GREENLEES-MAY DUALITY ON FORMAL SCHEMES

LEOVIGILDO ALONSO TARRÍO, ANA JEREMÍAS LÓPEZ, AND JOSEPH LIPMAN

Abstract. For a closed subscheme Z of a noetherian separated scheme X, let $\varGamma_Z$ be the functor of sections with support in Z, taking $\mathcal{O}_X$-modules to $\mathcal{O}_X$-modules. Inspired by a theorem of Greenlees and May [GM] about duality between local cohomology and local homology for modules over a commutative ring, we gave in [AJL] the result that on quasi-coherent complexes $\mathcal{F}$ the "homology localization" functor $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathcal{F})$ is a left-derived functor of $\Lambda_Z :=$ completion along Z, the corresponding map to $\Lambda_Z\mathcal{F}$ being such that its composition with the natural map $\mathcal{F} \to \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathcal{F})$ is the completion map $\mathcal{F} \to \Lambda_Z\mathcal{F}$. We also showed how this unifies and generalizes several other recorded duality theorems. Here we extend the result to an arbitrary noetherian separated formal scheme $\mathfrak{X}$, with "quasi-coherent" replaced by "$\varinjlim$ of coherent."

Given an open $\mathcal{O}_{\mathfrak{X}}$-ideal $\mathcal{I}$, with $\kappa\colon \hat{\mathfrak{X}}_{\mathcal{I}} \to \mathfrak{X}$ the completion by $\mathcal{I}$ and $\varGamma_{\mathcal{I}}$ the functor $\varinjlim \mathcal{H}om^\bullet(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^n, -)$, we deduce for any $\mathcal{O}_{\mathfrak{X}}$-complex $\mathcal{F}$ having coherent homology sheaves a canonical duality isomorphism

$$\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}, \kappa_*\kappa^*\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E}, \mathcal{F}) \qquad (\mathcal{E} \in \mathbf{D}(\mathfrak{X})).$$

The special case where $\mathcal{I}$ is an ideal of definition—so that $\kappa =$ identity—plays an important part in the duality theory of coherent sheaves on formal schemes.

Introduction

We use the terminology of [DFS, Section 1], and basic facts about formal schemes found in [GD, Chapter 1, §10]. In particular, we assume familiarity with the notion of triangle-preserving functors (Δ-functors) between triangulated categories (Δ-categories), and of (Δ-functorial) maps between such functors.

Throughout, $\mathfrak{X}$ is a separated noetherian formal scheme, $\mathcal{A}$ is the category $\mathcal{A}(\mathfrak{X})$ of $\mathcal{O}_{\mathfrak{X}}$-modules, $\mathbf{K}$ is the homotopy category of $\mathcal{O}_{\mathfrak{X}}$-complexes, $\mathbf{D}$ is its derived category (i.e., its localization with respect to quasi-isomorphisms [Hrt, pp. 28–35]), and $q\colon \mathbf{K} \to \mathbf{D}$ is the canonical functor. $\mathcal{A}_{\vec{c}} \subset \mathcal{A}$ is the plump subcategory (see [DFS, Section 1]) whose objects are $\varinjlim$'s of coherent $\mathcal{O}_{\mathfrak{X}}$-modules. The derived category $\mathbf{D}_{\vec{c}}$ of the homotopy category $\mathbf{K}_{\vec{c}}$ of $\mathcal{A}$-complexes whose homology sheaves are all in $\mathcal{A}_{\vec{c}}$ is a Δ-subcategory of $\mathbf{D}$ ([Hrt, p. 50, Example 3]).

We work with *affine-acyclic* complexes, i.e., $\mathcal{O}_{\mathfrak{X}}$-complexes $\mathcal{E}$ such that for each affine open $\mathfrak{U} \subset \mathfrak{X}$, the natural map $\Gamma(\mathfrak{U}, \mathcal{E}) \to \mathbf{R}\Gamma(\mathfrak{U}, \mathcal{E})$ is an isomorphism in the derived category of abelian groups. In other words, if $\mathcal{E} \to \mathcal{J}$ is a quasi-isomorphism from $\mathcal{E}$ to a K-injective complex $\mathcal{J}$—a K-injective resolution [Spn, 4.5])—then $\Gamma(\mathfrak{U}, \mathcal{E}) \to \Gamma(\mathfrak{U}, \mathcal{J})$ is a quasi-isomorphism. (Observe, using the functor "extension by 0," that the restriction of a K-injective complex to an open subset remains K-injective; and that $\mathbf{R}\Gamma$ can be realized via K-injective resolutions [Spn, 5.12, 6.4].)

For example, every $\mathcal{A}_{\vec{c}}$-complex is affine-acyclic (Example 2.2.1(c)).

First two authors partially supported by Xunta de Galicia research project XUGA20702B98 and Spain's DGES grant PB97-0530. They also thank the Mathematics Department of Purdue University for its hospitality, help and support.

Third author partially supported by the National Security Agency.

The homotopy category $\mathbf{K}^{\mathrm{a}}_{\vec{c}}$ of affine-acyclic complexes with $\mathcal{A}_{\vec{c}}$-homology is a Δ-subcategory of $\mathbf{K}$, see Remark 2.1.1(4). Since every complex admits a quasi-isomorphism into a K-injective (clearly affine-acyclic) one, the derived category $\mathbf{D}^{\mathrm{a}}_{\vec{c}}$ of $\mathbf{K}^{\mathrm{a}}_{\vec{c}}$ can be identified with a Δ-subcategory of $\mathbf{D}_{\vec{c}}$, whose inclusion $\boldsymbol{j}^{\mathrm{a}}\colon \mathbf{D}^{\mathrm{a}}_{\vec{c}} \hookrightarrow \mathbf{D}_{\vec{c}}$ is an *equivalence of Δ-categories.* The canonical functor $q^{\mathrm{a}}\colon \mathbf{K}^{\mathrm{a}}_{\vec{c}} \to \mathbf{D}^{\mathrm{a}}_{\vec{c}}$ is then the restriction of q.

Fix a coherent $\mathcal{O}_{\mathfrak{X}}$-ideal $\mathcal{I}$. The *torsion functor* $\varGamma_{\mathcal{I}}\colon \mathbf{K} \to \mathbf{K}$ is described by

$$\varGamma_{\mathcal{I}}\mathcal{G} := \varinjlim_{n>0} \mathcal{H}om^{\bullet}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^n, \mathcal{G}) \qquad (\mathcal{G} \in \mathbf{K}),$$

(see [DFS, Section 1.2.1]). The *completion functor* $\Lambda_{\mathcal{I}}\colon \mathbf{K} \to \mathbf{K}$ is described by

$$\Lambda_{\mathcal{I}}(\mathcal{G}) := \varprojlim_{n>0} \big((\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^n) \otimes \mathcal{G}\big) \qquad (\mathcal{G} \in \mathbf{K}).$$

By the universal property of right-derived functors, there is a unique functorial map $\mathbf{R}\varGamma_{\mathcal{I}} \to \mathbf{1}$ whose composition with the canonical map $\varGamma_{\mathcal{I}} \to \mathbf{R}\varGamma_{\mathcal{I}}$ is the inclusion map $\varGamma_{\mathcal{I}} \hookrightarrow \mathbf{1}$. For any K-injective resolution $\mathcal{E} \to \mathcal{J}$ as above, $\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E} \to \mathcal{E}$ is naturally isomorphic to the canonical map $\varGamma_{\mathcal{I}}\mathcal{J} \to \mathcal{J}$.

The functor $\mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, -)$ is right-adjoint to $\mathbf{R}\varGamma_{\mathcal{I}}$ (via (4.1.1.2), (4.1.1.1) below). So the following **main result** establishes a *duality* (= adjunction) between the right-derived torsion and the left-derived completion functors associated to $\mathcal{I}$.

Theorem 0.1. *There exists a unique Δ-functorial $\mathbf{D}$-morphism*

$$\zeta_{\mathcal{F}}\colon \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F}) \to \Lambda_{\mathcal{I}}\mathcal{F} \qquad (\mathcal{F} \in \mathbf{K}^{\mathrm{a}}_{\vec{c}})$$

such that:

(i) *the pair* $\big(\mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, -), \zeta\big)$ *is a* left-derived functor *of* $q\Lambda_{\mathcal{I}}|_{\mathbf{K}^{\mathrm{a}}_{\vec{c}}}$ (*i.e., a final object in the category of pairs* (S, ϱ) *with* $S\colon \mathbf{D}^{\mathrm{a}}_{\vec{c}} \to \mathbf{D}$ *a Δ-functor and* $\varrho\colon Sq^{\mathrm{a}} \to q\Lambda_{\mathcal{I}}|_{\mathbf{K}^{\mathrm{a}}_{\vec{c}}}$ *a map of Δ-functors*), *and*

(ii) *the composition of $\zeta_{\mathcal{F}}$ with the natural map*

$$\rho_{\mathcal{F}}\colon \mathcal{F} \cong \mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{O}_{\mathfrak{X}}, \mathcal{F}) \to \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})$$

is the canonical completion map $\mathcal{F} \to \Lambda_{\mathcal{I}}\mathcal{F}$.

Moreover, this $\zeta_{\mathcal{F}}$ is an isomorphism whenever $\mathcal{F}$ is a K-flat *$\mathcal{A}_{\vec{c}}$-complex.*[1]

The uniqueness of ζ is shown as follows. It results from (i) and (ii) that any two choices of ζ are the same modulo a functorial automorphism $\theta_{\mathcal{F}}$ of $\boldsymbol{\Lambda}_{\mathcal{I}}\mathcal{F} := \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})$ respecting $\rho_{\mathcal{F}}$. But just as in setting up the isomorphism (B) in [DFS, Remark 6.3.1](1), we can see that composition with $\rho_{\mathcal{F}}$ is an *isomorphism* $\mathrm{Hom}(\boldsymbol{\Lambda}_{\mathcal{I}}\mathcal{F}, \boldsymbol{\Lambda}_{\mathcal{I}}\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}, \boldsymbol{\Lambda}_{\mathcal{I}}\mathcal{F})$, whence $\theta_{\mathcal{F}} =$ identity.

0.2. Existence of ζ in Theorem 0.1 is proved along lines similar to those in [AJL], which deals with ordinary schemes. Here there are new technical problems. For example, the canonical functor $\mathbf{D}(\mathcal{A}_{\vec{c}}) \to \mathbf{D}_{\vec{c}}$ might not be an equivalence (as it is for separated quasi-compact ordinary schemes). Another—related—problem is that for an open immersion $i\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$, the functor i_* doesn't necessarily take $\mathcal{A}_{\vec{c}}(\mathfrak{U})$ into $\mathcal{A}_{\vec{c}}$ (as it does for separated quasi-compact ordinary schemes.) To get around such obstacles we need properties of sheaves on formal schemes established in [DFS].

[1] An $\mathcal{O}_{\mathfrak{X}}$-complex is *K-flat* if its tensor product with every exact $\mathcal{O}_{\mathfrak{X}}$-complex remains exact, cf. [Spn, pp. 139–140]. For example, any $\varinjlim$ of bounded-above flat complexes is K-flat (cf. e.g., [Lpm, Example (2.5.4)]).

In outline, we proceed as follows. After setting up some preliminaries on Čech functors in Section 1, we show in Section 2 that the completion functor $\Lambda_{\mathcal{I}}|_{\mathbf{K}^{\mathrm{a}}_{\vec{c}}}$ has a left-derived functor, for which the property corresponding to the last assertion in Theorem 0.1 holds. Then in Section 3 we identify this left-derived functor with the pair $(\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, -), \zeta)$, in such a way that Theorem 0.1(ii) is satisfied. In so doing we describe only the modifications needed in the corresponding proof in [AJL], referring to *loc. cit.* for the remaining details.

It must be noted that the main results in [AJL] are inadequately packaged, at least for non-affine $\mathfrak{X}$. There $\mathfrak{X}$ is an ordinary scheme. The completion functor $\Lambda_{\mathcal{I}}$ is said to have a left-derived functor (denoted $\mathbf{L}\Lambda_Z$, where $Z = \mathrm{Supp}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I})$) on $\mathbf{D}_{\vec{c}}(\mathfrak{X}) = \mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$. However, that holds only on some *equivalent subcategory* $\mathbf{D}^{\mathrm{a}}_{\mathrm{qc}}(\mathfrak{X})$, among whose objects are all the quasi-coherent complexes. (The error lies in the unduly facile paragraph preceding (1.2) on page 10 of [AJL]. To fill the gap we have had to work with the functor $\check{c}^\infty$ of Section 1.2 below.) Thus the map

$$\mathcal{F} \to \mathbf{L}\Lambda_Z\mathcal{F} \qquad (\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathfrak{X}),\ \kappa\colon \mathfrak{X}_{/Z} \to \mathfrak{X} \text{ the completion map})$$

and its factorization via $\mathcal{F} \to \kappa_*\kappa^*\mathcal{F}$, which play an important role in Theorem (0.3) and Proposition (0.4.1) of *loc. cit.*, make sense, as defined there, only on $\mathbf{D}^{\mathrm{a}}_{\mathrm{qc}}(\mathfrak{X})$. But they *can* be extended to all of $\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$, by means of its equivalence with $\mathbf{D}^{\mathrm{a}}_{\mathrm{qc}}(\mathfrak{X})$ or (better) as described above in Theorem 0.1, and then the rest is okay.

0.3. The oft-used Proposition 6.2.1 in [DFS] follows from Theorem 0.1. In fact we will use Theorem 0.1 to show that a certain Δ-bifunctorial map, more general than the one in *loc. cit.*, is an isomorphism. To wit:

Suppose $\mathcal{I}$ is open, i.e., contains an ideal of definition of $\mathfrak{X}$. Let

$$\kappa = \kappa_{\mathcal{I}}\colon \hat{\mathfrak{X}}_{\mathcal{I}} \to \mathfrak{X}$$

be the *completion by* $\mathcal{I}$. Thus $\hat{\mathfrak{X}}_{\mathcal{I}}$ is the topological space $\mathrm{Supp}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I})$ together with the sheaf of topological rings $\varprojlim \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^n$; and κ is the obvious ringed-space map. By [GD, p. 412, Proposition (10.6.3)], $\hat{\mathfrak{X}}_{\mathcal{I}}$ is a formal scheme, and by [GD, p. 422, Corollaire (10.8.9)], [DFS, Lemma 7.1.1], and [Brb, p. 103, Corollaire], the map κ is *flat*. (Both assertions are local, so need only be verified for affine $\mathfrak{X}$.)

In particular, if $\mathcal{I}$ itself is an ideal of definition then $\hat{\mathfrak{X}}_{\mathcal{I}} = \mathfrak{X}$ and κ is the identity.

Denote by $\mathbf{D}_{\mathrm{c}} \subset \mathbf{D}_{\vec{c}}$ the Δ-subcategory having as objects the complexes whose homology sheaves are all coherent. The following proposition is proved in Section 4.

Proposition 0.3.1. (i) *For each pair* $\mathcal{E}$, $\mathcal{F} \in \mathbf{D}$ *the map induced by the canonical map* $\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E} \to \mathcal{E}$ *factors* uniquely *as*

$$\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F}) \xrightarrow{\text{natural}} \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}, \kappa_*\kappa^*\mathcal{F}) \xrightarrow{\lambda(\mathcal{E},\mathcal{F})} \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E}, \mathcal{F}).$$

(ii) *If* $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}$ *then* $\lambda(\mathcal{E}, \mathcal{F})$ *is an* isomorphism.

Remark. Explicitly, $\lambda(\mathcal{E}, \mathcal{F})$ is the composition $s^{-1}t$ in the following natural commutative diagram, where the isomorphisms p and p' can be established by imitating [DFS, Corollary 5.2.3], and r (hence s) is an isomorphism by Lemma 4.1:

$$\begin{array}{ccccc}
\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}, \kappa_*\kappa^*\mathcal{F}) & \xrightarrow[t]{} & \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E}, \kappa_*\kappa^*\mathcal{F}) & \xleftarrow[p]{\simeq} & \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E}, \mathbf{R}\varGamma_{\mathcal{I}}\kappa_*\kappa^*\mathcal{F}) \\
\uparrow & & {\scriptstyle s}\uparrow{\scriptstyle \simeq} & & {\scriptstyle r}\uparrow{\scriptstyle \simeq} \\
\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E}, \mathcal{F}) & \xleftarrow[p']{\simeq} & \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E}, \mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F})
\end{array}$$

This description makes it clear that $\lambda(\mathcal{E}, \mathcal{F})$ is Δ-bifunctorial.

1. Preliminaries on Čech functors and left-derivability

1.1. Let $\mathcal{U} = (\mathfrak{U}_\alpha)_{1\le\alpha\le t}$ be an open cover of the separated formal scheme $\mathfrak{X}$. Let $\mathfrak{P}_t$ be the set of subsets of $\{1,2,\dots,t\}$. For $i \in \mathfrak{P}_t$ set

$$\mathfrak{U}_i := \bigcap_{\alpha\in i} \mathfrak{U}_\alpha, \quad \text{and} \quad \mathcal{O}_i := \mathcal{O}_{\mathfrak{U}_i}.$$

For $i \supset j$ in $\mathfrak{P}_t$ let $\lambda_{ij}\colon \mathfrak{U}_i \hookrightarrow \mathfrak{U}_j$ be the inclusion. Set $\lambda_i := \lambda_{i\phi}\colon \mathfrak{U}_i \hookrightarrow \mathfrak{X}$.

A *$\mathcal{U}$-module* is a family $\mathcal{F} = (\mathcal{F}_i)_{i\in\mathfrak{P}_t}$ such that $\mathcal{F}_i$ is an $\mathcal{O}_i$-module, together with a family of sheaf homomorphisms

$$\varphi_{jk}\colon \lambda^*_{jk}\mathcal{F}_k \to \mathcal{F}_j \qquad (j \supset k)$$

such that φ_{jk} is an $\mathcal{O}_j$-homomorphism, φ_{jj} is the identity map of $\mathcal{F}_j$, and whenever $i \supset j \supset k$ we have the transitivity relation $\varphi_{ik} = \varphi_{ij} \circ \big(\varphi_{jk}|_{\mathfrak{U}_i}\big)$, i.e., φ_{ik} factors as

$$\lambda^*_{ik}\mathcal{F}_k = \lambda^*_{ij}\lambda^*_{jk}\mathcal{F}_k \xrightarrow{\lambda^*_{ij}(\varphi_{jk})} \lambda^*_{ij}\mathcal{F}_j \xrightarrow{\varphi_{ij}} \mathcal{F}_i.$$

We say that the $\mathcal{U}$-module $\mathcal{F}$ is *quasi-coherent* (resp. *flat*, resp. ...) if each one of the $\mathcal{O}_i$-modules $\mathcal{F}_i$ is such. So, for example, $\mathcal{A}_{\vec{c}}(\mathcal{U})$ denotes the category of $\mathcal{U}$-modules $\mathcal{F}$ such that $\mathcal{F}_i \in \mathcal{A}_{\vec{c}}(\mathfrak{U}_i)$ for all $i \in \mathfrak{P}_t$.

The $\mathcal{U}$-modules and their morphisms (defined in the obvious manner) form an abelian category $\mathcal{A}(\mathcal{U})$ having $\varinjlim$'s and $\varprojlim$'s. For instance one checks that the $\varinjlim$ of a direct system $(\mathcal{F}^m, \varphi^m_{jk})_{m\in I}$ in $\mathcal{A}(\mathcal{U})$ is $(\varinjlim_m \mathcal{F}^m_i, \varinjlim_m \varphi^m_{jk})$.

The *Čech functor* $\check{C}^\bullet$ from the homotopy category $\mathbf{K}(\mathcal{U})$ to $\mathbf{K} := \mathbf{K}(\mathfrak{X})$ is defined as follows. With

$$|i| := (\text{cardinality of } i) - 1 \qquad (i \in \mathfrak{P}_t),$$

set, for $\mathcal{F} \in \mathcal{A}(\mathcal{U})$,

$$\check{C}^s(\mathcal{F}) := \bigoplus_{|i|=s} \lambda_{i*}\mathcal{F}_i \ \text{ if } 0 \le s < t, \text{ and } 0 \text{ otherwise.}$$

Whenever k is obtained from $j = \{j_0 < j_1 < \cdots < j_{s+1}\} \in \mathfrak{P}_t$ by removing a single element j_a, set $\epsilon_{jk} := (-1)^a$; and otherwise set $\epsilon_{jk} := 0$. (Thus if k is fixed then $\epsilon_{jk} = 0$ for all but finitely many j.) We specify the differential $\delta^s\colon \check{C}^s(\mathcal{F}) \to \check{C}^{s+1}(\mathcal{F})$ by requiring its restriction to $\lambda_{k*}\mathcal{F}_k$ ($|k| = s$) to be the natural composition

$$\lambda_{k*}\mathcal{F}_k \longrightarrow \lambda_{k*}\lambda_{jk*}\lambda^*_{jk}\mathcal{F}_k = \lambda_{j*}\lambda^*_{jk}\mathcal{F}_k \xrightarrow{\oplus\lambda_{j*}(\epsilon_{jk}\varphi_{jk})} \bigoplus_{|j|=s+1} \lambda_{j*}\mathcal{F}_j.$$

Then $\delta^{s+1}\circ\delta^s = 0$ for all s, and so we get a functor $\check{C}^\bullet$ from $\mathcal{U}$-modules to $\mathcal{O}_{\mathfrak{X}}$-complexes. Next, for a complex $\mathcal{F}^\bullet \in \mathbf{K}(\mathcal{U})$, $\check{C}^\bullet(\mathcal{F}^\bullet)$ is defined to be the total complex associated to the double complex $\check{C}^p(\mathcal{F}^q)$:

$$\check{C}^s(\mathcal{F}^\bullet) := \oplus_{p+q=s}\check{C}^p(\mathcal{F}^q),$$

and the differential $\check{C}^s(\mathcal{F}^\bullet) \to \check{C}^{s+1}(\mathcal{F}^\bullet)$ restricts on $\check{C}^p(\mathcal{F}^q)$ to

$$\big(\check{C}^p(d^q) \oplus (-1)^q\check{d}^p\big)\colon \check{C}^p(\mathcal{F}^q) \longrightarrow \big(\check{C}^p(\mathcal{F}^{q+1}) \oplus \check{C}^{p+1}(\mathcal{F}^q)\big)$$

where d^q (resp. $\check{d}^p$) is the differential in the complex $\mathcal{F}^\bullet$ (resp. $\check{C}^\bullet(\mathcal{F}^q)$). One checks that $\check{C}^\bullet$ commutes with translation and with forming mapping cones, so that $\check{C}^\bullet$ can (and will) be regarded as a Δ-functor from $\mathbf{K}(\mathcal{U})$ to $\mathbf{K}$.

For instance, if $\mathcal{M}$ is an $\mathcal{O}_{\mathfrak{X}}$-module and $\mathcal{M}^*$ is the "pulled-back" $\mathcal{U}$-module such that $\mathcal{M}^*_i := \lambda^*_i\mathcal{M}$ and φ_{jk} is the identity map of $\mathcal{M}^*_j = \lambda^*_{jk}\mathcal{M}^*_k$ for all $j \supset k$, then $\check{C}^\bullet(\mathcal{M}^*)$ is the usual $\mathcal{U}$-Čech resolution of $\mathcal{M}$ [Gdm, p. 206, Théorème 5.2.1]. More generally, composing $\check{C}^\bullet$ with the pullback functor $\mathbf{K} \to \mathbf{K}(\mathcal{U})$ gives the functor

$$\check{c} = \check{c}_{\mathcal{U}} : \mathbf{K} \to \mathbf{K}$$

sending each $\mathcal{O}_{\mathfrak{X}}$-complex $\mathcal{E}$ to its $\mathcal{U}$-Čech resolution $\check{c}\mathcal{E}$.

1.2. There is a functorial quasi-isomorphism $\chi_{\mathcal{E}} : \mathcal{E} \to \check{c}\mathcal{E}$ ($\mathcal{E} \in \mathbf{K}$), giving rise to an inductive system of functors $\big(\chi_{\check{c}^m} : \check{c}^m \to \check{c}^{m+1}\big)_{m>0}$. Set

$$\check{c}^\infty := \varinjlim_m \check{c}^m.$$

There is then a functorial quasi-isomorphism

$$\chi^\infty_{\mathcal{E}} : \mathcal{E} \to \check{c}^\infty\mathcal{E} \qquad (\mathcal{E} \in \mathbf{K}).$$

Since $\mathfrak{X}$ is noetherian, the functor $\check{c}^\infty$ commutes with $\varinjlim$, and hence is *idempotent*—$\chi^\infty_{\check{c}^\infty\mathcal{E}} : \check{c}^\infty\mathcal{E} \to \check{c}^\infty\check{c}^\infty\mathcal{E}$ is an isomorphism of complexes for all $\mathcal{E}$.

Lemma 1.2.1. *$\Lambda_{\mathcal{I}}(\chi^\infty_{\mathcal{E}})$ is a quasi-isomorphism. In other words, $q(\Lambda_{\mathcal{I}}(\chi^\infty_{\mathcal{E}}))$ is a* **D***-isomorphism.*

Proof. The question is readily seen to be local; and over any member $\mathfrak{U}$ of the open covering $\mathcal{U}$, $\chi^\infty_{\mathcal{E}}$ is, by the following Corollary 1.2.3, a *homotopy isomorphism,* whence so is $\Lambda_{\mathcal{I}}(\chi^\infty_{\mathcal{E}})$. (Observe that $(\check{c}\mathcal{E})|_{\mathfrak{U}}$ is the Čech resolution of $\mathcal{E}|_{\mathfrak{U}}$ with respect to the cover $(\mathfrak{U}\cap\mathfrak{U}_\alpha)_{1\le\alpha\le t}$ of $\mathfrak{U}$, a cover of which $\mathfrak{U}$ itself is a member). □

Lemma 1.2.2. *Let $\mathcal{U} = (\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_t)$ be an open cover of $\mathfrak{X}$, with $\mathfrak{U}_1 = \mathfrak{X}$. Then for any $\mathcal{O}_{\mathfrak{X}}$-complex $\mathcal{E}$, the natural map $\chi = \chi_{\mathcal{E}} : \mathcal{E} \to \check{c}\mathcal{E}$ has a left inverse whose kernel is homotopically trivial (i.e., its identity map is homotopic to 0).*

Proof. Let $\mathcal{E}^*$ be the pullback of $\mathcal{E}$ to a $\mathcal{U}$-complex, so that $\check{c}\mathcal{E} = \check{C}^\bullet\mathcal{E}^*$. Then

$$\check{C}^\bullet_0\mathcal{E}^* := \mathcal{E} \oplus \lambda_{2*}\lambda^*_2\mathcal{E} \oplus \cdots \oplus \lambda_{t*}\lambda^*_t\mathcal{E} \cong \check{C}^\bullet\mathcal{E}^*/\check{C}^\bullet_{>0}\mathcal{E}^*$$

where $\check{C}^\bullet_{>0}\mathcal{E}^*$ is the total complex associated to the double complex $\check{C}^p(\mathcal{E}^{*q})_{p>0;\, q\in\mathbb{Z}}$. Composing the natural map $\check{C}^\bullet\mathcal{E}^* \twoheadrightarrow \check{C}^\bullet_0\mathcal{E}^*$ and the projection $\check{C}^\bullet_0\mathcal{E}^* \twoheadrightarrow \mathcal{E}$ we get a left inverse $\pi = \pi_{\mathcal{E}}$ of χ.

Furthermore, π is a right *homotopy* inverse of χ. Indeed, if $h : \check{c}\mathcal{E} \to \check{c}\mathcal{E}[-1]$ is the map of graded sheaves defined on the summand $\lambda_{i*}\lambda^*_i\mathcal{E}^p$ ($i \in \mathfrak{P}_t,\ p \in \mathbb{Z}$) by:

if $1 \in i$, and $i' := i - \{1\}$, then the restriction of h to $\lambda_{i*}\lambda^*_i\mathcal{E}^p$ is

$$\lambda_{i*}\lambda^*_i\mathcal{E}^p \xrightarrow{(-1)^p} \lambda_{i*}\lambda^*_i\mathcal{E}^p = \lambda_{i'*}\lambda^*_{i'}\mathcal{E}^p,$$

and otherwise the restriction is 0,

then one calculates, with d the differential in $\check{c}\mathcal{E}$, that $hd + dh = 1 - \chi\pi$.

As $\pi\chi$ is the identity map of $\mathcal{E}$, we have a functorial direct-sum decomposition

$$\check{c}\mathcal{E} = \mathcal{E} \oplus \check{c}_0\mathcal{E} \tag{1.2.2.1}$$

where $\check{c}_0\mathcal{E}$ is the kernel of $\pi_{\mathcal{E}}$. If $\nu : \check{c}_0\mathcal{E} \hookrightarrow \check{c}\mathcal{E}$ is the inclusion, then in $\mathbf{K}$,

$$\nu = (\chi\pi) \circ \nu = \chi \circ (\pi\nu) = 0,$$

and since ν has a left inverse therefore $\check{c}_0\mathcal{E}$ is homotopically trivial. □

Corollary 1.2.3. *With $\mathcal U$ as in Lemma 1.2.2 the natural map of complexes*

$$\chi^\infty_{\mathcal E}\colon \mathcal E \to \check c^\infty\mathcal E := \varinjlim_m \check c^m\mathcal E$$

has a left inverse whose kernel is homotopically trivial.

Proof. From (1.2.2.1) we deduce via induction that

$$\check c^\infty\mathcal E \cong \mathcal E \oplus \bigoplus_{m=0}^{\infty} \check c_0\big(\check c^m\mathcal E\big).$$

Then we need only note that any direct sum of homotopically trivial complexes is homotopically trivial. □

1.3. The category $\mathbf K^{\mathrm a}_{\vec c}$ in Theorem 0.1 satisfies $\check c^\infty\mathbf K^{\mathrm a}_{\vec c} \subset \mathbf K^{\mathrm a}_{\vec c}$ (Corollary 2.2.5).

Let $\check{\mathbf K}^{\mathrm a}_{\vec c}$ be the essential image of $\check c^\infty|_{\mathbf K^{\mathrm a}_{\vec c}}$, i.e., the full subcategory of $\mathbf K^{\mathrm a}_{\vec c}$ whose objects are those complexes which are isomorphic to one of the form $\check c^\infty\mathcal E$ ($\mathcal E \in \mathbf K^{\mathrm a}_{\vec c}$). Then $\check{\mathbf K}^{\mathrm a}_{\vec c}$ *is a Δ-subcategory of* $\mathbf K_{\vec c}$. For if T is a $\mathbf K_{\vec c}$-triangle, with summit $\mathcal E$, whose base is a map $\mathcal E_1 \to \mathcal E_2$ with $\mathcal E_i \in \check{\mathbf K}^{\mathrm a}_{\vec c}$, then $\mathcal E \in \mathbf K^{\mathrm a}_{\vec c}$ (Remark 2.1.1(4)); and considering the map of triangles $\chi^\infty_T\colon T \to \check c^\infty T$ in light of the idempotence of $\chi^\infty_{\mathcal E_i}$ (see §1.2), we find that $\chi^\infty_{\mathcal E}$ is a $\mathbf K$-isomorphism, whence $\mathcal E \cong \check c^\infty\mathcal E \in \check{\mathbf K}^{\mathrm a}_{\vec c}$.

The functorial quasi-isomorphism $\chi^\infty\colon \mathbf 1 \to \check c^\infty$ shows that the additive functor $\check c^\infty\colon \mathbf K^{\mathrm a}_{\vec c} \to \check{\mathbf K}^{\mathrm a}_{\vec c}$ takes quasi-isomorphisms to quasi-isomorphisms, so induces a Δ-functor $\boldsymbol\rho\colon \mathbf D^{\mathrm a}_{\vec c} \to \check{\mathbf D}^{\mathrm a}_{\vec c}$ between the corresponding derived categories. Let $\boldsymbol j\colon \check{\mathbf D}^{\mathrm a}_{\vec c} \to \mathbf D^{\mathrm a}_{\vec c}$ be the natural Δ-functor. Then χ^∞ induces a functorial isomorphism from the identity of $\mathbf D^{\mathrm a}_{\vec c}$ to $\boldsymbol{j\rho}$, and from the identity of $\check{\mathbf D}^{\mathrm a}_{\vec c}$ to $\boldsymbol{\rho j}$ (see [Hrt, p. 33, Proposition 3.4]), so that $\boldsymbol\rho$ *and* $\boldsymbol j$ *are quasi-inverse equivalences of* Δ*-categories.*

We have then the natural commutative diagram of functorial maps:

$$\begin{array}{ccccc}
\check{\mathbf D}^{\mathrm a}_{\vec c} & \underset{\approx}{\xrightarrow{\ \boldsymbol j\ }} & \mathbf D^{\mathrm a}_{\vec c} & \underset{\approx}{\xrightarrow{\ \boldsymbol\rho\ }} & \check{\mathbf D}^{\mathrm a}_{\vec c} \\
{\scriptstyle \check q}\big\uparrow & & \big\uparrow{\scriptstyle q^{\mathrm a}} & & \big\uparrow{\scriptstyle \check q} \\
\check{\mathbf K}^{\mathrm a}_{\vec c} & \underset{j}{\longrightarrow} & \mathbf K^{\mathrm a}_{\vec c} & \underset{r:=\check c^\infty}{\longrightarrow} & \check{\mathbf K}^{\mathrm a}_{\vec c}
\end{array} \tag{1.3.1}$$

1.3.2. In the next section, we will identify a Δ-subcategory $\check{\mathbf L}^{\mathrm a}_{\vec c} \subset \check{\mathbf K}^{\mathrm a}_{\vec c}$ such that:

(a) for every exact complex $\mathcal P$ in $\check{\mathbf L}^{\mathrm a}_{\vec c}$, the complex $\Lambda_{\mathcal I}\mathcal P$ is exact (Lemma 2.1.2),

(b) every complex $\mathcal E$ in $\check{\mathbf K}^{\mathrm a}_{\vec c}$ is the target of a quasi-isomorphism with source in $\check{\mathbf L}^{\mathrm a}_{\vec c}$ (Proposition 2.1.3), and

(c) if $\mathcal E$ is a K-flat $\mathcal A_{\vec c}$-complex then $\check c^\infty\mathcal E \in \check{\mathbf L}^{\mathrm a}_{\vec c}$ (Corollary 2.3.5).

From (a) and (b) it follows that $\check\Lambda := \Lambda_{\mathcal I}|_{\check{\mathbf K}^{\mathrm a}_{\vec c}}$ has a left-derived functor $\mathbf L\check\Lambda\colon \check{\mathbf D}^{\mathrm a}_{\vec c} \to \mathbf D$ such that the canonical map $\mathbf L\check\Lambda\mathcal E \to q\check\Lambda\mathcal E$ is an isomorphism for all $\mathcal E \in \check{\mathbf L}^{\mathrm a}_{\vec c}$ [Hrt, p. 53, Theorem 5.1]. In view of (c), the following formal lemma shows then[2] that $\Lambda := \Lambda_{\mathcal I}|_{\mathbf K^{\mathrm a}_{\vec c}}$ *has a left-derived functor* $\mathbf L\Lambda\colon \mathbf D^{\mathrm a}_{\vec c} \to \mathbf D$ *such that the canonical map* $\mathbf L\Lambda\mathcal F \to q\Lambda\mathcal F$ *is an isomorphism for every K-flat $\mathcal A_{\vec c}$-complex $\mathcal F$*, thereby completing the first main step in the proof of Theorem 0.1.

[2]This point was overlooked in [AJL] (p. 10, just before §1.2). The resulting deficiency can be corrected by the arguments in this paper, which apply *mutatis mutandis* to any separated quasi-compact scheme.

Lemma 1.3.3. *With reference to diagram* (1.3.1), *if the restriction* $\check\Lambda := \Lambda_{\mathcal I}|_{\check{\mathbf K}^{\mathrm a}_{\vec c}}$ *has a left-derived functor* $(\mathbf L\check\Lambda, \check\xi)$ *then the functor* $(\mathbf L\check\Lambda)\boldsymbol\rho\colon \mathbf D^{\mathrm a}_{\vec c} \to \mathbf D$ *together with the* Δ*-functorial map*

$$\xi\colon (\mathbf L\check\Lambda)\boldsymbol\rho q^{\mathrm a} = (\mathbf L\check\Lambda)\check q r \xrightarrow{\ \check\xi\ } q\check\Lambda r \xrightarrow[(1.2.1)]{q(\Lambda_{\mathcal I}(\chi^\infty))^{-1}} q\Lambda_{\mathcal I}|_{\mathbf K^{\mathrm a}_{\vec c}}$$

is a left-derived functor of $\Lambda_{\mathcal I}|_{\mathbf K^{\mathrm a}_{\vec c}}$.

Proof. The assertion is that for any Δ-functor $S\colon \mathbf D^{\mathrm a}_{\vec c} \to \mathbf D$, with "Hom" denoting the group of morphisms of Δ-functors, and with $\Lambda := \Lambda_{\mathcal I}|_{\mathbf K^{\mathrm a}_{\vec c}}$, the composed map

$$\alpha\colon \mathrm{Hom}(S, (\mathbf L\check\Lambda)\boldsymbol\rho) \xrightarrow{\text{natural}} \mathrm{Hom}(Sq^{\mathrm a}, (\mathbf L\check\Lambda)\boldsymbol\rho q^{\mathrm a}) \xrightarrow{\text{via } \xi} \mathrm{Hom}(Sq^{\mathrm a}, q\Lambda)$$

is *bijective.*

In fact, $(\mathbf L\check\Lambda, \check\xi)$ being a left-derived functor of $\check\Lambda$, the composition

$$\gamma\colon \mathrm{Hom}(S\boldsymbol j, \mathbf L\check\Lambda) \xrightarrow{\text{natural}} \mathrm{Hom}(S\boldsymbol j\check q, (\mathbf L\check\Lambda)\check q) \xrightarrow{\text{via } \check\xi} \mathrm{Hom}(S\boldsymbol j\check q, q\check\Lambda)$$

is bijective; and we show now that the following natural composition, denoted β, is an inverse of α:

$$\begin{aligned}\mathrm{Hom}(Sq^{\mathrm a}, q\Lambda) &\longrightarrow \mathrm{Hom}(Sq^{\mathrm a}j, q\Lambda j)\\ &=\!=\; \mathrm{Hom}(S\boldsymbol j\check q, q\check\Lambda)\\ &\xrightarrow[\gamma^{-1}]{\ \sim\ } \mathrm{Hom}(S\boldsymbol j, \mathbf L\check\Lambda) \xrightarrow{\ \sim\ } \mathrm{Hom}(S, (\mathbf L\check\Lambda)\boldsymbol\rho).\end{aligned}$$

(The last isomorphism reflects the fact that $\boldsymbol j$ and $\boldsymbol\rho$ are quasi-inverse equivalences.)

First, we claim, $\beta\alpha =$ identity, i.e., the following natural diagram commutes:

$$\begin{array}{ccccc}\mathrm{Hom}(S, (\mathbf L\check\Lambda)\boldsymbol\rho) & \xrightarrow{\ \alpha\ } & \mathrm{Hom}(Sq^{\mathrm a}, q\Lambda) & \longrightarrow & \mathrm{Hom}(Sq^{\mathrm a}j, q\Lambda j)\\ \simeq\big\downarrow & & & & \big\|\\ \mathrm{Hom}(S\boldsymbol j, \mathbf L\check\Lambda) & & \xrightarrow[\gamma]{\ \sim\ } & & \mathrm{Hom}(S\boldsymbol j\check q, q\check\Lambda)\end{array}$$

Indeed, going from the top left corner to the bottom right corner either clockwise or counterclockwise has the same effect on any $\varrho \in \mathrm{Hom}(S, (\mathbf L\check\Lambda)\boldsymbol\rho)$, as can be verified in a mechanical way with the aid of the commutative diagram:

$$\begin{array}{ccccccc}Sq^{\mathrm a}j & \xrightarrow{\text{via } \varrho} & (\mathbf L\check\Lambda)\boldsymbol\rho q^{\mathrm a}j & =\!=\!= & (\mathbf L\check\Lambda)\check q rj & \xrightarrow{\text{via } \check\xi} & q\check\Lambda rj\\ \big\| & & \big\| & & {\scriptstyle \check q(\chi^\infty)^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle q\check\Lambda(\chi^\infty)^{-1}}\\ S\boldsymbol j\check q & \xrightarrow[\text{via } \varrho]{} & (\mathbf L\check\Lambda)\boldsymbol\rho\boldsymbol j\check q & \longrightarrow & (\mathbf L\check\Lambda)\check q & \xrightarrow[\text{via } \check\xi]{} & q\check\Lambda\end{array}$$

To see now that β and α are inverse to each other we need only check that β *is injective.* But this injectivity clearly follows from the fact that *any functorial map* $\varepsilon\colon S' \to q\Lambda$ *which vanishes on* $\check{\mathbf K}^{\mathrm a}_{\vec c}$ *must itself vanish,* a fact resulting from the commutative diagram (where $q\Lambda(\chi^\infty_{\mathcal E})$ is an isomorphism, see Lemma 1.2.1):

$$\begin{array}{ccc}S'\mathcal E & \xrightarrow{\ \varepsilon(\mathcal E)\ } & q\Lambda\mathcal E\\ {\scriptstyle S'(\chi^\infty_{\mathcal E})}\big\downarrow & & \simeq\big\downarrow{\scriptstyle q\Lambda(\chi^\infty_{\mathcal E})}\\ S'r\mathcal E & \xrightarrow[\varepsilon(r\mathcal E)=0]{} & q\Lambda r\mathcal E\end{array}\qquad (\mathcal E \in \mathbf K^{\mathrm a}_{\vec c}).$$

□

2. Affine-acyclicity

The purpose of this section is to establish that $\Lambda_{\mathcal{I}}|_{\mathbf{K}^{\mathrm{a}}_{\vec{\mathrm{c}}}}$ has a left-derived functor, by showing that the full subcategory $\check{\mathbf{L}}^{\mathrm{a}}_{\vec{\mathrm{c}}} \subset \check{\mathbf{K}}^{\mathrm{a}}_{\vec{\mathrm{c}}}$ whose objects are the K-flat, locally $\mathcal{I}$-acyclic complexes in $\check{\mathbf{K}}^{\mathrm{a}}_{\vec{\mathrm{c}}}$ (Definition 2.1) is a Δ-subcategory (Remark 2.1.1(4)) satisfying the conditions in Section 1.3.2.

Notation remains as before. Denote by $\mathfrak{q}\colon \mathbf{K}(\mathfrak{Ab}) \to \mathbf{D}(\mathfrak{Ab})$ the natural functor from the homotopy category of complexes of abelian groups to its derived category.

Definition 2.1. An $\mathcal{O}_{\mathcal{X}}$-complex $\mathcal{E}$ is *affine-acyclic* if for each affine open $\mathcal{U} \subset \mathcal{X}$ the natural map is a $\mathbf{D}(\mathfrak{Ab})$-isomorphism

$$\mathfrak{q}\Gamma(\mathcal{U},\mathcal{E}) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(\mathcal{U},\mathcal{E}).$$

$\mathcal{E}$ is *affine-$\mathcal{I}$-acyclic* if for all $n > 0$, $\mathcal{E}_n := \mathcal{E}/\mathcal{I}^n\mathcal{E}$ is affine-acyclic and for any affine open $\mathcal{U} \subset \mathcal{X}$ the natural map is a surjection

$$\Gamma(\mathcal{U},\mathcal{E}_n) \twoheadrightarrow \Gamma(\mathcal{U},\mathcal{E}_{n-1}).$$

$\mathcal{E}$ is *locally $\mathcal{I}$-acyclic* if $\mathcal{X}$ has an open cover $\mathcal{U} = (\mathcal{U}_\alpha)$ such that the restriction $\mathcal{E}|_{\mathcal{U}_\alpha}$ is an affine-$\mathcal{I}|_{\mathcal{U}_\alpha}$-acyclic $\mathcal{O}_{\mathcal{U}_\alpha}$-complex for every α.

Remarks 2.1.1. (1) Affine-acyclicity is affine-$\mathcal{I}$-acyclicity for $\mathcal{I} = (0)$.

(2) If $\mathcal{E}$ and $\mathcal{F}$ are $\mathbf{K}$-isomorphic complexes and $\mathcal{E}$ is affine-acyclic then so is $\mathcal{F}$.

(3) An $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$ (considered as an $\mathcal{O}_{\mathcal{X}}$-complex concentrated in degree 0) is affine-acyclic iff for each affine open $\mathcal{U} \subset \mathcal{X}$, $\mathrm{H}^i(\mathcal{U},\mathcal{E}) = 0$ for all $i > 0$. In particular, any flasque $\mathcal{O}_{\mathcal{X}}$-module is affine-acyclic.

(4) The cone $\mathcal{C}_u$ of any map $u\colon \mathcal{E} \to \mathcal{E}'$ of locally $\mathcal{I}$-acyclic complexes is itself locally $\mathcal{I}$-acyclic—an appropriate cover consisting of all intersections $\mathcal{U}_\alpha \cap \mathcal{U}'_\beta$ where $(\mathcal{U}_\alpha)$ (resp. $(\mathcal{U}'_\beta)$) is a cover over which $\mathcal{E}$ (resp. $\mathcal{E}'$) is affine-$\mathcal{I}$-acyclic.

(*Proof.* Let $u_n\colon \mathcal{E}_n \to \mathcal{E}'_n$ be the map induced by u, and consider the map of $\mathbf{D}(\mathfrak{Ab})$-triangles obtained by applying $\mathfrak{q}\Gamma(\mathcal{U},-) \to \mathbf{R}\Gamma(\mathcal{U},-)$ ($\mathcal{U} \subset \mathcal{U}_\alpha \cap \mathcal{U}'_\beta$) to the standard $\mathbf{K}$-triangle $\mathcal{E}_n \xrightarrow{u_n} \mathcal{E}'_n \to \mathcal{C}_u/\mathcal{I}^n\mathcal{C}_u \to \mathcal{E}_n[1], \dots$)

Hence if $\mathbf{K}_0$ is a Δ-subcategory of $\mathbf{K}$, then the full subcategory of locally $\mathcal{I}$-acyclic complexes in $\mathbf{K}_0$ is also a Δ-subcategory of $\mathbf{K}$.

Similarly, the full subcategory $\mathbf{K}_0^{\mathrm{a}} \subset \mathbf{K}_0$ whose objects are the affine-acyclic complexes is a Δ-subcategory of $\mathbf{K}$. For example, as was noted in the Introduction, *the inclusion $\mathbf{K}^{\mathrm{a}}_{\vec{\mathrm{c}}} \hookrightarrow \mathbf{K}_{\vec{\mathrm{c}}}$ induces an equivalence of derived categories $\mathbf{D}^{\mathrm{a}}_{\vec{\mathrm{c}}} \xrightarrow{\approx} \mathbf{D}_{\vec{\mathrm{c}}}$.*

Lemma 2.1.2. *If $\mathcal{E}$ is an exact K-flat locally $\mathcal{I}$-acyclic complex then $\Lambda_{\mathcal{I}}\mathcal{E}$ is exact.*

Proof. Since $\mathcal{E}$ is K-flat the complexes $\mathcal{E}_n := \mathcal{E}/\mathcal{I}^n\mathcal{E}$ ($n \geq 0$) are all exact ([Spn, p. 140, Proposition 5.7]), and being affine-acyclic, remain exact after application of any functor $\Gamma(\mathcal{U},-)$ with $\mathcal{U}$ an affine open subset of any $\mathcal{U}_\alpha$ as in Definition 2.1. Moreover, the natural maps $\Gamma(\mathcal{U},\mathcal{E}_n) \to \Gamma(\mathcal{U},\mathcal{E}_{n-1})$ are surjections. So by [EGA, p. 66, (13.2.3)], the complex

$$\Gamma(\mathcal{U},\Lambda\mathcal{E}) = \varprojlim_n \Gamma(\mathcal{U},\mathcal{E}_n)$$

is exact for any such $\mathcal{U}$, whence the assertion. □

This takes care of condition (a) in Section 1.3.2. Condition (b) is given by the following proposition.

Proposition 2.1.3. *Any $\mathcal{E} \in \check{\mathbf{K}}^{\mathrm{a}}_{\vec{\mathrm{c}}}$ admits a quasi-isomorphism $\mathcal{P}_{\mathcal{E}} \to \mathcal{E}$ where $\mathcal{P}_{\mathcal{E}} \in \check{\mathbf{K}}^{\mathrm{a}}_{\vec{\mathrm{c}}}$ is flat, K-flat and locally $\mathcal{I}$-acyclic.*

The *proof* will occupy the rest of section 2.

2.2. We need various examples of affine-acyclic and affine-$\mathcal{I}$-acyclic complexes.

Examples 2.2.1. (a) If $\mathcal{E} \in \mathcal{A}_{\vec{\mathrm{c}}}$, then $\mathcal{I}^n\mathcal{E} \in \mathcal{A}_{\vec{\mathrm{c}}}$ for all $n \geq 0$: this results from [DFS, Proposition 3.2.2], because $\mathcal{I}^n\mathcal{E}$ is the image of the natural map $\mathcal{I}^n \otimes \mathcal{E} \to \mathcal{E}$ whose source and target are both in $\mathcal{A}_{\vec{\mathrm{c}}}$. [DFS, Proposition 3.2.2] shows further that $\mathcal{I}^m\mathcal{E}/\mathcal{I}^n\mathcal{E} \in \mathcal{A}_{\vec{\mathrm{c}}}$ whenever $0 \leq m \leq n$. It follows then from [DFS, Corollary 3.1.8] that $\mathcal{E}$ *is affine-$\mathcal{I}$-acyclic.*

(b) *Any $\varinjlim$ of affine-acyclic complexes with $\mathcal{A}_{\vec{\mathrm{c}}}$-homology is affine-acyclic and, of course, has $\mathcal{A}_{\vec{\mathrm{c}}}$-homology.*

For, by [DFS, Proposition 3.2.2], $\mathcal{A}_{\vec{\mathrm{c}}}$ is plump in $\mathcal{A}$, and by [DFS, Proposition 3.4.3], the functor $\mathbf{R}\Gamma(\mathcal{U}, -)$ is bounded-above on $\mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{U})$ for any open $\mathcal{U} \subset \mathcal{X}$. One can, then, adapt the proof of [Lpm, Corollary (3.9.3.2)], *mutatis mutandis,* to show that for any small directed system $\mathcal{E}_\eta$ of $\mathcal{O}_{\mathcal{X}}$-complexes with $\mathcal{A}_{\vec{\mathrm{c}}}$-homology, and each $i \in \mathbb{Z}$, the bottom row of the natural commutative diagram

$$\begin{array}{ccc}
\varinjlim\limits_{\eta} \mathrm{H}^i\Gamma(\mathcal{U}, \mathcal{E}_\eta) & \xrightarrow{\ \sim\ } & \mathrm{H}^i\Gamma(\mathcal{U}, \varinjlim\limits_{\eta} \mathcal{E}_\eta) \\
{\scriptstyle \mu_i}\big\downarrow & & \big\downarrow{\scriptstyle \nu_i} \\
\varinjlim\limits_{\eta} \mathrm{H}^i\mathbf{R}\Gamma(\mathcal{U}, \mathcal{E}_\eta) & \longrightarrow & \mathrm{H}^i\mathbf{R}\Gamma(\mathcal{U}, \varinjlim\limits_{\eta} \mathcal{E}_\eta)
\end{array}$$

is an *isomorphism.*[3] Thus if every $\mathcal{E}_\eta$ is affine-acyclic, so that each μ_i is an isomorphism, then each ν_i is an isomorphism, and so $\varinjlim \mathcal{E}_\eta$ is affine-acyclic, as asserted.

(c) *Any complex $\mathcal{E}$ in $\mathcal{A}_{\vec{\mathrm{c}}}$ is affine-$\mathcal{I}$-acyclic.*

Indeed, for all $n > 0$ and affine open $\mathcal{U} \subset \mathcal{X}$ one sees as in (a) above that $\mathrm{H}^1(\mathcal{U}, \mathcal{I}^{n-1}\mathcal{E}/\mathcal{I}^n\mathcal{E}) = 0$, whence the natural map is a surjection

$$\Gamma(\mathcal{U}, \mathcal{E}_n) \twoheadrightarrow \Gamma(\mathcal{U}, \mathcal{E}_{n-1}) \qquad (\mathcal{E}_n := \mathcal{E}/\mathcal{I}^n\mathcal{E});$$

and that each component $\mathcal{E}_n^m$ ($m \in \mathbb{Z}$) of $\mathcal{E}_n$ is affine-acyclic. To see that $\mathcal{E}_n$ is affine-acyclic when $\mathcal{E}$ is bounded below, represent the complex $\mathbf{R}\Gamma(\mathcal{U}, \mathcal{E}_n)$ as $\Gamma(\mathcal{U}, \mathcal{T}_n)$, where $\mathcal{T}_n$ is the total complex associated to a Cartan-Eilenberg resolution of $\mathcal{E}_n$; and use standard arguments to deduce from affine-acyclicity of $\mathcal{E}_n^m$ that the natural map $\Gamma(\mathcal{U}, \mathcal{E}_n) \to \Gamma(\mathcal{U}, \mathcal{T}_n)$ is a quasi-isomorphism. Then for the unbounded case let $\mathcal{E}_{n,\eta}$ be the complex

$$\cdots \to 0 \to 0 \to \mathcal{E}_n^{-\eta} \to \mathcal{E}_n^{-\eta+1} \to \cdots \qquad (\eta \in \mathbb{N}).$$

and apply (b) to $\mathcal{E}_n = \varinjlim\limits_{\eta} \mathcal{E}_{n,\eta}$.

In particular, as quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules are locally in $\mathcal{A}_{\vec{\mathrm{c}}}$ ([DFS, Corollary 3.1.4]), *any bounded quasi-coherent $\mathcal{O}_{\mathcal{X}}$-complex is locally $\mathcal{I}$-acyclic.*

The next intermediate goal is Lemma 2.2.7, a generalization of Example (c).

[3] Roughly, boundedness of $\mathbf{R}\Gamma(\mathcal{U}, -)$ enables reduction to where the $\mathcal{E}_\eta$ are uniformly bounded below, and then $\mathbf{R}\Gamma(\mathcal{U}, \varinjlim \mathcal{E}_\eta)$ can be represented via $\varinjlim$'s of canonical flasque resolutions ...

Lemma 2.2.2. *Let $\nu\colon \mathcal{V} \hookrightarrow \mathcal{X}$ be an open immersion with $\mathcal{V}$ affine. Then for any affine-acyclic $\mathcal{O}_{\mathcal{V}}$-complex $\mathcal{F}$ the natural map is a* **D***-isomorphism.*

$$\theta\colon \nu_*\mathcal{F} \xrightarrow{\ \sim\ } \mathbf{R}\nu_*\mathcal{F}.$$

Proof. Let $\rho\colon \mathcal{F} \to \mathcal{J}$ be a K-injective resolution (a quasi-isomorphism from $\mathcal{F}$ to a K-injective $\mathcal{O}_{\mathcal{V}}$-complex). Then θ can be identified with the map $\nu_*\rho\colon \nu_*\mathcal{F} \to \nu_*\mathcal{J}$. A **D**-map being an isomorphism if and only if it induces homology isomorphisms, Lemma 2.2.2 just asserts that for all $i \in \mathbb{Z}$, the resulting map Θ^i of presheaves

$$\begin{aligned}\Theta^i(\mathcal{U})\colon \mathrm{H}^i\Gamma(\nu^{-1}\mathcal{U},\mathcal{F}) &= \mathrm{H}^i\Gamma(\mathcal{U},\nu_*\mathcal{F})\\ &\to \mathrm{H}^i\Gamma(\mathcal{U},\nu_*\mathcal{J}) = \mathrm{H}^i\Gamma(\nu^{-1}\mathcal{U},\mathcal{J}) \qquad (\mathcal{U} \text{ open in } \mathcal{X})\end{aligned}$$

induces an isomorphism of the associated sheaves. When $\mathcal{U}$ is affine, $\nu^{-1}\mathcal{U}$ is affine (since $\mathcal{X}$ is separated), so by Definition 2.1 the natural map is a **D**-isomorphism

$$\Gamma(\nu^{-1}\mathcal{U},\mathcal{F}) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(\nu^{-1}\mathcal{U},\mathcal{F}) = \Gamma(\nu^{-1}\mathcal{U},\mathcal{J}),$$

i.e., $\Theta^i(\mathcal{U})$ itself is an isomorphism, whence the conclusion. □

Corollary 2.2.3. *If $\nu\colon \mathcal{V} \hookrightarrow \mathcal{X}$ is an open immersion with $\mathcal{V}$ affine, then for any affine-acyclic $\mathcal{O}_{\mathcal{V}}$-complex $\mathcal{F}$, $\nu_*\mathcal{F}$ is also affine-acyclic.*

Proof. For the inclusion $\lambda\colon \mathcal{U} \hookrightarrow \mathcal{X}$ of an affine open subset, we have a commutative diagram of open immersions where, $\mathcal{X}$ being separated, $\nu^{-1}\mathcal{U}$ is affine:

$$\begin{array}{ccc} \nu^{-1}\mathcal{U} & \xrightarrow{\ \lambda'\ } & \mathcal{V} \\ {\scriptstyle \nu'}\big\downarrow & & \big\downarrow{\scriptstyle \nu} \\ \mathcal{U} & \xrightarrow[\ \lambda\ ]{} & \mathcal{X} \end{array}$$

Let $\rho\colon \mathcal{F} \to \mathcal{J}$ be a K-injective resolution. Since ν_* has an exact left adjoint (namely ν^*), therefore $\nu_*\mathcal{J}$ is K-injective; and so by Lemma 2.2.2, $\nu_*\rho\colon \nu_*\mathcal{F} \to \nu_*\mathcal{J}$ is a K-injective resolution. As $\mathcal{F}$ is affine-acyclic, the map

$$\Gamma(\mathcal{U},\nu_*\mathcal{F}) = \Gamma(\nu^{-1}\mathcal{U},\mathcal{F}) \xrightarrow{\text{via } \rho} \Gamma(\nu^{-1}\mathcal{U},\mathcal{J}) = \Gamma(\mathcal{U},\nu_*\mathcal{J})$$

is a quasi-isomorphism; and the conclusion follows. □

Lemma 2.2.4. *Let $\mathscr{U} = (\mathcal{U}_\alpha)_{1\le\alpha\le t}$ be a finite affine open cover of $\mathcal{X}$, and let $\check{C}^\bullet$ be the corresponding Čech functor from $\mathscr{U}$-complexes to $\mathcal{O}_{\mathcal{X}}$-complexes. If $\mathcal{F}$ is a $\mathscr{U}$-complex such that $\mathcal{F}_j$ is an affine-acyclic $\mathcal{O}_{\mathcal{U}_j}$-complex for all $j \in \mathfrak{P}_t$ then $\check{C}^\bullet\mathcal{F}$ is an affine–acyclic $\mathcal{O}_{\mathcal{X}}$-complex. Moreover, if $\mathcal{F}$ is exact then so is $\check{C}^\bullet\mathcal{F}$.*

Proof. The complex $\check{C}^\bullet\mathcal{F}$ has a finite filtration

$$\check{C}^\bullet\mathcal{F} = C_0 \supset C_1 \supset C_2 \cdots \supset C_t = 0,$$

with C_s the total complex associated to the double complex $\check{C}^p(\mathcal{F}^q)_{p\ge s;\, q\in\mathbb{Z}}$. Since C_{s+1} is, as graded module, a direct summand of C_s, we have triangles in $\mathbf{K}(\mathcal{X})$

$$C_{s+1} \longrightarrow C_s \longrightarrow C_s/C_{s+1} = \oplus_{|j|=s}\lambda_{j*}\mathcal{F}_j \to C_{s+1}[1] \qquad (0 \le s < t)$$

(see e.g., [Lpm, Example (1.4.3)]).

Since $\lambda_{j*}\mathcal{F}_j$ is affine-acyclic (Corollary 2.2.3), descending induction on s, starting with $s = t-1$, yields that each C_s is affine-acyclic. Thus $\check{C}^\bullet\mathcal{F}$ $(= C_0)$ is affine-acyclic.

If $\mathcal{F}$ is exact then for $k \in \mathfrak{P}_t$, Lemma 2.2.2 shows that $\lambda_{k*}\mathcal{F}_k$ is exact. Hence for $p \geq 0$, the complex $\check{C}^p\mathcal{F}$ is exact. The family $(\check{C}^p\mathcal{F})_{0\leq p<t}$ contains all the non-vanishing columns of the double complex of which $\check{C}^\bullet\mathcal{F}$ is the total complex, and the assertion results. □

Corollary 2.2.5. *Let $\mathcal{U} = (\mathfrak{U}_\alpha)_{1\leq\alpha\leq t}$ be a finite affine open cover of $\mathfrak{X}$. Let $\check{c} := \check{c}_{\mathcal{U}}$ and $\check{c}^\infty := \varinjlim \check{c}^m$ be the corresponding functors (see §1.2). If $\mathcal{E} \in \mathbf{K}_{\vec{c}}(\mathfrak{X})$ is an affine-acyclic $\mathcal{O}_\mathfrak{X}$-complex, then so are $\check{c}\mathcal{E}$ and $\check{c}^\infty\mathcal{E}$.*

Proof. Affine-acyclicity of $\check{c}\mathcal{E}$ is given by Lemma 2.2.4, applied to the $\mathcal{U}$-pullback $\mathcal{F}$ of $\mathcal{E}$ ($\mathcal{F}_j := \lambda_j^*\mathcal{E}$ for all $j \in \mathfrak{P}_t$ ... ; and $\check{c}\mathcal{E} = \check{C}^\bullet\mathcal{F}$).

By induction, $\check{c}^m\mathcal{E}$ is affine-acyclic for all $m > 0$. Being the target of a quasi-isomorphism from $\mathcal{E}$, $\check{c}^m\mathcal{E}$ is in $\mathbf{K}_{\vec{c}}(\mathfrak{X})$. It results then from Example 2.2.1(b) that $\check{c}^\infty\mathcal{E}$ is affine-acyclic. □

Lemma 2.2.6. *Let $\lambda\colon \mathfrak{U} \to \mathfrak{X}$ be an affine map of locally noetherian formal schemes. Then for any $\mathcal{F} \in \mathcal{A}_{\vec{c}}(\mathfrak{U})$ the natural map is an isomorphism*

$$\pi_\mathcal{G}\colon \lambda_*\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{G} \xrightarrow{\sim} \lambda_*(\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{U}} \lambda^*\mathcal{G}) \qquad \big(\mathcal{G} \in \mathcal{A}_{\mathrm{qc}}(\mathfrak{X})\big).$$

Proof. The map $\pi_\mathcal{G}$ is, by definition, adjoint to the natural map

$$\lambda^*(\lambda_*\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{G}) \xrightarrow{\sim} \lambda^*\lambda_*\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{U}} \lambda^*\mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_\mathfrak{U}} \lambda^*\mathcal{G}.$$

It follows that $\pi_\mathcal{G}$ "commutes" (in the obvious sense) with open immersions into $\mathfrak{X}$. Hence we may assume that $\mathfrak{X}$ and $\mathfrak{U}$ are affine, and that there is an exact sequence

$$\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G} \to 0$$

with $\mathcal{G}_2$ and $\mathcal{G}_1$ direct sums of copies of $\mathcal{O}_\mathfrak{X}$.

Then both $\mathcal{F}\otimes\lambda^*\mathcal{G}_1$ and $\mathcal{F}\otimes\lambda^*\mathcal{G}_2$, being direct sums of copies of $\mathcal{F}$, are in $\mathcal{A}_{\vec{c}}(\mathfrak{U})$. So [DFS, Proposition 3.2.2] yields that the kernels of the maps a and b in the natural exact sequence

$$\mathcal{F} \otimes \lambda^*\mathcal{G}_2 \xrightarrow{a} \mathcal{F} \otimes \lambda^*\mathcal{G}_1 \xrightarrow{b} \mathcal{F} \otimes \lambda^*\mathcal{G} \to 0$$

are in $\mathcal{A}_{\vec{c}}(\mathfrak{U})$. From [DFS, Lemma 3.4.2] it follows then that the bottom row in the natural commutative diagram

$$\begin{array}{ccccccc}
\lambda_*\mathcal{F} \otimes \mathcal{G}_2 & \longrightarrow & \lambda_*\mathcal{F} \otimes \mathcal{G}_1 & \longrightarrow & \lambda_*\mathcal{F} \otimes \mathcal{G} & \longrightarrow & 0 \\
{\scriptstyle \pi_{\mathcal{G}_2}}\big\downarrow & & {\scriptstyle \pi_{\mathcal{G}_1}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\mathcal{G}} & & \\
\lambda_*(\mathcal{F} \otimes \lambda^*\mathcal{G}_2) & \longrightarrow & \lambda_*(\mathcal{F} \otimes \lambda^*\mathcal{G}_1) & \longrightarrow & \lambda_*(\mathcal{F} \otimes \lambda^*\mathcal{G}) & \longrightarrow & 0
\end{array}$$

is exact, and of course the top row is exact.

Since $\mathfrak{U}$ is noetherian, the functor λ_* commutes with direct sums, whence $\pi_{\mathcal{G}_2}$ and $\pi_{\mathcal{G}_1}$ are isomorphisms; so $\pi_\mathcal{G}$ is an isomorphism too. □

Lemma 2.2.7. *Let $\mathcal{U} = (\mathfrak{U}_\alpha)_{1\leq\alpha\leq t}$ be a finite affine open cover of $\mathfrak{X}$, and let $\check{C}^\bullet$ be the corresponding Čech functor from $\mathcal{U}$-complexes to $\mathcal{O}_\mathfrak{X}$-complexes. Let $\mathcal{F}$ be a $\mathcal{U}$-complex such that for every $j \in \mathfrak{P}_t$, $\mathcal{F}_j$ is a direct sum of complexes of the form $\lambda_{ij*}\mathcal{F}_{ij}$ $(i \supset j)$ with $\mathcal{F}_{ij}$ an $\mathcal{A}_{\vec{c}}(\mathfrak{U}_i)$-complex. Then $\check{C}^\bullet\mathcal{F}$ is affine-$\mathcal{I}$-acyclic; and $\check{c}^\infty\check{C}^\bullet\mathcal{F}$ is locally $\mathcal{I}$-acyclic.*

Proof. Lemma 2.2.6 (with $\lambda = \lambda_{ij}, \mathcal{F} = \mathcal{F}_{ij}$ and $\mathcal{G} = \mathcal{O}_{\mathcal{U}_j}/\mathcal{I}^n\mathcal{O}_{\mathcal{U}_j}$) provides an isomorphism

$$\lambda_{ij*}\mathcal{F}_{ij}/\mathcal{I}^n\lambda_{ij*}\mathcal{F}_{ij} \xrightarrow{\sim} \lambda_{ij*}\big(\mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij}\big).$$

Hence $\mathcal{F}_j/\mathcal{I}^n\mathcal{F}_j \cong \oplus_{i\supset j}\lambda_{ij*}\big(\mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij}\big)$. As in Example 2.2.1(a), $\mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij}$ is an $\mathcal{A}_{\vec{c}}(\mathcal{U}_i)$-complex for all $n > 0$, so that by Example 2.2.1(c), $\mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij}$ is affine-acyclic. By Corollary 2.2.3, then, $\mathcal{F}_j/\mathcal{I}^n\mathcal{F}_j$ is an affine-acyclic $\mathcal{O}_{U_j}$-complex; and so by Lemma 2.2.4, $\check{C}^\bullet(\mathcal{F}/\mathcal{I}^n\mathcal{F})$ is an affine-acyclic $\mathcal{O}_{\mathcal{X}}$-complex.

Now the natural map $\check{C}^\bullet\mathcal{F}/(\mathcal{I}^n\check{C}^\bullet\mathcal{F}) \to \check{C}^\bullet(\mathcal{F}/\mathcal{I}^n\mathcal{F})$ is an isomorphism, since it is a direct sum of maps of the form

$$\begin{aligned}\lambda_{j*}\lambda_{ij*}\mathcal{F}_{ij}/\mathcal{I}^n\lambda_{j*}\lambda_{ij*}\mathcal{F}_{ij} &= \lambda_{i*}\mathcal{F}_{ij}/\mathcal{I}^n\lambda_{i*}\mathcal{F}_{ij}\\ &\underset{2.2.6}{\xrightarrow{\sim}} \lambda_{i*}\big(\mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij}\big) = \lambda_{j*}\lambda_{ij*}\big(\mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij}\big).\end{aligned}$$

So $\check{C}^\bullet\mathcal{F}/(\mathcal{I}^n\check{C}^\bullet\mathcal{F})$ is affine-acyclic.

Also, for any affine open $\mathcal{U} \subset \mathcal{X}$ and $n > 0$, the natural map is a *surjection*

$$\Gamma\big(\mathcal{U}, \check{C}^\bullet\mathcal{F}/(\mathcal{I}^{n+1}\check{C}^\bullet\mathcal{F})\big) \twoheadrightarrow \Gamma\big(\mathcal{U}, \check{C}^\bullet\mathcal{F}/(\mathcal{I}^n\check{C}^\bullet\mathcal{F})\big),$$

because it is isomorphic to a direct sum of surjective maps of the form

$$\begin{aligned}\Gamma\big(\mathcal{U}, \lambda_{i*}(\mathcal{F}_{ij}/\mathcal{I}^{n+1}\mathcal{F}_{ij})\big) &= \Gamma\big(\lambda_i^{-1}\mathcal{U}, \mathcal{F}_{ij}/\mathcal{I}^{n+1}\mathcal{F}_{ij}\big)\\ &\twoheadrightarrow \Gamma\big(\lambda_i^{-1}\mathcal{U}, \mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij}\big) = \Gamma\big(\mathcal{U}, \lambda_{i*}(\mathcal{F}_{ij}/\mathcal{I}^n\mathcal{F}_{ij})\big),\end{aligned}$$

surjectivity holding by [DFS, Lemma 3.1.8], since $\mathcal{I}^n\mathcal{F}_{ij}/\mathcal{I}^{n+1}\mathcal{F}_{ij} \in \mathcal{A}_{\vec{c}}(\mathcal{U}_i)$ (see Example 2.2.1(a)). Thus $\check{C}^\bullet\mathcal{F}$ is indeed affine-$\mathcal{I}$-acyclic.

Next, the pullback of $\check{C}^\bullet\mathcal{F}$ to a $\mathcal{U}$-module $\mathcal{F}^*$ is such that for every $\ell \in \mathfrak{P}_t$, $\mathcal{F}^*_\ell$ is a direct sum of complexes of the form $\lambda_{k\ell*}\mathcal{F}^*_{k\ell}$ $(k \supset \ell)$ with $\mathcal{F}^*_{k\ell}$ an $\mathcal{A}_{\vec{c}}(\mathcal{U}_k)$-complex: that follows from the relation

$$\lambda^*_\ell\lambda_{j*}\lambda_{ij*}\mathcal{F}_{ij} = \lambda_{(i\cup\ell)\ell*}\lambda^*_{(i\cup\ell)i}\mathcal{F}_{ij} \qquad (i \supset j \in \mathfrak{P}_t,\ \ell \in \mathfrak{P}_t).$$

Hence $\check{c}\check{C}^\bullet\mathcal{F} = \check{C}^\bullet\mathcal{F}^*$ is again affine-$\mathcal{I}$-acyclic. And as above, the natural map is an isomorphism

$$\check{c}\check{C}^\bullet\mathcal{F}/(\mathcal{I}^n\check{c}\check{C}^\bullet\mathcal{F}) = \check{C}^\bullet\mathcal{F}^*/(\mathcal{I}^n\check{C}^\bullet\mathcal{F}^*) \xrightarrow{\sim} \check{C}^\bullet(\mathcal{F}^*/\mathcal{I}^n\mathcal{F}^*) = \check{c}(\check{C}^\bullet\mathcal{F}/\mathcal{I}^n\check{C}^\bullet\mathcal{F}).$$

By induction, we find that $\check{c}^m\check{C}^\bullet\mathcal{F} = \check{c}^{m-1}\check{C}^\bullet\mathcal{F}^*$ is affine-$\mathcal{I}$-acyclic for all $m > 0$. So for all $n > 0$ and any affine open $\mathcal{U} \subset \mathcal{X}$, the natural map is a surjection

$$\Gamma(\mathcal{U}, \check{c}^m\check{C}^\bullet\mathcal{F}/\mathcal{I}^{n+1}\check{c}^m\check{C}^\bullet\mathcal{F}) \twoheadrightarrow \Gamma(\mathcal{U}, \check{c}^m\check{C}^\bullet\mathcal{F}/\mathcal{I}^n\check{c}^m\check{C}^\bullet\mathcal{F});$$

and applying $\varinjlim_m$, we get the same with ∞ in place of m.

By induction again, there is a natural isomorphism

$$\begin{aligned}\check{c}^m\check{C}^\bullet\mathcal{F}/(\mathcal{I}^n\check{c}^m\check{C}^\bullet\mathcal{F}) &= \check{c}^{m-1}\check{C}^\bullet\mathcal{F}^*/(\mathcal{I}^n\check{c}^{m-1}\check{C}^\bullet\mathcal{F}^*)\\ &\xrightarrow{\sim} \check{c}^{m-1}(\check{C}^\bullet\mathcal{F}^*/\mathcal{I}^n\check{C}^\bullet\mathcal{F}^*) \xrightarrow{\sim} \check{c}^m(\check{C}^\bullet\mathcal{F}/\mathcal{I}^n\check{C}^\bullet\mathcal{F}).\end{aligned}$$

So for all $n > 0$, the restriction to any $\mathcal{U}_\alpha$ of

$$\check{c}^\infty\check{C}^\bullet\mathcal{F}/(\mathcal{I}^n\check{c}^\infty\check{C}^\bullet\mathcal{F}) = \varinjlim_m \check{c}^m\check{C}^\bullet\mathcal{F}/(\mathcal{I}^n\check{c}^m\check{C}^\bullet\mathcal{F}) \cong \varinjlim_m \check{c}^m(\check{C}^\bullet\mathcal{F}/\mathcal{I}^n\check{C}^\bullet\mathcal{F})$$

is homotopy-isomorphic to the restriction of $\check{C}^\bullet\mathcal{F}/\mathcal{I}^n\check{C}^\bullet\mathcal{F}$ (cf. end of Section 1.2), so that $(\check{c}^\infty\check{C}^\bullet\mathcal{F}/\mathcal{I}^n\check{c}^\infty\check{C}^\bullet\mathcal{F})|_{\mathcal{U}_\alpha}$ is affine-acyclic (Remark 2.1.1(2)).

Thus $\check{c}^\infty\check{C}^\bullet\mathcal{F}$ is locally $\mathcal{I}$-acyclic (over the cover $\mathcal{U}$), as asserted. ∎

2.3. We also need a few results concerning flatness of complexes.

Lemma 2.3.1. *Let $\lambda\colon \mathcal{U} \to \mathcal{X}$ be a flat affine map of locally noetherian formal schemes. If $\mathcal{F} \in \mathcal{A}_{\vec{c}}(\mathcal{U})$ is $\mathcal{O}_{\mathcal{U}}$-flat, then $\lambda_*\mathcal{F}$ is $\mathcal{O}_{\mathcal{X}}$-flat.*

Proof. The question being local, we may assume that $\mathcal{U} = \mathrm{Spf}(B)$ and $\mathcal{X} = \mathrm{Spf}(A)$ where A and B are noetherian adic rings, and that $\lambda = \mathrm{Spf}(\phi)$ where $\phi\colon A \to B$ is a continuous ring homomorphism. Then by [DFS, Proposition 3.1.1], there is a B-module F such that $\mathcal{F} \cong \kappa^*\widetilde{F}$, where $\kappa\colon \mathrm{Spf}(B) \to \mathrm{Spec}(B)$ is the canonical map and $\widetilde{F}$ is the quasi-coherent sheaf on $\mathrm{Spec}(B)$ corresponding to F.

We claim that F *is B-flat.* Indeed, if J is any B-ideal and N is the kernel of the natural map $\tau_J\colon J \otimes_B F \to F$, then $\kappa^*\widetilde{N}$ is the kernel of the induced map $\kappa^*\widetilde{J} \otimes_{\mathcal{O}_{\mathcal{X}}} \kappa^*\widetilde{F} \to \kappa^*\widetilde{F}$, a map which is injective since $\kappa^*\widetilde{F} \cong \mathcal{F}$ is flat, so that $\kappa^*\widetilde{N} = 0$, whence by [DFS, Proposition 3.1.1], $\widetilde{N} = 0$, and so $N = 0$. Thus τ_J is injective for all J, i.e., F is B-flat.

Now for each $f \in A$, with $\kappa_f\colon \mathrm{Spf}(B_{\{f\}}) \to \mathrm{Spec}(B_{\{f\}})$ the natural map, and for any $B_{\{f\}}$-module M with corresponding quasi-coherent $\mathcal{O}_{\mathrm{Spec}(B_{\{f\}})}$-module $M^{\approx}$, it follows from [DFS, Proposition 3.1.1] (since the "quasi-coherator" $Q = Q_{\mathrm{Spec}(B_{\{f\}})}$ can be taken to be $\Gamma(\mathrm{Spec}(B_{\{f\}}), -)^{\sim}$) that

$$\Gamma\big(\mathrm{Spf}(B_{\{f\}}), \kappa_f^*(M^{\approx})\big) = \Gamma\big(\mathrm{Spec}(B_{\{f\}}), Q\kappa_{f*}\kappa_f^*(M^{\approx})\big) \cong \Gamma\big(\mathrm{Spec}(B_{\{f\}}), M^{\approx}\big) \cong M.$$

Taking $M = F \otimes_B B_{\{f\}}$—so that with $i_f\colon \mathrm{Spec}(B_{\{f\}}) \to \mathrm{Spec}(B)$ the natural map, $M^{\approx} = i_f^*\widetilde{F}$—we conclude that

$$\Gamma\big(\mathrm{Spf}(A_{\{f\}}), \lambda_*\kappa^*\widetilde{F}\big) = \Gamma\big(\mathrm{Spf}(B_{\{f\}}), \kappa^*\widetilde{F}\big) = \Gamma\big(\mathrm{Spf}(B_{\{f\}}), \kappa_f^*(M^{\approx})\big) \cong F \otimes_B B_{\{f\}}.$$

Hence, since $B_{\{f\}}$ is $A_{\{f\}}$-flat ([DFS, Lemma 7.1.1]), $\Gamma\big(\mathrm{Spf}(A_{\{f\}}), \lambda_*\kappa^*\widetilde{F}\big)$ is flat over $A_{\{f\}} = \Gamma\big(\mathrm{Spf}(A_{\{f\}}), \mathcal{O}_{\mathcal{X}}\big)$ for any $f \in A$; and so $\lambda_*\kappa^*\widetilde{F} \cong \lambda_*\mathcal{F}$ is $\mathcal{O}_{\mathcal{X}}$-flat. □

From the definitions of $\check{C}^\bullet$ and $\check{c}^m$ it follows now that:

Corollary 2.3.2. *If $\mathcal{U}$ is a finite affine open cover of $\mathcal{X}$ and $\mathcal{P}$ is a flat $\mathcal{A}_{\vec{c}}(\mathcal{U})$-complex, then $\check{c}^m\check{C}^\bullet\mathcal{P}$ $(0 \le m \le \infty)$ is a flat $\mathcal{O}_{\mathcal{X}}$-complex.*

If $\lambda\colon \mathcal{U} \hookrightarrow \mathcal{X}$ is the inclusion of an open subset then since $\mathcal{X}$ is noetherian, λ_* preserves $\varinjlim$. Hence $\check{C}^\bullet$ and $\check{c}^m$ $(0 \le m \le \infty)$ preserve $\varinjlim$, and so:

Corollary 2.3.3. *If $\mathcal{U}$ is a finite affine open cover of $\mathcal{X}$ and $\mathcal{P}$ is a $\varinjlim$ of bounded-above flat $\mathcal{A}_{\vec{c}}(\mathcal{U})$-complexes, then $\check{c}^m\check{C}^\bullet\mathcal{P}$ $(0 \le m \le \infty)$ is a $\varinjlim$ of bounded-above flat $\mathcal{O}_{\mathcal{X}}$-complexes, and hence is flat and K-flat.*

Corollary 2.3.4. (i) *If an $\mathcal{O}_{\mathcal{X}}$-complex $\mathcal{E}$ is K-flat then so is $\check{c}^m\mathcal{E}$ $(0 \le m \le \infty)$.*

(ii) *If an $\mathcal{A}_{\vec{c}}(\mathcal{X})$-complex $\mathcal{E}$ is flat then so is $\check{c}^m\mathcal{E}$ $(0 \le m \le \infty)$.*

Proof. (i) The restriction of a K-flat complex to an open subset is still K-flat, as one sees using "extension by 0." Thus for K-flatness the question is local, and we need only recall that $\check{c}^m\mathcal{E}$ is locally homotopy-isomorphic to $\mathcal{E}$ (cf. end of Section 1.2).

(ii) Apply Corollary 2.3.2 with $\mathcal{P}$ the pullback to $\mathcal{U}$ of $\mathcal{E}$. □

Applying Lemma 2.2.7 to the $\mathcal{U}$-pullback $\mathcal{F}$ of $\mathcal{E}$, we get (c) in Section 1.3.2:

Corollary 2.3.5. *For any K-flat $\mathcal{A}_{\vec{c}}$-complex $\mathcal{E}$, $\check{c}^\infty\mathcal{E}$ is K-flat and locally $\mathcal{I}$-acyclic.*

Lemma 2.4.1. *Let $Q_{\mathcal{X}}$ be right-adjoint to the inclusion functor $\mathcal{A}_{\vec{c}} \hookrightarrow \mathcal{A}$ (see [DFS, Proposition 3.2.3]). If $\mathcal{X}$ is affine then for any affine-acyclic $\mathcal{E} \in \mathbf{K}_{\vec{c}}$ the canonical map $Q_{\mathcal{X}}\mathcal{E} \to \mathcal{E}$ is a quasi-isomorphism.*

Proof. Let A be a noetherian adic ring such that $\mathcal{X} \cong \mathrm{Spf}(A)$ (e.g., $A := \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$), and let $\kappa\colon \mathcal{X} \to X := \mathrm{Spec}(A)$ be the canonical map. Then with $^\sim$ denoting the usual functor from A-modules to $\mathcal{O}_X$-modules, [DFS, Proposition 3.2.3] yields

$$Q_{\mathcal{X}}\mathcal{E} \cong \kappa^*\big(Q_X\kappa_*\mathcal{E}\big) = \kappa^*\big(\Gamma(X, \kappa_*\mathcal{E})^\sim\big) = \kappa^*\big(\Gamma(\mathcal{X}, \mathcal{E})^\sim\big).$$

By assumption, any K-injective resolution $i\colon \mathcal{E} \to \mathcal{J}$ induces a quasi-isomorphism $\Gamma(\mathcal{X}, \mathcal{E}) \to \Gamma(\mathcal{X}, \mathcal{J})$; and so since κ^* and $^\sim$ are exact, $Q_{\mathcal{X}}(i)\colon Q_{\mathcal{X}}\mathcal{E} \to Q_{\mathcal{X}}\mathcal{J}$ is a quasi-isomorphism.

Furthermore, [DFS, Corollary 3.3.4] implies that the natural map $Q_{\mathcal{X}}\mathcal{J} \to \mathcal{J}$ is a quasi-isomorphism. Conclude via the natural commutative diagram

$$\begin{array}{ccc} Q_{\mathcal{X}}\mathcal{E} & \longrightarrow & \mathcal{E} \\ {\scriptstyle Q_{\mathcal{X}}(i)}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ Q_{\mathcal{X}}\mathcal{J} & \longrightarrow & \mathcal{J} \end{array}$$

□

Proposition 2.4.2. *Let $\mathcal{U} = (\mathcal{U}_\alpha)_{1\le\alpha\le t}$ be an affine open cover of $\mathcal{X}$. Let $\mathcal{G}$ be a $\mathcal{U}$-complex with $\mathcal{A}_{\vec{c}}(\mathcal{U})$-homology such that for each $k \in \mathfrak{P}_t$, $\mathcal{G}_k$ is affine-acyclic. Then $\mathcal{G}$ receives a quasi-isomorphism from a $\varinjlim$ of bounded-above flat complexes of $\mathcal{A}_{\vec{c}}(\mathcal{U})$-modules.*

Proof. If $i \supset j$ are in $\mathfrak{P}_t$, then $\lambda_{ij}^* Q_{\mathcal{U}_j}\mathcal{G}_j$ is an $\mathcal{A}_{\vec{c}}(\mathcal{U}_i)$-complex, and so the natural composition

$$\lambda_{ij}^* Q_{\mathcal{U}_j}\mathcal{G}_j \to \lambda_{ij}^*\mathcal{G}_j \to \mathcal{G}_i$$

factors naturally as

$$\lambda_{ij}^* Q_{\mathcal{U}_j}\mathcal{G}_j \xrightarrow{\phi_{ij}} Q_{\mathcal{U}_i}\mathcal{G}_i \to \mathcal{G}_i\,.$$

The maps ϕ_{ij} make the family $(Q_{\mathcal{U}_k}\mathcal{G}_k)_{k\in\mathfrak{P}_t}$ into an $\mathcal{A}_{\vec{c}}(\mathcal{U})$-complex. Lemma 2.4.1 lets us replace $\mathcal{G}$ by this complex, i.e., we may assume that $\mathcal{G}$ is an $\mathcal{A}_{\vec{c}}(\mathcal{U})$-complex.

After noting that if $\mathcal{U}$ is an affine noetherian formal scheme then, by [DFS, Corollary 3.1.4], any $\mathcal{A}_{\vec{c}}(\mathcal{U})$-module is a homomorphic image of a free $\mathcal{O}_{\mathcal{U}}$-module, we can conclude as in the proof of [AJL, p. 11, Corollary 1.2.2]. □

2.5. We can now complete the proof of Proposition 2.1.3.

Let $\mathcal{U} = (\mathcal{U}_\alpha)_{1\le\alpha\le t}$ be an affine open cover of $\mathcal{X}$. For any $\mathcal{E} \in \check{\mathbf{K}}^{\mathrm{a}}_{\vec{c}}$, the pullback is a $\mathcal{U}$-module $\mathcal{G}$ satisfying the hypotheses of Proposition 2.4.2; so there is a quasi-isomorphism $\mathcal{P} \to \mathcal{G}$ with $\mathcal{P}$ a $\varinjlim$ of bounded-above flat complexes of $\mathcal{A}_{\vec{c}}(\mathcal{U})$-modules. Since $\mathcal{A}_{\vec{c}}$-complexes are affine-acyclic (Example 2.2.1(c) and Remark 2.1.1(1)), the (exact) mapping cone $\mathcal{F}$ of this quasi-isomorphism satisfies the hypotheses of Lemma 2.2.4, so $\check{C}^\bullet\mathcal{F}$ is exact, and hence the induced map $\check{C}^\bullet\mathcal{P} \to \check{C}^\bullet\mathcal{G} = \check{c}\mathcal{E}$ is a quasi-isomorphism, as is the resulting map

$$\mathcal{P}_{\mathcal{E}} := \check{c}^\infty\check{C}^\bullet\mathcal{P} \to \check{c}^\infty\check{c}\mathcal{E} \cong \check{c}^\infty\mathcal{E}.$$

Since $\check{C}^\bullet\mathcal{P}$, $\check{c}\mathcal{E}$, and $\mathcal{E}$ have isomorphic homology, therefore $\check{C}^\bullet\mathcal{P} \in \mathbf{K}_{\vec{c}}(\mathfrak{X})$; and $\check{C}^\bullet\mathcal{P}$ is affine-acyclic (Lemma 2.2.7, with $\mathcal{I}$=0), so $\mathcal{P}_\mathcal{E} \in \check{\mathbf{K}}^{\mathrm{a}}_{\vec{c}}$. (Look at the actual construction of $\mathcal{P}$—as given in [AJL, p. 11, Corollary 1.2.2]—to see that the hypotheses of Lemma 2.2.7 are satisfied.) By Lemma 2.2.7, furthermore, $\mathcal{P}_\mathcal{E}$ is locally $\mathcal{I}$-acyclic. From Corollary 2.3.3, it follows that $\mathcal{P}_\mathcal{E}$ is flat and K-flat.

Finally, since $\mathcal{E} \in \check{\mathbf{K}}_{\vec{c}}(\mathfrak{X})$ the natural map of complexes $\mathcal{E} \to \check{c}^\infty\mathcal{E}$ has an inverse, which can be composed with the preceding quasi-isomorphism $\mathcal{P}_\mathcal{E} \to \check{c}^\infty\mathcal{E}$ to give the desired quasi-isomorphism $\mathcal{P}_\mathcal{E} \to \mathcal{E}$. □

3. A functorial isomorphism

Denote by Λ the restricted completion functor $\Lambda_\mathcal{I}|_{\mathbf{K}^{\mathrm{a}}_{\vec{c}}}$, and by $\mathbf{L}\Lambda\colon \mathbf{D}^{\mathrm{a}}_{\vec{c}} \to \mathbf{D}$ its left-derived functor, whose existence has now been proved according to Section 1.3.2. The proof shows we may assume $\mathbf{L}\Lambda\mathcal{P} = \Lambda\mathcal{P}$ for any K-flat locally $\mathcal{I}$-acyclic $\mathcal{P} \in \check{\mathbf{K}}^{\mathrm{a}}_{\vec{c}}$.

3.1. We first construct a Δ-functorial map

$$\Phi\colon \mathbf{L}\Lambda\mathcal{E} \to \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_\mathcal{I}\mathcal{O}_\mathfrak{X}, \mathcal{E}) \qquad (\mathcal{E} \in \mathbf{D}^{\mathrm{a}}_{\vec{c}}).$$

Using the equivalence $\boldsymbol{\rho}\colon \mathbf{D}^{\mathrm{a}}_{\vec{c}} \to \check{\mathbf{D}}^{\mathrm{a}}_{\vec{c}}$ (Section 1.3), we can replace $\mathbf{D}^{\mathrm{a}}_{\vec{c}}$ by its Δ-subcategory $\check{\mathbf{D}}^{\mathrm{a}}_{\vec{c}}$. Because of Proposition 2.1.3, we can further restrict to the equivalent Δ-subcategory of $\check{\mathbf{D}}^{\mathrm{a}}_{\vec{c}}$ whose objects are the flat, K-flat, locally $\mathcal{I}$-acyclic complexes in $\check{\mathbf{K}}_{\vec{c}}(\mathfrak{X})$—this is the derived category of the homotopy category $\mathbf{H}$ of such complexes. Thus, with $q\colon \mathbf{K} \to \mathbf{D}$ the natural functor and q' the natural functor from $\mathbf{H}$ to its derived category, [Hrt, p. 33, Proposition 3.4] shows it sufficient to construct a map of functors

$$(\mathbf{L}\Lambda)q'\mathcal{P} = q\Lambda\mathcal{P} \to \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_\mathcal{I}\mathcal{O}_\mathfrak{X}, q'\mathcal{P}) \qquad (\mathcal{P} \in \mathbf{H}).$$

Hence if $\mathcal{O}_\mathfrak{X} \to \mathcal{R}$ is a K-injective resolution, it will suffice to construct a map

$$\Lambda\mathcal{P} \to \mathcal{H}om^\bullet(\varGamma_\mathcal{I}\mathcal{R}, \mathcal{P} \otimes \mathcal{R}) \qquad (\mathcal{P} \in \mathbf{H}) \tag{3.1.1}$$

between functors from $\mathbf{H}$ to $\mathbf{K}$.

For any $\mathcal{O}_\mathfrak{X}$-complexes $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, the natural map

$$\big(\mathcal{P} \otimes \mathcal{Q}\big) \otimes \big(\mathcal{H}om^\bullet(\mathcal{Q}, \mathcal{R})\big) \cong \mathcal{P} \otimes \big(\mathcal{Q} \otimes \mathcal{H}om^\bullet(\mathcal{Q}, \mathcal{R})\big) \to \mathcal{P} \otimes \mathcal{R}$$

induces (via $\otimes$–$\mathcal{H}om$ adjunction) a functorial map

$$\mathcal{P} \otimes \mathcal{Q} \to \mathcal{H}om^\bullet\big(\mathcal{H}om^\bullet(\mathcal{Q}, \mathcal{R}), \mathcal{P} \otimes \mathcal{R}\big).$$

Letting $\mathcal{Q}$ run through the inverse system $\mathcal{O}_\mathfrak{X}/\mathcal{I}^n$ $(n > 0)$ one gets the desired natural map

$$\begin{aligned}\Lambda\mathcal{P} = \varprojlim\,(\mathcal{P} \otimes \mathcal{O}_\mathfrak{X}/\mathcal{I}^n) \to {} & \varprojlim\,\mathcal{H}om^\bullet\big(\mathcal{H}om^\bullet(\mathcal{O}_\mathfrak{X}/\mathcal{I}^n, \mathcal{R}), \mathcal{P} \otimes \mathcal{R}\big)\\ \cong {} & \mathcal{H}om^\bullet\big(\varinjlim\,\mathcal{H}om^\bullet(\mathcal{O}_\mathfrak{X}/\mathcal{I}^n, \mathcal{R}), \mathcal{P} \otimes \mathcal{R}\big)\\ \cong {} & \mathcal{H}om^\bullet\big(\varGamma_\mathcal{I}\mathcal{R}, \mathcal{P} \otimes \mathcal{R}\big).\end{aligned}$$

Now comes the principal result.

Proposition 3.2. *The Δ-functorial map Φ is an isomorphism.*

Before describing its proof, we note that Proposition 3.2 allows us to complete the proof of Theorem 0.1, as follows.

There is a canonical functorial map $\xi\colon \mathbf{L}\Lambda \to q\Lambda$. With $\zeta := \xi\Phi^{-1}$, property (i) in Theorem 0.1 results. The functorial map $\mathcal{F} \to q\Lambda\mathcal{F}$ ($\mathcal{F} \in \mathbf{D}^{\mathrm{a}}_{\vec{\mathrm{c}}}$) factors uniquely as $\mathcal{F} \xrightarrow{\iota_{\mathcal{F}}} \mathbf{L}\Lambda\mathcal{F} \xrightarrow{\xi_{\mathcal{F}}} q\Lambda\mathcal{F}$ (thanks to the defining property of "left-derived functor"), and Theorem 0.1(ii) says that $\Phi_{\mathcal{F}} \circ \iota(\mathcal{F})$ is the natural map $\mathcal{F} \to \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})$. One checks that this comes down to the straightforward verification (left to the reader) that the following natural diagram commutes for all $\mathcal{P} \in \mathbf{H}$:

$$\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \Lambda\mathcal{P} \\
\big\downarrow & & \big\downarrow {\scriptstyle (3.1.1)} \\
\mathcal{H}om^{\bullet}(\mathcal{R}, \mathcal{P} \otimes \mathcal{R}) & \longrightarrow & \mathcal{H}om^{\bullet}(\varGamma_{\mathcal{I}}\mathcal{R}, \mathcal{P} \otimes \mathcal{R})
\end{array}$$

Finally, for the last assertion in Theorem 0.1 see Section 1.3.2.

Proof of Proposition 3.2. The proof is similar to that of the main Theorem (0.3) in [AJL]. We just indicate the non-trivial modifications.

Everything in sight commutes with restriction to open subsets, so we may assume that $\mathfrak{X}$ is affine and that $\mathcal{I}$ is generated by a finite number of global sections $t_1, \dots, t_m$. Recalling that every sheaf in $\mathcal{A}_{\vec{\mathrm{c}}}$ is then a homomorphic image of a free $\mathcal{O}_{\mathfrak{X}}$-module ([DFS, Corollary 3.1.4]), we have a "way-out" reduction, as in [AJL, p. 15], to the case where $\mathcal{E}$ is a single flat $\mathcal{O}_{\mathfrak{X}}$-module, denoted $\mathcal{P}$, in $\mathcal{A}_{\vec{\mathrm{c}}}$.

This case is dealt with as in [AJL, §4]. Let R be the adic ring $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ (topologized by the global sections of a defining ideal of $\mathfrak{X}$), so that $\mathfrak{X} \cong \mathrm{Spf}(R)$. Since R is noetherian, the sequence $\mathbf{t} := (t_1, \dots, t_m)$ in R is *proregular* (see [AJL, p. 16], Example (a) following Definition (3.0.1)). Throughout [AJL, §4], replace the scheme $(X, \mathcal{O}_X)$ by the noetherian affine formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and interpret "quasi-coherent" to mean "$\varinjlim$ of coherent." Also, replace $\mathbf{R}\varGamma_Z\mathcal{O}_X$ by $\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}$, which is $\mathbf{D}(\mathfrak{X})$-isomorphic to the complex

$$\mathcal{K}^{\bullet}_{\infty}(\mathbf{t}) := \varinjlim_{r>0} \mathcal{K}^{\bullet}(\mathbf{t}^r)$$

where $\mathcal{K}^{\bullet}(\mathbf{t}^r)$ is the Koszul $\mathcal{O}_{\mathfrak{X}}$-complex determined by the sequence $(t_1^r, \dots, t_m^r)$, see [AJL, p. 18, Lemma (3.1.1)].

To validate the sequence of isomorphisms (4.1.3) of [AJL, p. 28] in the present context, use the following observations:

(i) If $\mathcal{P} \to \mathcal{J}$ is an injective resolution, then the resulting map $\Gamma(\mathfrak{X}, \mathcal{P}) \to \Gamma(\mathfrak{X}, \mathcal{J})$ is a *quasi-isomorphism.*

This is because $\mathcal{P}$ is affine-acyclic (Example 2.2.1(a)).

(ii) If $\mathcal{E}$ is any complex in $\mathcal{A}_{\vec{\mathrm{c}}}$, then the natural maps

$$\mathrm{H}^i\Gamma(\mathfrak{X}, \mathcal{E}) \to \Gamma(\mathfrak{X}, H^i\mathcal{E}) \qquad (i \in \mathbb{Z})$$

are all isomorphisms.

This is because the functor $\Gamma := \Gamma(\mathfrak{X}, -)$ on the abelian category $\mathcal{A}_{\vec{\mathrm{c}}}$ is *exact* ([DFS, Proposition 3.2.2], [DFS, Corollary 3.1.8]). In particular, since $\mathcal{K}^{\bullet}(\mathbf{t}^r)$ is a free, finite-rank $\mathcal{O}_{\mathfrak{X}}$-complex and $R = \Gamma\mathcal{O}_{\mathfrak{X}}$, we have natural isomorphisms

$$\mathrm{H}^i\mathrm{Hom}^{\bullet}_R(\Gamma\mathcal{K}^{\bullet}(\mathbf{t}^r), \Gamma\mathcal{P}) \xrightarrow{\ \sim\ } \mathrm{H}^i\mathrm{Hom}^{\bullet}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{K}^{\bullet}(\mathbf{t}^r), \mathcal{P}) \xrightarrow{\ \sim\ } \Gamma H^i\mathcal{H}om^{\bullet}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{K}^{\bullet}(\mathbf{t}^r), \mathcal{P}).$$

The rest of the translation of [AJL, §4] to the present context is straightforward, except for the last half of the proof of Lemma (4.3) in *loc. cit.* For that, one needs [DFS, Corollary 3.3.4] (to replace the reference to [BN]), and [DFS, Corollary 3.1.8] (for imitating the proof of [Spn, p. 134, Proposition 3.13]); and one needs to keep in mind that $\mathcal{A}_{\vec{c}}$-complexes are affine-acyclic (Example 2.2.1(c)). □

4. Proof of Proposition 0.3.1.

Refer to Section 0.3 for notation, and for the statement of Proposition 0.3.1.

Lemma 4.1. *The canonical map* $\mathcal{F} \to \kappa_*\kappa^*\mathcal{F}$ *induces an isomorphism*

$$\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F} \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathcal{I}}\kappa_*\kappa^*\mathcal{F} \qquad (\mathcal{F} \in \mathbf{D}).$$

Proof. The question being local, we may assume that $\mathfrak{X}$ is the completion of an affine scheme $X = \operatorname{Spec}(A)$ along a closed subscheme Z, and that $\hat{\mathfrak{X}}_{\mathcal{I}}$ is the completion of X along a closed subscheme $Y \subset Z$, so that there is a natural factorization of the completion map $\kappa_Y \colon \hat{\mathfrak{X}}_{\mathcal{I}} \to X$ as

$$\hat{\mathfrak{X}}_{\mathcal{I}} \xrightarrow{\kappa} \mathfrak{X} \xrightarrow{\kappa_Z} X.$$

Now let $\mathcal{I}' \subset \mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. Then for any $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{E}$ such that $\mathbf{R}\varGamma_{\mathcal{I}'}\mathcal{E} = \mathcal{E}$, we have $\mathcal{E} \cong \kappa_Z^*E$ for some $\mathcal{O}_X$-module E [DFS, Proposition 5.2.4]. In particular, since $\mathbf{R}\varGamma_{\mathcal{I}'}\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F} = \mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F}$ (cf. [AJL, p. 20, Corollary 3.1.3]), therefore $\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F} \cong \kappa_Z^*F$ for some $\mathcal{O}_X$-module F.

By [DFS, Proposition 5.2.4] again, the natural maps are isomorphisms

$$\begin{aligned}\operatorname{Hom}_X(E,F) &\xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{X}}(\kappa_Z^*E, \kappa_Z^*F) = \operatorname{Hom}_{\mathfrak{X}}(\mathcal{E}, \mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F}),\\ \operatorname{Hom}_X(E,F) &\xrightarrow{\sim} \operatorname{Hom}_{\hat{\mathfrak{X}}_{\mathcal{I}}}(\kappa_Y^*E, \kappa_Y^*F) = \operatorname{Hom}_{\hat{\mathfrak{X}}_{\mathcal{I}}}(\kappa^*\mathcal{E}, \kappa^*\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F}).\end{aligned}$$

Furthermore, arguing as in the proof of [DFS, Proposition 5.2.4] we get a natural isomorphism

$$\kappa_*\kappa^*\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F} \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathcal{I}}\kappa_*\kappa^*\mathcal{F}$$

whose composition with the natural map $\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F} \to \kappa_*\kappa^*\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F}$ is the natural map $\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F} \to \mathbf{R}\varGamma_{\mathcal{I}}\kappa_*\kappa^*\mathcal{F}$.

Thus for any $\mathcal{E}$ such that $\mathbf{R}\varGamma_{\mathcal{I}'}\mathcal{E} = \mathcal{E}$, we have a composed isomorphism

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{X}}(\mathcal{E}, \mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F}) &\xrightarrow{\sim} \operatorname{Hom}_X(E,F) \xrightarrow{\sim} \operatorname{Hom}_{\hat{\mathfrak{X}}_{\mathcal{I}}}(\kappa^*\mathcal{E}, \kappa^*\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F})\\ &\xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{X}}(\mathcal{E}, \kappa_*\kappa^*\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{F}) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{X}}(\mathcal{E}, \mathbf{R}\varGamma_{\mathcal{I}}\kappa_*\kappa^*\mathcal{F})\end{aligned}$$

which (one checks) is induced by $\mathcal{F} \to \kappa_*\kappa^*\mathcal{F}$, whence the conclusion. □

Corollary 4.1.1. *The canonical map* $\mathcal{F} \to \kappa_*\kappa^*\mathcal{F}$ *induces an isomorphism*

$$\operatorname{Hom}\big(\kappa_*\kappa^*\mathcal{F}, \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})\big) \xrightarrow{\sim} \operatorname{Hom}\big(\mathcal{F}, \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})\big).$$

Proof. Using that $\mathbf{R}\varGamma_{\mathcal{I}}$ is right-adjoint to the inclusion into $\mathbf{D}$ of the derived category of the category of $\mathcal{I}$-torsion sheaves ([DFS, Lemma 5.2.2]), one finds that there is a unique functorial map

$$\psi_{\mathcal{E}} \colon \mathcal{E} \underset{=}{\otimes} \mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}} \to \mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E} \qquad (\mathcal{E} \in \mathbf{D}) \tag{4.1.1.1}$$

making the following natural diagram commute:

$$\begin{array}{ccc}
\mathcal{E} \underset{=}{\otimes} \mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}} & \xrightarrow{\ \psi_{\mathcal{E}}\ } & \mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{E} \\
\big\downarrow & & \big\downarrow \\
\mathcal{E} \underset{=}{\otimes} \mathcal{O}_{\mathfrak{X}} & \xrightarrow{\ \sim\ } & \mathcal{E}
\end{array}$$

It follows from [AJL, p. 20, Corollary (3.1.2)] that $\psi_{\mathcal{E}}$ is an *isomorphism.*

Let us also recall the basic *adjoint associativity* isomorphism (see, e.g., [Lpm, Proposition (2.6.1)*]):

$$\text{(4.1.1.2)}\quad \mathbf{R}\mathcal{H}om^{\bullet}\big(\mathcal{D} \underset{=}{\otimes} \mathcal{E},\mathcal{F}\big) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^{\bullet}\big(\mathcal{D}, \mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E},\mathcal{F})\big) \qquad (\mathcal{D},\mathcal{E},\mathcal{F} \in \mathbf{D}).$$

The isomorphism also holds with the outer $\mathbf{R}\mathcal{H}om^{\bullet}$'s replaced by $\mathrm{Hom} := \mathrm{Hom}_{\mathbf{D}}$, as one sees by applying the functor $\mathrm{H}^0\mathbf{R}\Gamma(\mathfrak{X}, -)$.

Corollary 4.1.1 results then from the natural commutative diagram, in which all the vertical arrows are isomorphisms, as is the bottom horizontal one:

$$\begin{array}{ccc}
\mathrm{Hom}\big(\kappa_*\kappa^*\mathcal{F}, \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})\big) & \longrightarrow & \mathrm{Hom}\big(\mathcal{F}, \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})\big) \\
\simeq\big\downarrow & & \big\downarrow\simeq \\
\mathrm{Hom}(\kappa_*\kappa^*\mathcal{F} \underset{=}{\otimes} \mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F}) & \longrightarrow & \mathrm{Hom}(\mathcal{F} \underset{=}{\otimes} \mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F}) \\
\simeq\big\uparrow & & \big\uparrow\simeq \\
\mathrm{Hom}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\kappa_*\kappa^*\mathcal{F}, \mathcal{F}) & \underset{\text{Lemma 4.1}}{\xrightarrow{\ \sim\ }} & \mathrm{Hom}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{F}, \mathcal{F})
\end{array}$$

□

In the following natural diagram, where "H" stands for "Hom," the isomorphisms a and a^* are obtained by adjoint associativity and multiple use of ψ (see (4.1.1.1)), and the isomorphisms b and b^* are obtained by imitating [DFS, Remark 5.2.10](4). The diagram commutes, so c is an isomorphism.

Assertion (i) in Proposition 0.3.1 results.

$$\begin{array}{ccc}
\mathrm{H}(\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \kappa_*\kappa^*\mathcal{F}), \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{E}, \mathcal{F})) & \xrightarrow{\ c\ } & \mathrm{H}(\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \mathcal{F}), \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{E}, \mathcal{F})) \\
a^*\big\downarrow\simeq & & \simeq\big\downarrow a \\
\mathrm{H}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \kappa_*\kappa^*\mathcal{F}) \underset{=}{\otimes} \mathcal{E}, \mathcal{F})) & \longrightarrow & \mathrm{H}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \mathcal{F}) \underset{=}{\otimes} \mathcal{E}, \mathcal{F})) \\
b^*\big\downarrow\simeq & & \simeq\big\downarrow b \\
\mathrm{H}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\kappa_*\kappa^*\mathcal{F}) \underset{=}{\otimes} \mathcal{E}, \mathcal{F})) & \underset{4.1.1}{\xrightarrow{\ \sim\ }} & \mathrm{H}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{F}) \underset{=}{\otimes} \mathcal{E}, \mathcal{F}))
\end{array}$$

4.2. What remains to be shown is that $\lambda(\mathcal{E},\mathcal{F})$ is an isomorphism for all $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}$.

We can reduce the problem to the case $\mathcal{E} = \mathcal{O}_{\mathfrak{X}}$ by factoring $\lambda(\mathcal{E},\mathcal{F})$ as

$$\begin{aligned}
\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \kappa_*\kappa^*\mathcal{F}) &\xrightarrow{\text{via } \lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})} \mathbf{R}\mathcal{H}om^{\bullet}\big(\mathcal{E}, \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F})\big) \\
&\underset{(4.1.1.2)}{\xrightarrow{\ \sim\ }} \mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E} \underset{=}{\otimes} \mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}, \mathcal{F}) \underset{(4.1.1.1)}{\xrightarrow{\ \sim\ }} \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\mathit{\Gamma}_{\mathcal{I}}\mathcal{E}, \mathcal{F}).
\end{aligned}$$

To check that this composition is in fact $\lambda(\mathcal{E},\mathcal{F})$, using the characterization in Proposition 0.3.1(i), consider the following natural diagram:

$$
\begin{array}{ccc}
\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\kappa_*\kappa^*\mathcal{F}) \\
\Big\downarrow{\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle \text{via } \lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})} \\
\mathbf{R}\mathcal{H}om^\bullet\big(\mathcal{E},\mathbf{R}\mathcal{H}om^\bullet(\mathcal{O}_{\mathfrak{X}},\mathcal{F})\big) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet\big(\mathcal{E},\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}},\mathcal{F})\big) \\
{\scriptstyle (4.1.1.2)}\Big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\Big\downarrow{\scriptstyle (4.1.1.2)} \\
\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}\underset{=}{\otimes}\mathcal{O}_{\mathfrak{X}},\mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}\underset{=}{\otimes}\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}},\mathcal{F}) \\
\Big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\Big\downarrow{\scriptstyle (4.1.1.1)} \\
\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{E},\mathcal{F})
\end{array}
$$

The top rectangle commutes by the characterization of $\lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})$, the middle one commutes by functoriality, and commutativity of the bottom one is given by the commutative diagram following (4.1.1.1). It suffices then to verify that the composition of the vertical maps on the left is the identity, which is easy to do after assuming (as one may) that $\mathcal{F}$ is K-injective, so that each $\mathbf{R}\mathcal{H}om^\bullet$ can be replaced by $\mathcal{H}om^\bullet$.

To show that $\lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})$ is an isomorphism when $\mathcal{F}\in\mathbf{D}_{\mathrm{c}}$ we may assume, after replacing $\mathcal{F}$ by a K-injective complex into which $\mathcal{F}$ maps quasi-isomorphically, that $\mathcal{F}\in\mathbf{D}^{\mathrm{a}}_{\vec{\mathrm{c}}}$. For such an $\mathcal{F}$ we will now give another construction of $\lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})$, adapted to application of Theorem 0.1.

For any $\mathcal{O}_{\mathfrak{X}}$-complex $\mathcal{F}$, we have the canonical completion map

$$\mathcal{F}\to\Lambda_{\mathcal{I}}\mathcal{F}=\kappa_*\varprojlim_n\big((\mathcal{F}/\mathcal{I}^n\mathcal{F})|_{\hat{\mathfrak{X}}_{\mathcal{I}}}\big),$$

whence a natural factorization $\mathcal{F}\longrightarrow\kappa_*\kappa^*\mathcal{F}\xrightarrow{\gamma_{\mathcal{F}}}\Lambda_{\mathcal{I}}\mathcal{F}$. The exact functor $\kappa_*\kappa^*$ induces a functor $\mathbf{D}^{\mathrm{a}}_{\vec{\mathrm{c}}}\to\mathbf{D}$, so for $\mathcal{F}\in\mathbf{D}^{\mathrm{a}}_{\vec{\mathrm{c}}}$ Theorem 0.1(i) yields a factorization $\gamma_{\mathcal{F}}=\zeta_{\mathcal{F}}\circ\lambda_0(\mathcal{F})$, as depicted:

$$\mathcal{F}\longrightarrow\kappa_*\kappa^*\mathcal{F}\xrightarrow{\lambda_0(\mathcal{F})}\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}},\mathcal{F})\xrightarrow{\zeta_{\mathcal{F}}}\Lambda_{\mathcal{I}}\mathcal{F}\qquad(\mathcal{F}\in\mathbf{D}^{\mathrm{a}}_{\vec{\mathrm{c}}}).$$

In view of Theorem 0.1, (i) and (ii), *the composition of* $\lambda_0(\mathcal{F})$ *with the canonical map* $\mathcal{F}\to\kappa_*\kappa^*\mathcal{F}$ *is the natural map* $\mathcal{F}\to\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}},\mathcal{F})$, since both this composition and the natural map give the same result when composed with $\zeta_{\mathcal{F}}$. Proposition 0.3.1(i) shows then that $\lambda_0(\mathcal{F})=\lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})$.

The question of whether $\lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})$ is an isomorphism is local. One can therefore assume that $\mathfrak{X}$ is affine, so that every coherent $\mathcal{O}_{\mathfrak{X}}$-module is a homomorphic image of a finite-rank free one [GD, p. 427, Théorème (10.10.2)]. Now the exact functor $\kappa_*\kappa^*$ is bounded on $\mathbf{D}_{\vec{\mathrm{c}}}$, and so is the functor $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}},-)$, by [AJL, p. 30, Lemma (4.3)], suitably modified (see discussion at the end of Section 3 above). Hence we have a "way-out" reduction [Hrt, p. 68, Proposition 7.1] to the case where $\mathcal{F}$ is a single finite-rank free $\mathcal{O}_{\mathfrak{X}}$-module. In this case it is straightforward to see that $\gamma_{\mathcal{F}}=\zeta_{\mathcal{F}}\lambda_0(\mathcal{F})$ is an isomorphism; and by the last assertion in Theorem 0.1, so is $\zeta_{\mathcal{F}}$. Thus $\lambda(\mathcal{O}_{\mathfrak{X}},\mathcal{F})=\lambda_0(\mathcal{F})$ *is* an isomorphism, and Proposition 0.3.1 is proved.

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Universidade de Santiago de Compostela, Facultade de Matemáticas, E-15771 Santiago de Compostela, SPAIN

E-mail address: `lalonso@zmat.usc.es`

Universidade de Santiago de Compostela, Facultade de Matemáticas, E-15771 Santiago de Compostela, SPAIN

E-mail address: `jeremias@zmat.usc.es`

Dept. of Mathematics, Purdue University, W. Lafayette IN 47907, USA

E-mail address: `lipman@math.purdue.edu`