INTEGRALLY CLOSED IDEALS
IN TWO-DIMENSIONAL REGULAR LOCAL RINGS
ARE MULTIPLIER IDEALS

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ABSTRACT. Multiplier ideals in commutative rings are certain integrally closed ideals with properties that lend themselves to highly interesting applications. How special are they among integrally closed ideals in general? We show that in a two-dimensional regular local ring with algebraically closed residue field there is in fact no difference between "multiplier" and "integrally closed" (or "complete.") But among multiplier ideals arising from an *integer* multiplying constant (also known as *adjoint* ideals), and primary for the maximal ideal, the only simple complete ideals are those of order one.

Introduction. There has arisen in recent years a substantial body of work on *multiplier ideals* in commutative rings (see [La]). Multiplier ideals are integrally closed ideals with properties that lend themselves to highly interesting applications. One is tempted then to ask just how special multiplier ideals are among integrally closed ideals in general.

In this note we show that in a two-dimensional regular local ring R with maximal ideal $\mathfrak{m}$ such that the residue field $R/\mathfrak{m}$ is algebraically closed[1] there is actually no difference between multiplier ideals and integrally closed ideals. In fact it turns out more convenient to do this for *fractionary* R-ideals, i.e., nonzero finitely-generated R-submodules of the fraction field L of R.

Main Result. *Every integrally closed fractionary R-ideal is a multiplier ideal.*

After this paper was first submitted, we learned that independently of us C. Favre and M. Jonsson had found a related proof [FJ, §6]. Their argument is given in the context of a novel treatment of valuations of R. Though we thought initially that our proof applied only to $\mathfrak{m}$-primary ideals, Favre and Jonnson had no such restriction. This prompted us to reexamine our proof, which we then found could be made to apply to the general case as well.

[1]It most likely suffices that $R/\mathfrak{m}$ be infinite, but we want to avoid additional technicalities.

Throughout, $(R, \mathfrak{m})$ and its fraction field L will be as above. "Ideal" will mean "fractionary R-ideal." An ideal is *integral* if it is contained in R; and *of finite colength* if it is either (integral and) $\mathfrak{m}$-primary or the unit ideal. For brevity we use the classical term "complete" instead of "integrally closed in L." Any complete ideal I is uniquely the product of a principal ideal and a finite-colength complete ideal: $I = (I^{-1})^{-1}(II^{-1})$ where the inverse of an ideal J is $J^{-1} := \{\, x \in L \mid xJ \subset R \,\}$.

1. Geometric formulation of the problem. The goal of this section is to develop the geometric criterion Corollary 1.4.2 for an ideal to be a multiplier ideal, while laying the groundwork for the proof in the next section that every complete ideal satisfies that criterion.

We begin by recalling some preliminary definitions and known results. (For some historical pointers to the development of the theory of complete ideals see the second paragraph on the first page of [L3].)

For any complete ideal I there exists a *log resolution,* that is, a proper birational map $f : X \to \mathrm{Spec}(R)$ where X is a regular scheme such that for any closed point $x \in X$ there exist t_1, t_2 generating the maximal ideal of $S := \mathcal{O}_{X,x}$ such that $IS = t_1^{a_1} t_2^{a_2} S$ for some integers a_1, a_2. (In other words, $I\mathcal{O}_X$ is invertible, with normal-crossing support.) To see quickly that there is an f, composed of maps obtained by blowing up closed points, and such that the $\mathcal{O}_X$-ideal $I\mathcal{O}_X$ is at least *invertible*—assuming, as one clearly may, that I is $\mathfrak{m}$-primary—one can use the Hoskin-Deligne formula [L2, p. 222, Thm. 3.1], which shows that the length of the "transform" of I can be successively lowered by suitable closed-point blowups, until it vanishes, at which point I generates an invertible ideal sheaf. Then well-known facts about embedded resolution of curves in two-dimensional regular schemes ensure that with further closed-point blowups one can reach the desired normal-crossing situation.

The group $\mathrm{Div}(X)$ of *X-divisors* is, by definition, the free abelian group on the set of reduced irreducible one-dimensional subschemes of X. These subschemes, also called *prime divisors,* are of two kinds, *affine* resp. *exceptional* according as f maps their generic point to a non-closed resp. closed point of $\mathrm{Spec}(R)$. A divisor D can be represented as a formal sum with integer coefficients, $D = \sum_E d_E E$ where E runs through all prime divisors and only finitely many of the integers d_E are nonzero; an E for which $d_E \neq 0$ will be called an *irreducible component* of D. One has then the decomposition $D = D_{\mathrm{aff}} + D_{\mathrm{exc}}$ where D_{aff} (resp. D_{exc}), the *affine* (resp. *exceptional*) *part* of D is obtained by replacing all the d_E where E is exceptional (resp. affine) by 0. We say D is affine (resp. exceptional) if $D = D_{\mathrm{aff}}$ (resp. $D = D_{\mathrm{exc}}$). Exceptional divisors will also be called *f-divisors.* They make up a subgroup $\mathrm{Div}_{\mathrm{e}}(X) \subset \mathrm{Div}(X)$.

Each prime divisor E gives rise to the discrete valuation v_E whose valuation ring is the local ring on X of the generic point of E (which may be assumed to be a subring of L). The divisor (t) of a nonzero $t \in L$ is defined to be $(t) := \sum_E v_E(t)E$. One has $t \in R \Leftrightarrow v_E(t) \geq 0$ for all affine $E \Leftrightarrow (t)_{\mathrm{aff}} \geq 0$.

For $D = \sum_E d_E E$, the sheaf $\mathcal{O}_X(D)$ is the invertible $\mathcal{O}_X$-module sending any open $U \subset X$ to

$$\Gamma\big(U, \mathcal{O}_X(D)\big) := \{\, t \in L \mid v_E(t) \geq -d_E \text{ for all } E \text{ meeting } U \,\}.$$

In particular, $\Gamma\big(X, \mathcal{O}_X(-D)\big) := \{\, t \in L \mid (t) \geq D\}$. This is a complete ideal, integral iff $d_E \geq 0$ for all affine E, and of finite colength iff D is exceptional. Every invertible $\mathcal{O}_X$-submodule of the constant sheaf L is $\mathcal{O}_X(D)$ for a unique D.

The abelian group $\mathrm{Div}(X)$ is ordered, the positive (or *effective*) divisors being those D such that $d_E \geq 0$ for all E, or equivalently, $\mathcal{O}_X(-D) \subset \mathcal{O}_X$. An effective D can be regarded as a one-dimensional subscheme of X, with structure sheaf $\mathcal{O}_D$ fitting in a natural exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

If, moreover, D is exceptional, then this subscheme of X is projective over $R/\mathfrak{m}^n$ for some $n > 0$. In particular, the exceptional prime divisors $E^1, E^2, \ldots, E^s$ are isomorphic to the projective line $\mathbb{P}^1_{R/\mathfrak{m}}$; and any two of them intersect transversally. (This can be shown by induction on the number of blowups making up f.)

For any invertible $\mathcal{O}_X$-module $\mathcal{L}$ the *intersection number* $\mathcal{L} \cdot E$ of $\mathcal{L}$ with an effective f-divisor E is the degree of the invertible $\mathcal{O}_E$-module $\mathcal{L}_E := \mathcal{L} \otimes \mathcal{O}_E$:

$$\mathcal{L} \cdot E := \chi_E \mathcal{L}_E - \chi_E \mathcal{O}_E$$

where with λ the length function on R-modules,

$$\chi_E \mathcal{M} := \lambda \mathrm{H}^0(E, \mathcal{M}) - \lambda \mathrm{H}^1(E, \mathcal{M})$$

is the Euler characteristic of a coherent $\mathcal{O}_E$-module $\mathcal{M}$.

For any X-divisor F set $F \cdot E := \mathcal{O}_X(F) \cdot E$. This intersection number extends uniquely to a $\mathbb{Z}$-valued symmetric bilinear form on $\mathrm{Div}_e(X)$ (see e.g., [L1, §13]). If $F' = (t) + F$ for some $t \in L$ then $\mathcal{O}_X(F') \cong \mathcal{O}_X(F)$, and hence $F' \cdot E = F \cdot E$.

An X-divisor F is said to be numerically effective, *nef* for short, if $F \cdot E^i \geq 0$ for all $i = 1, 2, \ldots, s$ $(\Rightarrow F \cdot E \geq 0$ for all effective $E \in \mathrm{Div}_e(X))$. F is said to be *antinef* if $-F$ is nef.

The following basic result is contained in [L1, p. 220, Thm. (12.1)].

Theorem 1.1. *An X-divisor D is nef if and only if $\mathcal{O}_X(D)$ is generated by its global sections.*

Corollary 1.1.1. *If D is antinef and $D_{\mathrm{aff}} \geq 0$ then D is effective.*

That's because, as above, $\Gamma\big(X, \mathcal{O}_X(-D)\big) \subset R$, and so by the Theorem,

$$\mathcal{O}_X(-D) = \Gamma\big(X, \mathcal{O}_X(-D)\big)\mathcal{O}_X \subset \mathcal{O}_X.$$

(A simpler proof, not using Theorem 1.1, can be found in [L1, p. 238].)

If I is a complete ideal and $I\mathcal{O}_X$ is invertible then $I = \Gamma(X, I\mathcal{O}_X)$, whence:

Corollary 1.1.2. (Cf. [L1, §18].) *Sending E to $\Gamma\big(X, \mathcal{O}_X(-E)\big)$ defines an isomorphism from the (additive) monoid of antinef X-divisors to the (multiplicative) monoid of those complete ideals I such that $I\mathcal{O}_X$ is invertible.*[2] *Under this isomorphism antinef f-divisors correspond to finite-colength complete ideals.*

It is simple to show, by induction on the number of blowups making up f, that the intersection matrix $(E^i \cdot E^j)$ has determinant ± 1. Hence for each i there is a unique f-divisor G_i such that $G_i \cdot E^j = 0$ unless $j = i$, in which case $G_i \cdot E^j = -1$; and for any f-divisor E it holds that $-E = \sum_{i=1}^{s}(E.E^i)G_i$. Thus the monoid of antinef f-divisors is freely generated by these G_i. In other words, "unique factorization" holds in this monoid—and therefore in the monoid of finite-colength complete ideals to which, by Corollary 1.1.2, it is isomorphic.

An integral ideal $P \neq R$ is *simple* if whenever $P = IJ$ (I, J integral ideals) then either I or J is the unit ideal. For example, if G_i is as above then the $\mathfrak{m}$-primary ideal $P_i := \Gamma\big(X, \mathcal{O}_X(-G_i)\big)$ is simple, since if P_i is the product of I and J then it is also the product of the integral closures of I and J.

Corollary 1.1.3. (Zariski, [ZS, p. 386, Thm. 3].) *Every complete integral ideal is, in a unique way, the product of simple complete ideals.*

(As R is a unique factorization domain, one reduces at once to the finite-colength case. Also, it helps to note that for any ideals I, J, if $IJ\mathcal{O}_X$ is invertible then $I\mathcal{O}_X$ and $J\mathcal{O}_X$ are both invertible.)

Corollary 1.1.4. *If I is a complete ideal with $I\mathcal{O}_X$ invertible, $I\mathcal{O}_X = \mathcal{O}_X(-E)$, and $P_i := \Gamma\big(X, \mathcal{O}_X(-G_i)\big)$ is the simple $\mathfrak{m}$-primary complete ideal corresponding to the above G_i, then $II^{-1} = \prod_{i=1}^{s} P_i^{-E.E^i}$. Thus P_i divides $II^{-1} \Leftrightarrow E \cdot E^i \neq 0$.*

Moreover, the valuations v_{E^i} associated to those E^i such that $E \cdot E^i \neq 0$ are precisely the Rees valuations of II^{-1} (i.e., those valuations whose valuation ring is the local ring of the generic point of some reduced irreducible component of the closed fiber of the normalized blowup of I). (See [L4, p. 300, Prop. (4.4)].)

The following Lemma[3] will be needed.

Lemma 1.2. *Let E be an X-divisor and I the complete ideal $\Gamma(X, \mathcal{O}_X(-E))$. Then $I\mathcal{O}_X$ is invertible. Equivalently (see (1.1.2)), there is an antinef $E^- \geq E$, such that $I = \Gamma(X, \mathcal{O}_X(-E^-))$; this E^- must be the* least *antinef divisor $\geq E$.*

Proof. Since R is a unique factorization domain, there exists a $t \in L$ such that $(t)_{\mathrm{aff}} = E_{\mathrm{aff}}$, and one can replace (E, I) by $(E - (t), t^{-1}I)$; so one may assume that E is exceptional, say $E = \sum a_i E^i$, and that I has finite colength. One may also assume $X \neq \operatorname{Spec}(R)$. Then there exists an antinef $F \geq E$: pick n such that $\mathfrak{m}^n \subset I$, and define F by $\mathfrak{m}^n \mathcal{O}_X = \mathcal{O}_X(-F)$. Among all antinef $F = \sum b_i E^i \geq E$ choose one—call it F_E—for which $\sum_i (b_i - a_i)$ has minimal value, denoted σ_E.

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[2]It is a theorem of Zariski that a product of two complete ideals is still complete [ZS, p. 385, Thm. 2′], [L1, p. 209, Thm. (7.1)].

[3]related to Enriques's "principle of discharge" [Z, p. 28],

Procede by induction on σ_E. Suppose $\sigma_E > 0$ (otherwise there is nothing to prove), and that the Lemma holds for all E' with $\sigma_{E'} < \sigma_E$. With $F := F_E$ as above, there is an i such that $E \cdot E^i > 0 \geq F \cdot E^i$, and since $E^j \cdot E^i \geq 0$ when $j \neq i$ (clearly) and $E^i \cdot E^i < 0$,[4] therefore $b_i > a_i$, whence $F \geq E + E^i$. So $\sigma_{E+E^i} < \sigma_E$, and therefore $\Gamma(X, \mathcal{O}_X(-E - E^i))\mathcal{O}_X$ is invertible. It suffices then to verify that $\Gamma\big(X, \mathcal{O}_X(-E - E^i)\big) = I$, by applying the left-exact functor $\Gamma(X, -)$ to the natural exact sequence

$$0 \longrightarrow \mathcal{O}_X(-E - E^i) \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}$$

and observing that since $\mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}$ has degree $-E \cdot E^i < 0$, therefore

$$\Gamma\big(X, \mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}\big) = \Gamma\big(E^i, \mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}\big) = 0. \qquad \square$$

(1.3). (Canonical divisors.) Let $Y \xrightarrow{g} X \xrightarrow{f} \operatorname{Spec}(R)$ be proper birational maps with Y and X regular schemes. By a theorem of Zariski and Abhyankar (see, e.g., [L1, p. 204, Thm. (4.1)]) both f and g are compositions of point blowups.

Let $F^1, F^2, \ldots, F^t$ be the reduced irreducible components of $(fg)^{-1}\{\mathfrak{m}\}$. As before, the intersection matrix $(F^i \cdot F^j)$ has determinant ± 1, and so there is a unique fg-divisor K_g such that

(1.3.1) $$K_g \cdot F^i = \begin{cases} -F^i \cdot F^i - 2 & \text{if } g(F^i) \text{ is a point} \\ 0 & \text{otherwise.} \end{cases}$$

This K_g is called the *canonical divisor of g*.[5]

For an X-divisor D, g^*D denotes the Y-divisor whose coefficient at any prime divisor F on Y is $v_F(\mathcal{O}_X(-D)_x)$, where $x \in X$ is the g-image of the generic point of F. There is a natural isomorphism $g^*\mathcal{O}_X(D) \xrightarrow{\sim} \mathcal{O}_Y(g^*D)$.

(1.3.2). The following easily-checked properties characterize K_g for all g:
- If g is the blowup of a closed point $x \in X$ then $K_g = g^{-1}\{x\}$.
- If $Z \xrightarrow{h} Y \xrightarrow{g} X \xrightarrow{f} \operatorname{Spec}(R)$ are proper birational maps with Z, Y and X regular schemes, then
$$K_{gh} = h^*K_g + K_h.$$

(1.4). (Multiplier ideals.) For $D = \sum_E d_E E \in \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ set

$$[D] := \sum_E [d_E]E \in \operatorname{Div}(X)$$

where $[d_E]$ is the greatest integer $\leq d_E$.

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[4]It is well-known, going back to Du Val, that the intersection matrix $(E^i \cdot E^j)$ is negative definite, see e.g., [L1, p. 224, Lemma (14.1)].
[5]One has $\mathcal{O}(K_g) = H^0(g^!\mathcal{O}_X)$ with $g^!$ as in Grothendieck duality theory [LS, p. 206, (2.3)].

Definition 1.4.1. Let I be a complete ideal, $h : Y \to \mathrm{Spec}(R)$ a log resolution of I, say $I\mathcal{O}_Y = \mathcal{O}_Y(-G)$, and let c be a positive real number. The *multiplier ideal* $\mathcal{J}(I^c)$ is defined to be

$$\mathcal{J}(I^c) := \Gamma\big(Y, \mathcal{O}(K_h - [cG])\big).$$

Thus, by Lemma 1.2,

$$\mathcal{J}(I^c)\mathcal{O}_Y = \mathcal{O}_Y\big(-F\big)$$

where $F := \big([cG] - K_h\big)^-$ is the least antinef h-divisor $\geq [cG] - K_h$.

For the blowup $h_1 : Y_1 \to Y$ of a closed point $y \in Y$ one finds via (1.3.2) that the log resolution h can be replaced by the log resolution $h \circ h_1$ without affecting $\mathcal{J}(I^c)$. (When c is not an integer, the log-resolution hypothesis that G be a "normal crossing divisor" is important.) As any two log resolutions are dominated by a third, obtained from each of the two by a sequence of point blowups,[6] it follows that $\mathcal{J}(I^c)$ does not depend on the choice of the log resolution h.

Corollary 1.4.2. *A complete ideal J satisfies $J = \mathcal{J}(I^c)$ for some c, I iff for some log resolution $h : Y \to \mathrm{Spec}(R)$ of J, say $J\mathcal{O}_Y = \mathcal{O}_Y(-F)$, there is an antinef h-divisor G and a real $c > 0$ such that*

$$(1.4.2.1) \qquad\qquad F = \big([cG] - K_h\big)^- .$$

2. Proof of Main Result. Let J be a complete ideal. To show that $J = \mathcal{J}(I^c)$ for some c, I, we will describe a log resolution $h : Y \to \mathrm{Spec}(R)$ of J, and a G and c as in Corollary 1.4.2, such that if $J\mathcal{O}_Y = \mathcal{O}_Y(-F)$ then (1.4.2.1) holds. (The number of suitable (h, G, c) will be enormous.)

Factor JJ^{-1} as $JJ^{-1} = \prod_{\ell=1}^u P_\ell^{e_\ell}$ (P_ℓ simple complete, $e_\ell > 0$)—see Corollary 1.1.3 and the paragraph preceding it. Let $f : X \to \mathrm{Spec}(R)$ be any log resolution of J, say $J\mathcal{O}_X = \mathcal{O}_X(-F^0)$. As in Corollary 1.1.4, one has for each ℓ,

$$(2.0) \qquad\qquad e_\ell = -F^0 \cdot E^\ell.$$

We will first construct, for each u-tuple $N := (n_1, n_2, \ldots, n_u)$ of non-negative integers, a proper birational map $g_N : Y_N \to X$, to be realized as a composition of closed point blowups, so that $h_N := f \circ g_N$ will be a log resolution of J.

For ease of expression we say "blow up a closed point $x \in X$ generically, n times" to mean "blow up $x_0 := x$ to get $g_1 : X_1 \to X$, then blow up a closed point x_1 on $g_1^{-1}x_0$ but not on any other irreducible component of $g_1^* F^0$ to get $g_2 : X_2 \to X_1$, then blow up a closed point x_2 on $g_2^{-1}x_1$ but not on ... then blow up a closed point x_{n-1} on $g_{n-1}^{-1}x_{n-2}$ but not on any other irreducible component of $(g_1 \circ g_2 \circ \cdots \circ g_{n-1})^* F^0$ to get $g_n : X_n \to X_{n-1}$."

Then with $g := g_1 \circ g_2 \circ \cdots \circ g_n$ it holds that:

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[6]By the above-mentioned theorem of Zariski and Abhyankar, it suffices to principalize some ideal sheaf on one of the log resolutions by a sequence of point blowups ("elimination of indeterminacies"), which can be done e.g., via the Hoskin-Deligne formula, as before.

(2.1) $g^{-1}x$ is a chain of n integral curves $D_1, D_2, \ldots, D_n$ such that for $0 < i < n$, $D_i \cdot D_{i+1} = 1$ and $D_i \cdot D_i = -2$, while $D_n \cdot D_n = -1$; and if $|j - i| > 1$ then $D_i \cdot D_j = 0$.

(For the proof one can use, e.g., [L1, p. 229, middle, and p. 227, α) and β)]. Here, and subsequently, the reader may find it useful to do some rough sketches.)

As in Corollary 1.1.4, there corresponds to each D_i an $\mathfrak{m}$-primary simple complete ideal Q_i; and, we claim, *these Q_i form a strictly decreasing sequence* $Q_1 > Q_2 > \cdots > Q_n$, with Q_1 strictly contained in each of the simple ideals corresponding to the (one or two) prime f-divisors E^j passing through x.

Indeed, let G_j be the f-divisor such that $G_j \cdot E^j = -1$ and $G_j \cdot E = 0$ for every other prime f-divisor E, and let $\widetilde{Q}_j := \Gamma(X, \mathcal{O}_X(-G_j))$ be the corresponding simple complete ideal. It follows from, e.g., [L1, p. 227, α) and β)] that $g_1^* G_j$ is antinef; and the corresponding simple complete ideal is

$$\Gamma\big(X_1, \mathcal{O}_{X_1}(-g_1^* G_j)\big) = \Gamma\big(X, \mathcal{O}_X(-G_j)\big) = \widetilde{Q}_j.$$

Further, with $E' := g_1^{-1} x$, let G' be the fg_1-divisor such that $G' \cdot E' = -1$ and $G' \cdot E'' = 0$ for every other prime fg_1-divisor E''. Since $g_1^* G_j + E'$ has intersection number -1 with E' and ≥ 0 with each E'', therefore $G' - g_1^* G_j - E'$ is antinef, hence effective (Corollary 1.1.1); and consequently $G' > g_1^* G_j$. Thus the simple complete ideal $\Gamma\big(X_1, \mathcal{O}_{X_1}(-G')\big)$ is strictly contained in $\widetilde{Q}_j$.

Continuing in this way we establish the claim.

Now for each $\ell = 1, 2, \ldots, u$, pick e_ℓ distinct closed points $x_1, \ldots, x_{e_\ell}$ which lie on E^ℓ but on no $E \neq E^\ell$ and blow up all of these points generically, n_ℓ times. Then Y_N is the resulting surface, and g_N is the composition of all the blowups. It is easily seen that (Y_N, g_N) does not depend (up to isomorphism) on the order in which the chosen points are blown up—though that won't be important.[7]

To simplify notation, fix N and set $(Y, g) := (Y_N, g_N)$ and $F := g^* F^0$, so that $J\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Also, set $h := fg : Y \to \mathrm{Spec}(R)$.

For an X-divisor D, we denote by $D^\#$ the *proper transform* of D on Y, obtained from $g^* D =: \sum_E a_E E$ (where E runs through all prime divisors on Y) by replacing a_E by 0 whenever E is *g-exceptional*, i.e., $g(E)$ is a closed point.

For each $\ell = 1, 2, \ldots, u$ and $x_{j_\ell} \in E^\ell$ $(j_\ell = 1, 2, \ldots, e_\ell)$ let

$$\{ E^\ell_{j_\ell k_\ell} \mid k_\ell = 1, 2, \ldots, n_\ell \}$$

be the family of prime Y-divisors whose g-image is x_{j_ℓ}, the ordering of these curves by the index k_ℓ conforming to the ordering of the D's in (2.1). These curves are all isomorphic to the projective line $\mathbb{P}^1_{R/\mathfrak{m}}$.

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[7]The initial $\sum_\ell e_\ell$ points might be taken to be the intersection of the closed fiber on X with a sufficiently generic curve C in the linear system $|-F^0|$ (i.e., a divisor—having no component in the closed fiber—of the form $(j) - F^0$ with j a sufficiently generic element of J). Then at each stage the point to be blown up could be taken to be a specialization of some nonclosed point of C.

If a_ℓ is the E^ℓ-coefficient of the divisor F^0, and b_ℓ of the divisor K_f, then one finds (using (1.3.2)) that

$$F = g^* F^0 = F^{0\#} + \sum_\ell \sum_{j_\ell, k_\ell} a_\ell E^\ell_{j_\ell k_\ell},$$

$$g^* K_f = K_f^\# + \sum_\ell \sum_{j_\ell, k_\ell} b_\ell E^\ell_{j_\ell k_\ell},$$

$$K_g = \sum_\ell \sum_{j_\ell, k_\ell} k_\ell E^\ell_{j_\ell k_\ell}.$$

Set $G := F + K_g$. Noting that $F \cdot E^\ell_{j_\ell k_\ell} = g^* F^0 \cdot E^\ell_{j_\ell k_\ell} = 0$ [L1, p. 227, β)], and using (2.1) together with the preceding expansion of K_g (or together with (1.3.1) and (1.3.2)), one finds that for every g-exceptional prime divisor E, $G \cdot E = 0$ unless E is one of the curves $E^\ell_{j_\ell n_\ell}$ at the end of the chains emanating from the $\sum_\ell e_\ell$ originally chosen points (i.e., $g(E)$ is a point and $E \cdot E = -1$), in which case $G \cdot E = -1$. Moreover, for any f-exceptional prime divisor D, $g^* F^0 \cdot D^\# = F^0 \cdot D$ [L1, p. 227, α)], and $E^\ell_{j_\ell k_\ell} \cdot D^\# = 0$ if $k_\ell > 1$ (since then $E^\ell_{j_\ell k_\ell} \cap D^\#$ is empty), so using (2.0) one finds that $G \cdot D^\# = 0$. Thus G is an antinef Y-divisor.

By Corollary 1.1.4, if I is the corresponding complete ideal $\Gamma\big(Y, \mathcal{O}_Y(-G)\big)$ then II^{-1} is the product of the simple complete ideals corresponding to the $\sum_\ell e_\ell$ curves $E^\ell_{j_\ell n_\ell}$ having self-intersection -1.

Here is a key technical point:

Lemma 2.2. *For all sufficiently small $\epsilon > 0$ there exists N such that*

(2.2.1) $$(1 + \epsilon) G - K_h = F + A \qquad \big([A] \le 0\big)$$

where the coefficient of $[A]$ at each $E^\ell_{j_\ell n_\ell}$ and at each affine prime Y-divisor is 0.

Proof. Using (1.3.2), one transforms (2.2.1) into the equality

$$\epsilon(F + K_g) - g^* K_f = A.$$

More explicitly (see above)

$$\epsilon\Big(F^{0\#} + \sum_\ell \sum_{j_\ell, k_\ell} (a_\ell + k_\ell) E^\ell_{j_\ell k_\ell}\Big) - \Big(K_f^\# + \sum_\ell \sum_{j_\ell, k_\ell} b_\ell E^\ell_{j_\ell k_\ell}\Big) = A.$$

So to get (2.2.1) we can choose any $\epsilon > 0$ such that the coefficients of the X-divisor $\epsilon F^0 - K_f$ are all < 1, and then look for n_ℓ such that $\epsilon(a_\ell + k_\ell) - b_\ell < 1$ for all ℓ and for all $k_\ell \le n_\ell$, while $\epsilon(a_\ell + n_\ell) - b_\ell \ge 0$. These conditions mean precisely that n_ℓ satisfies the inequalities

$$1/\epsilon + b_\ell/\epsilon - a_\ell > n_\ell \ge b_\ell/\epsilon - a_\ell \quad (\ell = 1, 2, \ldots, u).$$

Clearly, such integers n_ℓ can be found if $\epsilon < 1$. $\square$

For $c = 1 + \epsilon$ and N satisfying Lemma 2.2, and with $h : Y \to \mathrm{Spec}(R)$ and F, G, as before, we have

$$F' := \left([cG] - K_h\right)^- \le F,$$

so that

$$J' := \Gamma\left(Y, \mathcal{O}(-F')\right) \supset \Gamma\left(Y, \mathcal{O}(-F)\right) = J.$$

Let us verify that $J' = J$ $\left(= (J^{-1})^{-1} J J^{-1}\right)$, thereby proving the main result.

Since $G = F + K_g$ and F have the same affine part, the affine part of $[A]$ must be 0, and hence F' and F have the same affine part. This means that $(J'^{-1})^{-1} = (J^{-1})^{-1}$. So $J'J'^{-1} \supset JJ^{-1}$, and we need only show that these two $\mathfrak{m}$-primary ideals are equal.

Recall that the valuations $v_\ell := v_{E^\ell}$ are just the Rees valuations of JJ^{-1}. (See the remark following Corollary 1.1.4). So

$$JJ^{-1} = \{\,\rho \in R \mid v_\ell(\rho) \ge v_\ell(JJ^{-1}) \text{ for all } \ell = 1, 2, \ldots, u\,\}.$$

Thus we need only show that for each ℓ, the $E^{\ell\#}$-coefficient a'_ℓ of F' is the same as that of F (namely a_ℓ). Let us say that ℓ is "good" if $a'_\ell = a_\ell$ and "bad" if $a'_\ell < a_\ell$.

If ℓ is good then since $F' \le F$ therefore

$$F' \cdot E^{\ell\#} \le F \cdot E^{\ell\#} = F^0 \cdot E^\ell \overset{(2.0)}{=} -e_\ell.$$

Corollary 1.1.4 shows then that $J'J'^{-1}$ is divisible by $P_\ell^{e_\ell}$.

Suppose ℓ is bad. For $j \in [1, e_\ell]$ and with a'_{jk} the E^ℓ_{jk}-coefficient of F' it is easily seen that $a'_\ell =: a'_{j0} \le a'_{j1} \le a'_{j2} \le \cdots \le a'_{jn_\ell} = a_\ell$.[8] So there is a $k \in [1, n_\ell]$ such that $a'_{j,k-1} < a'_{jk} = a'_{j,k+1} = \cdots = a_\ell$. Then

$$F' \cdot E^\ell_{jk} = \begin{cases} a'_{j,k-1} - 2a'_{jk} + a'_{j,k+1} < 0 & \text{if } k < n_\ell, \\ a'_{j,n_\ell-1} - a'_{jn_\ell} < 0 & \text{if } k = n_\ell. \end{cases}$$

From Corollary 1.1.4 and the remarks after 2.1, one deduces that $J'J'^{-1}$ is divisible by a simple complete ideal $P'_{\ell j} < P_\ell$. This being so for all j, and the $P'_{\ell j}$ being distinct (Corollary 1.1.2), it follows from Corollary 1.1.3 that $J'J'^{-1}$ is divisible by $P'_{\ell 1} P'_{\ell 2} \cdots P'_{\ell e_\ell} < P_\ell^{e_\ell}$. Thus (by Corollary 1.1.3 again) the existence of a bad ℓ leads to a factorization of $J'J'^{-1}$ which contradicts $J'J'^{-1} \supset JJ^{-1}$.

So every ℓ is good, and $J'J'^{-1} = JJ^{-1}$. $\square$

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[8]With $v := v_{E^\ell_{j,k+1}}$ and $S \supset R$ the regular local ring blown up to give $E^\ell_{j,k+1}$, one finds:
$$a'_{j,k+1} = v(J'S) = v\left(((J'S)^{-1})^{-1}\right) + v\left(J'S(J'S)^{-1}\right) = a'_{jk} + v\left(J'S(J'S)^{-1}\right).$$

Remarks. 1. By the choice of ϵ, the E^ℓ-coefficient of $\epsilon F^0 - K_f$ is < 1, i.e., $\epsilon a_\ell - b_\ell < 1$, i.e., $b_\ell/\epsilon - a_\ell > -1/\epsilon$. It could happen that $b_\ell/\epsilon - a_\ell < 0$ for all ℓ. In this case one can take $N = (0, 0, \dots, 0)$, and then $J = \mathcal{J}((1 + \epsilon)J)$.

2. The proof shows that if J is a simple complete $\mathfrak{m}$-primary ideal then there is a simple complete $\mathfrak{m}$-primary ideal $P \subset J$ and a $c > 0$ such that $J = \mathcal{J}(cP)$.

3. By way of illustration of our method, let J be a simple $\mathfrak{m}$-primary ideal of order 16, whose successive transforms have orders $(8, 8, 4, 4, 2, 2, 1, 1, 0, 0 \dots)$. (For the existence of such a J see e.g., [L4, p. 298, Cor. (3.1)].) Here $\ell = 1$, and one calculates that $a_1 = 426$, $b_1 = 46$. Then any $\epsilon \in (0, 5/48)$ will do. Since $46/(5/48) - 426 > 15$, the least possible value of N is 16, which is attained, e.g., when $\epsilon = 23/221$. One has then that $J = \mathcal{J}(P^{244/221})$, where $P \subset J$ is a simple complete ideal of order 16 with successive transforms of orders $(8, 8, 4, 4, 2, 2, 1, 1, ..., 1, 0, 0 \dots)$ (18 ones).

A simpler permissible choice of ϵ would be $1/12$. But one would need $N = 126$ blowups to unearth a simple $P \subset J$ such that $J = \mathcal{J}(P^{13/12})$.

There might be other methods of finding a representation $J = \mathcal{J}(I^c)$ with I "closer" to J than here. But $I = J$ can never occur, because the inequality $b_1 < \epsilon a_1$ of example 1 could then be deduced. (Exercise.) On the other hand, this inequality does hold for any of the successive transforms of the present J.

4. Since the $c = 1 + \epsilon$ we have considered can be arbitrarily close to 1, one may ask if it is possible for c actually to be 1. (This would be the case studied in [L5], where $\mathcal{J}(I)$ is called the *adjoint ideal* of I.)

For simple complete $\mathfrak{m}$-primary J, the answer is given by:

Proposition 2.3. *A simple complete $\mathfrak{m}$-primary ideal J is of the form $\mathcal{J}(I)$ for some $I \iff J \not\subset \mathfrak{m}^2 \iff J = (a, b^n)R$ for some integer $n > 0$ and $a, b \in R$ such that $(a, b)R = \mathfrak{m}$.*

Proof. The last $\iff$ holds because $J \not\subset \mathfrak{m}^2$ means that J contains an element a such that R/aR is a discrete valuation ring. Moreover, if $(a, b)R = \mathfrak{m}$ and $z \in R$ is integral over $J = (a, b^n)R$ then the canonical image of z in R/aR is integral over—and hence is a multiple of—that of b^n, whence $z \in J$, and thus J is complete (and clearly simple). It is an easy exercise to show that for such a J, $J = \mathcal{J}(J^2)$. (One could use [L5, p. 749, Prop. (3.1.2)].)

For a simple complete $\mathfrak{m}$-primary J, let $f : X \to \mathrm{Spec}(R)$ be a log resolution obtained by successively blowing up *base points of J*—closed points at which J does not generate an invertible ideal—for as long as such points are available. (As noted before, the Hoskin-Deligne formula guarantees that this process terminates.) From [L1, p. 199, Prop. 1.2 and p. 203, Prop. 3.1] it follows that this f is the minimal desingularization of the blowup of J: for any log resolution h of J there exists a map $g : Y \to X$ composed of point blowups such that $h = fg$.

We have seen before that there is a unique exceptional prime X-divisor E such that $J\mathcal{O}_X \cdot E \neq 0$. This E satisfies $E \cdot E = -1$: for, there is at least one exceptional prime X-divisor E' such that $E' \cdot E' = -1$, namely the closed fibre $f'^{-1}\{x'\}$ for the blowup $f' : X \to X'$ of $x' \in X'$ coming last in the sequence

of blowups composing to f (see [L1, middle of p. 229]); and $J' := J\mathcal{O}_{X', x'}$ is not invertible, from which one sees, with $\mathfrak{m}'$ the maximal ideal of $R' := \mathcal{O}_{X', x'}$, that $J' = d\mathfrak{m}'^s$ for some $d \in R'$ and $s > 0$, whence $J\mathcal{O}_X \cdot E' = s$, so that $E' = E$.

We claim that *if $F \cdot F = -2$ for all exceptional prime divisors $F \neq E$ then $J \not\subset \mathfrak{m}^2$.* (The converse is part of the exercise at the end of the first paragraph.) Indeed, this condition on the F's means that among the base points of J no two are "proximate" to the same one, and the conclusion follows from [L4, p. 301, (3)].

Assume now that $J = \mathcal{J}(I)$. From J being $\mathfrak{m}$-primary it follows easily that so is I. (I and $\mathcal{J}(I)$ have the same gcd.) Let $h : Y \to \operatorname{Spec}(R)$ be a log resolution of I, obtained as above by blowing up base points of I, and say $I\mathcal{O}_Y = \mathcal{O}_Y(-G)$. Let F be an exceptional prime Y-divisor. As above, we find that $F \cdot F = -1 \Rightarrow G \cdot F < 0$. (We may assume that F is the closed fiber for the last blowup in some sequence of blowups composing to g, because if g is the blowup of a point lying on an exceptional prime divisor F_1 then $0 > F_1 \cdot F_1 = g^* F_1 \cdot F_1^\# = F_1^\# \cdot F_1^\# + 1$.) Since $K_h \cdot F = -F \cdot F - 2$ and G is antinef, we see that $G - K_h$ is antinef, and hence $\mathcal{O}_Y(K_h - G) = J\mathcal{O}_Y$.

So $J\mathcal{O}_Y$ is invertible, i.e., h is a log resolution of J, and as above there exists a $g : Y \to X$ composed of point blowups such that $h = fg$. Let $F^{1\#}, F^{2\#}, \ldots, F^{n\#}$ be the proper transforms on Y of the prime X-divisors $F^1, F^2, \ldots, F^n$ other than the above E. Then $F^{i\#} \cdot F^{i\#} \leq F^i \cdot F^i \leq -2$ and so $K_h \cdot F^{i\#} \geq 0$. But by [L1, p. 227, β)],

$$K_h \cdot F^{i\#} \leq (K_h - G) \cdot F^{i\#} = J\mathcal{O}_Y \cdot F^{i\#} = J\mathcal{O}_X \cdot F^i = 0,$$

and thus $K_h \cdot F^{i\#} = 0$, i.e., $F^{i\#} \cdot F^{i\#} = -2$, whence, finally, $F^i \cdot F^i = -2$. The above claim shows then that $J \not\subset \mathfrak{m}^2$. $\square$

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