THE SOURCE DOUBLE-POINT CYCLE OF A FINITE MAP OF CODIMENSION ONE

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Abstract. Let X, Y be smooth varieties of dimensions n, $n+1$ over an algebraically closed field, and $f\colon X \to Y$ a finite map, birational onto its image Z. The source double-point set supports two natural positive cycles: (1) the fundamental cycle of the divisor M_2 defined by the conductor of X/Z, and (2) the direct image of the fundamental cycle of the residual scheme X_2 of the diagonal in the product $X \times_Y X$. Over thirteen years ago, it was conjectured that the two cycles are equal if the characteristic is 0 or if f is "appropriately generic." That conjecture will be established in a more general form.

1. Introduction

Let X and Y be smooth varieties over an algebraically closed field, and assume that $\dim Y - \dim X = 1$. Let $f\colon X \to Y$ be a finite map that is birational onto its image Z. For example, X might be a projective variety, and f a general central projection onto a hypersurface Z in $Y := \mathbf{P}^{n+1}$. Consider the *source double-point scheme* M_2 of f. By definition, M_2 is the effective divisor whose ideal is the conductor $\mathcal{C}_X$ of X/Z. Its underlying set consists of the points x of X whose fiber $f^{-1}f(x)$ is a scheme of length at least 2. Consider also the residual scheme X_2 of the fiber product $X \times_Y X$ with respect to the diagonal. By definition, $X_2 := \mathbf{P}(\mathcal{I}(\Delta))$ where $\mathcal{I}(\Delta)$ is the ideal of the diagonal. Consider finally the map $f_1\colon X_2 \to X$ induced by the second projection. Its image f_1X_2 too consists of the x whose fiber $f^{-1}f(x)$ has length at least 2. Thus there are two natural source double-point cycles: the fundamental cycle $[M_2]$, and the direct image $f_{1*}[X_2]$. Are the two cycles equal? For over thirteen years, the equation

$$[M_2] = f_{1*}[X_2] \tag{1.1}$$

has been known if $\dim X = 1$, and conjectured if $\dim X$ is arbitrary provided also the characteristic is zero or f is "appropriately generic" [**19**, p. 383; **9**, p. 95]. This article will establish that conjecture in a more general form.

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In arbitrary characteristic, Equation (1.1) holds if and only if

$$\operatorname{cod}(\overline{\Sigma}_2, X) \geq 2, \tag{1.2}$$

where $\overline{\Sigma}_2$ is the locus of points x in X such that $\dim \Omega^1_f(x) \geq 2$; see (3.11). (In other words, $\overline{\Sigma}_2$ is the "Thom–Boardman" locus of points where the "kernel rank," or the "differential corank," of f is at least 2. It is also the locus where the fibers of f are not "curvilinear.") If f is a "generic" map, such as a generic projection [**19**, pp. 365–366], then the ramification locus $\overline{\Sigma}_1$, the locus where $\dim \Omega^1_f(x) \geq 1$, is of codimension 2 or empty, and $\overline{\Sigma}_2$ is of codimension 6 or empty. In many important cases in practice, $\overline{\Sigma}_1$ is, however, of codimension 1, but (1.2) holds nevertheless. For example, f might be a central projection of a smooth curve X onto a plane curve Z with cusps. Indeed, Condition (1.2) holds automatically if $\dim X = 1$ or if the characteristic is zero, but not always if the characteristic is positive; see (2.6) and (2.7). In any event,

$$[M_2] = f_{1*}[X_2] + D, \tag{1.3}$$

where $D \geq 0$; moreover, the components of D are exactly the components of codimension 1 of $\overline{\Sigma}_2$; see (3.10).

¿From the point of view of the enumerative theory of singularities of mappings, $[M_2]$ is the right double-point cycle whether or not Condition (1.2) is satisfied. Indeed, its rational equivalence class is given by the double-point formula,

$$f^*f_*[X] - c_1(f)[X], \tag{1.4}$$

where $c_1(f)$ is the first Chern class of the virtual normal sheaf $\mathcal{V}_f = f^*\mathcal{T}_Y - \mathcal{T}_X$. That statement follows from Grothendieck duality theory; see (2.3). However, if f is not finite or if f is of codimension s greater than 1, then the cycle class defined by the conductor need not be given by the general double-point formula $f^*f_*[X] - c_s(f)[X]$; Fulton [**9**, 2.4, 2.5, pp. 95–96] gave examples. For an introduction to some classical instances of the double-point formula, see [**19**, pp. 312–315 and 366–368] and [**10**, pp. 167–170].

Suppose that f is *appropriately generic* in the sense that $\dim X - \dim X_2 = 1$. For example, f is appropriately generic if it is a general central projection [**19**, pp. 388]. Then Condition (1.2) holds; see the proof of (3.12). Moreover, the class of $f_{1*}[X_2]$ too is given by the double-point formula (1.4); that statement follows from residual-intersection theory, and the proof works whenever the map f_1 is of the same codimension s as f [**19**, pp. 377–384; **20**, pp. 46]. Since the cycles $f_{1*}[X_2]$ and $[M_2]$ have the same class, it was reasonable to conjecture that they are equal.

In practice, there are three important cases where it is too restrictive to assume that X and Y are smooth over a field: (1) iterative multiple-point theory [**19**, pp. 384-391; **20**], where rth-order theory for $f\colon X \to Y$ is derived from $(r-1)$th-order theory for the "iteration" map $f_1\colon X_2 \to X$; (2) Catanese's theory of "quasi-generic canonical projections" [**6**, **7**], where Y is a (singular) weighted projective space; and (3) van Straten and de Jong's deformation theory of "normalizations" [**32**, §3], where the base is an Artin ring. However, that restrictive assumption can be suitably relaxed. In fact, a priori, it is natural to assume instead that f is Gorenstein and Y is (S_2). For example, Y could be a normal scheme or the flat deformation of a normal scheme. (Coincidentally, Avramov and Foxby [**2**] are now developing a local algebraic theory of Gorenstein maps.) On the other hand, multiple-point theory of higher order or of higher codimension requires an assumption of intermediate strength, namely, that X is a local complete intersection in a smooth Y-scheme and Y is Cohen-Macaulay.

The theory in this article is part of a larger body of theory, which has had a remarkable history over the last fifteen to twenty years. On the very day (in June 1976) that Equation (1.1) was conjectured, Fulton solved the first case, where X is a smooth curve and Y a smooth surface. He proceeded by analyzing the effect of blowing up a singular point of the image Z of X in Y. Fulton's proof appears in [**9**, pp. 98–99]. Two months later, Teissier told Fulton that, the previous year, he [**30**, pp. 118–121] had been led to discover virtually the same equality and proof, while studying the equisingularity of curves over the complex numbers. At the same time, Teissier [**30**, pp. 121–123] gave a second proof, based on deforming Z.

It was a theorem whose time had come; indeed, closely related work had already been done independently. In 1974, Gusein-Zade [**16**, p. 23], as part of a study of vanishing cycles, proved Equation (1.1) for a smooth curve mapping into a smooth surface over the complex numbers; he used blowups in about the same way as Fulton and Teissier. In 1973, Fischer [**11**] studied the module of jets of a unibranched map from a smooth curve into a smooth surface by considering the same blowups. In that case, he obtained a length formula that is equivalent to Equation (1.1). The equivalence holds because the module of jets and the structure sheaf of X_2 are locally isomorphic as $\mathcal{O}_X$-modules, for example, because of (3.2)(2). Later, in 1976, Brown [**3**] generalized Fischer's work, eliminating the hypothesis of unibranchedness. Brown also found that Fisher's proofs were mildly incomplete in the case of positive characteristic, leading Fischer to publish an improved version [**12**] in 1978. Of those five authors, only Fulton mentioned the case of higher dimensional X and Y.

In 1972 Artin and Nagata [**1**, (5.8), pp. 322], inspired by some unpublished results and questions of Mumford, proved a version of Equation (1.1) in the case that X is a smooth curve, Y is a smooth surface, $f\colon X \to Y$ is any map birational onto its image, and the base is a field of any characteristic. Their version, like Fischer's and Brown's, is a statement about the ideal of the diagonal of $X \times_Y X$. Their proof, like Teissier's second, involves deforming f into a map whose image has simple nodes at worst. Artin and Nagata also gave an example that shows that their version of Equation (1.1) does not generalize to the case that X is a smooth surface and $Y := \mathbf{P}^4$.

In a nutshell, the proof of (1.3) runs as follows; see (3.9). First, it is shown that the direct image of $[X_2]$ on $X \times_Y X$ is equal to the fundamental cycle of the ideal of the diagonal, $[\mathcal{I}(\Delta)]$, diminished by a positive cycle C whose components lie in the diagonal and correspond precisely to the components of codimension 1 of $\overline{\Sigma}_2$. In fact, off the image of $\overline{\Sigma}_2$ in the diagonal subscheme, the structure map $p\colon X_2 \to X \times_Y X$ is a closed embedding, and its ideal is $\mathcal{A}nn(\mathcal{I}(\Delta))$; see (3.4)(2). Moreover, off the image of $\overline{\Sigma}_2$, locally $\mathcal{I}(\Delta)$ is generated by one element, and hence is isomorphic to $p_*\mathcal{O}_{X_2}$; see (3.3). (So, in particular, X_2 is equal, off the image of $\overline{\Sigma}_2$, to the double-point scheme X'_2 considered by Mond [**26**, § 3, pp. 368–371] and Marar and Mond [**25**, 1.1, pp. 554-555], which is defined by

$\mathcal{A}nn(\mathcal{I}(\Delta))$. Moreover, as $\mathcal{I}(\Delta)$ is locally generated by one element,

$$\mathcal{A}nn_{X\times_Y X}(\mathcal{I}(\Delta)) = \mathcal{F}itt^0_{X\times_Y X}(\mathcal{I}(\Delta)).$$

Mond [**26**, 3.2(i), p. 369] made a note of that equation because the Fitting ideal is "more readily calculable" [**25**, bottom p. 554]. Furthermore [**25**, bottom p. 554; **31**], if X is a local complete intersection in a smooth Y-scheme and Y is Cohen–Macaulay — for example, if X and Y are smooth over a field — then that equation continues to hold, and X_2' is Cohen–Macaulay and is of finite flat dimension over Y; see (3.13).)

On the other hand, at any point w of the diagonal in the image of $\overline{\Sigma}_2$, the fiber $p^{-1}w$ has dimension at least 1; see (3.2)(1). Therefore, the components of X_2 lying over the image of $\overline{\Sigma}_2$ do not contribute to $p_*[X_2]$, and the other components of X_2 contribute with the same multiplicity to both $p_*[X_2]$ and $[\mathcal{I}(\Delta)]$. (However, Ulrich [**31**] has proved that, if X is a local complete intersection in a smooth Y-scheme and Y is (S_2), then $[X_2'] = [\mathcal{I}(\Delta)]$; see (3.13).) The preceding considerations (including those in parentheses) are valid in great generality; in particular, f may have any codimension s (provided that, in the more sophisticated statements, it is assumed that M_2 is of codimension at least s in X — whence it follows that M_2 and X_2' are of codimension exactly s; see (3.13)).

It now suffices to prove that the direct image of $[\mathcal{I}(\Delta)]$ is equal to $[M_2]$; in other words, at each generic point ξ of M_2, the length of $\mathcal{I}(\Delta)$ is equal to the colength of $\mathcal{C}_X$. The latter statement follows from these equations:

$$\mathcal{F}itt^0_X(\mathcal{I}(\Delta)) = \mathcal{F}itt^0_Y(f_*\mathcal{O}_X/\mathcal{O}_Z)\,\mathcal{O}_X = \mathcal{C}_X. \tag{1.5}$$

Indeed, it will be proved that $\mathcal{C}_X$ is invertible as f is Gorenstein; see (2.3). Hence, $\mathcal{I}(\Delta)$ is of flat dimension 1 over X, and so the desired length-colength equation holds.

The first equation in (1.5) is proved via rather simple and general considerations, which require no special hypotheses; see (3.4)(1). The key lemma (3.3) was apparently known to Artin and Nagata, and perhaps to Mumford; see the statement in parentheses on line 6 of p. 322 in [**1**]. The second equation in (1.5) is an immediate consequence of the following equation on Y:

$$\mathcal{F}itt^0_Y(f_*\mathcal{O}_X/\mathcal{O}_Z) = \mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_Z). \tag{1.6}$$

Equation (1.6) follows from a general theorem of Buchsbaum and Eisenbud [**4**, p. 232]; see (3.7) and (3.5). However, only a special case of the general theorem is needed here, and in that case, the theorem's proof simplifies to a few lines involving the Hilbert–Burch theorem; Eisenbud showed that short proof to the authors (on 26 May 1989), and it too is given in (3.5).

The *target double-point scheme* N_2 is, by definition, the subscheme of Y of the adjoint ideal $\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_Z)$. So, Equation (1.6) says, in other words, that N_2 is determinantal, cut out locally by the maximal minors of any matrix presenting $f_*\mathcal{O}_X/\mathcal{O}_Z$. Now, the proof in (3.5) of (1.6) also shows that N_2 is of flat dimension 2 in Y; hence, it is of pure codimension 2 in Y by the Intersection Theorem of Peskine–Szpiro and Roberts, and it is Cohen–Macaulay if Y is by the Auslander-Buchsbaum Theorem.

Equation (1.6) was already known, however. Mond and Pellikaan, in their March 1988 preprint [**27**, p. 121], had obtained it independently and also from Buchsbaum and Eisenbud's theorem. They prove (1.6) in the course of proving the equation,

$$\mathcal{F}itt^0_Y(f_*\mathcal{O}_X/\mathcal{O}_Z) = \mathcal{F}itt^1_Y(X). \tag{1.7}$$

That equation interested them because it, together with (1.6), says that the target double-point scheme N_2 is also defined by the Fitting ideal $\mathcal{F}itt^1_Y(X)$.

Another proof of (1.7) is found, as Mond and Pellikaan indicated, in a June 1988 preprint of van Straten and de Jong, who used the Hilbert–Burch theorem directly, [**32**; combine (4.8), (4.12), and (4.13)]. They used (1.7) to compare the deformation theory of the pair (X, Z) with that of (N_2, Z). Mond and Pellikaan, and van Straten and de Jong gave credit to Catanese [**6**] (see [**7**] also) for introducing the key ideas in 1982. (In turn, Catanese said that he drew inspiration from work of Arbarello, Sernesi, and Ciliberto.) Catanese's purpose was to study "pluriregular varieties of free general type" via "quasi-generic canonical projections."

Independently, in 1981, Gruson and Peskine [**15**] were led to Equations (1.6) and (1.7) while studying the scheme of r-secants of a smooth space curve C. They viewed the secant scheme as the target r-fold locus of the map $f\colon X \to Y$, where Y is the Grassmannian of lines L and where X is the variety of pairs (P, L) with $P \in L \cap C$. They did not prove (1.6) directly, but first proved a form [**15**, 1.5, p. 5] of the equation,

$$\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_Z) = \mathcal{F}itt^1_Y(X). \tag{1.8}$$

Their proof is simple and direct, and does not involve the Hilbert–Burch theorem or anything like it. They do prove a form [**15**, 1.3, p. 4] of (1.7), but their proof needs an additional hypothesis, which, as it turns out, amounts to the assumption that $\overline{\Sigma}_2$ is empty. On the other hand, under that assumption, they prove a more general statement, involving the higher-order Fitting ideals.

The higher-order multiple-point loci of $f\colon X \to Y$ are also of some interest. The first job is to find a reasonable scheme-theoretic definition of them. Assume, as always, that $f\colon X \to Y$ is finite and birational onto its image Z. Assume also that $\overline{\Sigma}_2$ is empty. This hypothesis is not that much of a restriction in the rth-order theory for $r \le 6$, because the expected codimension in X of $\overline{\Sigma}_2$ is 6. Moreover, there are many applications where, in fact, $\overline{\Sigma}_2$ is empty. Assume finally that X is a local complete intersection in a smooth Y-scheme and that Y is Cohen-Macaulay. Under roughly those hypotheses, Gruson and Peskine [**15**] and Mond and Pellikaan [**27**] independently strove to show that the Fitting ideal $\mathcal{F}itt^{r-1}_Y(X)$ defines a reasonable scheme N_r of target r-fold points. For example, for $r = 1$, that Fitting ideal defines the scheme-theoretic image Z; see (2.2). For $r = 2$, the Fitting ideal is equal to the adjoint ideal by (1.8); so the new definition of N_2 agrees with the old. In the work [**23**] under preparation, the present authors will develop the following additional evidence for the reasonableness of this definition of N_r.

Following the iterative approach to multiple-point theory of [**20**], define the scheme M_r of source r-fold points of f as the scheme of target $(r-1)$-fold points of f_1; in other words, define M_r as the scheme with ideal $\mathcal{F}itt^{r-2}_X(X_2)$. For example, for $r = 2$, that

Fitting ideal is equal to the conductor $\mathcal{C}_X$; see (3.4) and (3.7). So the new definition of M_2 agrees with the old. Now, if the definitions of M_r and N_r are indeed reasonable, then these schemes should be compatible under pullback:

$$M_r = f^{-1}N_r.$$

That compatibility equation will be proved in [**23**]; the proof is similar to the proof of (3.4).

Assume that M_r and N_r have the expected codimensions, $r-1$ and r, everywhere. Then a general point of N_r has an inverse image of length r; so the cycle relation

$$f_*[M_r] = r[N_r] \tag{1.9}$$

should hold. For example, for $r=2$, this relation is equivalent to the usual Gorenstein formula, because then M_2 and N_2 are defined by the conductors on X and Z. Relation (1.9) will be proved for arbitrary r in [**23**].

There is another generalization of the usual Gorenstein formula, due to Gruson and Peskine [**15**, Prop. 2.6, p. 13]. For $r=2$, it reduces to the other form of the usual formula: the colength of the conductor in a 1-dimensional Gorenstein domain is equal to the colength of the domain in its normalization. For arbitrary r, the generalization says intuitively that a general $(r+1)$-fold point counts as $r+1$ r-fold points. In [**23**], following the approach to multiple-point theory based on the Hilbert scheme, which is developed in [**22**], it will be shown how to interpret Gruson and Peskine's generalization of the Gorenstein formula as a statement about the Hilbert scheme Hilb^r_f, and how to derive it from Relation (1.9). The key step is to prove that Hilb^r_f is equal to the blowup of N_r along N_{r+1}.

2. The double-point schemes

(2.1) *Setup.* Let $f\colon X \to Y$ be a finite map of locally Noetherian schemes. Assume that f is birational onto its image; more precisely, assume that there is an open subset U of Y such that its preimage $f^{-1}U$ is dense in X and the restriction $f^{-1}U \to U$ is an embedding. Assume that f is of pure codimension 1; that is, if ξ is the generic point of an arbitrary component of X, then $\dim \mathcal{O}_{Y,f\xi} = 1$. Assume that f is of flat dimension 1. Finally, assume that Y satisfies Serre's condition (S_2) [**14**, (5.7.2), p. 103]: for every $y \in Y$,

$$\operatorname{depth}(\mathcal{O}_{Y,y}) \geq \inf(2, \dim \mathcal{O}_{Y,y}).$$

These conditions will be assumed without further mention throughout Section 2.

If, in the derived category, $f^!\mathcal{O}_Y$ is isomorphic to a (shifted) invertible sheaf ω_f, then f is called *Gorenstein* [**17**, p. 144]. If f is Gorenstein, define its first Chern class as that of ω_f:

$$c_1(f) := c_1(\omega_f).$$

For example, if there is a factorization $f = \pi i$ where $i\colon X \hookrightarrow P$ is a regular embedding and $\pi\colon P \to Y$ is smooth, then f is Gorenstein and

$$\omega_f = \det(\mathcal{V}_i) \otimes \det(\mathcal{T}_\pi)^{-1}$$

where $\mathcal{V}_i$ is the normal sheaf and $\mathcal{T}_\pi$ is the tangent sheaf. For instance, if X and Y are smooth over some base scheme S, then the product $P := X \times_S Y$, the graph map $i\colon X \to P$, and the projection $\pi\colon X \times_S Y \to Y$ will work; in this case,

$$\omega_f = \det(\mathcal{T}_{X/Y})^{-1} \otimes \det(\mathcal{T}_{Y/S}).$$

The ideal $\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{I}m\,\mathcal{O}_Y)$ is called the *adjoint ideal.* The scheme it defines is denoted by N_2 and called the *target double-point scheme.* Its underlying set consists of the points y of Y whose fiber $f^{-1}(y)$ is a scheme of length at least 2 over $k(y)$. The adjoint ideal is an $f_*\mathcal{O}_X$-module, and the associated sheaf on X

$$\mathcal{C}_X := \widetilde{\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{I}m\,\mathcal{O}_Y)}$$

is an ideal, called the *conductor on X*. The corresponding scheme is denoted by M_2 and called the *source double-point scheme.* Obviously, $M_2 = f^{-1}N_2$ as schemes, and the restriction $M_2 \to N_2$ is finite and surjective.

The Fitting ideal $\mathcal{F}itt_X^{r-1}(\Omega_f^1)$ defines a scheme, denoted $\overline{\Sigma}_r$. Its underlying set consists of the points x of X such that $\dim \Omega_f^1(x) \geq r$. Obviously, $\overline{\Sigma}_0 = X$, and

$$M_2 \supseteq \overline{\Sigma}_1 \supseteq \overline{\Sigma}_2 \supseteq \cdots .$$

The formation of $\overline{\Sigma}_r$ commutes with base change as the formation of Ω_f^1 does and the formation of a Fitting ideal does.

Denote by Z the scheme-theoretic image of X in Y. By definition [**13**, (6.10.1), p. 324], Z is the smallest closed subscheme of Y through which f factors. Because f is quasi-compact and quasi-separated, Z exists and is defined by the ideal $\mathcal{A}nn_Y(f_*\mathcal{O}_X)$. Obviously, N_2 is a closed subscheme of Z; its ideal is the sheaf

$$\mathcal{C}_Z := \mathcal{A}nn_Z(f_*\mathcal{O}_X/\mathcal{O}_Z),$$

and $\mathcal{C}_Z$ is called the *conductor on Z*.

Proposition (2.2) *The scheme-theoretic image Z of X in Y is a divisor, and its ideal is equal to the Fitting ideal $\mathcal{F}itt_Y^0(X)$. In other words, locally Z is defined by the determinant of any square matrix presenting $\mathcal{O}_X$ over $\mathcal{O}_Y$; such matrices exist, and their determinants are regular elements. Moreover, the formation of Z commutes with base change.*

Proof. Because f is finite, the Fitting ideal $\mathcal{F}itt_Y^0(X)$ exists. Because f is of flat dimension 1 and of codimension 1, locally $\mathcal{O}_X$ is presented over $\mathcal{O}_Y$ by a square matrix whose determinant is regular and generates $\mathcal{F}itt_Y^0(X)$. Let W denote the corresponding divisor. Then W has no embedded components because Y satisfies (S_2).

The schemes W and Z have the same support, and $Z \subseteq W$ because

$$\mathcal{A}nn_X(Y)^n \subseteq \mathcal{F}itt_Y^0(X) \subseteq \mathcal{A}nn_X(Y)$$

for some integer n. Moreover, W and Z are generically equal because f is generically an embedding. Therefore, W and Z are equal because W has no embedded components.

The formation of Z commutes with base change because the formation of a Fitting ideal does.

Theorem (2.3) *The conductor $\mathcal{C}_X$ is an invertible sheaf if and only if f is a Gorenstein map. In either case, the double-point cycle $[M_2]$ is given by the double-point formula,*

$$[M_2] = f^*f_*[X] - c_1(f)[X],$$

which holds modulo rational equivalence.

Proof. The proof is a version of that in [**19**, pp. 365–366]; this version uses more abstract, but nevertheless standard Grothendieck duality theory [**17**, **24**].

Say $f = jg$ where $g\colon X \to Z$ and $j\colon Z \hookrightarrow Y$. Work in the derived category of quasi-coherent sheaves. Trivially

$$f^!\mathcal{O}_Y = \mathbf{R}\mathcal{H}om_X(\mathcal{O}_X,\, f^!\mathcal{O}_Y).$$

Since g and j are finite, $\mathbf{R}g_* = g_*$ and $\mathbf{R}j_* = j_*$. Also, $f^! = g^!j^!$ and $f_* = j_*g_*$. Hence, duality yields the equations,

$$\begin{aligned}(g_*)f^!\mathcal{O}_Y &= \mathbf{R}\mathcal{H}om_Z(g_*\mathcal{O}_X,\, j^!\mathcal{O}_Y),\\ j_*\mathbf{R}\mathcal{H}om_Z(g_*\mathcal{O}_X,\, j^!\mathcal{O}_Y) &= \mathbf{R}\mathcal{H}om_Z(f_*\mathcal{O}_X,\, \mathcal{O}_Y).\end{aligned}$$

The latter complex has all its cohomology concentrated in degree 1, because f is finite and of flat dimension 1 and because Z is nowhere dense in Y and Y has no embedded components as it satisfies (S_2). Hence, $f^!\mathcal{O}_Y$ does too. Say $f^!\mathcal{O}_Y[1]$ is isomorphic to the quasi-coherent sheaf ω_f.

By (2.2), Z is a divisor in Y. So $j^!\mathcal{O}_Y = \mathcal{O}_Z(Z)[-1]$. Hence

$$g_*\omega_f = \mathcal{H}om_Z(g_*\mathcal{O}_X,\, \mathcal{O}_Z(Z)) = \mathcal{H}om_Z(g_*\mathcal{O}_X,\, \mathcal{O}_Z) \otimes \mathcal{O}_Z(Z).$$

Now, X has no embedded component because f is of flat dimension 1 and Y satisfies (S_2); hence, $g_*\mathcal{O}_X$ is contained in the sheaf of total quotient rings of $\mathcal{O}_Z$ (that condition is not implied by the definition of birationality adopted in (2.1)). Therefore, standard elementary considerations show that evaluation at 1 defines an isomorphism,

$$\mathcal{H}om_Z(g_*\mathcal{O}_X,\, \mathcal{O}_Z) = \mathcal{A}nn_Z(g_*\mathcal{O}_X/\mathcal{O}_Z);$$

its inverse sends a local section to multiplication by that section. Therefore, taking associated sheaves yields the following equation on X:

$$\omega_f = \mathcal{C}_X \otimes g^*\mathcal{O}_Z(Z).$$

The assertions follow immediately.

Proposition (2.4) *The source double-point scheme M_2 is of pure codimension* 1 *in X, and the target double-point scheme N_2 is of pure codimension* 1 *in Z and of pure codimension* 2 *in Y.*

Proof. Since $X \to Z$ is finite and birational and since $M_2 = f^{-1}N_2$, it suffices to treat N_2. By (2.1), Z is a divisor in Y. So it suffices to prove that N_2 is of pure codimension 2 in Y. Now, by definition, N_2 is the support of the $\mathcal{O}_Y$-module $f_*\mathcal{O}_X/\mathcal{O}_Z$. That module is of flat dimension at most 2 because $\mathcal{O}_X$ and $\mathcal{O}_Z$ are both of flat dimension 1. Hence N_2 is everywhere of codimension at most 2 by virtue of the Intersection Theorem; it is well known that the case needed here may be derived easily from the work of Peskine and Szpiro [**28**], but the general case was proved by P. Roberts [**29**]. On the other hand, N_2 is of codimension at least 2, because $X \to Z$ is birational. Therefore, N_2 is of pure codimension 2.

The Intersection Theorem is not needed here if f is Gorenstein, for then $\mathcal{C}_X$ is invertible by (2.3).

Lemma (2.5) *Let ξ be a generic point of a component of M_2. Suppose that X is regular at ξ and that the field extension $k(\xi)/k(f\xi)$ is separable. Then $\xi \notin \overline{\Sigma}_2$.*

Proof. Denote the reduced scheme $(M_2)_{\text{red}}$ by D. Then D is of pure codimension 1 in X by (2.4). So D is a divisor in X at ξ because X is regular there. At ξ, consider the standard exact sequence,

$$\mathcal{O}_D(-D) \longrightarrow \Omega^1_f|D \longrightarrow \Omega^1_{D/Y} \longrightarrow 0.$$

The first term is invertible, and the third term vanishes because $k(\xi)/k(f\xi)$ is finite and separable. Hence, $\dim_{k(\xi)} \Omega^1_f(\xi) \le 1$. In other words, $\xi \notin \overline{\Sigma}_2$.

Proposition (2.6) *Suppose that X and Y are of finite type over a field k, and that X is regular in codimension 1 (for example, normal). Suppose either* (a) $\dim X = 1$ *and k is perfect or* (b) *k is of characteristic zero. Then* $\operatorname{cod}(\overline{\Sigma}_2, X) \ge 2$.

Proof. Let ξ be a generic point of a component of M_2. Then X is regular at ξ because M_2 is of pure codimension 1 in X by (2.4). If $\dim X = 1$ and k is perfect, then $k(\xi)/k$ is finite and separable; whence, then $k(\xi)/k(f\xi)$ is separable. Of course, $k(\xi)/k(f\xi)$ is separable if the characteristic of k is 0. Therefore, $\xi \notin \overline{\Sigma}_2$ by (2.5). Thus the assertion holds.

(2.7) *Example.* Here is an example where $\overline{\Sigma}_2$ has codimension 1. Fix an algebraically closed field of positive characteristic p. Let X be a closed, reduced, and irreducible surface in $\mathbf{P}^3$, and consider its Gauss map $f\colon X_0 \to Y$, where X_0 is the smooth locus of X, and Y is the dual $\mathbf{P}^3$. Fix a point P of X_0, and choose affine coordinates x, y, z such that x, y are regular parameters of X at P. Then

$$\dim \Omega^1_f(P) = 2 - \operatorname{rank} \frac{\partial^2 z}{\partial(x,\, y)^2}$$

by [**18**, (2.6.1), p. 153; **21**, § I-5, 175–177]. Therefore, the point P lies in $\overline{\Sigma}_2$ if and only if the Hessian $\frac{\partial^2 z}{\partial(x,\, y)^2}$ vanishes at P.

Suppose $p = 2$. Then $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ vanish identically near P. Hence $\overline{\Sigma}_2$ is defined near P by the vanishing of $\frac{\partial^2 z}{\partial y \partial x}$. Suppose that X is smooth of degree d at least 2. Then $f: X \to Y$ is finite. That fact is well known, and holds in any characteristic. It holds, for example, because $f^*\mathcal{O}_Y(1)$ is ample, as it is equal to $\mathcal{O}_X(d-1)$ [**19**, middle of p. 360; **21**, § II-2, p.190]. Suppose that X is general of its degree. Then f is birational onto its image [**18**, (5.6), p. 176; **21**, (21), p. 180]. In particular, $\overline{\Sigma}_2 \neq X$. Therefore, $\operatorname{cod}(\overline{\Sigma}_2, X) = 1$.

Suppose $p \geq 3$. Then $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ vanish identically if, for instance,

$$z = xy(x+y)^p + x^{p+1} + y^{p+1}.$$

Then, moreover, $\frac{\partial^2 z}{\partial y \partial x} = (x+y)^p$. So, if X is the surface with that equation, then its Gauss map f is birational onto its image by the Hessian Criterion [**18**, (3.3), p. 155; **21**, (12), p. 176]. Unfortunately, X has a (unique) singular point at infinity, (0,0,1,0). However, that point corresponds to a curve C in Y, and a computation shows that the (reducible) curve D of X corresponding to C contains the entire curve at infinity of X, but does not contain the locus $\{x + y = 0\}$. Hence, the restriction $(X - D) \to (Y - C)$ is finite, and its $\overline{\Sigma}_2$ is of codimension 1.

3. The residual double-point cycle

Definition (3.1) Let $f: X \to Y$ be a separated map of schemes. Form the residual scheme of the diagonal and the corresponding map,

$$X_2 := \mathbf{P}(\mathcal{I}(\Delta)) \quad \text{and} \quad f_1: X_2 \xrightarrow{p} X \times_Y X \xrightarrow{p_2} X,$$

where $\mathcal{I}(\Delta)$ is the ideal of the diagonal, p is the structure map, and p_2 is the second projection. Then X_2 is called the *iteration*, or *derived*, scheme, and f_1 is called the *iteration*, or *derived*, map [**20**, 4.1, pp. 36–37; **22**, (2.10)]. If f is proper, then $f_{1*}[X_2]$ is defined and will be called the *residual double-point cycle* of f.

Lemma (3.2) *Let $f: X \to Y$ be a separated map locally of finite type between locally Noetherian schemes. Let w be a point of $X \times_Y X$.*

(1) *The following four conditions are equivalent:*

(a) *The structure map $p: X_2 \to X \times_Y X$ is a closed embedding at w.*
(b) *The fiber $p^{-1}w$ is empty or of dimension* 0.
(c) *Either w lies off the diagonal, or w lies on the diagonal and* $\dim \Omega^1_f(p_2 w) \leq 1$.
(d) *The ideal $\mathcal{I}(\Delta)$ of the diagonal is generated by one element at w.*

(2) *Let U be an open subset of $X \times_Y X$ on which $\mathcal{I}(\Delta)$ is generated by a single element of $\Gamma(U, \mathcal{I}(\Delta))$. Then the restriction $p^{-1}U \to U$ is a closed embedding, its ideal is $\mathcal{A}nn(\mathcal{I}(\Delta))|U$, and there is an isomorphism of $\mathcal{O}_U$-modules,*

$$p_* \mathcal{O}_{X_2}|U \simeq \mathcal{I}(\Delta)|U.$$

Proof. Trivially (a) implies (b). For convenience, set $\mathcal{I} := \mathcal{I}(\Delta)$. Then $X_2 = \mathbf{P}(\mathcal{I})$ by (3.1). So the fiber $p^{-1}w$ is equal to $\mathbf{P}((\mathcal{I}/\mathcal{I}^2)(w))$. Hence (b) implies (c) because $\mathcal{I}/\mathcal{I}^2$

is isomorphic to the direct image under the diagonal map of Ω^1_f. Also because of that isomorphism and by Nakayama's lemma, (c) implies (d).

Let U be an open subset of $X \times X$ on which $\mathcal{I}$ is generated by a single section. That section defines a surjection on U from $\mathcal{O}_{X\times X}$ to $\mathcal{I}$, and its kernel is obviously $\mathcal{A}nn(\mathcal{I})$; in other words, there is an exact sequence,

$$0 \longrightarrow \mathcal{A}nn(\mathcal{I})|U \longrightarrow \mathcal{O}_{X\times X}|U \longrightarrow \mathcal{I}|U \longrightarrow 0. \tag{3.2.1}$$

In general, if $\mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of quasi-coherent sheaves on an arbitrary scheme, then the ideal of $\mathbf{P}(\mathcal{G})$ in $\mathbf{P}(\mathcal{F})$ is equal to the image of $\mathcal{E}(-1)$ in $\mathcal{O}_{\mathbf{P}(\mathcal{F})}$, because the following sequence is well known to be exact [**5**, Ch. III, §6, no. 2, Prop. 4, p. 499]:

$$\mathcal{E}\,\mathcal{S}ym(\mathcal{F})[-1] \longrightarrow \mathcal{S}ym(\mathcal{F}) \longrightarrow \mathcal{S}ym(\mathcal{G}) \longrightarrow 0.$$

Since $\mathbf{P}(\mathcal{O}_U) \xrightarrow{\sim} U$ and $\mathcal{O}_{\mathbf{P}(\mathcal{O}_U)}(-1) = \mathcal{O}_{\mathbf{P}(\mathcal{O}_U)}$, therefore $p^{-1}U \to U$ is a closed embedding, with ideal $\mathcal{A}nn(\mathcal{I})|U$. A second look at (3.2.1) now reveals that $p_*\mathcal{O}_{X_2}|U$ is isomorphic to $\mathcal{I}|U$. Thus (2) holds. Hence, (d) implies (a).

Lemma (3.3) *Let R be a (commutative) ring, B an R-algebra, and I the kernel of the multiplication map $B\otimes_R B \to B$. View $B\otimes_R B$ as a B-algebra via the homomorphism u given by $u(b) := 1\otimes b$. Then there is an isomorphism of B-modules,*

$$I \simeq (B/\operatorname{Im} R) \otimes_R B.$$

Proof. The homomorphism u splits the following exact sequence of B-modules:

$$0 \longrightarrow I \longrightarrow B\otimes_R B \longrightarrow B \longrightarrow 0.$$

So I is isomorphic to the cokernel of u. On the other hand, tensoring the exact sequence $R \to B \to B/\operatorname{Im} R \to 0$ with B yields the exact sequence,

$$B \longrightarrow B\otimes_R B \longrightarrow (B/\operatorname{Im} R)\otimes_R B \longrightarrow 0,$$

in which the first map is u. So the cokernel of u is also isomorphic to $(B/\operatorname{Im} R)\otimes_R B$. Thus the assertion holds.

Proposition (3.4) *Let $f\colon X \to Y$ be a finite map of locally Noetherian schemes.*

(1) *View the ideal $\mathcal{I}(\Delta)$ of the diagonal of $X\times_Y X$ as an $\mathcal{O}_X$-module via the second projection p_2. Then*

$$\mathcal{F}itt^0_X(\mathcal{I}(\Delta)) = \mathcal{F}itt^0_Y(f_*\mathcal{O}_X/\mathcal{I}m\,\mathcal{O}_Y)\,\mathcal{O}_X.$$

(2) *Off the image of $\overline{\Sigma}_2$ under the diagonal map, the structure map $p\colon X_2 \to X\times_Y X$ is a closed embedding, and its ideal is $\mathcal{A}nn(\mathcal{I})$. Off $\overline{\Sigma}_2$, the iteration map $f_1\colon X_2 \to X$ is finite, and*

$$\mathcal{F}itt^0_X(X_2) = \mathcal{F}itt^0_X(\mathcal{I}(\Delta)).$$

Proof. (1) The two Fitting ideals are defined because f is finite. The asserted equation holds locally because of (3.3), as the formation of a Fitting ideal commutes with base change. Therefore the equation holds globally.

(2) Off $\overline{\Sigma}_2$, the structure map p is a closed embedding by (3.2)(1). So $f_1 := p_2 p$ is finite, because p_2 is as f is.

It suffices to establish the asserted equality of ideals locally at each point $x \notin \overline{\Sigma}_2$. By (3.2)(1), $\mathcal{I}(\Delta)$ is generated by one element at each point w of $p_2^{-1}x$. Since p_2 is finite, there is, therefore, a neighborhood V of x such that, if $U := p_2^{-1}V$, then $\mathcal{I}(\Delta)|U$ is generated by a single element of $\Gamma(U, \mathcal{I}(\Delta))$. So (3.2)(2) yields the asserted equality on V.

Lemma (3.5) *Let R be a Noetherian local ring. Let $A := R/\delta$ where δ is a regular element (non-zero-divisor). Let F be a finitely generated A-module such that $A \subset F \subset K$, where K is the total fraction ring of A. Suppose that $F/A \neq 0$ and that the flat dimension $\mathrm{f.d}_R F$ is 1. Then*

$$\mathit{Ann}_R(F/A) = \mathit{Fitt}_R^0(F/A),$$

and $R/\mathit{Ann}_R(F/A)$ is an R-module of flat dimension 2, grade 2, and codimension 2.

Proof. Choose elements $x_1, \ldots, x_h$ of F whose images in $F/m_A F$ form a basis, and let E be the submodule of $R^{\oplus h}$ of relations among the x_i. Then E is free because $\mathrm{f.d}_R F = 1$, and E is of rank h because $F \subset K$. It is now easy to see that F/A is presented by a h by $h+1$ matrix.

The codimension of the R-module F/A is at least 2 because $F \subset K$. Furthermore, the grade of F/A is at least 2; that is, $\mathit{Ann}_R(F/A)$ contains a regular sequence of two elements. Indeed, if $\mathit{Ann}_R(F/A)/\delta R$ consisted of entirely of zero-divisors, then $\mathit{Ann}_R(F/A)$ would lie in the union of the associated primes of δ, so in one of them, say P. However, $(F/A)_P = 0$ because $A_P = K_P$.

Since $(h+1)-h+1 = 2$, it follows from a general theorem of Eagon and Hochster that the grade of F/A is exactly 2, and from a general theorem of Buchsbaum and Eisenbud that $\mathit{Ann}_R(F/A)$ is equal to $\mathit{Fitt}_R^0(F/A)$; for both conclusions, see [**4**, top p. 232]. The general theorems may be avoided in the case at hand by the following argument, which also yields the remaining two assertions.

Since $\mathrm{f.d}_R F = 1$ and $\mathrm{f.d}_R A = 1$, obviously $\mathrm{f.d}_R F/A \leq 2$. So, since F/A is presented by an h by $h+1$ matrix, there is an exact sequence

$$0 \longrightarrow R \longrightarrow R^{\oplus(h+1)} \longrightarrow R^{\oplus h} \longrightarrow F/A \longrightarrow 0.$$

Since the grade of F/A is at least 2, dualizing that exact sequence yields this one,

$$0 \longrightarrow R^{\oplus h} \longrightarrow R^{\oplus(h+1)} \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where I is an ideal. In particular, $R/I = \mathrm{Ext}_R^2(F/A, R)$. Since any element of R that kills F/A also kills $\mathrm{Ext}_R^2(F/A, R)$, therefore

$$\mathit{Ann}_R(F/A) \subseteq I.$$

Now, the dual of the second exact sequence is obviously the first. Hence

$$\mathrm{Ext}^i_R(R/I,\, R) = \begin{cases} 0, & \text{if } i < 2; \\ F/A, & \text{if } i = 2. \end{cases}$$

Therefore, the grade of R/I is exactly 2. Moreover, I lies in $Ann_R(F/A)$, so the two ideals are equal. Finally, the Hilbert–Burch Theorem [**8**, Thm. 1, p. 122] yields that $I = Fitt^0_R(F/A)$.

The preceding argument also shows that $R/Ann_R(F/A)$ has flat dimension 2, as asserted, because $Ann_R(F/A) = I$. Hence its codimension is at most 2 by the Intersection Theorem of Peskine–Szpiro and Roberts (see the proof of (2.4)). Since, as was noted above, its codimension is at least 2, it is exactly 2, as asserted.

Proposition (3.6) *Under the conditions of* (2.1)*, the target double-point scheme N_2 is of pure flat dimension* 2*, grade* 2*, and codimension* 2 *in Y.*

Proof. By (2.2), Z is a divisor in Y. So the assertion follows (3.5) applied locally.

Proposition (3.7) *If the conditions of* (2.1) *hold, then*

$$\begin{aligned} \mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_Z) &= \mathcal{F}itt^0_Y(f_*\mathcal{O}_X/\mathcal{O}_Z) \\ \mathcal{C}_Z &= \mathcal{F}itt^0_Z(f_*\mathcal{O}_X/\mathcal{O}_Z) \\ \mathcal{C}_X &= \mathcal{F}itt^0_Z(f_*\mathcal{O}_X/\mathcal{O}_Z)\,\mathcal{O}_X. \end{aligned}$$

Proof. By (2.2), Z is a divisor in Y. So the first equation holds at each point of Y by (3.5). The second and third equations are easily derived from the first.

Proposition (3.8) *Under the conditions of* (2.1)*, the formation of the double-point schemes, M_2 and N_2, commutes with base change.*

Proof. The assertion follows directly from (3.7) because the formation of a Fitting ideal commutes with base change.

Lemma (3.9) *Under the conditions of* (2.1)*, suppose f is Gorenstein. Then*

$$p_{2*}[\mathcal{I}(\Delta)] = [M_2]$$

where $[\mathcal{I}(\Delta)]$ is the fundamental cycle of the ideal of the diagonal, viewed simply as an $\mathcal{O}_X$-module.

Proof. By (3.4)(1) and (3.7), $\mathcal{F}itt^0_X(\mathcal{I}(\Delta))$ is equal to $\mathcal{C}_X$. By (2.3), $\mathcal{C}_X$ is invertible. Hence, by standard algebra, $\mathcal{I}(\Delta)$ is of flat dimension 1 over X. Hence, at each generic point ξ of M_2, the length of $\mathcal{I}(\Delta)$ is equal to the colength of $\mathcal{C}_X$ [**10**, A.2.3, p. 411]. Therefore, $p_{2*}[\mathcal{I}(\Delta)]$ and M_2 are equal at ξ.

It remains to note that, if η is the generic point of a component of the support of $\mathcal{I}(\Delta)$, then $\dim \mathcal{O}_{X,p_2\eta} = 1$. However, the completion of the stalk $\mathcal{I}(\Delta)_\eta$ is a module of finite length and of flat dimension 1 over the completion of $\mathcal{O}_{X,p_2\eta}$. Hence its *0*th Fitting ideal is invertible, and is primary for the maximal ideal. Therefore, $\dim \mathcal{O}_{X,p_2\eta} = 1$.

Theorem (3.10) *Under the conditions of* (2.1), *suppose f is Gorenstein. Then*

$$[M_2] = f_{1*}[X_2] + D$$

where $D \geq 0$, and the components of D are exactly the components of codimension 1 *of $\overline{\Sigma}_2$.*

Proof. It follows from (3.2) that $[\mathcal{I}(\Delta)] = p_*[X_2] + C$ where C is a positive cycle whose components are exactly those components of the support of $\mathcal{I}(\Delta)$ that are images under the diagonal map of the components of $\overline{\Sigma}_2$. So, the assertion follows from (3.9) because $f_1 := p_2 p$ and because of (2.4).

Corollary (3.11) *Under the conditions of* (2.1), *suppose f is Gorenstein. Then*

$$[M_2] = f_{1*}[X_2]$$

off $\overline{\Sigma}_2$, and that equation holds everywhere if and only if also $\operatorname{cod}(\overline{\Sigma}_2, X) \geq 2$.

Proof. The assertion follows immediately from (3.10).

Corollary (3.12) *Under the conditions of* (2.1), *suppose that X and Y are of finite type over a base scheme S, and that f is a Gorenstein S-map. Then*

$$[M_2] = f_{1*}[X_2]$$

provided also one of the following two conditions is satisfied:

(1) *If $\xi \in M_2$ is the generic point of an arbitrary component, then X/S is smooth at ξ, and either* (a) $\dim_\xi(X/S) = 1$, *or* (b) $k(\xi)$ *is of characteristic 0, or simply,* (c) $k(\xi)/k(f\xi)$ *is separable.*

(2) *The map f is appropriately generic in the sense that, if $\eta \in X_2$ is the generic point of an arbitrary component, then*

$$\dim_\eta(X_2/S) = \dim_{f_1\eta}(X/S) - 1.$$

Proof. The assertion follows from (3.11) as $\operatorname{cod}(\overline{\Sigma}_2, X) \geq 2$. Indeed, by (2.4), M_2 is of pure codimension 1 in X. Let $\xi \in M_2$ be the generic point of a component. If (1) holds, then $\xi \notin \overline{\Sigma}_2$ by (2.5) applied to the geometric fiber of f over the image of ξ in S; compare with the proof of (2.6). If (2) holds, then $f_1^{-1}\xi$ is of dimension 0; so $\xi \notin \overline{\Sigma}_2$ by (3.2)(1).

(3.13) *Remark.* Ulrich [**31**] has proved a complement to (3.10), which suggests that the subscheme X_2' of $X \times_Y X$ defined by $\mathcal{A}nn(\mathcal{I}(\Delta))$ is a better external scheme of source double-points than X_2. Namely, under the conditions of (3.10),

$$[M_2] = f_{1*}[X_2']$$

provided that X is a complete intersection over Y at the generic point of every component of M_2.

Ulrich derives that assertion from the following one, in which the restriction to codimension 1 has been dropped: Let $f\colon X \to Y$ be a finite map of locally Noetherian schemes that is birational onto its image. Assume that X is locally a complete intersection of codimension s over Y, that Y satisfies S_{2s+1}, and that M_2 is of codimension at least s. Then X_2' is a perfect Y-scheme of grade $2s$, its ideal $\mathcal{A}nn(\mathcal{I}(\Delta))$ is equal to $\mathcal{F}itt^0_{X\times_Y X}(\mathcal{I}(\Delta))$, and its fundamental cycle $[X_2']$ is equal to $[\mathcal{I}(\Delta)]$. In particular, M_2 and X_2' are of pure codimension s, and X_2' is of flat dimension $2s$ over Y; moreover, X_2' has no embedded components, and it is Cohen–Macaulay if Y is. The assertion in the preceding paragraph follows because of (3.9) and (2.4).

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