced to the case $q = 1$; and to settle this case, after choosing an $A'$-linear section $\sigma$ of the natural map

$$R/I^2 = R'/(I\,R')^2 \to R'/I\,R' = R/I = P$$

we need only note (in view of (1.4.1)) that for any $r \in R$, with natural image $r'$ in $R'$,

$$r\sigma - \sigma r = r'\sigma - \sigma r' \in \mathrm{Hom}_{A'}(P, I\,R'/(I\,R')^2) \subseteq \mathrm{Hom}_A(P, I/I^2).$$

PROPOSITION (2.3). *Suppose we have a commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{\ \phi\ } & R' \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h'} \\ A & \xrightarrow[\ \psi\ ]{} & A' \end{array}$$

*as in (2.1). Let* $I \subset R$, $I' \subset R'$ *be two-sided ideals such that* $\phi(I) \subset I'$, *set* $P = R/I$, $P' = R'/I'$, *and suppose that the natural map* $P \otimes_A A' \to P'$ *induced by* $\phi$ *and* $\psi$ *is* bijective. *Assume further that as an* A*-module (via* h), P *is finitely generated and projective. Let* $\alpha_1, \ldots, \alpha_q \in \mathrm{Hom_P}(I/I^2, P)$ *and* $\alpha'_1, \ldots, \alpha'_q \in \mathrm{Hom_{P'}}(I'/I'^2, P')$ *(all* left *module homomorphisms) be such that for each* $i = 1, 2, \ldots, q$, *the diagram*

$$\begin{array}{ccc} I/I^2 & \xrightarrow{\ \bar{\phi}\ } & I'/I'^2 \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha'_i} \\ P & \xrightarrow[\ \chi\ ]{} & P' \end{array}$$

*(with horizontal arrows induce by* $\phi$*) commutes. Let* $\omega \in H_q(R, R)$ *and let* $\omega' = t(\omega)$ *(cf. (2.1.2)). Then, with notation as in (1.8.4), we have :*

$$\text{(2.3.1)} \qquad \mathrm{Res}_{A',R',P'}\begin{bmatrix} \omega' \\ \alpha'_1, \ldots, \alpha'_q \end{bmatrix} = \psi\left(\mathrm{Res}_{A,R,P}\begin{bmatrix} \omega \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}\right)$$

For the *proof*, we need:

LEMMA (2.3.2). *With the preceding assumptions, and the notation of (2.1.1) and (1.8.3), we have*

$$vu[\alpha'_1 \cdots \alpha'_q] = w[\alpha_1 \cdots \alpha_q].$$

*Proof.* We proceed by induction on $q$. For $q = 1$, set $\alpha_1 = \alpha$, $\alpha'_1 = \alpha'$, and let

$$\bar{\phi}: R/I^2 \to R'/I'^2$$

$$\chi: P = R/I \to R'/I' = P'$$

be the maps induced by $\phi$, so that $\chi\alpha = \alpha'\bar{\phi} \mid (I/I^2)$. Choose an A-linear section $\sigma$ of the natural map $\pi: R/I^2 \to P$, and an $A'$-linear section $\sigma'$ of $\pi': R'/I'^2 \to P'$. Then $w[\alpha]$ is represented by the A-derivation $D: R \to \mathrm{Hom}_A(P, P')$ given by

$$D(r) = \chi \circ \alpha \circ (r\sigma - \sigma r) = \alpha' \circ \bar{\phi} \circ (r\sigma - \sigma r);$$

and $vu[\alpha']$ is represented by the derivation $D': R \to \mathrm{Hom}_A(P, P')$ given by

$$D'(r) = \alpha' \circ (r\sigma' - \sigma' r) \circ \chi$$

$$= \alpha' \circ (r\sigma'\chi - \sigma'\chi r)$$

(where $P'$ and $R'/I'^2$ are considered as R-modules via $\phi$). Hence

$$(D - D')(r) = \alpha' \circ \{r[\bar{\phi}\sigma - \sigma'\chi] - [\bar{\phi}\sigma - \sigma'\chi]r\}$$

$$= r\alpha' \circ [\bar{\phi}\sigma - \sigma'\chi] - \alpha' \circ [\bar{\phi}\sigma - \sigma'\chi]r$$

(apply $\pi'$ to see that $\bar{\phi}\sigma - \sigma'\chi \in \mathrm{Hom}_A(P, I'/I'^2)$.) Thus $D - D'$ is an *inner* derivation, and so $w[\alpha] = vu[\alpha']$.

To treat the case $q > 1$, note that

$$\chi \in \mathrm{Hom}_R(P, P') = H^0(R, \mathrm{Hom}_A(P, P))$$

(cf. (1.0.2)) and that, denoting the cohomology product of (1.8) by $*$, we have

$$\begin{aligned} w[\alpha_1 \cdots \alpha_q] &= \chi * [\alpha_1 \cdots \alpha_q] \\ &= \chi * [\alpha_1 \cdots \alpha_{q-1}] * [\alpha_q] \\ &= w[\alpha_1 \cdots \alpha_{q-1}] * [\alpha_q] \\ &= vu[\alpha'_1 \cdots \alpha'_{q-1}] * [\alpha_q] && \text{(inductive assumption)} \\ &= u[\alpha'_1 \cdots \alpha'_{q-1}] * \chi * [\alpha_q] \\ &= u[\alpha'_1 \cdots \alpha'_{q-1}] * w[\alpha_q] \\ &= u[\alpha'_1 \cdots \alpha'_{q-1}] * vu[\alpha'_q] && \text{(case } q = 1) \\ &= u[\alpha'_1 \cdots \alpha'_{q-1}] * u[\alpha'_q] * \chi \\ &= u[\alpha'_1 \cdots \alpha'_{q-1}\alpha'_q] * \chi \\ &= vu[\alpha'_1 \cdots \alpha'_q] && \text{Q.E.D.} \end{aligned}$$

*Proof of* (2.3). As in (1.5.2), let $f$ (resp. $f'$) be a q-cocycle representing $[\alpha_1 \cdots \alpha_q]$ (resp. $[\alpha'_1 \cdots \alpha'_q]$), and let $1 \otimes x$ be a q-cycle representing $\omega$. Let $x'$ be the image of $x$ under the natural map of complexes $\mathbf{B}.(h) \to \mathbf{B}.(h')$, so that $1 \otimes x'$ is a q-cycle representing $\omega'$. We have to show then that

$$\mathrm{Tr}_{P'/A'}(f'(x')) = \psi \mathrm{Tr}_{P/A}(f(x)).$$

Considering the image under the map

$$H^q(R, \mathrm{Hom}_A(P, P')) \otimes H_q(R, R) \xrightarrow{(1.1)} H_0(R, \mathrm{Hom}_A(P, P'))$$

of the element

$$vu[\alpha'_1 \cdots \alpha'_q] \otimes \omega = w[\alpha_1 \cdots \alpha_q] \otimes \omega$$

(cf. (2.3.2)) we see that $\chi \circ f(x) - f'(x') \circ \chi$ lies in the kernel of the natural map $\mathrm{Hom}_A(P, P') \to H_0(R, \mathrm{Hom}_A(P, P'))$, i.e. (cf. (1.0.1))

$$\chi \circ f(x) - f'(x') \circ \chi = \sum_{i=1}^{n} r_i\phi_i - \phi_i r_i$$

for suitable $r_i \in R$, $\phi_i \in \mathrm{Hom}_A(P, P')$. Recalling that $P' = P \otimes_A A'$, we can identify $\mathrm{Hom}_A(P, P')$ with $\mathrm{Hom}_{A'}(P', P')$ and conclude (since $\mathrm{Tr}(r\phi - \phi r) = 0$) that

$$\mathrm{Tr}_{P'/A'}(f'(x')) = \mathrm{Tr}_{P'/A'}(\chi \circ f(x)) = \psi \mathrm{Tr}_{P/A}(f(x))$$

where the second equality is given by the well-known (and easily proved) commutativity of "Trace" and "base change". Q.E.D.

*Remark* (2.3.3). Note in particular the case $\{A = A', \psi = \text{identity}\}$, which generalizes the case $\{A = A', \psi = \text{identity}\}$ of (2.2.1).

COROLLARY (2.4). ("Compatibility of residues and base change"). *Let* $h: A \to R$ *be as usual, let* $I \subset R$ *be a two-sided ideal such that the* A-*module* $P = R/I$ *is finitely generated and projective, and let* $\alpha_1, \ldots, \alpha_q \in \mathrm{Hom}_P(I/I^2, P)$. *Let* $\psi: A \to A'$ *be a ring homomorphism with* $A'$ *commutative, let* $R'$ *be the* $A'$-*algebra* $R \otimes_A A'$, *and let* $I'$ *be the* $R'$-*ideal* $I R' = R' I$ (i.e. *the image of the obvious map* $j: I \otimes_A A' \to R \otimes_A A' = R'$). *Then the natural maps*

$$P \otimes_A A' \to R'/I' = (\text{say})\, P'$$

$$I/I^2 \otimes_A A' \to I'/I'^2$$

*are bijective, and* (2.3.1) *holds with* $\alpha'_i = \alpha_i \otimes 1$ $(1 \le i \le q)$.

*Proof.* From the natural exact sequence

$$I \otimes_A A' \xrightarrow{j} R \otimes_A A' \to P \otimes_A A' \to 0$$

we see that $P \otimes_A A' = R'/I'$, so that the hypotheses of (2.3) hold. Moreover since $P$ is $A$-flat, therefore $j$ is *injective*, so that $I' = I \otimes_A A'$, and $I'^2$ is the image of the obvious map $I^2 \otimes_A A' \to I \otimes_A A'$, a map whose cokernel is $I/I^2 \otimes_A A'$. Thus

$$I/I^2 \otimes_A A' = I'/I'^2,$$

and we can indeed take $\alpha'_i = \alpha_i \otimes 1$ in (2.3.1). Q.E.D.

* * *

(2.5) We show next how our maps $\mathrm{Res}^q$ lead to the residue maps defined in [H]. The precise statement is given in (2.6) below, after the following preliminary remarks.

Recall (with $h:A \to R$ as usual) that for any two left R-modules M, N, there is an obvious R-R bimodule structure on $\mathrm{Hom}_A(M, N)$, and natural maps

(2.5.1) $$\gamma = \gamma^q(M, N): H^q(R, \mathrm{Hom}_A(M, N)) \to \mathrm{Ext}^q_R(M, N) \qquad q \geq 0.$$

The maps $\gamma^q$ arise as follows: $H^{\cdot}(R, \mathrm{Hom}_A(M, N))$ is the homology of the complex

$$\mathrm{Hom}_{R^e}(B.(h), \mathrm{Hom}_A(M, N)) = \mathrm{Hom}_R(B.(h) \otimes_R M, N)$$

(cf. (1.0)) while $\mathrm{Ext}^{\cdot}_R(M, N)$ is the homology of the complex

$$\mathrm{Hom}_R(X., N)$$

where $X. \to M$ is an R-projective resolution of M; but $B.(h) \otimes_R M$ is a resolution of M (denoted by $B(R, M)$ in [M, p.281, Thm. 2.1]), and so there is a homotopy unique lifting of the identity map of M to a map of complexes $X. \to B.(h) \otimes_R M$, whence the maps $\gamma^q$.

(2.5.2) *If both* R *and* M *are projective* A*-modules, then the maps* $\gamma^q$ *are all bijective* (since then the resolution $B.(h) \otimes_R M$ of M is R-projective).

PROPOSITION (2.6). *There is a unique family of* A*-linear maps*

$$\mathrm{Res}^{*q}_{A,R,I}: \mathrm{Ext}^q_R(R/I, R/I) \otimes_R \Omega^q_{R/A} \to A \qquad (q \geq 0)$$

*indexed by triples* (A,R,I) *with* R *a commutative* A*-algebra and* I *an* R*-ideal such that the* A*-module* R/I *is finitely generated and projective, and satisfying:*

(i) *for all* (A,R,I), *the diagram*

$$\begin{array}{ccc} H^q(R, \mathrm{Hom}_A(R/I, R/I)) \otimes_R \Omega^q_{R/A} & \xrightarrow{(1.10.2)} & H^q(R, \mathrm{Hom}_A(R/I, R/I)) \otimes_R H_q(R, R) \\ \downarrow{\scriptstyle \gamma \otimes 1} & & \downarrow{\scriptstyle \mathrm{Res}^q} \\ \mathrm{Ext}^q_R(R/I, R/I) \otimes_R \Omega^q_{R/A} & \xrightarrow[\mathrm{Res}^{*q}]{} & A \end{array}$$

*commutes; and*

(ii) *if* (A,R,I), (A,R′,I′) *are triples as above; if* $\phi: R \to R'$ *is an* A*-algebra homomorphism such that* $\phi(I) \subset I'$ *and the resulting map* $\bar{\phi}: R/I \to R'/I'$ *is bijective; and if*

$$\alpha{:}\mathrm{Ext}^q_{R'}(R'/I', R'/I') \to \mathrm{Ext}^q_R(R'/I', R'/I') = \mathrm{Ext}^q_R(R/I, R/I)$$

*and*

$$\beta{:}\Omega^q_{R/A} \to \Omega^q_{R'/A}$$

*are the natural maps, then, for all*

$$\xi' \in \mathrm{Ext}^q_{R'}(R'/I', R'/I'), \qquad \eta \in \Omega^q_{R/A}$$

*we have*

$$\mathrm{Res}^{*q}_{A,R,I}(\alpha(\xi') \otimes \eta) = \mathrm{Res}^{*q}_{A,R',I'}(\xi' \otimes \beta(\eta)).$$

*In fact, with the notation of* [H, p. 513] (*and cf.* [*ibid.*, p. 512, Lemma 1.1]), $\mathrm{Res}^{*q}$ *is the composition*

$$\mathrm{Ext}^q_R(R/I, R/I) \otimes_R \Omega^q_{R/A} \xrightarrow{\ \mu \cdot \sigma^q\ } \mathrm{Hom}_A(R/I, A) \xrightarrow{\ \text{evaluation at } 1\ } A.$$

*Proof.* Let $\phi{:}R \to R'$ be as in (ii) above, and let $X.$ (respectively $X'.$) be an R-projective (respectively R′-projective) resolution of $R/I = R'/I'$. From

$$\begin{array}{ccc} R/I & \xrightarrow{\ \bar{\phi}\ } & R'/I' \\ \| & & \| \\ R/I & \xrightarrow{\ \bar{\phi}\ } & R'/I' \end{array}$$

we derive a homotopy-commutative diagram of R-homomorphisms of complexes

$$\begin{array}{ccc} X. & \longrightarrow & X'. \\ \downarrow & & \downarrow \\ \mathbf{B}.(h) \otimes_R (R/I) & \xrightarrow{\ (\text{via } \phi)\ } & \mathbf{B}.(\phi \circ h) \otimes_{R'} (R'/I') \end{array}$$

from which we deduce a commutative diagram (cf. (2.5))

$$
(2.6.1)\qquad
\begin{array}{ccc}
\mathrm{Ext}^q_R(R/I, R/I) & \xleftarrow{\ \alpha\ } & \mathrm{Ext}^q_{R'}(R'/I', R',I') \\
\uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\gamma} \\
H^q(R, \mathrm{Hom}_A(R/I, R/I)) & \xleftarrow[(2.1.1)]{\ u\ } & H^q(R', \mathrm{Hom}_A(R'/I', R'/I'))
\end{array}
$$

If we confine ourselves to triples (A,R,I) for which R is *projective* as an A-module, then the corresponding maps $\gamma$ are bijective (2.5.2), so that $\mathrm{Res}^{*q}$ is uniquely determined by (i) above; and in view of the remark immediately following (2.1.2), and of commutativity in (2.6.1), if R and R′ are both A-projective then (ii) follows easily from (2.2) (with $A' = A$, $P = P' = R/I$).

Now for arbitrary $R', I'$ with $R'/I'$ finite and projective over A, let $\xi' \in \mathrm{Ext}^q_{R'}(R'/I', R'/I')$ and $\eta' \in \Omega^q_{R'/A}$. Clearly there exists a polynomial ring $R = A[T_1, \ldots, T_n]$ and an A-homomorphism $\phi: R \to R'$ such that the composition $R \xrightarrow{\phi} R' \to R'/I'$ is surjective (so that if $I = \phi^{-1}(I')$ then $\bar{\phi}: R/I \to R'/I'$ is bijective) and such that furthermore $\eta' = \beta(\eta)$ for some $\eta \in \Omega^q_{R/A}$. Since R is A-projective, there is (as already noted) a unique map $\mathrm{Res}^{*q}_{A,R,I}$ making the diagram in (i) commute. If (ii) is to be true, then we must have

$$\mathrm{Res}^{*q}_{A,R',I'}(\xi' \otimes \eta') = \mathrm{Res}^{*q}_{A,R,I}(\alpha(\xi') \otimes \eta).$$

This proves uniqueness for $\mathrm{Res}^{*q}$, and indicates a proof for existence. Indeed, if $A[T_1, \ldots, T_n] \to R'$, $A[U_1, \ldots, U_m] \to R'$ are A-algebra homomorphisms such as we have just considered, then both can be "dominated" by a third such homomorphism $A[T_1, \ldots, T_n, U_1, \ldots, U_m] \to R'$, and it follows in a straightforward way that the above procedure for determining $\mathrm{Res}^{*q}(\xi' \otimes \eta')$ gives a result which does not depend on the choice of the polynomial ring R. Thus we get a definition for $\mathrm{Res}^{*q}$, and the rest of this existence proof may be left to the reader.

It remains to prove the last assertion in (2.6) (which then gives another more constructive proof that $\mathrm{Res}^{*q}$ exists).

By a slight modification of [H, p.253, Prop. 2.4], we see that the map

$$\mathrm{Res}^{*q} = (\text{evaluation at } 1) \circ \mu \circ \sigma^q$$

does satisfy (ii) of (2.6).

To prove that this $\mathrm{Res}^{*q}$ also satisfies (i), i.e. that the maps $\mathrm{Res}^q \circ (1.10.2)$ and $\mathrm{Res}^{*q} \circ (\gamma \otimes 1)$ coincide, it is enough to check that both maps have the same effect on elements of the form $\xi \otimes dr_1 dr_2 \cdots dr_q$, with $\xi \in H^q(R, \mathrm{Hom}_A(R/I, R/I))$, and $r_1, r_2, \ldots, r_q \in R$.

We first determine the effect of $\mathrm{Res}^q \circ (1.10.2)$ on $\xi \otimes dr_1 \cdots dr_q$.

We have the *Koszul complex* $\mathbf{K}.(r_i \otimes 1 - 1 \otimes r_i)$ over $R \otimes_A R$ determined by the sequence $(r_i \otimes 1 - 1 \otimes r_i)_{1 \le i \le q}$; and with $h: A \to R$ and $\mathbf{B}.(h)$ as usual, there is an $(R \otimes_A R)$-linear map of complexes

$$\Lambda: \mathbf{K}.(r_i \otimes 1 - 1 \otimes r_i) \to \mathbf{B}.(h)$$

defined as follows. For each i, if $\mathbf{K}_{(i)}$ is the complex

$$\underset{(\text{degree } 1)}{R \otimes_A R} \xrightarrow{\quad r_i \otimes 1 - 1 \otimes r_i \quad} \underset{(\text{degree } 0)}{R \otimes_A R}$$

then there is a unique $(R \otimes_A R)$-linear map of complexes

$$\mathbf{K}_{(i)} \to \mathbf{B}.(h)$$

which is the identity map of $R \otimes_A R$ in degree 0, and which in degree 1 takes $1 \in R \otimes_A R$ to

$$[r_i] = 1 \otimes r_i^* \otimes 1 \in R \otimes_A (R/A) \otimes_A R = B_1$$

(cf. (1.0)). By tensoring these maps we obtain the map of complexes

$$\Lambda: \mathbf{K}.(r_i \otimes 1 - 1 \otimes r_i) = \mathbf{K}_{(1)} \otimes \cdots \otimes \mathbf{K}_{(q)} \to \mathbf{B}.(h) \otimes \cdots \otimes \mathbf{B}.(h) \to \mathbf{B}.(h)$$

where the last map is given by the "shuffle product" of (1.10). Moreover it is easily checked via definitions that if

$$1_q \in (\mathbf{K}.(r_i \otimes 1 - 1 \otimes r_i))_q = R \otimes_A R$$

is the identity element, then the homology class of the q-cycle

$$1 \otimes \Lambda(1_q) \in R \otimes_{R^e} B_q$$

is just

$$\theta_q(dr_1 \cdots dr_q) \in H_q(R, R)$$

where $\theta_q$ is as in (1.10.2).

Now $\xi$ is represented by an $(R \otimes_A R)$-linear map

$$f: B_q \to \mathrm{Hom}_A(R/I, R/I)$$

or, equivalently, by an R-linear map

$$f^*: B_q \otimes_R (R/I) \to R/I.$$

And, by (1.5.2), $\mathrm{Res}^q(\xi \otimes \theta_q(dr_1 \cdots dr_q))$ is the trace of the map $f(\Lambda(1_q))$, a map satisfying (for $\overline{r} = r + I \in R/I$):

$$[f(\Lambda(1_q)](\overline{r}) = f^*(\Lambda(1_q) \otimes \overline{r}) = f^*(\lambda(1_q \otimes \overline{r})) \tag{2.6.2}$$

where

$$\lambda = \Lambda \otimes_R R/I: K.(r_i \otimes 1 - 1 \otimes r_i) \otimes_R R/I \to B.(h) \otimes_R R/I.$$

Note here that (with $\overline{r}_i = r_i + I \in R/I$)

$$K.(r_i \otimes 1 - 1 \otimes r_i) \otimes_R (R/I) = K.(r_i \otimes 1 - 1 \otimes \overline{r}_i),$$

the Koszul complex over

$$S = R \otimes_A (R/I)$$

determined by the sequence $(r_i \otimes 1 - 1 \otimes \overline{r}_i)_{1 \le i \le q}$.

Next we determine the effect of $\mathrm{Res}^{*q} \circ (\gamma \otimes 1)$ on $\xi \otimes dr_1 \cdots dr_q$.

Let $X. \to R/I$ be an $R$-projective resolution of $R/I$. Since $R/I$ is $A$-projective, therefore the above $S$ is $R$-projective, so that $K.(r_i \otimes 1 - 1 \otimes \overline{r}_i)$ is an $R$-projective complex, mapping to $R/I$ via the obvious (multiplication) map

$$(K.(r_i \otimes 1 - 1 \otimes \overline{r}_i))_0 = R \otimes_A R/I \to R/I.$$

Since $B.(h) \otimes_R (R/I)$ is a resolution of $R/I$ (cf. (2.5.1)), it follows that we have a homotopy-commutative diagram of $R$-homomorphisms of complexes

$$\begin{array}{ccc} K. = K.(r_i \otimes 1 - 1 \otimes \overline{r}_i) & \xrightarrow{\ \lambda\ } & B.(h) \otimes_R (R/I) \\ & \searrow^{\phi} \quad \nearrow_{\psi} & \\ & X. & \end{array} \tag{2.6.3}$$

($\lambda = \Lambda \otimes_R (R/I)$ as above). By definition of $\gamma$ (2.5.1), and of $\mathrm{Res}^{*q}$ [H, pp.512-516], we find then that $\mathrm{Res}^{*q}(\gamma(\xi) \otimes dr_1 \cdots dr_q)$ is the trace of the map $g$ given by

$$g(\overline{r}) = f^*(\psi\phi(1_q \otimes \overline{r})). \tag{2.6.2$'$}$$

In view of (2.6.2) and (2.6.2)′, to complete the proof we need to show that the map $\varsigma: R/I \to R/I$ given by

$$\varsigma(\bar{r}) = [f^* \circ (\lambda - \psi\phi)](1_q \otimes \bar{r})$$

satisfies

$$\mathrm{Tr}_{(R/I)/A}(\varsigma) = 0. \tag{2.6.4}$$

Since (2.6.3) is homotopy-commutative, there exist R-linear maps

$$\mu: K_q \to B_{q+1} \otimes_R (R/I)$$

$$\nu: K_{q-1} \to B_q \otimes_R (R/I)$$

such that

$$\varsigma(\bar{r}) = [f^* \circ (\partial\mu + \nu\delta)](1_q \otimes \bar{r})$$

where $\partial$ (respectively $\delta$) is the boundary map in $\mathbf{B}.(h) \otimes_R (R/I)$ (respectively in $\mathbf{K}.$). But $f^*$ (which represents the homology class $\xi$) is a q-*cocycle* in the complex $\mathrm{Hom}_R(\mathbf{B}.(h) \otimes_R R/I, R/I)$, i.e. $f^* \circ \partial = 0$. Moreover, using the definition of $\delta$, and the R-linearity of $f^*$, $\nu$, and $\delta$, we find that $f^*\nu\delta(1_q \otimes \bar{r})$ is a sum of elements of the form

$$r_i f^* \nu((1 \otimes \bar{r})k_i) - f^* \nu((1 \otimes \bar{r}_i\bar{r})k_i) = \bar{r}_i \varsigma_i(\bar{r}) - \varsigma_i(\bar{r}_i r)$$

where $k_i \in K_{q-1}$ does not depend on $\bar{r}$, and where $\varsigma_i: R/I \to R/I$ is given by

$$\varsigma_i(\bar{r}) = f^* \nu((1 \otimes \bar{r})k_i).$$

Thus

$$\varsigma = \sum_{i=1}^{q} (\bar{r}_i \varsigma_i - \varsigma_i \bar{r}_i),$$

and since

$$\mathrm{Tr}_{(R/I)/A}(\bar{r}_i \varsigma_i - \varsigma_i \bar{r}_i) = 0$$

therefore (2.6.4) holds. Q.E.D.

(2.7). We retain the notation of (2.6), and assume further that the R-ideal I is generated by a sequence $(f_1, \ldots, f_q)$ such that the Koszul complex $\mathbf{K}_R(f_i) = \mathbf{K}(f_i)$ over R determined by $(f_1, \ldots, f_q)$ is *exact* except in degree zero.

Then $\mathbf{K}(f_i)$ is an R-projective resolution of R/I. Using this resolution we see that the natural map

$$\text{(2.7.1)} \qquad \mathrm{Ext}^q_R(R/I, R/I) \otimes_R \Omega \to \mathrm{Ext}^q_R(R/I, \Omega/I\Omega) \qquad (\Omega = \Omega^q_{R/A})$$

is an *isomorphism*. For $\omega \in \Omega$, we denote by

$$\left\{ {\omega \atop f_1, \ldots, f_q} \right\} \in \mathrm{Ext}^q_R(R/I, \Omega/I\Omega)$$

the cohomology class of the q-cocycle

$$\xi \in \mathrm{Hom}_R((K(f_i))_q, \Omega/I\Omega)$$

determined by

$$\xi(1_q) = \omega + I\Omega \in \Omega/I\Omega$$

where $1_q$ is the identity element of $R = (K(f_i))_q$. Via the identification (2.7.1), we have then the element

$$\mathrm{Res}^{*q}_{A,R,I} \left\{ {\omega \atop f_1, \ldots, f_q} \right\} \in A,$$

which by (2.6) is the same as the element denoted in [H, pp. 516-517] by

$$\mathrm{Res}^q_{R/A} \left[ {\omega \atop f_1, \ldots, f_q} \right].$$

Agreement of Hopkins' residue symbol with the one defined in (1.9) and (1.10) above is then given by:

Corollary (2.7.2).. *In the preceding situation we have, for all* $\omega \in \Omega = \Omega^q_{R/A}$,

$$\mathrm{Res} \left[ {\omega \atop f_1, \ldots, f_q} \right] = \mathrm{Res}^{*q} \left\{ {\omega \atop f_1, \ldots, f_q} \right\}.$$

*More generally, if*

$$\alpha_1, \ldots, \alpha_q \in \mathrm{Hom}_{R/I}(I/I^2, R/I)$$

*and*

$$\bar{f}_j = f_j + I^2 \in I/I^2 \qquad\qquad 1 \le j \le q$$

*and if* $\Delta(\alpha,\mathbf{f})$ *is any element in* R *whose natural image in* R/I *is the determinant* $\det(\alpha_i(\bar{f}_j))$, *then*

$$\operatorname{Res}\begin{bmatrix} \omega \\ \alpha_1, \ldots, \alpha_q \end{bmatrix} = \operatorname{Res}^{*q}\begin{Bmatrix} \Delta(\alpha,\mathbf{f})\omega \\ f_1, \ldots, f_q \end{Bmatrix}.$$

*Proof.* Set $P = R/I$. In view of (2.6.1), we need only show that

$$\gamma[\alpha_1\alpha_2 \cdots \alpha_q] \in \operatorname{Ext}^q_R(P, P)$$

is the cohomology class of the q-cocycle

$$\varsigma \in \operatorname{Hom}_R(K(f_i)_q, P)$$

determined (with $1_q$ as above) by

$$\varsigma(1_q) = \det(\alpha_i(\bar{f}_j)).$$

For this, we use (cf. (2.5)) the explicit map of complexes

$$\Phi: K(f_i) \to B.(h) \otimes_R P$$

which is the composition

$$K_R(f_i) \xrightarrow{\text{natural}} K_S(f_i \otimes 1) = K_S(f_i \otimes 1 - 1 \otimes \bar{f}_i) \xrightarrow{\lambda} B.(h) \otimes_R P$$

where $S = R \otimes_A P$, and $\lambda$ is as in the proof of (2.6) (cf. (2.6.2) etc., replacing $r_i$ by $f_i$). One checks again via definitions that

$$\text{(2.7.3)} \qquad \Phi(1_q) = \sum_\tau (-1)^{|\tau|} \otimes f_{\tau(1)} \otimes \cdots \otimes f_{\tau(q)} \otimes \bar{1} \in B_q \otimes_R P$$

where $\tau$ runs through all permutations of $\{1,2,\ldots, q\}$ and $|\tau| = 0$ (resp. $|\tau| = 1$) if the permutation $\tau$ is even (resp. odd).

Now according to (2.5), all we have to do is to take a cocycle

$$\varsigma': B_q \to \operatorname{Hom}_A(P, P)$$

representing $[\alpha_1\alpha_2 \cdots \alpha_q]$, reinterpret it as an R-linear map

$$\varsigma'': B_q \otimes_R P \to P,$$

and show that

$$\varsigma''(\Phi(1_q)) = \det(\alpha_i(\bar{f}_j)). \tag{2.7.4}$$

Such a $\varsigma'$ is described in (1.8.3). Since $f_j$ annihilates P, we see for any A-linear section $\sigma$ of the natural map $R/I^2 \to R/I = P$ that the map

$$\alpha_i \circ (f_j\sigma - \sigma f_j) \in \mathrm{Hom}_A(P, P)$$

is just multiplication by $\alpha_i(\bar{f}_j)$. In view of (2.7.3), the relation (2.7.4) follows at once.

COROLLARY (2.8) ("Transition formula"). *Let* $\mathbf{f} = (f_1, \ldots, f_q)$, $\mathbf{g} = (g_1, \ldots, g_q)$ *be sequences in* R *such that the Koszul complexes* $K(f_i)$ *and* $K(g_i)$ *are exact except in degree zero, and such that the* A*-modules* $R/\mathbf{f}R$, $R/\mathbf{g}R$ *are finitely generated and projective. Suppose also that* $\mathbf{f}R \subset \mathbf{g}R$, *say*

$$f_i = \sum_{j=1}^{q} r_{ij}g_j \qquad r_{ij} \in R, \quad 1 \le i \le q.$$

*Then for any* $\nu \in \Omega$, *we have*

$$\mathrm{Res}\begin{bmatrix} \nu \\ g_1, \ldots, g_q \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \det(r_{ij})\nu \\ f_1, \ldots, f_q \end{bmatrix}.$$

*Proof.* In view of (2.7.2), this is just [H, p.522, Corollary 2.2] (slightly generalized).

(2.9) *Exercises* (not used elsewhere).

1. Let R be a commutative A-algebra, I an ideal in R, $P = R/I$, and assume that the P-modules $I/I^2$, $I^2/I^3$ are free of ranks $q$, $q(q+1)/2$ respectively. Assume also that the natural map $R/I^3 \to R/I = P$ has an A-linear section. Show that for every $\xi \in H^1(R, \mathrm{Hom}_A(P, P))$ we have $\xi^2 = 0$ (cf. (1.8)); and deduce that then the residue symbol

$$\mathrm{Res}\begin{bmatrix} \omega \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}.$$

(defined if P is *perfect* as an A-module) is an *alternating* A-multilinear function of $\alpha_1, \ldots, \alpha_q$. (When I is generated by an R-regular sequence, this last assertion follows from the statement of (2.7.2).)

2. (a) Let R be a commutative A-algebra, let I be an ideal in R, let $P = R/I$, and set

$$\Lambda^{\bullet}{}_P(I/I^2)^{*gr} = \bigoplus_{q\geq 0} \mathrm{Hom}_P(\Lambda^q{}_P(I/I^2), P),$$

with its graded anticommutative P-algebra structure [B, Ch. III, §11.5]. Expanding on the technique used in the proof of (2.7.2), define a natural *homomorphism of graded algebras* (cf. (1.8))

$$\bigoplus_{q\geq 0} H^q(R, \mathrm{Hom}_A(P, P)) \to \Lambda^{\bullet}{}_P(I/I^2)^{*gr}$$

agreeing in degree 1 with the map $\psi_P$ of (1.4).

(b) With notation as in (a), show that the maps $\gamma^q$ of (2.5) give a *homomorphism of graded algebras*

$$\bigoplus_{q\geq 0} H^q(R, \mathrm{Hom}_A(P, P)) \to \bigoplus_{q\geq 0} \mathrm{Ext}^q{}_R(P, P)$$

where multiplication in the "Ext" algebra is given by Yoneda composition.

(c) Show that the "fundamental local homomorphism" (cf. e.g. [L, p.111])

$$\bigoplus_{q\geq 0} \mathrm{Ext}^q{}_R(P, P) \to \Lambda^{\bullet}{}_P(I/I^2)^{*gr}$$

is a *homomorphism of graded algebras.*

(d) Show that the following diagram *commutes*:

$$\begin{array}{ccc}
\bigoplus_{q\geq 0} H^q(R, \mathrm{Hom}_A(P,P)) & \xrightarrow{(b)} & \bigoplus_{q\geq 0} \mathrm{Ext}^q_R(P,P) \\
& {\scriptstyle (a)}\searrow \qquad \swarrow{\scriptstyle (c)} & \\
& \Lambda^{\bullet}_P(I/I^2)^{*gr} &
\end{array}$$

## §3. QUASI-REGULAR SEQUENCES

In this section we generalize some familiar formulas, involving residues with respect to a sequence of variables in a power series ring over a commutative ring A, to quasi-regular sequences $\mathbf{f} = (f_1, \ldots, f_q)$ in a commutative A-algebra R. Roughly speaking, the idea is first to map R into a power series ring in $(f_1, \ldots, f_q)$ with coefficients in the algebra of endomorphisms $E = \mathrm{Hom}_A(R/\mathbf{f}R, R/\mathbf{f}R)$. The main result, for which (3.1)–(3.5) are preparatory, is the somewhat technical Theorem (3.6), of which the formulas in question are immediate consequences (cf. (3.7) and its corollaries (3.8) and (3.9)). Moreover we obtain from (3.6) a "trace formula I " (3.10)[1], from which we deduce in Appendix A how the residues defined in this paper give rise to residues on algebraic varieties, as described in [L]. Finally, in Appendix B, we generalize a well-known residue formula involving exterior differentiation; the proof is rather straightforward in the power series case, but for arbitrary quasi-regular sequences it appears to need a lot of machinery.

* * *

(3.1). Let $\mathbf{f} = (f_1, \ldots, f_q)$ be a sequence in a commutative ring R. For any q-tuple $M = (m_1, \ldots, m_q)$ of non-negative integers we set

$$\mathbf{f}^M = f_1^{m_1} f_2^{m_2} \cdots f_q^{m_q}.$$

Let $\hat{R}$ be the $\mathbf{f}R$-adic completion of R, and let

$$\sigma : R/\mathbf{f}R = \hat{R}/\mathbf{f}\hat{R} \to \hat{R}$$

be a section of the canonical map $\pi : \hat{R} \to \hat{R}/\mathbf{f}\hat{R}$ (i.e. $\sigma$ is any map of sets such that $\pi \circ \sigma =$identity). Assume for simplicity that $\sigma(0) = 0$. Then any element r of $\hat{R}$ can be represented as a power series

[1] "trace formula II" is given in §4.7

$$r = \sum_M \sigma(r_M)f^M \qquad (r_M \in R/fR)$$

where the summation is over all q-tuples M as above, and where, by abuse of notation, the natural image of $(f_1, \ldots, f_q)$ in $\hat{R}$ is still denoted by $(f_1, \ldots, f_q)$.

We recall that the sequence $\mathbf{f} = (f_1, \ldots, f_q)$ in R is said to be *quasi-regular* if, with $I = \mathbf{f}R$, the R/I-algebra homomorphism

$$(R/I)[X_1, \ldots, X_q] \to gr_I R = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

which sends the indeterminate $X_i$ to $(f_i + I^2) \in I/I^2$ $(1 \leq i \leq q)$ is *bijective.*

LEMMA (3.1.1). *Let* $\mathbf{f}$, $\hat{R}$, *and* $\sigma: R/\mathbf{f}R \to \hat{R}$ *be as above. Then* $\mathbf{f}$ *is quasi-regular if and only if for every*

$$r = \sum_M \sigma(r_M)f^M \in \hat{R} \qquad (r_M \in R/\mathbf{f}R)$$

*the* $r_M$ *are all uniquely determined by* r.

The *proof* is left to the reader.

*Examples* (3.2). (a) Any *regular* sequence $\mathbf{f}$ in R is quasi-regular [EGA $0_{IV}$, (15.1.9)]; and the converse holds if R is $\mathbf{f}R$-adically complete. (*Idea of proof* : show, for $j \leq q$, that any $r \in (f_1, \ldots, f_j)R$ has an expansion $r = \Sigma\sigma(r_M)f^M$ where for each $M = (m_1, \ldots, m_q)$, $r_M \neq 0 \Rightarrow m_i \neq 0$ for some i with $1 \leq i \leq j$. Using (3.1.1), conclude that

$$(\Sigma\sigma(r'_M)f^M)f_{j+1} \in (f_1, \ldots, f_j)R \Rightarrow \Sigma\, \sigma(r'_M)f^M \in (f_1, \ldots, f_j)R.)$$

It follows that *a sequence* $\mathbf{f}$ *is quasi-regular in* R *if and only if its natural image in the* $\mathbf{f}R$-*adic completion* $\hat{R}$ *is regular.*

(b) A sequence $\mathbf{f}$ in R is quasi-regular if and only if the image of $\mathbf{f}$ in the localization $R_p$ is quasi-regular for all prime ideals (or all maximal ideals) $p \supset \mathbf{f}R$. In case R is *noetherian,* a sequence of non-units in $R_p$ is quasi-regular if and only if it is regular [EGA $0_{IV}$, (15.1.9)]; and hence $\mathbf{f}$ *is quasi-regular in* R *if and only if the Koszul complex over* R *determined by* $\mathbf{f}$ *is exact except in degree zero.*

(c) If the sequence $(f_1, \ldots, f_q)$ is quasi-regular in R, then so is $(f_1^{m_1}, \ldots, f_q^{m_q})$ for any q-tuple $(m_1, \ldots, m_q)$ of positive integers. (Proof left to reader.)

(3.3) Let $h: A \to R$ be a homomorphism of commutative rings, and let $\mathbf{f} = (f_1, \dots, f_q)$ be a quasi-regular sequence in R. The fR-adic completion $\hat{R}$ is then an A-algebra via the composition

$$\hat{h}: A \xrightarrow{h} R \xrightarrow{\text{natural}} \hat{R},$$

and in fact an algebra over the formal power series ring $A[[\mathbf{X}]]$ in q indeterminates $\mathbf{X} = (X_1, \dots, X_q)$, via the homomorphism

$$h_{\mathbf{f}}: A[[\mathbf{X}]] \to \hat{R}$$

given by

$$h_{\mathbf{f}}\,(\sum_M a_M X^M) = \sum_M a_M f^M \;\; (= \sum_M \hat{h}(a_M) f^M) \qquad (a_M \in A)$$

(notation as in (3.1)). Moreover if

$$\sigma : P = R/\mathbf{f}R \to \hat{R}$$

is an A-linear section (= right inverse) of the natural map $\pi : \hat{R} \to \hat{R}/\mathbf{f}\hat{R} = P$, then we obtain, by extension of scalars, an $A[[\mathbf{X}]]$-linear map

$$\sigma^* : P \otimes_A A[[\mathbf{X}]] \to \hat{R}. \tag{3.3.1}$$

Lemma (3.3.2). *If the* A-*module* $P = R/\mathbf{f}R$ *is finitely presented, then the above map* $\sigma^*$ *is bijective.*

*Proof* (communicated in essence by M. Hochster). Lemma (3.1.1) gives us an obvious identification of the A-module $\hat{R}$ with a direct product – indexed by the q-tuples M – of copies of P. Then $\sigma^*$ is identified with the natural map

$$P \otimes_A (\prod_M A_M) \to \prod_M (P \otimes_A A_M) \qquad (A_M = A).$$

Hence (3.3.2) is a special case of [B′, Ch. I, §2, Exercise 9(a)].

*Remark* (3.3.3). Let

$$A[[\mathbf{f}]] = h_{\mathbf{f}}(A[[\mathbf{X}]]) \subset \hat{R},$$

so that $A[[\mathbf{f}]]$ consists of all power series in $f_1, \dots, f_q$ with coefficients in $\hat{h}(A)$.

The existence of an A-linear section $\sigma$ as above implies that $A[[f]]$ is actually a *formal power series ring* in $(f_1, \ldots, f_q)$ over $\hat{h}(A)$ (i.e. if $\sum_M \alpha_M f^M = 0$ with $\alpha_M \in \hat{h}(A)$ for all M, then $\alpha_M = 0$ for all M). This will be clear from (3.1.1) if we can find a section $\sigma': P \to \hat{R}$ such that $\hat{h}(A) \subset \sigma'(P)$. But if $\pi: \hat{R} \to P$ is the natural map, then $\pi\sigma(1) = 1$, i.e. $\sigma(1) \in 1 + f\hat{R}$, so that $\sigma(1)$ is a unit in $\hat{R}$; and $\sigma' = \sigma(1)^{-1}\sigma$ is an A-linear section with $\sigma'(1) = 1$, whence for all $a \in A$:

$$\hat{h}(a) = \hat{h}(a)\sigma'(1) = \sigma'(\pi\hat{h}(a).1) \subset \sigma'(P),$$

as desired.

(3.4) Let $h: A \to R$ and $f$ be as in (3.3), assume that the A-module $P = R/fR$ is finitely presented, and let $\sigma: P \to \hat{R}$ be an A-linear section of the natural map $\hat{R} \to \hat{R}/f\hat{R} = P$, so that we have, by (3.3.2), an isomorphism of $A[[X]]$-modules

$$\sigma^*: P \otimes_A A[[X]] \xrightarrow{\sim} \hat{R}.$$

Let H be the $A[[X]]$-algebra

$$H = \mathrm{Hom}_{A[[X]]}(\hat{R}, \hat{R}) \cong \mathrm{Hom}_A(\sigma(P), \hat{R})$$

(where the last isomorphism is given by restriction of maps), and let

$$E = \mathrm{Hom}_A(P, P).$$

For any $\phi: \sigma(P) \to \hat{R}$ in H, and $p \in P$, we have

$$\phi(\sigma(p)) = \sum_M \sigma(\phi_M(p))f^M$$

where for each M, $\phi_M \in E$ is well-defined because of (3.1.1). Thus we have a map

$$\sigma^\#: H \to E[[X]]$$

(where $E[[X]]$ is the $A[[X]]$-algebra consisting of formal power series with coefficients in the A-algebra E ) given by

$$\sigma^\#(\phi) = \sum_M \phi_M X^M \qquad (\phi \in H).$$

It is easily seen that $\sigma^\#$ *is* $A[[X]]$-*linear and bijective.* Thus H *is complete and separated in its* X-*adic topology.*

After embedding $\hat{R}$ in H by identifying $r \in \hat{R}$ with "multiplication by r", we have that the natural image of $X_i$ in H is $f_i$ $(1 \le i \le q)$; and so the X-adic topology on H coincides with the $\mathfrak{f}$-adic topology (defined by the powers of the two-sided ideal $\mathfrak{f}$H).

Note further that, for $\psi \in E \subset E[[X]]$, the map $\psi^{\#} = (\sigma^{\#})^{-1}(\psi)$ is the unique element of H such that for all $p \in P$:

$$\psi^{\#}(\sigma(p)) = \sigma(\psi(p)).$$

It follows at once that $\psi_1^{\#}\psi^{\#} = (\psi_1\psi)^{\#}$ for any $\psi_1 \in E$; i.e. $(\sigma^{\#})^{-1}$ *maps* E *isomorphically onto an* A-*subalgebra of* H; *and consequently* $\sigma^{\#}$ *is an isomorphism of* A[[X]]-*algebras.*

In other words, having thus identified E with an A-subalgebra of H (the identification depending on $\sigma$ ) we have that *each element* $\phi$ *of* H *is uniquely of the form*

$$\phi = \sum_M \phi_M f^M \qquad \phi_M \in E;$$

*and if* $\psi \in H$ *is given by*

$$\psi = \sum_N \psi_N f^N \qquad \psi_N \in E$$

*then*

$$\phi \circ \psi = \sum_{M,N} (\phi_M \circ \psi_N) f^{M+N}.$$

Thus we can think of $H = E[[\mathfrak{f}]]$ as being the ring of formal power series in $f_1, \ldots, f_q$, with coefficients in E.

The natural map $A[[X]] \to H$ then takes $\sum_M a_M X^M$ to $\sum_M \bar{a}_M f^M$ where $\bar{a}_M \in E$ is "multiplication by $a_M$ in P".

And $\hat{R}$ is naturally embedded as an A[[X]]-subalgebra of H.

(3.4.1) If P is finitely generated and projective over A, then

$$\hat{R} \cong P \otimes_A A[[X]] \qquad \text{(cf.(3.3.2))}$$

is finitely generated and projective over A[[X]], and we have trace maps

$$\mathrm{Tr}_{P/A}: E \to A$$

$$\mathrm{Tr}_{\hat{R}/A[[X]]}: H \to A[[X]] .$$

I claim that then:

$$\text{(3.4.2)} \qquad \mathrm{Tr}_{\hat{R}/A[[X]]}\Big(\sum_M \phi_M f^M\Big) = \sum_M (\mathrm{Tr}_{P/A}\,\phi_M)X^M.$$

Indeed, the map taking

$$\phi = \sum_M \phi_M f^M \in H$$

to

$$\sum_M (\mathrm{Tr}_{P/A}\,\phi_M)X^M \in A[[X]]$$

is $A[[X]]$-linear, and therefore it suffices to verify (3.4.2) for $\phi \in E$ (i.e. when $\phi_M = 0$ for all $M \neq (0,0,\dots,0)$). But the above described embedding $(\sigma^{\#})^{-1}$ of $E$ into $H$ simply takes $\phi \in E$ to

$$\phi^{\#} = \phi \otimes 1 \in \mathrm{Hom}_{A[[X]]}(P \otimes_A A[[X]], P \otimes_A A[[X]]) = \mathrm{Hom}_{A[[X]]}(\hat{R}, \hat{R}) = H,$$

and so the assertion follows from the commutativity of "Trace" with "base change".

*Remark.* Note that for given $\phi = \Sigma\, \phi_M f^M \in H$, the coefficients $\phi_M$ depend on the choice of $\sigma$, but (3.4.2) shows that their traces do not.

(3.5) Before stating the central result (3.6) of this section, we need some more preliminaries. We fix as above a homomorphism of commutative rings $h{:}A \to R$ and a quasi-regular sequence $\mathbf{f} = (f_1, \dots, f_q)$ in $R$, and denote by $\hat{R}$ the $\mathbf{f}R$-adic completion of $R$. Assume that the $A$-module $P = R/\mathbf{f}R$ is finitely generated and projective; and let $\sigma{:}\, P \to \hat{R}$ be an $A$-linear section (= right inverse) of the natural map $\hat{R} \to \hat{R}/\mathbf{f}\hat{R} = P$. As in (3.4), we set

$$E = \mathrm{Hom}_A(P, P)$$

$$H = \mathrm{Hom}_{A[[X]]}(\hat{R}, \hat{R})$$

and identify $H$ with a formal power series ring

$$H = E[[\mathbf{f}]]$$

(an identification depending on the choice of $\sigma$). We have, as in (3.4.1), the trace map

$$\mathrm{Tr}_{\hat{R}/A[[X]]}{:}\, H \to A[[X]]$$

factoring through $H_0(R, H)$ (cf. (1.5)).

(3.5.1) Recall that for any positive integers $m_1, \dots, m_q$, the sequence $(f_1^{m_1}, \dots, f_q^{m_q})$ is also quasi-regular, as is its natural image $(\hat{f}_1^{m_1}, \dots, \hat{f}_q^{m_q})$ in the $\mathbf{f}R$-adic completion $\hat{R}$ (cf. (3.2)). Moreover if $J$ is the ideal $(f_1^{m_1}, \dots, f_q^{m_q})R$, then from (3.1.1) one deduces that $R/J = \hat{R}/J\hat{R}$ is again a finitely generated projective A-module, isomorphic to a direct sum of copies of R/fR (one copy for each monomial $f^N$, $N = (n_1, \ldots, n_q)$ such that $0 \le n_i < m_i$ for all i). Hence for any $\omega \in H_q(R, R)$ and $\hat{\omega} \in H_q(\hat{R}, \hat{R})$ the residue symbols

$$\operatorname{Res}\begin{bmatrix} \omega \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix}, \qquad \operatorname{Res}\begin{bmatrix} \hat{\omega} \\ \hat{f}_1^{m_1}, \ldots, \hat{f}_q^{m_q} \end{bmatrix}$$

are defined (cf. (1.9)); and in fact by (2.3) (with $A = A'$, $R' = \hat{R}$, and $\phi, \psi$ the obvious maps) if $\hat{\omega}$ happens to be the natural image of $\omega$, then

$$\operatorname{Res}\begin{bmatrix} \omega \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} = \operatorname{Res}\begin{bmatrix} \hat{\omega} \\ \hat{f}_1^{m_1}, \ldots, \hat{f}_q^{m_q} \end{bmatrix}$$

As before, we will write "$f_i$" for "$\hat{f}_i$" if no confusion results.

(3.5.2). We have E-derivations $\dfrac{\partial}{\partial f_i}$ $(1 \le i \le q)$ of $H = E[[f]]$ into itself. Denote by $\partial_i$ the composed A-derivation

$$R \to \hat{R} \to H \xrightarrow{\partial/\partial f_i} H;$$

and by $[\partial_i] \in H^1(R, H)$, $[\partial_1 \cdots \partial_q] \in H^q(R, H)$ the associated cohomology classes (cf. (1.8.2), with $\mu\colon H \otimes_R H \to H$ given by composition of maps). One can (but need not, for present purposes) show that changing $\sigma$ changes each $\partial_i$ by an *inner* derivation, so that $[\partial_1 \cdots \partial_q]$ is actually independent of the choice of $\sigma$ (cf. (4.2.4) below).

THEOREM (3.6). *With the notations and assumptions of* (3.5), *let*

$$\rho = \rho_H^q : H^q(R, H) \otimes_R H_q(R, R) \to H_0(R, H)$$

*be as in* (1.1). *Then for any* $\omega \in H_q(R, R)$, *we have, in* $A[[X]]$:

$$\text{(3.6.1)} \qquad \operatorname{Tr}_{\hat{R}/A[[X]]}\, \rho([\partial_1 \cdots \partial_q] \otimes \omega) = \sum_{m_1, \ldots, m_q} \operatorname{Res}\begin{bmatrix} \omega \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} X_1^{m_1 - 1} \cdots X_q^{m_q - 1}$$

*where* $(m_1, \ldots, m_q)$ *runs through all* q-*tuples of positive integers.*

Before proving (3.6), we state some consequences.

COROLLARY (3.7). *For any* $r \in R$, *let* $r^{\#}$ *be its image under the natural composition* $R \to \hat{R} \to H = E[[f]]$. *For* $r_1, \ldots, r_q \in R$, *set*

$$r^{\#}\det\left(\left(\frac{\partial}{\partial f_i}\right)r_j^{\#}\right) = \sum_M \delta_M f^M \in E[[f]]$$

("det" = "determinant", cf. (1.10.3)), *where* $M$ *runs through all* $q$*-tuples of non-negative integers. Then, for any positive integers* $m_1, \ldots, m_q$, *we have*

$$\operatorname{Res}\begin{bmatrix} r\,dr_1 \cdots dr_q \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} = \operatorname{Tr}_{P/A}(\delta_{m_1-1,\ldots,m_q-1}).$$

*In particular, if*

$$r^{\#} = \sum \gamma_M f^M$$

*then*

$$\operatorname{Res}\begin{bmatrix} r\,df_1 \cdots df_q \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} = \operatorname{Tr}_{P/A}(\gamma_{m_1-1,\ldots,m_q-1}).$$

*Proof of* (3.7). As in (1.10.3), we see that $\rho([\partial_1 \cdots \partial_q] \otimes \theta_q(r\,dr_1 \cdots dr_q))$ is the natural image in $H_0(R, H)$ of the map $r^{\#}\det((\frac{\partial}{\partial f_i})r_j^{\#}) \in H$. In view of (3.4.2), the assertion follows from (3.6).

*Remark* (3.7.1). Proposition 2.11 in [H, p. 529] is not always valid. But it is if the A-linear section $\sigma$ preserves multiplication (so that $\hat{R}$ is actually a formal power series ring over P), since then, if $r \in R$ and $\hat{r} = \Sigma\, \sigma(c_M) f^M$ is its image in $\hat{R}$, we have for any $p \in P$ that

$$\hat{r}\sigma(p) = \sum \sigma(c_M p) f^M$$

so that in (3.7), $\gamma_M$ is just "multiplication by $c_M$", and

$$\operatorname{Res}\begin{bmatrix} r\,df_1 \cdots df_q \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} = \operatorname{Tr}_{P/A}(c_{m_1-1,\ldots,m_q-1}).$$

COROLLARY (3.8) (cf. also (1.10.6)).

$$\operatorname{Res}\begin{bmatrix} r\,df_1 \cdots df_q \\ f_1, \ldots, f_q \end{bmatrix} = \operatorname{Tr}_{P/A}(r')$$

*where* $r'$ *is the natural image of* $r \in R$ *in* $P = R/I$.

COROLLARY (3.9). *If any of the integers* $m_i$ *is* $> 1$, *then*

$$\operatorname{Res}\begin{bmatrix} df_1 \cdots df_q \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} = 0.$$

COROLLARY (3.10) ("Trace formula I"). *Let* $K$ *be the total ring of fractions of* $A[[X]]$, *and let*

$$T = \hat{R} \otimes_{A[[X]]} K,$$

*so that* ($\hat{R} \cong P \otimes_A A[[X]]$ *being* $A[[X]]$-*projective) we have*

$$\hat{R} \subset T \subset T' = \text{total ring of fractions of } \hat{R},^{(1)}$$

*and there is a* $K$-*linear trace map*

$$\operatorname{Tr}_{T/K} : R \to K$$

*whose restriction to* $\hat{R}$ *is* $\operatorname{Tr}_{\hat{R}/A[[X]]}$. *Suppose that* $T$ *is an unramified (hence étale)* $K$-*algebra, so that the derivations* $\dfrac{\partial}{\partial X_i}$ *of* $A[[X]]$ *extend to derivations* $D_i : T \to T$ $(1 \le i \le q)$. *Then for all* $r, r_1, \ldots, r_q \in \hat{R}$:

$$\operatorname{Tr}_{T/K}(r.\det(D_i r_j)) = \sum_{m_1,\ldots,m_q} \operatorname{Res}\begin{bmatrix} r\,dr_1 \cdots dr_q \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} X_1^{m_1-1} \cdots X_q^{m_q-1} \in A[[X]]$$

*where* $m_1, \ldots, m_q$ *runs through all* $q$-*tuples of positive integers.*

*Proof.* We may assume that $R = \hat{R}$; and then by (3.6) we need to show that

(3.10.1) $\quad \operatorname{Tr}_{T/K}(r.\det(D_i r_j)) = \operatorname{Tr}_{\hat{R}/A[[X]]}\, \rho([\partial_1 \cdots \partial_q] \otimes \theta_q(r\,dr_1 \cdots dr_q))$

with $\theta_q$ as in (1.10.2).

---

(1) In fact $T = T'$ because {x is regular in T} ⇒{ norm x is regular in K} (as can be seen e.g. by localizing and using [Amer. J. Math. 87 (1965), p. 888, Prop. 6.1]) ⇒{x is a unit in T} (since x divides its norm).

We work with the situation depicted by

$$\begin{array}{ccccc} K & \longrightarrow & T \circlearrowleft^{D_i} & \xrightarrow{\ \mu\ } & \overline{H} = H\otimes_{A[[X]]} K = \mathrm{Hom}_K(T,T) \\ \uparrow & & \uparrow & & \uparrow \\ A[[X]] & \longrightarrow & R & \longrightarrow & H \circlearrowleft^{\partial/\partial t_i} \end{array}$$

$\mu$ being the natural map.

The derivations $\partial_i\colon R \to H$ extend uniquely to derivations $\overline{\partial}_i\colon T \to \overline{H}$; and it is easily checked that

(3.10.2) $$\mathrm{Tr}_{\hat{R}/A[[X]]}\, \rho([\partial_1 \cdots \partial_q] \otimes \theta_q(r\,dr_1 \cdots dr_q))$$

$$= \mathrm{Tr}_{T/K}\, \overline{\rho}([\overline{\partial}_1 \cdots \overline{\partial}_q] \otimes \overline{\theta}_q(r\,dr_1 \cdots dr_q))$$

$$= \mathrm{Tr}_{T/K}(r.\det(\overline{\partial}_i r_j)) \qquad \text{(cf.(1.10.3))}$$

where

$$\overline{\rho}\colon H^q(T, \overline{H}) \otimes_T H_q(T, T) \to H_0(T, \overline{H})$$

is as in (1.1), and

$$\overline{\theta}_q\colon \Omega^q{}_{T/A} \to H_q(T, T)$$

as in (1.10.2). Thus we will be done if the derivations $\mu \circ D_i$ and $\overline{\partial}_i$ differ by an inner derivation. (For, we can then replace $\overline{\partial}_i$ in (3.10.2) by $\mu \circ D_i$, to get (3.10.1).)

Since the restrictions of $\mu \circ D_i$ and $\overline{\partial}_i$ to K coincide, we need only note now that any K-derivation of T into $\overline{H}$ is inner, i.e. that

$$H^1{}_K(T, \mathrm{Hom}_K(T, T)) = 0$$

(where the cohomology is calculated with T regarded as a K-algebra, not as an A-algebra). In fact, since T is a projective $(T \otimes_K T)$-module (because T is unramified over K, cf. [EGA IV, (18.3.1)]) therefore T is an allowable projective resolution of itself, so that by [M, p.261, Thm. 4.3], $\mathbf{B}.(T) \to T$ is a homotopy equivalence (over $T \otimes_K T$), and hence for any $(T \otimes_K T)$-module M and any $q > 0$ we have

$$H^q{}_K(T, M) = 0.$$

* * *

We return now to the *proof of Theorem* (3.6).

Fix a q-tuple $M = (m_1, \dots, m_q)$ of positive integers, and set

$$J = (f_1^{m_1}, \dots, f_q^{m_q})R.$$

Every element in $R/J$ (respectively $R/J^2$) has a *unique* representation as a sum of monomials of the form $\sigma(e)f^L$, where $e \in P$ and where $L = (\ell_1, \dots, \ell_q)$ runs through those finitely many q-tuples such that $f^L \notin J$ (resp. $f^L \notin J^2$). (Here we abuse notation by identifying $f_i$ with its image in $R/J$ or in $R/J^2$; and the product $\sigma(e)f^L$ in $R/J$ is defined via the natural R-algebra structure of $R/J$, and similarly for $R/J^2$.) We define an A-linear section $\tau: R/J \to R/J^2$ of the natural map $R/J^2 \to R/J$ by

$$\tau\Big(\sum_{L<M} \sigma(e_L)f^L\Big) = \sum_{L<M} \sigma(e_L)f^L$$

where $L < M$ means $\ell_i < m_i$ for all $i = 1,2, \dots, q$.

We also define $(R/J)$-linear maps $\alpha_i: J/J^2 \to R/J$ by letting $(\alpha_1, \dots, \alpha_q)$ be the dual basis of the basis $(f_1^{m_1}, \dots, f_q^{m_q})$ of $J/J^2$.

Recalling (1.5.2), (1.8.3), and (3.4.2), we see that it will be enough to show the following:

*If* $r_0, r_1, \dots, r_q \in R$, *and if*

$$r_0\partial_1(r_1)\partial_2(r_2)\cdots\partial_q(r_q) = \sum_N \gamma_N f^N \in E[[f]] = H$$

*then*

(3.6.2) $$\mathrm{Tr}_{P/A}(\gamma_{m_1-1, \dots, m_q-1}) = \mathrm{Tr}_{(R/J)/A}(r_0\alpha_1(r_1\tau - \tau r_1)\alpha_2(r_2\tau - \tau r_2)\cdots\alpha_q(r_q\tau - \tau r_q)).$$

Let us verify (3.6.2). Let $r_i^\#$ be the natural image of $r_i$ in H, say

$$r_i^\# = \sum_N \gamma_{iN} f^N \qquad 0 \le i \le q.$$

Then we have

$$r_0\partial_1(r_1)\partial_2(r_2)\cdots\partial_q(r_q) = \sum_{N_0, \dots, N_q} n_1^1 n_2^2 \cdots n_q^q\, \gamma_{0N_0}\gamma_{1N_1}\cdots\gamma_{qN_q}\, f^{N_0+N_1+\dots+N_q-(1,1,\dots,1)}$$

where

$$N_j = (n_1^j, \dots, n_q^j) \qquad 0 \le j \le q$$

runs through all q-tuples of non-negative integers. Thus

(3.6.3)
$$\gamma_{m_1-1,\ldots,m_q-1} = \sum_{N_0+N_1+\ldots+N_q=M} n_1^1 n_2^2 \cdots n_q^q \gamma_{0N_0} \cdots \gamma_{qN_q}.$$

Next, one checks, for $e \in P$ and $L = (\ell_1, \ldots, \ell_q) < M$, that

$$\alpha_i(r_i\tau - \tau r_i)(\sigma(e)f^L) = \sum_N{}^{(i)} \sigma(\gamma_{iN}(e)) f_1^{n_1+\ell_1} \cdots f_i^{n_i+\ell_i-m_i} \cdots f_q^{n_q+\ell_q}$$

where $\sum_N^{(i)}$ means "sum over those $N = (n_1, \ldots, n_q)$ such that

$$0 \leq n_i + \ell_i - m_i < m_i$$

and

$$n_j + \ell_j < m_j \qquad (j \neq i)".$$

With this in mind, one finds that

$$r_0\alpha_1(r_1\tau - \tau r_1)\alpha_2(r_2\tau - \tau r_2) \cdots \alpha_q(r_q\tau - \tau r_q)(\sigma(e)f^L)$$
$$= \sum_{N_0,\ldots,N_q}^{*} \sigma(\gamma_{0N_0}\gamma_{1N_1} \cdots \gamma_{qN_q}(e)) f^{N_0+N_1+\ldots+N_q+L-M}$$

where $\sum^*$ means "sum over those q-tuples $(N_0, \ldots, N_q)$ such that, with $N_j = (n_1^j, \ldots, n_q^j)$ $(0 \leq j \leq q)$, we have:

(3.6.4)
$$\begin{gathered} \ell_q < m_q \leq n_q^q + \ell_q \\ n_{q-1}^q + \ell_{q-1} < m_{q-1} \leq n_{q-1}^q + n_{q-1}^{q-1} + \ell_{q-1} \\ \vdots \\ n_1^q + n_1^{q-1} + \ldots + n_1^2 + \ell_1 < m_1 \leq n_1^q + n_1^{q-1} + \ldots + n_1^2 + n_1^1 + \ell_1 \end{gathered}$$

and such that

$$N_0 + N_1 + \ldots + N_q + L - M < M".$$

Now the A-module $R/J$ is the direct sum of its submodules

$$P_L = \{\sigma(e)f^L\} \qquad (L < M).$$

$P_L$ is clearly A-isomorphic to $P$, and we see from the foregoing that the contribution of $P_L$ to the right hand side of (3.6.2) is the trace of the A-endomorphism $\gamma$ of $P_L = P$ given by

(3.6.5) $$\gamma(e) = \sum^{*}_{N_0+N_1+\dots+N_q=M} \gamma_{0N_0}\gamma_{1N_1}\cdots\gamma_{qN_q}.$$

Comparing (3.6.5) and (3.6.3), we see that all that remains to be noted is the easily checked fact that *for given* $N_0, N_1, \dots, N_q$ *with* $N_0 + N_1 + \dots + N_q = M$, *the number of distinct* $L < M$ *for which the conditions* (3.6.4) *are satisfied is* $n_1^1 n_2^2 \cdots n_q^q$.

This completes the proof.

## APPENDIX A. RESIDUES ON ALGEBRAIC VARIETIES

(A.1). Suppose that $A$ is a perfect field, and that $R$ is a q-dimensional local domain which is a localization of a finitely generated A-algebra, and whose residue field $R/m$ ($m$ = maximal ideal of $R$) is finite over $A$. In other words, $R$ is A-isomorphic to the local ring of a closed point on a q-dimensional algebraic variety over $A$. In [L, p.97, Theorem 11.2], there is specified a family of A-linear maps, for $A$ fixed and $R$ (as above) variable:

$$\mathrm{res}_R : H^q_m(\Omega_R) \to A$$

where $H^q_m$ denotes local cohomology and $\Omega_R = \Omega^q{}_{R/A}$. According to *loc. cit.*, the family $\mathrm{res}_R$ is uniquely determined by two properties, which can be formulated as follows:

(i) If the local ring $R$ is regular, and $\mathbf{f} = (f_1, \ldots, f_q)$ generates $m$, so that any element $\xi$ of $H^q_m(\Omega_R)$ can be represented as a "generalized fraction"

$$\xi = r\,df_1 \cdots df_q/(f_1^{m_1}, \ldots, f_q^{m_q})$$

for suitable $r \in R$ and positive integers $m_1, \ldots, m_q$ (cf. [L, §7]), and if the natural image $\hat{r}$ of $r$ in the completion $\hat{R} = P[[f_1, \ldots, f_q]]$ ($P \cong R/m$) is given by

$$\hat{r} = \sum_M c_M f^M \qquad (c_M \in P)$$

then

$$\mathrm{res}_R(\xi) = \mathrm{Tr}_{P/A}(c_{m_1-1, \ldots, m_q-1}).$$

In view of (3.7.1), this equation can be rewritten as:

(i)′
$$\mathrm{res}_R(r\,df_1 \cdots df_q/(f_1^{m_1}, \ldots, f_q^{m_q})) = \mathrm{Res}\begin{bmatrix} r\,df_1 \cdots df_q \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} .$$

(ii) If $R$ and $m = \mathbf{f}R$ are as in (i), and $S \supset R$ is a q-dimensional localization of a finite R-algebra (so that $\mathbf{f}$ is a system of parameters in $S$), with $S$ a domain whose fraction field $T$ is separable over the fraction field $K$ of $R$, then, for all $s, s_1, \ldots, s_q$ in $S$, and all $q$-tuples $(m_1, \ldots, m_q)$:

$$\mathrm{res}_S(s\,ds_1 \cdots ds_q/(f_1^{m_1}, \ldots, f_q^{m_q})) = \mathrm{res}_{\hat{R}}(\mathrm{Tr}_{\hat{T}/\hat{K}}\{s \det(D_i s_j)\}df_1 \cdots df_q/(f_1^{m_1}, \ldots, f_q^{m_q}))$$

where $D_i$ is the unique extension to $T$ of the derivation $\dfrac{\partial}{\partial f_i}$ of $K$, $\hat{K}$ is the fraction field of $\hat{R}$, $\hat{T} = T \otimes_K \hat{K}$ is the total ring of fractions of $\hat{S}$, $\mathrm{Tr}_{\hat{T}/\hat{K}}$ is the trace map (so that, as is well-known, $\mathrm{Tr}_{\hat{T}/\hat{K}}(s \det(D_i S_j)) \in \hat{R}$) and

$$\mathrm{res}_{\hat{R}} = \mathrm{res}_R : H^q_m(\Omega_R) = H^q_m(\Omega_R \otimes_R \hat{R}) \to A.$$

In view of (i), we can deduce from (3.10) that *if* $S$ *is Cohen-Macaulay, so that the sequence* $\mathbf{f}$ *is regular in* $S$, *then the preceding equation is equivalent to*

(ii)′
$$\mathrm{res}_S(s\,ds_1 \cdots ds_q/(f_1^{m_1}, \ldots, f_q^{m_q})) = \mathrm{Res}\begin{bmatrix} s\,ds_1 \cdots ds_q \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix}.$$

What is indicated here is that *the residues defined in this paper may be used to give another proof of the existence of the family of maps* $\mathrm{res}_R$. This is what the present Appendix is about.

The point is that this proof will be entirely "intrinsic", not to mention more generally applicable. In contrast [L] uses the following procedure (which makes sense in the context of [L], where emphasis is placed on the connections between local and global duality, but which is otherwise outlandish): $R$ is realized as the local ring of a point $v$ on some proper $q$-dimensional $A$-variety $V$; and then $\mathrm{res}_R$ is defined to be the composition

$$H^q_m(\Omega_R) = H^q_v(\Omega^q_{V,v}) \xrightarrow{\text{natural}} H^q(V, \Omega^q_V) \xrightarrow{\text{via } c} H^q(V, \omega_V) \xrightarrow{\theta} A$$

where

- $\omega_V$ is a dualizing sheaf on $V$,
- $\theta$ is the canonical map, and
- $c : \Omega^q_V \to \omega_V$ is a certain canonical sheaf map (the "fundamental class" of $V$).

Then one must show that $\mathrm{res}_R$ is independent of all choices made, and that (i) and (ii) hold...(1)

* * *

(1) Cf. also the remarks on pages 13, 26, and 63 of [L].

The definition we have in mind is based on the following Lemma, valid for any algebra R over any commutative ring A.

LEMMA (A.2). *Let* $I \subset I'$ *be left ideals in* R *such that* $P = R/I$ *and* $P' = R/I'$ *are both finitely generated projective* A*-modules. Then the following diagram, induced by the natural maps* $R \xrightarrow{\lambda} P \xrightarrow{\mu} P'$, *commutes (the abbreviations* $H^q(\cdot) = H^q(R, \cdot)$, $H_q(\cdot) = H_q(R, \cdot)$, $\otimes = \otimes_{R^e}$, *are used; and* $\mathrm{Res}^q$ *is as in (1.5.1)).*

$$\begin{array}{ccccc}
H^q(\mathrm{Hom}_A(P', R)) \otimes H_q(R) & = & H^q(\mathrm{Hom}_A(P', R)) \otimes H_q(R) & \rightarrow & H^q(\mathrm{Hom}_A(P, R)) \otimes H_q(R) \\
\downarrow & & \downarrow & & \downarrow \\
H^q(\mathrm{Hom}_A(P', P')) \otimes H_q(R) & \leftarrow & H^q(\mathrm{Hom}_A(P', P)) \otimes H_q(R) & \rightarrow & H^q(\mathrm{Hom}_A(P, P)) \otimes H_q(R) \\
& \mathrm{Res}^q \searrow & & \swarrow \mathrm{Res}^q & \\
& & A & &
\end{array}$$

*Proof.* Start with an element

$$\xi \otimes \eta \in H^q(\mathrm{Hom}_A(P', R)) \otimes H_q(R)$$

where $\xi$ is represented by a q-cocycle

$$f \in \mathrm{Hom}_A(T_A^q(R/A), \mathrm{Hom}_A(P', R))$$

and $\eta$ is represented by a q-cycle

$$1 \otimes x \in R \otimes_{R^e} B_q.$$

Referring to (1.5.2), we find that (A.2) comes down to the equality

$$\mathrm{Tr}_{P'/A}(\mu \circ \lambda \circ f(x)) = \mathrm{Tr}_{P/A}(\lambda \circ f(x) \circ \mu),$$

which holds by [B, Ch. II, §4.3, Prop. 3].

COROLLARY (A.2.1) (cf. also [H, p. 523, Cor. 2.3].) *Suppose that there is a sequence of left ideals in* R,

$$R = I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$$

*such that* $R/I_n$ *is finitely generated and projective over* A *for all* n. *Then* $\mathrm{Res}^q$ *induces an* A*-linear map*

$$\varinjlim_n H^q(R, \mathrm{Hom}_A(R/I_n, R)) \otimes H_q(R, R) \to A.$$

EXAMPLE-DEFINITION (A.3). Suppose that A is a field, and that R is a commutative noetherian q-dimensional semi-local A-algebra with Jacobson radical $m$, such that $R/m$ is finite over A. Then by (2.5.2) we have a natural isomorphism

(A.3.1) $$\varinjlim_n H^q(R, \mathrm{Hom}_A(R/m^n, R)) \xrightarrow{\sim} \varinjlim_n \mathrm{Ext}^q_R(R/m^n, R) = H^q_m(R).$$

So we can define

$$\mathrm{res}_R : H^q_m(\Omega_R) \to A$$

to be the composition

$$H^q_m(\Omega_R) = H^q_m(R) \otimes_R \Omega_R \overset{(1.10.2)}{\to} H^q_m(R) \otimes H_q(R, R) \overset{(A.2.1)}{\to} A.$$

COROLLARY (A.3.2). *With* A, R *as in* (A.3), *assume further that* R *is Cohen-Macaulay. For any* $\omega \in \Omega_R = \Omega^q{}_{R/A}$ *and any system of parameters* $\mathbf{f} = (f_1, \ldots, f_q)$ *in* R, *let*

$$\omega/(f_1, \ldots, f_q) \in H^q_m(\Omega_R)$$

*be the natural image of*

$$\begin{array}{ccc} \omega + \mathbf{f}\Omega_R & \in & \Omega_R/\mathbf{f}\Omega_R \\ \| & & \|\wr \\ \left\{\begin{matrix} \omega \\ f_1, \ldots, f_q \end{matrix}\right\} & \in & \mathrm{Ext}^q_R(R/\mathbf{f}R, \Omega_R/\mathbf{f}\Omega_R) \cong \mathrm{Ext}^q_R(R/\mathbf{f}R, \Omega_R) \end{array}$$

(cf. (2.7)). *Then*

$$\mathrm{res}_R(\omega/(f_1, \ldots, f_q)) = \mathrm{Res}\begin{bmatrix} \omega \\ f_1, \ldots, f_q \end{bmatrix}.$$

*Hence, if* A *is a perfect field and* R *is a Cohen-Macaulay local domain essentially of finite type over* A, *then the map* $\mathrm{res}_R$ *of* (A.3) *agrees with the map* $\mathrm{res}_R$ *in* [L].

*(Idea of) proof.* The first assertion follows from (2.7.2),[1] and the second from the discussion in (A.1).

[1] For present purposes we only need the case when R is local.

* * *

Now we look at the general case, when R is not necessarily Cohen-Macaulay, the aim being to show that the map $\mathrm{res}_R$ of (A.3) agrees with that of [L, p. 97], where applicable. We need only show that property (ii) in (A.1) holds for $\mathrm{res}_R$; and we will do this by reduction to the Cohen-Macaulay case.

Let $S \supset R$ be as in (ii), with, say, $S = \overline{R}_p$, where $\overline{R}$ is a domain which is a finite R-algebra, with maximal ideals $p, p_1, \ldots, p_n$. Choose an element x in $p - \bigcup_{i=1}^n p_i$, and choose an element $y \in p \cap p_1 \cap \cdots \cap p_n$ such that $T = K[y]$. Then, R being infinite (we may assume $q > 0$), a standard argument shows that for suitable $r \in R$, we have $T = K[x + ry]$; and clearly

$$x + ry \in p - \textstyle\bigcup_{i=1}^n p_i.$$

It follows that if $S'$ is the localization of $R' = R[x + ry]$ at $p \cap R'$, then $S = \overline{R} \otimes_{R'} S'$ is a finite $S'$-module, both S and $S'$ have the same fraction field T, and $S'$ is Cohen-Macaulay.

Now for some $r \neq 0$ in R, $rsds_1 \cdots ds_q$ lies in the image of the natural map $\Omega_{S'} \to \Omega_S$. I claim that it suffices to prove (ii) with s replaced by rs. Indeed, bearing in mind that, with $n$ the maximal ideal of S, we have

$$H_n^q(\Omega_S) = H_m^q(\Omega_S) = H_m^q(R) \otimes_R \Omega_S,$$

we see that (ii) can be interpreted as asserting the commutativity of a certain diagram of R-linear maps

$$\begin{array}{ccc} \Omega_S & \longrightarrow & \Omega_R \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}_A(H_m^q(R), A) & \end{array}$$

But we have an exact sequence

$$H_m^q(R) \xrightarrow{r} H_m^q(R) \to H_m^q(R/rR) = 0$$

(since $R/rR$ has support of dimension $< q$), and applying the functor $\mathrm{Hom}_A(\cdot, A)$ we conclude that multiplication by r in $\mathrm{Hom}_A(H_m^q(R), A)$ is injective, whence the claim.

Let us assume then that $\nu = sds_1 \cdots ds_q$ lies in the image of $\Omega_{S'} \to \Omega_S$. Since (ii) holds for the pair $S' \supset R$ ($S'$ being Cohen-Macaulay), the following Lemma will complete the proof.

LEMMA (A.4). *Let* A *be a field and let* $S' \subset S$ *be* q-*dimensional noetherian semi-local* A-*algebras. Let* $m$ *be the Jacobson radical of* S′, *and assume that the* A-*algebra* $S/mS$ *is finite-dimensional. Assume further that for some* $s' \in S'$ *with* $\dim(S'/s'S') < q$ *we have* $s'S \subset S'$. *Then the following diagram commutes*

$$\begin{array}{ccc} \Omega_{S'} & \longrightarrow & \mathrm{Hom}_A(H^q_m(S'), A) \\ \downarrow & & \uparrow \\ \Omega_S & \longrightarrow & \mathrm{Hom}_A(H^q_m(S), A) \end{array}$$

*where the vertical arrows represent natural maps, and the horizontal arrows represent the maps corresponding to the maps* $\mathrm{res}_{S'}$ *and* $\mathrm{res}_S$ *of (A.3).*

*Proof.* Let $\nu' \in \Omega_{S'}$, with image $\nu \in \Omega_S$; and let $\eta' \in H^q_m(S')$, with image $\eta \in H^q_m(S)$. As in the preceding argument, multiplication by $s'$ in $H^q_m(S')$ is surjective, so it will be enough to show that

(A.4.1) $$\mathrm{res}_{S'}(s'\eta' \otimes \nu') = \mathrm{res}_S(s'\eta \otimes \nu).$$

Note that since multiplication by $s'$ is an S′-linear map of S into S′, we have a commutative diagram

(A.4.2) $$\begin{array}{ccc} H^q_m(S') & \xrightarrow{\text{natural}} & H^q_m(S) \\ & {\scriptstyle s'}\searrow \quad \swarrow{\scriptstyle s'} & \\ & H^q_m(S') & \end{array}$$

For any $n > 0$, set

$$S'_n = S'/m^n, \qquad S_n = S/(m^n)S.$$

We can choose, for sufficiently large n,

$$\eta'_n \in H^q(S', \mathrm{Hom}_A(S'_n, S'))$$

having natural image $\eta'$, cf. (A.3.1); and similarly (since $H^q_m(S) = H^q_{mS}(S)$)

$$\eta_n \in H^q(S, \mathrm{Hom}_A(S_n, S))$$

having natural image $\eta$.

Let $\alpha$ be the composed map

$$H^q(S, \mathrm{Hom}_A(S_n, S)) \to H^q(S', \mathrm{Hom}_A(S_n, S)) \xrightarrow{s'} H^q(S', \mathrm{Hom}_A(S_n, S')) \to H^q(S', \mathrm{Hom}_A(S'_n, S'))$$

where the unlabelled maps are natural. One checks, using (A.4.2), that the image of $\alpha(\eta_n)$ in $H^q_m(S')$ is $s'\eta = s'\eta'$. Hence, after enlarging n if necessary, we have

$$\alpha(\eta_n) = s'\eta'_n. \tag{A.4.3}$$

Now to prove (A.4.1), let

$$\rho : H^q(S, \mathrm{Hom}_A(S_n, S)) \otimes H_q(S, S) \to H_0(S, \mathrm{Hom}_A(S_n, S))$$

and

$$\rho' : H^q(S', \mathrm{Hom}_A(S'_n, S')) \otimes H_q(S', S') \to H_0(S', \mathrm{Hom}_A(S'_n, S'))$$

be as in (1.1), and let

$$\theta : \Omega_S \to H_q(S, S)$$

$$\theta' : \Omega_{S'} \to H_q(S', S')$$

be as in (1.10.2). Consider the following sequence

$$S'_n \xrightarrow{\iota} S_n \xrightarrow{\rho^*} S \xrightarrow{s'} S' \xrightarrow{\pi'} S'_n \xrightarrow{\iota} S_n$$

where

- $\iota$ and $\pi'$ are the natural maps
- $\rho^* = \rho(\eta_n \otimes \theta(\nu))$.

It is then a straightforward exercise to verify that

$$\begin{aligned}
\mathrm{res}_{S'}(s'\eta' \otimes \nu') &= \mathrm{Tr}_{S'_n/A}(\pi' \circ \rho'(s'\eta'_n \otimes \theta'(\nu'))) \\
&= \mathrm{Tr}_{S'_n/A}(\pi' \circ \rho'(\alpha(\eta_n) \otimes \theta'(\nu'))) && \text{(A.4.3)} \\
&= \mathrm{Tr}_{S'_n/A}(\pi' \circ s' \circ \rho^* \circ \iota) \\
&= \mathrm{Tr}_{S_n/A}(\iota \circ \pi' \circ s' \circ \rho^*) && \text{[B, Ch.II,§4.3, Prop. 3]} \\
&= \mathrm{Tr}_{S_n/A}(\pi \circ s'\rho^*) && \text{(where } \pi : S \to S_n \text{ is the natural map)} \\
&= \mathrm{res}_S(s'\eta \otimes \nu).
\end{aligned}$$

Q.E.D.

## APPENDIX B. EXTERIOR DIFFERENTIATION

Assumptions are as in (3.6).

PROPOSITION (B.1).[1] *Let* $\delta$ *denote exterior differentiation of differential forms. Then for any* $\nu \in \Omega^{q-1}_{R/A}$, *and positive integers* $m_1, \ldots, m_q$, *we have*

$$\operatorname{Res}\begin{bmatrix} \delta\nu \\ f_1^{m_1}, \ldots, f_q^{m_q} \end{bmatrix} = \sum_{k=1}^{q} m_k \operatorname{Res}\begin{bmatrix} df_k \wedge \nu \\ f_1^{m_1}, \ldots, f_k^{m_k+1}, \ldots, f_q^{m_q} \end{bmatrix}.$$

*Proof.* We may replace $R$ by $\hat{R}$ (cf. (3.5.1)) and so assume that all sequences like $(f_1^{m_1}, \ldots, f_q^{m_q})$ (or any of its permutations) are *regular* (cf. (3.2)). Then by (2.8) we have

$$\text{(B.2)}\quad \operatorname{Res}\begin{bmatrix} df_k \wedge \nu \\ f_1^{m_1}, \ldots, f_k^{m_k+1}, \ldots, f_q^{m_q} \end{bmatrix} = (-1)^{(k-1)(q-1)} \operatorname{Res}\begin{bmatrix} df_k \wedge \nu \\ f_k^{m_k+1}, f_{k+1}^{m_{k+1}}, \ldots, f_{k-1}^{m_{k-1}} \end{bmatrix}.$$

(The bottom row on the right is a cyclic permutation of the one on the left.)

Now, for proving (B.1), we may assume that $\nu = r_1\, dr_2 \cdots dr_q$, so that

$$\delta\nu = dr_1\, dr_2 \cdots dr_q, \qquad df_k \wedge \nu = r_1\, df_k\, dr_2 \cdots dr_q.$$

From (3.7), (3.4.2), and (B.2), we can deduce that (B.1) will follow from the identity (with $\mathrm{Tr} = \mathrm{Tr}_{\hat{R}/A[[X]]}$):

[1] For the case of residues on algebraic varieties, cf. [L, pp. 65-66, (7.3.3) and (7.3.4); and p. 99, Remark (iv)]. Cf. also [Bv , p.200, Remarque 1].

(B.3) $$\mathrm{Tr}\{\det((\frac{\partial}{\partial f_i})r_j^{\#})\} = \sum_{k=1}^{q} \mathrm{Tr}\{(\frac{\partial}{\partial f_k})(r_1^{\#}\Delta_k)\}$$

$$= \sum_{k=1}^{q} \mathrm{Tr}\{((\frac{\partial}{\partial f_k})r_1^{\#})\Delta_k\} + \mathrm{Tr}\{r_1^{\#} \sum_{k=1}^{q} (\frac{\partial}{\partial f_k})\Delta_k\}$$

where $\Delta_k$ is the following element of H (to avoid clutter, in the rest to this proof we will denote determinants by vertical bars, write "$D_i$" for "$\frac{\partial}{\partial f_i}$", and "$r_j$" for "$r_j^{\#}$"):

$$\Delta_k = (-1)^{(k-1)(q-1)} \begin{vmatrix} 1 & D_k r_2 & D_k r_q \\ 0 & D_{k+1} r_2 & D_{k+1} r_q \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & D_{k-1} r_2 & D_{k-1} r_q \end{vmatrix}$$

So we will prove (B.1) by showing that

(B.3)′ $$\mathrm{Tr}\{\det(D_i r_j)\} = \sum_{k=1}^{q} \mathrm{Tr}\{(D_k r_1)\Delta_k\}$$

and that

(B.3)″ $$\mathrm{Tr}\{r_1 \sum_{k=1}^{q} D_k \Delta_k\} = 0.$$

For clarity, we illustrate the case $q = 3$ of (B.3)′, leaving the general case to the reader. The right hand side of (B.3)′ is the trace of

(B.4) $$\begin{vmatrix} D_1r_1 & D_1r_2 & D_1r_3 \\ 0 & D_2r_2 & D_2r_3 \\ 0 & D_3r_2 & D_3r_3 \end{vmatrix} + (-1)^2 \begin{vmatrix} D_2r_1 & D_2r_2 & D_2r_3 \\ 0 & D_3r_2 & D_3r_3 \\ 0 & D_1r_2 & D_1r_3 \end{vmatrix} + (-1)^4 \begin{vmatrix} D_3r_1 & D_3r_2 & D_3r_3 \\ 0 & D_1r_2 & D_1r_3 \\ 0 & D_2r_2 & D_2r_3 \end{vmatrix}.$$

Since $\mathrm{Tr}(\alpha\beta\gamma) = \mathrm{Tr}(\gamma\alpha\beta)$ for $\alpha, \beta, \gamma \in H$, we can replace (B.4) by

$$\begin{vmatrix} D_1r_1 & D_1r_2 & D_1r_3 \\ 0 & D_2r_2 & D_2r_3 \\ 0 & D_3r_2 & D_3r_3 \end{vmatrix} + \begin{vmatrix} 0 & D_1r_2 & D_1r_3 \\ D_2r_1 & D_2r_2 & D_2r_3 \\ 0 & D_3r_2 & D_3r_3 \end{vmatrix} + \begin{vmatrix} 0 & D_1r_2 & D_1r_3 \\ 0 & D_2r_2 & D_2r_3 \\ D_3r_1 & D_3r_2 & D_3r_3 \end{vmatrix}.$$

This last sum is nothing but $\det(D_ir_j)$.

For proving (B.3)″, consider the (q–1)-cochain

$$\mathscr{D} = \sum_{k=1}^{q} (-1)^{(k-1)(q-1)} D_k \circ (D_{k+1} \otimes D_{k+2} \otimes \cdots \otimes D_{k-1}) \in \mathrm{Hom}_A(T_A^{q-1}(H/A), H)$$

(cf. (1.8)). We will show below that

(B.5) *$\mathscr{D}$ is a (q–1)-coboundary.*

Thus $\mathscr{D}$ is a (q–1)-cocycle whose cohomology class $[\mathscr{D}] \in H^{q-1}(H, H)$ vanishes. Hence if $\eta \in H_{q-1}(H, H)$ is the image of $r_1dr_2 \cdots dr_q$ under the composed map

$$\Omega^{q-1}{}_{R/A} \xrightarrow{(1.10.1)} H_{q-1}(R, R) \xrightarrow{\text{natural}} H_{q-1}(H, H),$$

and if

$$\rho\colon H^{q-1}(H, H) \otimes_{H^e} H_{q-1}(H, H) \to H_0(H, H)$$

is as in (1.1), then

$$\rho([\mathscr{D}] \otimes \eta) = 0.$$

But $\eta$ is the homology class of the cycle

$$r_1 \otimes \sum_{\tau} (-1)^{|\tau|} [r_{\tau(2)} | \cdots | r_{\tau(q)}] \in H \otimes_A T_A^{q-1}(H/A) = H \otimes_{H^e} B_{q-1}(H)$$

where $\tau$ runs through all permutations of $(2,3,\dots, q)$ (cf. (1.10.2)). So $0 = \rho([\mathscr{D}] \otimes \eta)$ is the natural image in $H_0(H, H) = H/\{h_1h_2 - h_2h_1\}$ (cf. (1.0.1)) of the element

$$r_1 \sum_{k=1}^{q} (-1)^{(k-1)(q-1)} D_k \sum_{\tau} (-1)^{|\tau|} (D_{k+1}r_{\tau(2)} \circ D_{k+2}r_{\tau(3)} \circ \cdots \circ D_{k-1}r_{\tau(q)})$$

$$= r_1 \sum_{k=1}^{q} D_k \Delta_k \in H.$$

Since the trace map $\mathrm{Tr}\colon H \to A[[X]]$ annihilates any element of the form $h_1h_2 - h_2h_1$, therefore it factors through $H_0(H, H)$, and we see then that (B.3)″ holds.

It remains to prove (B.5).

For convenience, set $S = A[[X]]$. Since $E = \mathrm{Hom}_A(P, P)$ is a finitely generated projective A-module, the natural map

$$E \otimes_A S = E \otimes_A A[[X]] \to H = E[[f]]$$

is bijective (cf. (3.3.2)). It follows that there is an A-linear homomorphism of complexes

$$\mathrm{Hom}_{S^e}(\mathbf{B}.(S), S) \to \mathrm{Hom}_{H^e}(\mathbf{B}.(H), H)$$

taking any p-cochain $\gamma: B_p(S) \to S$ to the unique p-cochain $\gamma': B_p(H) \to H$ satisfying (for $e_1, \ldots, e_p \in E \subset H$; $s_1, \ldots, s_p \in S$; and where, for any $s \in S$, $s'$ is its natural image in H):

$$\gamma'[e_1s_1' \mid e_2s_2' \mid \ldots \mid e_ps_p'] = e_1e_2 \cdots e_p(\gamma[s_1 \mid s_2 \mid \ldots \mid s_p])'.$$

We see then that $\mathscr{D} = \mathscr{D}_0'$, where $\mathscr{D}_0: B_{q-1}(S) \to S$ is given by the same formula as $\mathscr{D}$. Thus *it will suffice to show that* $\mathscr{D}_0$ *is a coboundary.*

(In other words we have reduced to the case when $H = A[[X]]$.)

Let us first prove the corresponding statement for the polynomial ring $S^* = A[X]$. In this case, both $\mathbf{B}.(S^*)$ and the Koszul complex $\mathbf{K}. = \mathbf{K}.(X_i \otimes 1 - 1 \otimes X_i)$ over $S^{*e} = S^* \otimes_A S^*$ determined by the sequence $(X_i \otimes 1 - 1 \otimes X_i)_{1 \le i \le q}$ are $S^{*e}$-projective resolutions of $S^*$, so that the $S^{*e}$-linear map

$$\Lambda: \mathbf{K}.(X_i \otimes 1 - 1 \otimes X_i) \to \mathbf{B}.(S^*)$$

described in the proof of (2.6) is a homotopy equivalence. But it is immediate from the definitions that

$$\mathscr{D}_0 \circ \Lambda = 0,$$

and hence the homology class

$$[\mathscr{D}_0] \in H^{q-1}(\mathrm{Hom}_{S^{*e}}(\mathbf{B}.(S^*), S^*)) = H^{q-1}(S^*, S^*)$$

vanishes, as desired.

We can say more. The complex $\mathbf{B}.(S^*)$ is a graded module over the graded (polynomial) ring $S^{*e}$, with

$$\mathrm{degree}(f_0[f_1 \mid \ldots \mid f_n]f_{n+1}) = \sum_{i=0}^{n+1} \mathrm{degree}(f_i)$$

for homogeneous polynomials $f_0, f_1, \ldots, f_{n+1}$ in $S^*$. Also the Koszul complex $\mathbf{K}.$ is a graded $S^{*e}$-algebra, the basis elements in the free module $K_1 = (S^{*e})^q$ having degree one. The boundary maps in both $\mathbf{K}.$ and $\mathbf{B}.(S^*)$ are homogeneous, of degree zero, as is the above map $\Lambda$. It is easily seen that a homotopy inverse $\Phi\colon \mathbf{B}.(S^*) \to \mathbf{K}.$ for $\Lambda$ (i.e. a lifting of the identity map of $S^*$) can also be chosen to be homogeneous of degree zero, as can a sequence of $S^{*e}$-linear maps $\phi_n\colon B_n(S^*) \to B_{n+1}(S^*)$ $(n \geq 0)$ such that

$$\partial_{n+1}\phi_n + \phi_{n-1}\,\partial_n = (1 - \Lambda\Phi)_n.$$

Finally, $\mathscr{D}_0$ is homogeneous, of degree $-q$; and since $\mathscr{D}_0$ is a cocycle, with $\mathscr{D}_0 \circ \Lambda_{q-1} = 0$, we have

$$\mathscr{D}_0 = \mathscr{D}_0(1 - \Lambda\Phi)_{q-1} = \mathscr{D}_0(\partial_q\phi_{q-1} + \phi_{q-2}\partial_{q-1}) = \mathscr{D}_0^{\sim}\partial_{q-1}$$

where

$$\mathscr{D}_0^{\sim} = \mathscr{D}_0\phi_{q-2}\colon B_{q-2}(S^*) \to S^*$$

is *homogeneous*, of degree $-q$.

Returning now to the power series ring $S = A[[\mathbf{X}]]$, with its $\mathbf{X}$-adic topology, we consider on each

$$B_n(S) = S \otimes_A S \otimes \cdots \otimes_A S \qquad (n + 1 \text{ times})$$

the topology for which a fundamental system of neighborhoods of 0 is given by the kernels of the natural maps

$$B_n(S) \to B_n(S/(\mathbf{X}S)^m) \qquad (m > 0).$$

For this topology, $B_{q-2}(S^*)$ is dense in $B_{q-2}(S)$, and $\mathscr{D}_0^{\sim}$, being homogeneous, is uniformly continuous. Hence $\mathscr{D}_0^{\sim}$ extends to a continuous map

$$\mathscr{D}_0^{\approx}\colon B_{q-2}(S) \to S$$

and we have (over S):

$$\mathscr{D}_0 = \mathscr{D}_0^{\approx}\,\partial_{q-1}.$$

(Both sides are continuous, and agree on the dense subset $B_{q-1}(S^*)$ of $B_{q-1}(S)$, hence are equal.) Thus $\mathscr{D}_0$ is a $(q-1)$-coboundary, as asserted in (B.5).

## § 4. TRACE AND COTRACE

The principal result in this section is the "Trace Formula II" given in §4.7, which asserts that for a finite projective R-algebra $R'$, $\xi \in H^q(R, \mathrm{Hom}_A(P, P))$ (P an R-module which is finite and projective over A), and $\omega' \in H_q(R', R')$, we have

$$\mathrm{Res}^q(\gamma^q(\xi) \otimes \omega') = \mathrm{Res}^q(\xi \otimes t_q'(\omega')).$$

Here

$$t_q' \colon H_q(R', R') \to H_q(R, R)$$

is a natural "trace" map, and

$$\gamma^q \colon H^q(R, \mathrm{Hom}_A(P, P)) \to H^q(R', \mathrm{Hom}_A(P', P'))$$

$(P' = R' \otimes_R P)$ is a natural "cotrace" map.

Most of the section is taken up with defining the trace and cotrace maps, giving examples, and developing the properties needed to prove the Trace Formula (4.7.1) and its corollaries (4.7.2) and (4.7.3). The definitions are based on a canonical (up to homotopy) $H^e$-linear map of complexes $(H = \mathrm{Hom}_R(R', R'))$:

$$\mathbf{B}.(H) \to \mathrm{Hom}_R(R', R' \otimes_R \mathbf{B}.(R))$$

described in (4.1).

As mentioned in the Introduction, the trace and cotrace maps defined here should be of interest in other contexts. In (4.6), for instance, we take a side trip to view several illustrations of the connection between the trace map and previously known trace maps for differential forms.

(4.1). Let $g \colon A \to R$ be a homomorphism of commutative rings. We will think of R-modules as being *right* R-modules. An R-module isomorphic to one of the form $N \otimes_A R$ (N an A-module) is said to be g-*free* ; and a direct summand of a g-free module is said to be g-*projective.* For example any R-free module is g-free, and any

R-projective module is g-projective.

Let F be a g-projective R-module, and let $H = \mathrm{Hom}_R(F, F)$, which is naturally an R-algebra. Then H is also an A-algebra via the composition

$$g_H: A \xrightarrow{g} R \xrightarrow{\text{natural}} H.$$

We will denote the bimodule bar resolutions $\mathbf{B}.(g)$, $\mathbf{B}.(g_H)$ (cf. (1.0)) by $\mathbf{B}.(R)$, $\mathbf{B}.(H)$ respectively.

For any R-R bimodule M, we will consider $F \otimes_R M$ to be an R-module via the right R-module structure on M. Then $\mathrm{Hom}_R(F, F \otimes_R M)$ is an H-H bimodule, with

$$(h_1 \phi\, h_2) = (h_1 \otimes 1) \circ \phi \circ h_2 \qquad\qquad h_1, h_2 \in H;\ \phi \in \mathrm{Hom}_R(F, F \otimes_R M).$$

A basic role in this section is played by natural R-linear cotrace maps

$$c^q: H^q(R, M) \to H^q(H, \mathrm{Hom}_R(F, F \otimes_R M)) \qquad (q \geq 0)$$

defined as follows.

Since the complex $\mathbf{B}.(R) \to R$ has a right R-linear splitting $(s_n)_{n \geq -1}$ (cf. (1.0)), the same is true, for any A-module N, of the complex

$$N \otimes_A \mathbf{B}.(R) = (N \otimes_A R) \otimes_R \mathbf{B}.(R) \to N \otimes_A R,$$

and we deduce, for F a direct summand of $N \otimes_A R$, that the complex

$$F \otimes_R \mathbf{B}.(R) \to F$$

has a right R-linear splitting. (Restrict the splitting of the complex $(N \otimes_A R) \otimes_R \mathbf{B}.(R)$ to its direct summand $F \otimes_R \mathbf{B}.(R)$, then project back down to the direct summand.) It follows that the complex of $H^e$ $(= H \otimes_A H^{op})$-modules

$$\mathrm{Hom}_R(F, F \otimes_R \mathbf{B}.(R)) \to \mathrm{Hom}_R(F, F) = H$$

has a right H-linear splitting, say $(\sigma_n)_{n \geq -1}$ ; and we may assume that $\sigma_n \sigma_{n-1} = 0$ for all $n \geq 0$ [**M**, p.264, Thm. 5.2]. Since $\mathbf{B}.(H) \to H$ is an $(A \to H^e)$-free resolution of H, we conclude, by [**M**, p.261, Thm. 4.3; and p.265, Corollary 5.3] that *there is a homotopy unique* $H^e$*-linear map of complexes*

(4.1.1) $$C_F: \mathbf{B}.(H) \to \mathrm{Hom}_R(F, F \otimes_R \mathbf{B}.(R))$$

*lifting the identity map of* H.

*For example* we could take for $C_F$ the "canonical comparison" of [**M**, p.267, Thm. 6.2], which can be seen (since all $\sigma_n$ are right H-linear) to satisfy

(4.1.2) $$C_F(h[h_1 \mid h_2 \mid \dots \mid h_q]h') = h\, \sigma_{q-1} h_1 \sigma_{q-2} \cdots h_q \sigma_{-1}(h').$$

Thus, if $(s'_n)_{n\geq -1}$ is a right R-linear splitting of $F\otimes_R \mathbf{B}.(R)\to F$, with $s'_n s'_{n-1}=0$ for all $n\geq 0$, then $C_F(h[h_1\mid h_2\mid \dots\mid h_q]h')$ is the composed map

(4.1.3) $$F \xrightarrow[h']{} F \xrightarrow[s'_{-1}]{} F\otimes B_0 \xrightarrow[h_q\otimes 1]{} F\otimes B_0 \xrightarrow[s'_0]{} F\otimes B_1 \to$$

$$\cdots \xrightarrow[h_1\otimes 1]{} F\otimes B_{q-1} \xrightarrow[s'_{q-1}]{} F\otimes B_q \xrightarrow[h\otimes 1]{} F\otimes B_q .$$

Now any

$$\xi \in H^q(R,M) = H^q(\mathrm{Hom}_{R^e}(\mathbf{B}.(R),M))$$

can be identified with a homotopy class of $R^e$-linear maps of complexes $\mathbf{B}.(R)\to M[q]$, where $M[q]$ is the complex which is $M$ in degree $-q$[1] and $0$ elsewhere. So $\xi$ induces a homotopy class of $H^e$-linear maps

$$\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)) \to \mathrm{Hom}_R(F, F\otimes_R M)[q]$$

which, composed with $C_F$, gives us a homotopy class of $H^e$-linear maps

$$\mathbf{B}.(H) \to \mathrm{Hom}_R(F, F\otimes_R M)[q],$$

i.e. an element

$$c^q(\xi) \in H^q(H, \mathrm{Hom}_R(F, F\otimes_R M)).$$

This, then, is how the cotrace $c^q$ is defined.

*Exercise* (4.1.4). If $\xi\in H^0(R,M)\subset M$, then

$$c^0(\xi)\in H^0(H,\mathrm{Hom}_R(F,F\otimes_R M)) = \mathrm{Hom}_H(F, F\otimes_R M)$$

is given by

$$[c^0(\xi)](f) = f\otimes\xi .$$

EXAMPLE (4.2). Suppose that F is h-free, i.e. that there exists an R-isomorphism $\psi: F \xrightarrow{\sim} N\otimes_A R$ for some A-module N. After identifying F with $N\otimes_A R$ via $\psi$, we see that in (4.1.3) we can take

$$\begin{array}{cccc} s'_n: & F\otimes_R B_n & \longrightarrow & F\otimes_R B_{n+1} \\ & \| & & \| \\ & N\otimes_A B_n & & N\otimes_A B_{n+1} \end{array}$$

to be the map given by

[1] We consider $B_q$ to be the component of $\mathbf{B}.$ of degree $-q$ (not $q$).

$$s'_{-1}(\nu \otimes r') = \nu \otimes [\ ]r' \in N \otimes_A B_0$$

$$s'_n(\nu \otimes r[r_1 \mid \dots \mid r_n]r') = \nu \otimes [r \mid r_1 \mid \dots \mid r_n]r' \qquad (n \geq 0).$$

*More explicitly* (and even less canonically) *we can describe* $C_F$ *as follows*.

Fix a family $(\nu_i)_{i \in L}$ of generators of N. For any

$$h \in H = \mathrm{Hom}_R(F, F) = \mathrm{Hom}_A(N, N \otimes_A R)$$

we can set

(4.2.1)
$$h(\nu_j) = \sum_{i \in L} \nu_i \otimes r_{ij}^h \qquad (j \in L)$$

where $r_{ij}^h \in R$ vanishes for all but finitely many i. In this way we associate a matrix

$$\mu^h = (r_{ij}^h) \qquad (i,j \in L)$$

to h. (Of course, $\mu^h$ depends on many choices – the isomorphism $\psi$, the generating family $(\nu_i)$, and coefficients $r_{ij}^h$ making (4.2.1) hold!). Similarly, if M is any R-R bimodule, then to any map in

$$\mathrm{Hom}_R(F, F \otimes_R M) = \mathrm{Hom}_A(N, N \otimes_A M)$$

we can associate a matrix with coefficients in M.

Now a simple calculation shows that a matrix associated with the map

$$(h \otimes 1) \circ s'_{q-1} \circ (h_1 \otimes 1) \circ \cdots \circ (h_q \otimes 1) \circ s'_{-1} \circ h' \in \mathrm{Hom}_R(F, F \otimes_R B_q)$$

($s'$ as above) is

(4.2.2)
$$\mu^h \otimes \tilde{\mu}^{h_1} \otimes \cdots \otimes \tilde{\mu}^{h_q} \otimes \mu^{h'}$$

where "$\tilde{\mu}$" means "replace each entry in the matrix $\mu$ by its natural image in R/A, the cokernel of $g: A \to R$", and where, for example, the "tensor product" of two matrices $\mu = (r_{ij})$, $\tilde{\mu}' = (\tilde{r}_{ij}')$ (with $r_{ij} \in R$, respectively $\tilde{r}_{ij}' \in R/A$, vanishing for all but finitely many $i \in L$) is given by

$$\mu \otimes \tilde{\mu}' = (s_{ij})$$

with

$$s_{ij} = \sum_{\ell \in L} r_{i\ell} \otimes \tilde{r}_{\ell j}' \in R \otimes (R/A).$$

Thus we find that:

(4.2.3). *For any* q-*cocycle* $\phi: B_q(R) \to M$ *representing an element* $\xi \in H^q(R, M)$, *a* *cocycle*

$$\Phi: B_q(H) \to \mathrm{Hom}_R(F, F \otimes_R M)$$

*representing* $c^q(\xi)$ *can be specified by*

$$\Phi(h[h_1 \mid \dots \mid h_q]h') = \begin{cases} \text{the map } F \to F \otimes_R M \text{ given by the matrix} \\ \text{(with entries in M) obtained from the tensor product matrix} \\ \text{(4.2.2) by applying } \phi \text{ to each of its entries.} \end{cases}$$

*Exercise* (4.2.4). As in (3.5) (and cf. (3.3.2)) let

$$H = \mathrm{Hom}_{A[[X]]}(P \otimes_A A[[X]], P \otimes_A A[[X]]) = E[[f]]$$

so that we have, as above, the cotrace

$$c^1: H^1(A[[X]], A[[X]]) \to H^1(H, H).$$

Show that, with notation as in (1.8.2),

$$c^1[\partial/\partial X_i] = [\partial/\partial f_i].$$

(Thus, while the derivation $\partial/\partial f_i$ depends on the choice of a section $\sigma: P \to \hat{R}$ as in (3.5), the cohomology class $[\partial/\partial f_i]$ doesn't).

EXAMPLE (4.3). Let $g: A \to R$ be, as before, a homomorphism of commutative rings, and let $R'$ be an $R$-algebra which is $g$-projective as an $R$-module. Let $P$ be an $R$-module and let $P' = R' \otimes_R P$, so that $P'$ is a left $R'$-module, and there is a natural $R$-homomorphism $P \to P'$ taking $p \in P$ to $1 \otimes p \in P'$. Set

$$H = \mathrm{Hom}_R(R', R').$$

We consider $R' \otimes_R \mathrm{Hom}_A(P, P)$ to be a right $R$-module, with

$$(r' \otimes \phi)r = r' \otimes \phi r \qquad r' \in R, \ \phi \in \mathrm{Hom}_A(P, P), \ r \in R.$$

This is consistent with what was done in (4.1), with $F = R'$, $M = \mathrm{Hom}_A(P, P)$. We also consider $\mathrm{Hom}_A(P, R' \otimes_R P)$ to be a right $R$-module with

$$(\psi r)(p) = \psi(rp) \qquad \psi \in \mathrm{Hom}_A(P, R' \otimes_R P), r \in R, p \in P.$$

Then there is a natural right-$R$-linear map

$$R' \otimes_R \mathrm{Hom}_A(P, P) \to \mathrm{Hom}_A(P, R' \otimes_R P)$$

and hence natural $H$-$H$ bimodule homomorphisms

$$(4.3.1) \qquad \mathrm{Hom}_R(R', R'\otimes_R \mathrm{Hom}_A(P,P)) \to \mathrm{Hom}_R(R', \mathrm{Hom}_A(P, R'\otimes_R P))$$

$$\xrightarrow{\sim} \mathrm{Hom}_A(R'\otimes_R P, R'\otimes_R P)$$

$$= \mathrm{Hom}_A(P', P').$$

Combining (4.3.1) with the map $c^q$ of (4.1), we obtain a composed map

$$(4.3.2) \qquad \gamma^q\colon H^q(R, \mathrm{Hom}_A(P,P)) \xrightarrow{c^q} H^q(H, \mathrm{Hom}_R(R', R'\otimes_R \mathrm{Hom}_A(P,P)))$$

$$\to H^q(H, \mathrm{Hom}_A(P', P') \to H^q(R', \mathrm{Hom}_A(P', P')$$

where the last map is induced by the A-algebra homomorphism $R' \to H$ taking $r' \in R'$ to "left multiplication by $r'$".

Proposition (4.3.3). *With preceding notation, the following diagram commutes:*

$$\begin{array}{ccc} H^q(R, \mathrm{Hom}_A(P,P) & \xrightarrow{\ \gamma^q\ } & H^q(R', \mathrm{Hom}_A(P', P')) \\ \Big\downarrow w & & \Big\downarrow u \\ H^q(R, \mathrm{Hom}_A(P, P')) & \xleftarrow{\ v\ } & H^q(R, \mathrm{Hom}_A(P', P')) \end{array}$$

*where* $u$ *is naturally induced by* $R \to R'$, *and* $v, w$ *are naturally induced by* $P \to P'$ (*cf. (2.1.1) with* $A = A'$).

Before giving the proof, we note the following interpretation in the case $q = 1$:

Corollary (4.3.4). *Assume that* $P = R/I$ *for some ideal* $I$ *in* $R$, *so that* $P' = R'/I'$ *with* $I' = I R' = R' I$. *Define the injective maps*

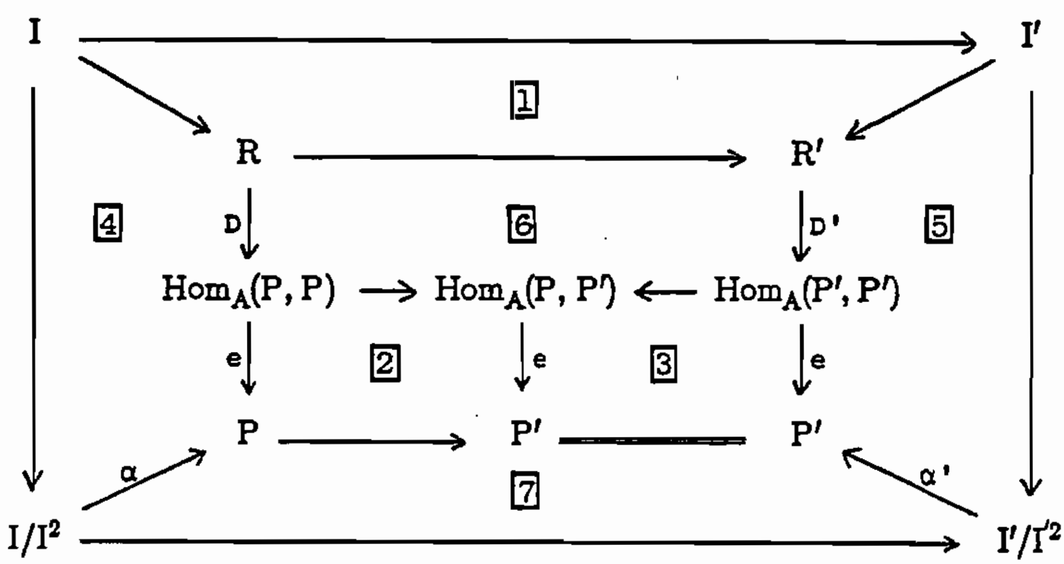
$$\bar{\psi}\colon H^1(R, \mathrm{Hom}_A(P,P)) \to \mathrm{Hom}_P(I/I^2, P)$$

$$\bar{\psi}'\colon H^1(R', \mathrm{Hom}_A(P', P')) \to \mathrm{Hom}_{P'}(I'/I'^2, P')$$

*as in (1.4). Then, for any* $\xi \in H^1(R, \mathrm{Hom}_A(P,P))$, *if* $\alpha = \bar{\psi}(\xi)$, *and* $\alpha' = \bar{\psi}'(\gamma^1(\xi))$ ($\gamma^1$ *as above), then* $\alpha'$ *is the unique* $P'$*-linear map such that the following diagram (with horizontal arrows representing obvious maps) commutes:*

$$\begin{array}{ccc} I/I^2 & \longrightarrow & I'/I'^2 \\ \alpha\downarrow & & \downarrow\alpha' \\ P & \longrightarrow & P'. \end{array}$$

*Proof of* (4.3.4). Let $\xi$ (respectively $\gamma^1(\xi)$) be the cohomology class of the derivation $D\colon R \to \mathrm{Hom}_A(P,P)$ (respectively $D'\colon R' \to \mathrm{Hom}_A(P',P')$), cf. (1.3.3). In the following diagram (where unlabelled arrows represent obvious maps, and "e" means "evaluate at 1", the subdiagrams [1], [2], [3], obviously commute, [4] and [5] commute by the very definition of $\overline{\psi}, \overline{\psi}'$, and (4.3.3) states that [6] commutes modulo inner derivations:




Since any inner derivation $R \to \mathrm{Hom}_A(P,P')$ vanishes on $I$, and since $I \to I/I^2$ is surjective, it follows easily that [7] commutes. Q.E.D.

*Proof of* (4.3.3). We begin with some preliminary remarks. For any R-R bimodule $M$, $\mathrm{Hom}_R(R', R'\otimes_R M)$ is, as in (4.1), an H-H bimodule, and hence (via $R \to H$) an R-R bimodule. Also $\mathrm{Hom}_A(R', R'\otimes_R M)$ is an R-R bimodule with

$$(r_1\phi r_2)r' = r_1\phi(r')r_2 \qquad\qquad r_1, r_2 \in R,\ \phi \in \mathrm{Hom}_A(\cdot\,,\cdot),\ r' \in R;$$

and the inclusion

$$\mathrm{Hom}_R(R', R'\otimes_R M) \subset \mathrm{Hom}_A(R', R'\otimes_R M)$$

is a homomorphism of R-R bimodules, i.e. it is $R^e$-linear.

We define an $R^e$-linear map of complexes

$$\theta: \mathbf{B}.(R) \to \mathrm{Hom}_A(R', R' \otimes_R \mathbf{B}.(R))$$

by

$$[\theta(x)](\rho) = \rho \otimes x \qquad\qquad x \in \mathbf{B}.(R),\ \rho \in R'.$$

As in (4.1), $R'$ being g-projective, the complex $R' \otimes_R \mathbf{B}.(R) \to R'$ has an A-linear splitting, whence so does the complex of $R^e$-modules

$$\mathrm{Hom}_A(R', R' \otimes_R \mathbf{B}.(R)) \to \mathrm{Hom}_A(R', R').$$

The $R^e$-linear map $\theta$ lifts the map $R \to \mathrm{Hom}_A(R', R')$ taking $r \in R$ to "multiplication by r". The same is true of the composed $R^e$-linear map

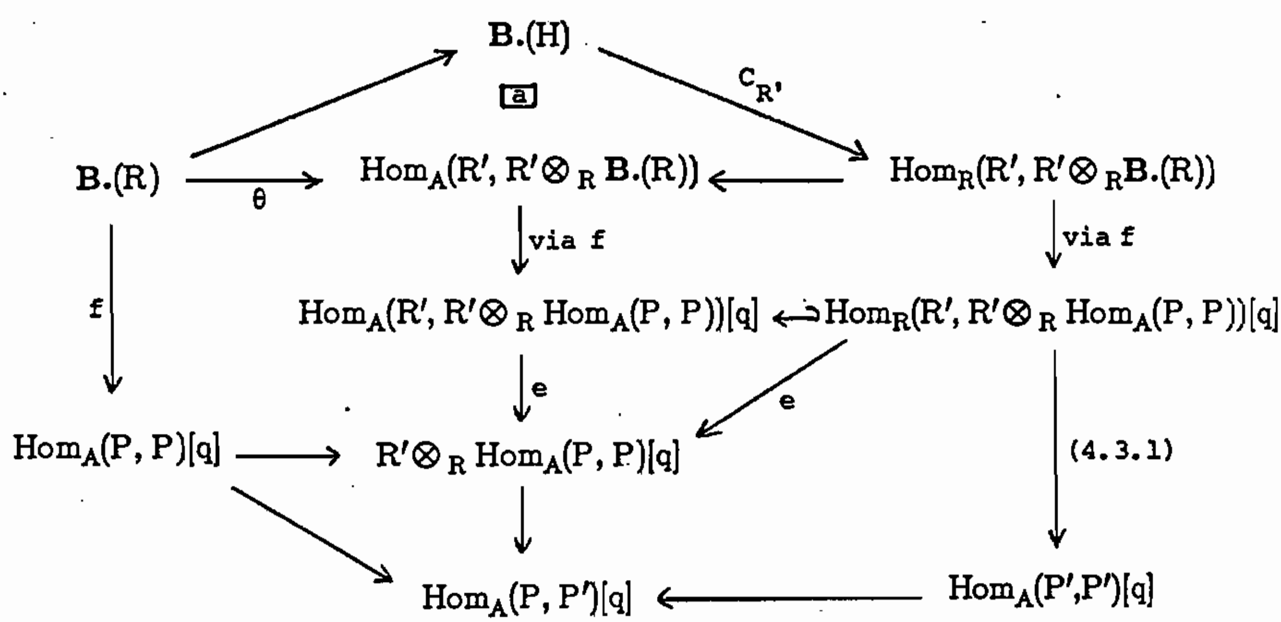
$$\theta_1: \mathbf{B}.(R) \xrightarrow{\text{natural}} \mathbf{B}.(H) \underset{(4.1.1)}{\xrightarrow{C_{R'}}} \mathrm{Hom}_R(R', R' \otimes_R \mathbf{B}.(R)) \to \mathrm{Hom}_A(R', R' \otimes_R \mathbf{B}.(R)).$$

Hence, by [M, p.261, Theorem (4.3)], $\theta$ and $\theta_1$ are homotopic.

Now let $\xi \in H^q(R, \mathrm{Hom}_A(P, P))$ be represented by a q-cocycle $f: B_q(R) \to \mathrm{Hom}_A(P, P)$, i.e. by a map (still denoted by f) of complexes

$$f: \mathbf{B}.(R) \to \mathrm{Hom}_A(P, P)[q]$$

(cf. remarks following (4.1.3)). Consider the following diagram of $R^e$-linear maps of complexes (where unlabelled arrows represent obvious natural maps, and "e" stands for "evaluation at 1"):




We are trying to show that $vu\gamma^q(\xi) = w(\xi)$ (cf. (4.3.3)), which means in other words that the two maps obtained by going from $\mathbf{B}.(R)$ to $\mathrm{Hom}_A(P, P')[q]$ around the outer border of the diagram, in the clockwise and counterclockwise directions respectively, are homotopic. We have already noted however that the subdiagram [a] is homotopy-commutative; and it is simple to check that all the other subdiagrams are commutative. The conclusion follows.

(4.4) We show next that *the cotrace maps* $c^q$ *of (4.1) "respect products".*

Let $g\colon A \to R$, $F$, and $H = \mathrm{Hom}_R(F,F)$ be as in (4.1). Let $M$ and $N$ be R-R bimodules, so that we have cohomology products (cf. (1.8)):

(4.4.1) $$H^p(R,M)\otimes_R H^q(R,N) \to H^{p+q}(R, M\otimes_R N).$$

Similarly we have cohomology products (with $H^c$ the center of H):

(4.4.2) $$H^p(H, \mathrm{Hom}_R(F, F\otimes_R M))\otimes_{H^c} H^q(H, \mathrm{Hom}_R(F, F\otimes_R N))$$
$$\to H^{p+q}(H, \mathrm{Hom}_R(F, F\otimes_R M)\otimes_H \mathrm{Hom}_R(F, F\otimes_R N)).$$

There is a unique $H^e$ $(= H\otimes H^{op})$-linear map

(4.4.3) $$\lambda\colon \mathrm{Hom}_R(F, F\otimes_R M)\otimes_H \mathrm{Hom}_R(F, F\otimes_R N) \to \mathrm{Hom}_R(F, F\otimes_R M\otimes_R N)$$

such that $\lambda(\phi\otimes\psi)$ is the composed map

$$F \xrightarrow{\psi} F\otimes_R N \xrightarrow{\phi\otimes 1} (F\otimes_R M)\otimes_R N.$$

Applying $\lambda$ to (4.4.2), and recalling that $H^c$ is an R-algebra, we obtain the products

(4.4.2)′ $$H^p(H, \mathrm{Hom}_R(F, F\otimes_R M))\otimes_R H^q(H, \mathrm{Hom}_R(F, F\otimes_R N))$$
$$\to H^{p+q}(H, \mathrm{Hom}_R(F, F\otimes_R M\otimes_R N)).$$

Proposition (4.4.4). *Denoting both of the preceding products (4.4.1), (4.4.2)′* by $*$ *and with the cotrace maps* $c^{\bullet}$ *of (4.1), we have, for any* $\xi \in H^p(R,M)$, $\eta \in H^q(R,N)$ :

$$c^{p+q}(\xi * \eta) = c^p(\xi) * c^q(\eta).$$

As a special case[1], we have:

Corollary (4.4.5). *With assumptions as in (4.3.4), let* $\xi_1, \ldots, \xi_q \in H^1(R, \mathrm{Hom}_A(P,P))$, *let*

[1] which, incidentally, in view of results in (4.3), implies (2.3.2).

$$\alpha_i = \bar{\psi}(\xi_i) \in \mathrm{Hom}_P(I/I^2, P) \qquad 1 \le i \le q$$

$$\alpha_i' = \bar{\psi}\gamma^1(\xi_i) \in \mathrm{Hom}_{P'}(I'/I'^2, P') \qquad 1 \le i \le q.$$

*(Note the relation between* $\alpha_i$ *and* $\alpha_i'$ *given by (4.3.4).) Then, with the notation of (1.8.3), we have*

$$\gamma^q[\alpha_1\alpha_2 \cdots \alpha_q] = [\alpha_1' \cdots \alpha_q']$$

*(where, again,* $\gamma^q$ *is the composition (4.3.2)).*

The *proof of* (4.4.5) is left to the reader.

*Proof of* (4.4.4). We first reexamine the definition of the cohomology product given in (1.8). There is a unique $R^e$-linear map of complexes

$$\mu: \mathbf{B}.(R) \to \mathbf{B}.(R) \otimes_R \mathbf{B}.(R)$$

such that

$$\mu([r_1 \mid r_2 \mid \dots \mid r_q]) = \sum_{i=0}^{q} [r_1 \mid \dots \mid r_i] \otimes [r_{i+1} \mid \dots \mid r_q].$$

(Verification left to the reader.) If $\xi \in H^p(R, M)$ is the homotopy class of a map $f: \mathbf{B}.(R) \to M[p]$ and $\eta \in H^q(R, N)$ is the homotopy class of $g: \mathbf{B}.(R) \to N[q]$ (cf. remarks following (4.1.3)), then

$$\xi * \eta \in H^{p+q}(R, M \otimes_R N)$$

is the homotopy class of the composed map

$$(f \otimes g) \circ \mu: \mathbf{B}.(R) \xrightarrow{\mu} \mathbf{B}.(R) \otimes_R \mathbf{B}.(R) \xrightarrow{f \otimes g} M[p] \otimes_R N[q] = (M \otimes_R N)[p+q].$$

To prove (4.4.4), it suffices therefore to show that the following diagram commutes up to homotopy:

$$
\begin{array}{ccc}
\mathbf{B}.(H) & \xrightarrow{\quad\mu\quad} & \mathbf{B}.(H)\otimes_H \mathbf{B}.(H) \\
\downarrow{\scriptstyle c_F} & & \\
\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)) & & \downarrow{\scriptstyle c_F\otimes c_F} \\
\downarrow{\scriptstyle \text{via } \mu} & & \\
\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)\otimes_R \mathbf{B}.(R)) & \xleftarrow[(4.4.3)]{\lambda} & \mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R))\otimes_H \mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)) \\
\downarrow{\scriptstyle \text{via } f\otimes g} & & \downarrow{\scriptstyle (\text{via } f)\otimes(\text{via } g)} \\
\mathrm{Hom}_R(F, F\otimes_R M[p]\otimes_R N[q]) & \xleftarrow[(4.4.3)]{\lambda} & \mathrm{Hom}_R(F, F\otimes_R M[p])\otimes_H \mathrm{Hom}_R(F, F\otimes_R N[q])
\end{array}
$$

The existence of the splitting $(s_n)_{n\geq -1}$ given in (1.0) shows that the resolution $\mathbf{B}.(R) \xrightarrow{\epsilon} R$ is a homotopy equivalence of complexes of right $R$-modules. Hence we have a composed homotopy equivalence

$$\mathbf{B}.(R)\otimes_R \mathbf{B}.(R) \xrightarrow{\epsilon\otimes 1} R\otimes_R \mathbf{B}.(R) = \mathbf{B}.(R) \xrightarrow{\epsilon} R;$$

i.e.

$$\mathbf{B}.(R)\otimes_R \mathbf{B}.(R) \xrightarrow{\epsilon\otimes\epsilon} R\otimes_R R = R$$

is a right $R$-split resolution of $R$. As in (4.1), it follows then that the corresponding complex of $H^e$-modules

$$\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)\otimes_R \mathbf{B}.(R)) \to \mathrm{Hom}_R(F, F\otimes_R R) = H$$

has a right $H$-linear splitting, whence by [M, p.261, Theorem 4.3] the top half of the above diagram is homotopy commutative.

The bottom half is easily checked to be commutative, and the conclusion results.

(4.5) Again let $g: A \to R$ be a homomorphism of commutative rings, let $F$ now be a finitely generated projective $R$-module, and let $H$ be the $A$-algebra $\mathrm{Hom}_R(F, F)$. We define **trace maps**

(4.5.1) $$t_q : H_q(H, H) \to H_q(R, R)$$

as follows.

Let $\mathbf{B}.(R, R)$ be the complex of $R$-modules

$$\mathbf{B}.(R, R) = R \otimes_{R^e} \mathbf{B}.(R),$$

whose homology is H.(R, R). Combining $C_F$ (cf. (4.1.1)) with the natural map $\mathbf{B}.(R) \to \mathbf{B}.(R, R)$, we get a homotopy class of $H^e$-linear maps

$$\text{(4.5.2)} \quad \mathbf{B}.(H) \to \mathrm{Hom}_R(F, F \otimes_R \mathbf{B}.(R, R)) = \mathrm{Hom}_R(F, F) \otimes_R \mathbf{B}.(R, R)$$

$$= H \otimes_{R^e} \mathbf{B}.(R)$$

and hence a homotopy class of maps

$$H \otimes_{H^e} \mathbf{B}.(H) \to (H \otimes_{H^e} H) \otimes_{R^e} \mathbf{B}.(R) = H_0(H, H) \otimes_{R^e} \mathbf{B}.(R).$$

Passing to homology, we obtain canonical maps

$$\text{(4.5.3)} \qquad H_q(H, H) \to H_q(R, H_0(H, H)) \qquad (q \geq 0)$$

which, combined with the usual trace map $\mathrm{Tr}_{F/R} : H_0(H, H) \to R$ (cf. (1.5)) give us the maps $t_q$ of (4.5.1).

*Example.* One checks that for $q = 0$, (4.5.3) is just the identity map of $H_0(H, H)$, so that

$$t_0 : H_0(H, H) \to H_0(R, R) = R$$

is the usual trace, i.e. $t_0 = \mathrm{Tr}_{F/R}$.

The following Proposition expresses a kind of adjointness between "trace" and "cotrace".

PROPOSITION (4.5.4). *Let* M *be an* R*-module (considered as an* R-R *bimodule in the natural way). Let*

$$t_q : H_q(H, H) \to H_q(R, R)$$

*be the trace map* (4.5.1), *let*

$$c^q : H^q(R, M) \to H^q(H, \mathrm{Hom}_R(F, F \otimes_R M)) = H^q(H, H \otimes_R M)$$

*be the cotrace* (cf. (4.1)), *and let*

$$\rho_{H\otimes M}: H^q(H, H\otimes_R M)\otimes_{H^e} H_q(H, H) \to H_0(H, H\otimes_R M)$$
$$= H\otimes_{H^e}(H\otimes_R M) = H_0(H, H)\otimes_R M$$

*and*

$$\rho_M: H^q(R, M)\otimes_R H_q(R, R) \to H_0(R, M) = M$$

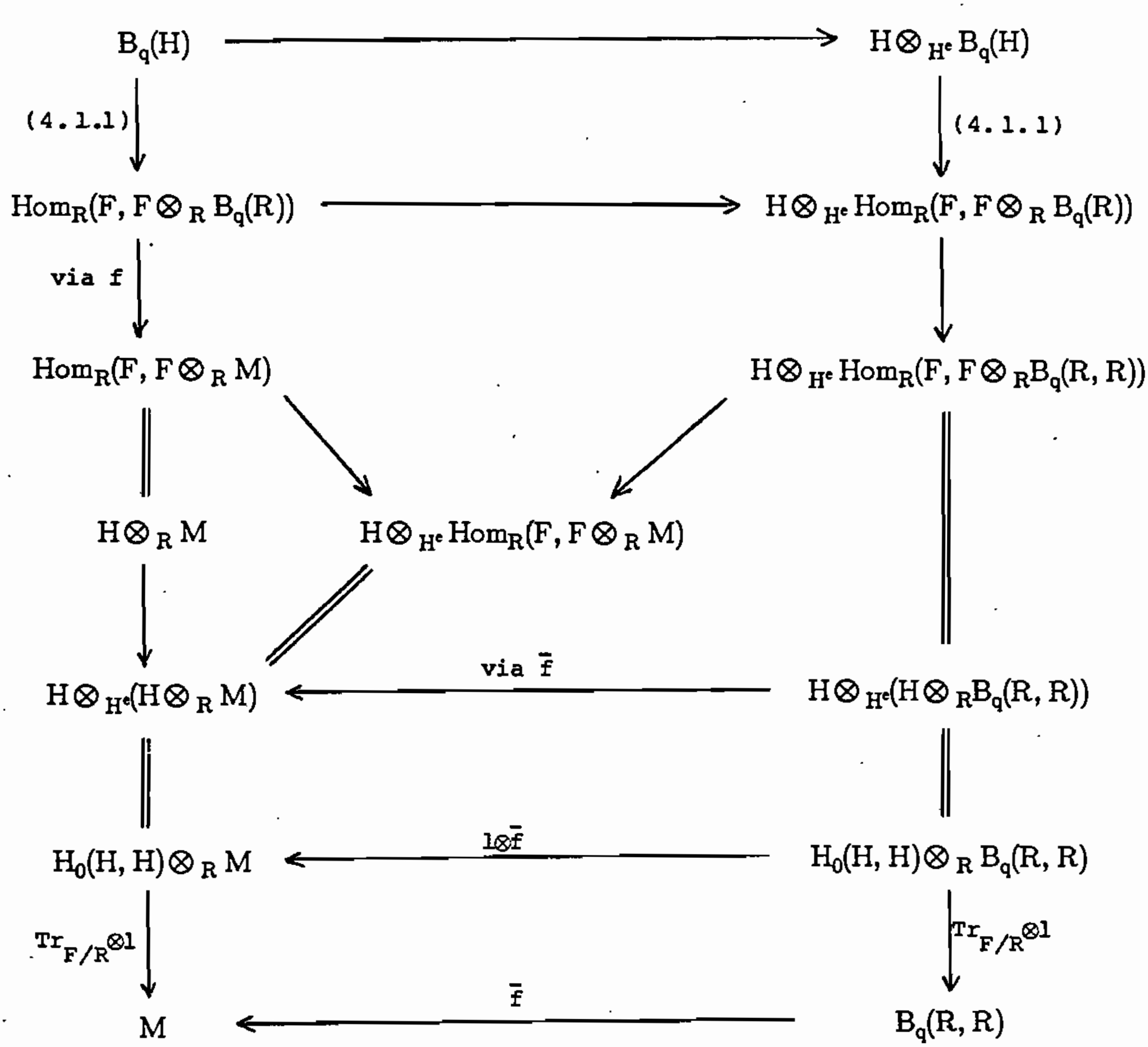
*be as in (1.1). Then for any* $\xi \in H^q(R, M)$, $\omega \in H_q(H, H)$, *we have*

$$\rho_M(\xi\otimes t_q(\omega)) = (\mathrm{Tr}_{F/R}\otimes 1)(\rho_{H\otimes M}(c^q(\xi)\otimes\omega)).$$

*Proof.* Let $\xi$ be represented by a q-cocycle $f: B_q(R) \to M$, and let

$$\overline{f} = 1\otimes f : B_q(R, R) = R\otimes_{R^e} B_q(R) \to R\otimes_{R^e} M = M.$$

Let $\omega$ be represented by the q-cycle $1\otimes x \in H\otimes_{H^e} B_q(H)$. Consider the following commutative diagram, where unlabelled arrows represent natural maps:




An examination of definitions reveals that going down the left side of the diagram takes $x \in B_q(H)$ to $(\mathrm{Tr}_{F/R} \otimes 1)(\rho_{H\otimes M}(c^q(\xi) \otimes \omega)) \in M$; while going around in the clockwise direction takes $x$ to $\rho_M(\xi \otimes t_q(\omega))$. The conclusion follows.

(4.6) To give more substance to the maps $t_q$, we give some *examples involving differential forms*. (Strictly speaking, in the final section (4.7) only Definition (4.6.2) will be needed.) A much more detailed discussion appears in notes of E. Kunz [K, §16] and R. Hübl (to appear).

Again, let $g: A \to R$ be a homomorphism of commutative rings and let $F$ be a finitely generated projective R-module. Let $S$ be a commutative R-algebra, and let

$$\psi: S \to H = \mathrm{Hom}_R(F, F)$$

be an R-algebra homomorphism. Then we have a diagram

$$\begin{array}{ccccc} \Omega^q{}_{S/A} & \xrightarrow{\theta_q^S} & H_q(S,S) & \xrightarrow{H_q(\psi)} & H_q(H,H) \\ \tau_q \downarrow ? & & & & \downarrow t_q \ (4.5) \\ \Omega^q{}_{R/A} & & \xrightarrow[\theta_q^R]{} & & H_q(R,R) \end{array} \tag{4.6.1}$$

and this suggests the question: *when does there exist a map* $\tau_q$ *making the diagram commute?*

DEFINITION (4.6.2). *A map* $\tau_q$ *making the diagram* (4.6.1) *commute is called a* $\psi$-*trace for differential forms of degree* q.

*Remarks*. (i) If $\theta_q^R$ is injective then of course there exists at most one $\psi$-trace.

(ii) Below (cf. (4.6.4.1)) we describe a map

$$\overline{\delta}_q\colon H_q(R,R) \to \Omega^q{}_{R/A}$$

such that

$$\overline{\delta}_q \circ \theta_q^R = q!(\text{identity}).$$

Hence if $\tau_q$ and $\tau'_q$ are two $\psi$-traces, then

$$q!(\tau_q - \tau'_q) = 0.$$

In particular, if q! is a unit in R, then $\tau_q = \tau'_q$.

In fact it will be seen below (4.6.5) that $q!\tau_q = q!\tau'_q$ is necessarily the "pretrace" constructed by Angéniol in [A, pp. 108 ff ]. However, even when q! is a unit, I do not know whether (1/q!) times Angéniol's pretrace is necessarily a $\psi$-trace.

Here are some examples of $\psi$-traces.

PROPOSITION (4.6.3) (cf. [HKR, p. 395]). *Assume that the* A-*algebra* R *is smooth* [EGA IV, (17.5.2)]. *Then* $\theta_q^R$ *is bijective; and hence there exists a unique* $\psi$-*trace, viz.*

$$\tau_q = (\theta_q^R)^{-1} \circ t_q \circ H_q(\psi) \circ \theta_q^S.$$

*Proof.* Set $E = R \otimes_A R$, so that R is, as usual, an E-algebra via the multiplication map $E = R \otimes_A R \to R$. For any E-projective resolution $\mathbf{P}_{\bullet}$ of R, there is a homotopy-unique lifting of the identity map $1_R$ of R to an E-linear map of complexes $\alpha: \mathbf{P.} \to \mathbf{B.} = \mathbf{B.}(R)$, and hence, for each $q \geq 0$, a canonical map

$$\bar{\alpha}_q\colon \mathrm{Tor}_q^E(R, R) = H_q(R \otimes_E \mathbf{P.}) \to H_q(R \otimes_E \mathbf{B.}) = H_q(R, R).$$

Since R is smooth – hence flat – over A, therefore **B.** is E-flat, so $\bar{\alpha}_q$ is *bijective.* Furthermore, a lifting of $1_R$ to a map of complexes $\mu\colon \mathbf{P.} \otimes_E \mathbf{P.} \to \mathbf{P.}$ induces a map $\bar{\mathbf{P}}. \otimes_R \bar{\mathbf{P}}. \to \bar{\mathbf{P}}.$ (where $\bar{\mathbf{P}}. = R \otimes_E \mathbf{P.}$), which gives, upon passage to homology, a canonical graded R-algebra structure on $T = \otimes_{q \geq 0} \mathrm{Tor}_q^E(R, R)$. The diagram

$$\begin{array}{ccc} \mathbf{P.} \otimes_E \mathbf{P.} & \xrightarrow{\ \mu\ } & \mathbf{P.} \\ {\scriptstyle \alpha\otimes\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbf{B.} \otimes_E \mathbf{B.} & \xrightarrow[\ (1.10)\ ]{\text{shuffle}} & \mathbf{B.} \end{array}$$

is homotopy-commutative, since all the maps in it lift $1_R$. Thus, after applying $R \otimes_E$ and passing to homology, we find that

$$\bigoplus_{q \geq 0} \bar{\alpha}_q\colon T = \bigoplus_{q \geq 0} \mathrm{Tor}_q^E(R, R) \to \bigoplus_{q \geq 0} H_q(R, R) = H$$

is an *isomorphism of graded* R-*algebras.*

In view of the definition of $\theta$ (cf. (1.10.1)), Proposition (4.6.3) asserts, in essence, the *bijectivity* of the canonical R-algebra map $\wedge H_1(R, R) \to H$ ($\wedge$ denotes "exterior algebra" ), i.e. (by the above) of the canonical E-algebra map $\wedge \mathrm{Tor}_1^E(R, R) \to T$. This latter map is bijective if and only if it is so after localization at each prime ideal Q containing the kernel J of the multiplication map $E \to R$. Set $F = E_Q$, $U = R_Q$. Then $J_Q$ is generated by an F-regular sequence $\mathbf{f} = (f_1, \ldots, f_m)$ (argue as in [EGA IV, (17.12.4)], using *ibid*, (17.12.1c)); and if **K.** is the corresponding Koszul complex over F, i.e. $\mathbf{K.} = \wedge(F^m)$ with differential determined by **f**, then **K.** is an F-projective resolution of U, $U \otimes_F \mathbf{K.} = \wedge(U^m)$ with *vanishing differential*, and, as a *graded group*:

$$T_Q = \bigoplus_{q \geq 0} \mathrm{Tor}_q^F(U, U) = \bigoplus_{q \geq 0} H_q(U \otimes_F \mathbf{K.}) = \wedge(U^m) = \wedge(\mathrm{Tor}_1^F(U, U)).$$

It remains therefore to be verified that the canonical product on $T_Q$ is identical with the exterior algebra product. But this follows from the fact (easily checked) that the exterior algebra product $\mathbf{K.} \otimes_F \mathbf{K.} \to \mathbf{K.}$ is a homomorphism of complexes lifting $1_U$, and hence (as above) inducing the canonical algebra structure on $T_Q$. Q.E.D.

*Remark* (4.6.4). Under the conditions of (4.6.3), one would like an explicit description of $(\theta_q^R)^{-1}$. In fact, we can describe a *left* inverse of $\theta_q^R$ under either of the following hypotheses:

(i) A is a **Q**-algebra.

(ii) $\Omega^1{}_{R/A}$ is a free R-module of finite rank.

Indeed, if we define

$$\delta_q': B_q(R, R) = R \otimes_{R^e} B_q(R) \to \Omega^q{}_{R/A}$$

by

$$\delta_q'(r_0 \otimes [r_1 \mid \dots \mid r_q]) = r_0\, dr_1\, dr_2 \cdots dr_q$$

then it is easily checked that $\delta_q'$ annihilates the image of the boundary map $B^{q+1}(R, R) \to B_q(R, R)$, whence $\delta_q'$ induces

(4.6.4.1) $$\overline{\delta}_q: H_q(R, R) \to \Omega^q{}_{R/A};$$

and one verifies by direct computation that

$$\overline{\delta}_q \circ \theta_q^R = q!(\text{identity}).$$

Thus if A is a **Q**-algebra (or, more generally, if q! is a unit in A) then

(4.6.4.2) $$\delta_q = (1/q!)\overline{\delta}_q$$

is a left inverse for $\theta_q^R$.

If $\Omega^q{}_{R/A}$ is free over R, with basis, say $(\omega_1, \dots, \omega_m)$, and if $D_i: R \to R$ is the derivation corresponding to the R-homomorphism $\Omega^1{}_{R/A} \to R$ taking $\omega_i$ to 1 and $\omega_j$ $(j \neq i)$ to 0, then (noting that $\Omega^q{}_{R/A}$ is R-free) we can define an element

$$\delta^{(q)} \in H^q(R, \Omega^q{}_{R/A}) = H^q(R, R) \otimes_R \Omega^q{}_{R/A}$$

by

$$\delta^{(q)} = \sum_{i_1 < i_2 < \dots < i_q} [D_{i_1} D_{i_2} \cdots D_{i_q}] \otimes \omega_{i_1} \omega_{i_2} \cdots \omega_{i_q}$$

where, as in (1.8.2), $[D_{i_1} \cdots D_{i_q}] \in H^q(R, R)$ is the product of the elements in $H^1(R, R)$ corresponding to the derivations $D_{i_1}, \dots, D_{i_q}$. Via the natural pairing

(4.6.4.3) $$H^q(R, \Omega^q{}_{R/A}) \otimes_R H_q(R, R) \to \Omega^q{}_{R/A}$$

(cf. (1.1)) the element $\delta^{(q)}$ gives rise to a map

$$\delta_q^*: H_q(R, R) \to \Omega^q{}_{R/A},$$

which is a left inverse for $\theta_q^R$.

It may be noted that

$$q!\delta^{(q)} = (\delta)^q \in H^q(R, \Omega^q_{R/A})$$

where $\delta \in H^1(R, \Omega^1{}_{R/A})$ corresponds to the universal derivation $d : R \to \Omega^1{}_{R/A}$, and the q-th power $(\delta)^q$ is defined via the cohomology product of (1.8). Moreover the mapping $H_q(R, R) \to \Omega^q{}_{R/A}$ corresponding to $(\delta)^q$ (via (4.6.4.3)) is just the map $\overline{\delta}_q$ of (4.6.4.1). Hence

(4.6.4.4) $$q!\delta_q^* = \overline{\delta}_q;$$

and so if $q!$ is a unit in A, then $\delta_q^*$ coincides with the map $\delta_q$ of (4.6.4.2).

Roughly speaking, then, finding a left inverse for $\theta_q^R$ involves finding a "divided power"

$$(\delta)^q/q! \in H^q(R, \Omega^q{}_{R/A}).$$

*Remark* (4.6.5). If $\tau_q$ is a $\psi$-trace and $\overline{\delta}_q$ is as above (4.6.4.1), then

$$q!\tau_q = \overline{\delta}_q \circ \theta_q^R \circ \tau_q = \overline{\delta}_q \circ t_q \circ H_q(\psi) \circ \theta_q^S$$

(cf. (4.6.1)), which is Angéniol's "pretrace" [A, pp. 108 ff.].

In other words,

(4.6.5.1) $$\overline{\delta}_q t_q H_q(\psi)\theta_q^S(s_0 ds_1 \cdots ds_q) \qquad (s_i \in S)$$

is found as follows (at least after localizing, so that F becomes R-free): pick a basis of F, and let $\mu_i$ $(0 \leq i \leq q)$ be the matrix corresponding to the R-endomorphism $\psi(s_i)$; then (4.6.5.1) is the trace (= sum of diagonal entries) of the matrix

$$\sum_\tau (-1)^{|\tau|} \mu_0 d\mu_{\tau(1)} d\mu_{\tau(2)} \cdots d\mu_{\tau(q)}$$

where $\sum_\tau$ is as in (1.10.2), and for a matrix $\mu$, $d\mu$ is the matrix, with entries (of degree one) in the exterior algebra $\bigoplus_{n \geq 0} \Omega^n{}_{R/A}$, obtained by applying the universal derivation d to the entries of $\mu$. This can be verified through a careful examination of the definitions of $\overline{\delta}_q$, $t_q$, $H_q(\psi)$, and $\theta_q^S$, and of example (4.2). Details are left to the reader.

Note that we have indicated here an intrinsic approach to Angéniol's pretrace, via Hochschild homology, which renders unnecessary all the computations in [A, pp. 109-113].

* * *

When S is étale over R, there is a very simple description of a $\psi$-trace (cf. (4.6.7) below).

PROPOSITION (4.6.6). *There is a natural structure on* $\bigoplus_{q\geq 0} H_q(H,H)$ *of graded module over the graded ring* $\bigoplus_{q\geq 0} H_q(R,R)$ ; *and for all* p, q *the resulting diagram*

(4.6.6.1)
$$\begin{array}{ccc} H_q(R,R)\otimes H_p(H,H) & \longrightarrow & H_{p+q}(H,H) \\ \Big\downarrow{\scriptstyle 1\otimes t_p} & & \Big\downarrow{\scriptstyle t_{p+q}} \\ H_q(R,R)\otimes H_p(R,R) & \xrightarrow[(1.10)]{} & H_{p+q}(R,R) \end{array}$$

*commutes.*

Before giving the proof, we note the following easy consequence of the special case $p=0$ of (4.6.6):

COROLLARY (4.6.7). *Given* $\psi\colon S\to H = \mathrm{Hom}_R(F,F)$ *as above, define the trace map* $\mathrm{tr}_\psi\colon S\to R$ *to be the composition*

$$\mathrm{tr}_\psi\colon S\xrightarrow{\psi} H\xrightarrow{\text{trace}} R$$

*(i.e.* $\mathrm{tr}_\psi$ *is the unique* $\psi$*-trace for* $q=0$*). Then for any* $q\geq 0$ *the following diagram commutes:*

$$\begin{array}{ccccccc} \Omega^q_{R/A}\otimes_R S & \xrightarrow{\text{natural}} & \Omega^q_{S/A} & \xrightarrow{\theta_q^S} & H_q(S,S) & \xrightarrow{H_q(\psi)} & H_q(H,H) \\ \Big\downarrow{\scriptstyle 1\otimes \mathrm{tr}_\psi} & & & & & & \Big\downarrow{\scriptstyle t_q} \\ \Omega^q_{R/A}\otimes_R R = \Omega^q_{R/A} & & & \xrightarrow[\theta_q^R]{} & & & H_q(R,R) \end{array}$$

*In particular, if* S *is étale over* R *(so that* $\Omega^q_{R/A}\otimes_R S\to\Omega^q_{S/A}$ *is bijective) then* $1\otimes\mathrm{tr}_\psi$ *is a* $\psi$*-trace .*

*Proof of* (4.6.6). Since H is an R-algebra, we can define a shuffle product

$$\mathbf{B}.(R)\otimes_A \mathbf{B}.(H) \to \mathbf{B}.(H) \tag{4.6.6.2}$$

by essentially the same formula used in (1.10); and there results a map of complexes

$$(R\otimes_{R^e}\mathbf{B}.(R))\otimes_R (H\otimes_{H^e}\mathbf{B}.(H)) \to H\otimes_{H^e}\mathbf{B}.(H)$$

which gives rise at the homology level to pairings

$$H_q(R, R)\otimes H_p(H, H) \to H_{p+q}(H, H)$$

which define the asserted graded module structure.

For the commutativity of (4.6.6.1) we consider the diagram of $(R^e\otimes_A H^e)$-linear maps of complexes:

$$\begin{array}{ccc}
\mathbf{B}.(R)\otimes_A \mathbf{B}.(H) & \xrightarrow{\ (4.6.6.2)\ } & \mathbf{B}.(H) \\
\downarrow{\scriptstyle 1\otimes c_F} & & \\
\mathbf{B}.(R)\otimes_A \mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)) & & \downarrow{\scriptstyle c_F} \\
\downarrow{\scriptstyle \text{natural}} & & \\
\mathrm{Hom}_R(F, \mathbf{B}.(R)\otimes_A F\otimes_R \mathbf{B}.(R)) & \xrightarrow[\text{shuffle}]{\text{via}} & \mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R))
\end{array} \tag{4.6.6.3}$$

This diagram lifts the commutative diagram

$$\begin{array}{ccc}
R\otimes_A H & \xrightarrow{\ \text{natural}\ } & H \\
\| & & \\
R\otimes_A \mathrm{Hom}_R(F, F) & & \| \\
\downarrow & & \\
\mathrm{Hom}_R(F, R\otimes_A F) & \xrightarrow[\text{via multiplication}]{} & \mathrm{Hom}_R(F, F)
\end{array}$$

But, as in (4.1), the complex $\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)) \to \mathrm{Hom}_R(F,F)$ splits over A; and hence by [M, p. 261, Theorem 4.3], (4.6.6.3) is homotopy-commutative, whence so is the diagram obtained from (4.6.6.3) by applying the functor $\otimes_{R^e\otimes H^e}(R\otimes H)$. It follows that the composition

$$H_q(R,R)\otimes H_p(H,H) \to H_{p+q}(H,H) \xrightarrow{t_{p+q}} H_{p+q}(R,R)$$

is obtained at the homology level from the composition

$$\begin{aligned}(4.6.6.4)\quad \mathbf{B}.(R,R)\otimes_A \mathbf{B}.(H,H) &\xrightarrow{\text{via } 1\otimes C_F} \mathbf{B}.(R,R)\otimes_A (H\otimes_{H^e}\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R)))\\ &\to H\otimes_{H^e}\mathrm{Hom}_R(F, \mathbf{B}.(R,R)\otimes_A F\otimes_R \mathbf{B}.(R,R))\\ &\xrightarrow{\text{shuffle}} H\otimes_{H^e}\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R,R)) = H_0(H,H)\otimes_R \mathbf{B}.(R,R)\\ &\xrightarrow{\text{trace}} \mathbf{B}.(R,R).\end{aligned}$$

But we have a commutative diagram

$$\begin{array}{ccc} H\otimes_{H^e}\mathrm{Hom}_R(F, \mathbf{B}.(R,R)\otimes_A F\otimes_R \mathbf{B}.(R,R)) & \longrightarrow & H\otimes_{H^e}\mathrm{Hom}_R(F, F\otimes_R \mathbf{B}.(R,R)) \\ \text{via}\downarrow\text{trace} & & \text{via}\downarrow\text{trace} \\ \mathbf{B}.(R,R)\otimes_A \mathbf{B}.(R,R) & \xrightarrow{\text{shuffle}} & \mathbf{B}.(R,R) \end{array}$$

Hence, it is easily checked that (4.6.6.4) also gives the composition

$$H_q(R,R)\otimes_R H_p(H,H) \underset{1\otimes t_p}{\to} H_q(R,R)\otimes H_p(R,R) \underset{(1.10)}{\to} H_{p+q}(R,R)$$

in (4.6.6.1). Thus (4.6.6.1) commutes, as asserted.

* * *

Our last example of a $\psi$-trace involves a variant of the "Cartier Operator".

*Example* (4.6.8). *Suppose that* A *contains a field of characteristic* $p > 0$. *Let* $r_1, \ldots, r_q \in R$, *let* $F = S$ *be the* R-*algebra*

$$S = R[X_1, \ldots, X_q]/(X_1^p - r_1, \ldots, X_q^p - r_q) = R[\xi_1, \ldots, \xi_q]$$

($X_1, \ldots, X_q$ *indeterminates), and let*

$$\psi: S \to \mathrm{Hom}_R(S, S)$$

*be the regular representation, i.e.,* $\psi(s) =$ "multiplication by s". *Then there is a unique* R-*linear map*

$$\tau_q': \Omega^q{}_{S/R} \to \Omega^q{}_{R/A}$$

*such that*

$$\tau_q'((\xi_1\xi_2 \cdots \xi_q)^{p-1} d\xi_1 \cdots d\xi_q) = dr_1 \cdots dr_q$$

*and*

$$\tau_q'(\xi_1^{a_1} \cdots \xi_q^{a_q} d\xi_1 \cdots d\xi_q) = 0$$

*for any* q-*tuple of integers* $(a_1, \ldots, a_q) \neq (p-1, \ldots, p-1)$ *with* $0 \leq a_i < p$ $(1 \leq i \leq q)$. *Moreover, the composition*

$$\tau_q: \Omega^q{}_{S/A} \xrightarrow{\text{natural}} \Omega^q{}_{S/R} \xrightarrow{\tau_q'} \Omega^q{}_{R/A}$$

*is a* $\psi$-*trace.*

*Proof.* Since $\Omega^1{}_{S/R}$ is the free S-module generated by $(d\xi_1, \ldots, d\xi_q)$, the existence and uniqueness of $\tau_q'$ is clear.

Since

$$\Omega^q{}_{S/A} = \bigoplus_{i=0}^{q} \Omega^i{}_{S/R} \otimes_R \Omega^{q-i}{}_{R/A},$$

and in view of (4.6.6), to show that $\tau_q$ is a $\psi$-trace, it will be enough to show, for $n \leq q$, that for all sequences $0 < i_1 < i_2 < \cdots < i_n < q$, and for all $(a_1, \ldots, a_q)$ with $0 \leq a_i < p$, we have

$$t_n H_n(\psi)\, \theta_n^S(\xi_1^{a_1} \cdots \xi_q^{a_q} d\xi_{i_1} \cdots d\xi_{i_n}) = 0$$

unless $(i_1, \ldots, i_n) = (1, 2, \ldots, q)$ and $(a_1, \ldots, a_q) = (p-1, \ldots, p-1)$, in which case we have

$$t_q H_q(\psi) \theta_q^S((\xi_1 \cdots \xi_q)^{p-1} d\xi_1 \cdots d\xi_q) = \theta_q^R(dr_1 \cdots dr_q).$$

As an R-module, $S = N \otimes_A R$, where N is the free A-module with basis $\{\xi_1^{b_1} \cdots \xi_q^{b_q}\}_{0 \leq b_i < p}$. We proceed then to compute as in example (4.2). We have

$$\xi_i\xi_1^{b_1}\cdots\xi_q^{b_q} = \xi_1^{b_1}\cdots\xi_i^{b_i+1}\cdots\xi_q^{b_q} \qquad \text{if } b_i < p-1$$

$$= r_1\xi_1^{b_1}\cdots\xi_{i-1}^{b_{i-1}}\xi_{i+1}^{b_{i+1}}\cdots\xi_q^{b_q} \qquad \text{if } b_i = p-1.$$

Thus if $\mu_i$ is the associated matrix, and $\tilde{\mu}_i$ is obtained from $\mu_i$ by replacing each entry $r$ in $\mu$ by its natural image $\tilde{r} = r + A$ in $R/A$, then $\tilde{\mu}_i$ has $p^{q-1}$ entries equal to $\tilde{r}_i$, (arising from members of the basis having $b_i = p-1$), and all other entries vanish. The reader can check that for $i_1 < \cdots < i_n$, the tensor product matrix $\tilde{\mu}_{i_1}\otimes\cdots\otimes\tilde{\mu}_{i_n}$ has $p^{q-n}$ entries equal to $\tilde{r}_{i_1}\otimes\cdots\otimes\tilde{r}_{i_n}$ (arising from members of the basis having $b_{i_1} = b_{i_2} = \cdots = b_{i_n} = p-1$), and no other non-zero entries. Hence the element of

$$\mathrm{Hom}_R(S, S\otimes_R B_n(R))$$

corresponding to the element

$$\sum_\tau (-1)^{|\tau|}[\xi_{i_{\tau(1)}}|\ldots|\xi_{i_{\tau(n)}}] \in B_n(\mathrm{Hom}_R(S,S))$$

(notation as in (1.10.2)) has $p^{q-n}$ non-zero entries, all equal to

$$\sum_\tau (-1)^{|\tau|}[\tilde{r}_{i_{\tau(1)}}|\ldots|\tilde{r}_{i_{\tau(n)}}].$$

Now if $\mu^a$ is the matrix corresponding to multiplication by $\xi^a = \xi_1^{a_1}\cdots\xi_q^{a_q}$ $(0 \le a_i < q)$, then the matrix

$$\mu^a\otimes\tilde{\mu}_{i_1}\otimes\cdots\otimes\tilde{\mu}_{i_n}\otimes 1$$

has non-zero diagonal entries only if $a_{i_1} = a_{i_2} = \cdots = a_{i_n} = p-1$, in which case there are $p^{q-n}$ such entries, all equal to

$$\sum_\tau (-1)^{|\tau|}[\tilde{r}_{i_{\tau(1)}}|\ldots|\tilde{r}_{i_{\tau(q)}}] \in B_q(R).$$

Hence the trace of this matrix (= sum of diagonal entries) vanishes unless $q = n$ and $a_1 = a_2 = \cdots = a_q = p-1$, in which case the trace is an element of $B_q(R)$ whose image in $B_q(R,R)$ is a cycle with homology class $\theta_q^R(dr_1\cdots dr_q)$ (cf. (1.10.2)). Q.E.D.

(4.7) We are now prepared to give the main result of this section. But first let us review the necessary notation.

Let $g: A \to R$ be, as before, a homomorphism of commutative rings, and let $R'$ be a finite projective $R$-algebra. Let $P$ be an $R$-module which as an $A$-module via $g$ is finitely generated and projective, and let $P' = R' \otimes_R P$ (so that the $A$-module $P'$ is also finitely generated and projective). Set $H = \mathrm{Hom}_R(R', R')$, fix an integer $q \geq 0$, let

$$\gamma^q: H^q(R, \mathrm{Hom}_A(P, P)) \to H^q(R', \mathrm{Hom}_A(P', P'))$$

be the map defined by (4.3.2), and let $t_q'$ be the composed map

$$t_q': H_q(R', R') \xrightarrow{H_q(\psi)} H_q(H, H) \xrightarrow{t_q} H_q(R, R)$$

where $\psi: R' \to H$ is the $A$-algebra homomorphism taking $r' \in R'$ to "left multiplication by $r'$", and where $t_q$ is the trace map of (4.5.1).

THEOREM (4.7.1) ("Trace Formula II"). *With the preceding notation, we have, for any* $\xi \in H^q(R, \mathrm{Hom}_A(P, P))$ *and* $\omega' \in H_q(R', R')$:

$$\mathrm{Res}^q(\gamma^q(\xi) \otimes \omega') = \mathrm{Res}^q(\xi \otimes t_q'(\omega'))\,.$$

COROLLARY (4.7.2). *Under the hypotheses of (4.4.5), we have*

$$\mathrm{Res}\begin{bmatrix} \omega' \\ \alpha_1', \dots, \alpha_q' \end{bmatrix} = \mathrm{Res}\begin{bmatrix} t_q'(\omega') \\ \alpha_1, \dots, \alpha_q \end{bmatrix}.$$

*In particular, if* $I$ *(as in (4.3.4)) is such that* $I/I^2$ *is* $R/I$*-free, with basis* $(f_i + I^2)_{1 \leq i \leq q}$ $(f_i \in I)$, *and if* $f_i' \in R'$ *is the natural image of* $f_i$, *then*

$$\mathrm{Res}\begin{bmatrix} \omega' \\ f_1', \dots, f_q' \end{bmatrix} = \mathrm{Res}\begin{bmatrix} t_q'(\omega') \\ f_1, \dots, f_q \end{bmatrix}.$$

Corollary (4.7.3). *If there exists a* $\psi$*-trace* $\tau_q: \Omega^q_{R'/A} \to \Omega^q_{R/A}$ *(cf. (4.6.2)) then for any* $\nu \in \Omega^q_{R'/A}$ *(cf. (1.10.4)):*

$$\operatorname{Res}\begin{bmatrix} \nu \\ \alpha_1', \ldots, \alpha_q' \end{bmatrix} = \operatorname{Res}\begin{bmatrix} \tau_q(\nu) \\ \alpha_1, \ldots, \alpha_q \end{bmatrix}.$$

*Proof of* (4.7.1). ((4.7.2) and (4.7.3) left to reader.) Let M be the R-module

$$M = H_0(R, \operatorname{Hom}_A(P, P)) = H_0(R, P \otimes_A \operatorname{Hom}_A(P, A))$$
$$= \operatorname{Hom}_A(P, A) \otimes_R P \qquad \text{(cf.(1.0.1))}.$$

Via the indicated identifications, the trace map

$$\operatorname{Tr}_{P/A}: H_0(R, \operatorname{Hom}_A(P, P)) \to A$$

gets transformed into the map

$$T_{P/A}: \operatorname{Hom}_A(P, A) \otimes_R P \to A$$

given by

$$T_{P/A}(\phi \otimes p) = \phi(p).$$

In view of remark (1.1.1) and the definition (1.5.1) of $\operatorname{Res}^q$, we see that

$$\operatorname{Res}^q(\xi \otimes t_q'(\omega')) = T_{P/A}\, \rho_M(\bar{\xi} \otimes t_q'(\omega'))$$

where $\bar{\xi}$ is the natural image in $H^q(R, M)$ of $\xi \in H^q(R, \operatorname{Hom}_A(P, P))$ (via the natural $R^e$-linear map

$$\operatorname{Hom}_A(P, P) \to H_0(R, \operatorname{Hom}_A(P, P)) = M).$$

And by (4.5.4), if $\omega \in H_q(H, H)$ is the natural image of $\omega' \in H_q(R', R')$, then

$$\rho_M(\bar{\xi} \otimes t_q'(\omega')) = \rho_M(\bar{\xi} \otimes t_q(\omega))$$
$$= (\operatorname{Tr}_{R'/R} \otimes 1)(\rho_{H \otimes M}(c^q(\bar{\xi}) \otimes \omega)).$$

On the other hand, if

$$\rho': H^q(R', \operatorname{Hom}_A(P', P')) \otimes_{R'^e} H_q(R', R')$$
$$\to H_0(R', \operatorname{Hom}_A(P', P')) = \operatorname{Hom}_A(P', A) \otimes_{R'} P'$$

is as in (1.1), then we have

$$\operatorname{Res}^q(\gamma^q(\xi)\otimes \omega') = T_{P'/A}\, \rho'(\gamma^q(\xi)\otimes \omega');$$

and furthermore, it is straightforward to check that the natural composed map

$$\begin{aligned}\pi\colon H_0(R', \operatorname{Hom}_A(P', P')) &= H_0(R', \operatorname{Hom}_R(R', R'\otimes_R \operatorname{Hom}_A(P, P))) \qquad \text{(cf.(4.3.1)}\\ &\to H_0(R', \operatorname{Hom}_R(R', R'\otimes_R M))\\ &\to H_0(H, \operatorname{Hom}_R(R', R'\otimes_R M))\\ &= H\otimes_{H^e}(H\otimes_R M)\\ &= H_0(H, H)\otimes_R M\end{aligned}$$

satisfies

$$\pi(\rho'(\gamma^q(\xi)\otimes\omega')) = \rho_{H\otimes M}(c^q(\overline{\xi})\otimes\omega).$$

Hence it will suffice to show that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}_A(P', A)\otimes_{R'} P' & = & H_0(R', \operatorname{Hom}_A(P', P')) \\ & & \Big\downarrow \pi \\ \Big\downarrow T_{P'/A} & & H_0(H, H)\otimes_R M \\ & & \Big\downarrow \mathrm{Tr}_{R'/R}\otimes 1 \\ A \underset{T_{P/A}}{\longleftarrow} & \operatorname{Hom}_A(P, A)\otimes_R P & = M \end{array}$$

As usual, one needs to expand the diagram according to the definitions of the maps involved, and go through a tedious verification. One such expanded diagram looks like this:

$$
\begin{array}{ccccc}
\mathrm{Hom}_A(P',A)\otimes_A P' & = & \mathrm{Hom}_A(P',P') & \longrightarrow & H_0(R',\mathrm{Hom}_A(P',P')) \\
\| & & \| & & \Big\downarrow \pi \\
\mathrm{Hom}_R(R',\mathrm{Hom}_A(P,A))\otimes_A P' & = & \mathrm{Hom}_R(R',R'\otimes_R \mathrm{Hom}_A(P,P)) & & \\
\| & & \downarrow & & \\
\mathrm{Hom}_R(R',R)\otimes_R \mathrm{Hom}_A(P,A)\otimes_A R'\otimes_R P & & \mathrm{Hom}_R(R',R'\otimes_R M) & & \\
\downarrow & & \| & & \\
\mathrm{Hom}_R(R',R)\otimes_R R'\otimes_R \mathrm{Hom}_A(P,A)\otimes_R P & = & \mathrm{Hom}_R(R',R')\otimes_R(\mathrm{Hom}_A(P,A)\otimes_R P) & \rightarrow & H_0(H,H)\otimes_R(\mathrm{Hom}_A(P,A)\otimes_R P) \\
\| & & & & \Big\downarrow \mathrm{Tr}_{R'/R}\otimes 1 \\
\mathrm{Hom}_R(R',\mathrm{Hom}_A(P,A))\otimes_R P' & & & & \\
\downarrow & & & & \\
\mathrm{Hom}_A(P',A)\otimes_{R'} P' & & & & \\
\Big\downarrow T_{P'/A} & & & & \\
A & & \xleftarrow{\quad T_{P/A} \quad} & & \mathrm{Hom}_A(P,A)\otimes_R P
\end{array}
$$

Remaining details are left to the devoted reader.

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