LECTURES ON LOCAL COHOMOLOGY AND DUALITY

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Abstract. In these expository notes derived categories and functors are gently introduced, and used along with Koszul complexes to develop the basics of local cohomology. Local duality and its far-reaching generalization, Greenlees-May duality, are treated. A canonical version of local duality, via differentials and residues, is outlined. Finally, the fundamental Residue Theorem, described here e.g., for smooth proper maps of formal schemes, marries canonical local duality to a canonical version of Grothendieck duality for formal schemes.

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Introduction

This is an expanded version of a series of lectures given during the Local Cohomology workshop at CIMAT, Guanajuoto, Mexico, Nov. 29–Dec. 3, 1999, and at the University of Mannheim, Germany, during May, 2000. I am grateful to these institutions for their support.

Rings are assumed once and for all to be *commutative* and *noetherian* (though noetherianness plays no role until midway through §2.3.) We deal for the most part with modules over such rings; but almost everything can be done, under suitable noetherian hypotheses, over commutative graded rings (see [BS, Chapters 12 and 13]), and globalizes to sheaves over schemes or even formal schemes (see [DFS]).

In keeping with the instructional intent of the lectures, prerequisites are relatively minimal: for the most part, familiarity with the language of categories and functors, with homology of complexes, and with some basics of commutative algebra should suffice, theoretically. (Little beyond the flatness property of completion is needed in the first four sections, and in §5 some use is made of power-series rings and exterior powers of modules of differentials. The final section 5.6, however, involves formal schemes.) Otherwise, I have tried to make the exposition self-contained, in the sense of comprehensibility of the main concepts and results. In proofs, significant underlying ideas are often indicated without technical details, but with ample references to where such details can be found.

These lectures are meant to complement foundational expositions which have full proofs and numerous applications to commutative algebra, like Grothendieck's classic [Gr2], the book of Brodmann and Sharp [BS], or the notes of Schenzel [Sch].

For one thing, the basic approach is different here. One goal is to present a quick, accessible introduction—inspiring, not daunting—to the use of derived categories. This we do in §1, building on the definition of local cohomology. Derived categories are a supple tool for working with homology, arising very naturally when one thinks about homology in terms of underlying defining complexes. They also foster conceptual simplicity. For example, the abstract Local Duality Theorem (2.3.1) is a framework for several disparate statements which appear in the literature under the name "Local Duality." The abstract theorem itself is almost trivial, following immediately from derived Hom-Tensor adjunction and compatibility of the derived local cohomology functor with derived tensor products. The nontrivial fun comes in deducing concrete consequences—see e.g., §2.4 and §5.3.

Section 3 shows how the basic properties of local cohomology, other than those shared by all right-derived functors, fall out easily from the fact that local cohomology with respect to an ideal I is, as a derived functor, isomorphic to tensoring with the direct limit of Koszul complexes on powers of a system of generators of I. A more abstract, more general approach is indicated in an appendix.

Section 4 deals with a striking generalization of local duality—and almost any other duality involving inverse limits—discovered in special cases in the 1970s, and then in full generality in the context of modules over rings by Greenlees and May in the early 1990s [GM1]. The derived-category formulation, that left-derived completion (= local homology) is canonically right-adjoint to right-derived power-torsion (= local cohomology), is conceptually very simple, see Theorem 4.1. One application, "Affine Duality," is discussed in §4.3; others can be found in §§5.4 and 5.5. While we stay with modules, the result extends to formal schemes [DGM], where it plays an important role in the duality theory of coherent sheaves (§5.6).

Further, it is through derived functors that the close relation of local duality with global Grothendieck duality on formal schemes is, from the point of view of these lectures, most transparently formulated. The latter part of Section 5 aspires to make this claim understandable, and perhaps even plausible. As before, however, the main challenge is to negotiate the passage between abstract functorial formulations and concrete canonical constructions. The principal result, one of the fundamental facts of duality theory, is the Residue Theorem. The final goal is to explain this theorem, at least for smooth maps (§5.6). A local version, which is a canonical form of Local Duality via differentials and residues, is given in Theorem 5.3.3. Then the connection with a canonical version of global duality is drawn via flat base change (§5.4)—for which, incidentally, Greenlees-May duality is essential—and the fundamental class (§5.5), developed first for power-series rings, then, finally, for smooth maps of formal schemes.

1. Local cohomology, derived categories and functors

1.1. **Local cohomology of a module.** Let R be a commutative ring and $\mathcal{M}(R)$ the category of R-modules. For any R-ideal I, let Γ_I be the *I-power-torsion* subfunctor of the identity functor on $\mathcal{M}(R)$: for any R-module M,

$$\Gamma_I M = \{\, m \in M \mid \text{for some } s > 0, I^s m = 0\,\}.$$

If J is an ideal containing I then $\Gamma_J \subset \Gamma_I$, with equality if $J^n \subset I$ for some $n > 0$.

Choose for each M an injective resolution, i.e., a complex of injective R-modules[1]

$$E_M^\bullet : \quad \cdots \to 0 \to 0 \to E_M^0 \to E_M^1 \to E_M^2 \to \cdots$$

[1] A complex $C^\bullet = (C^\bullet, d^\bullet)$ of R-modules (*R-complex*) is understood to be a sequence of R-homomorphisms

$$\cdots \xrightarrow{d^{i-2}} C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots \qquad (i \in \mathbb{Z})$$

such that $d^i d^{i-1} = 0$ for all i. The *differential* $d^\bullet$ is often omitted in the notation. The *i-th homology* $\mathrm{H}^i C^\bullet$ is $\ker(d^i)/\mathrm{im}(d^{i-1})$. The *translation* (or *suspension*) $C[1]^\bullet$ of $C^\bullet$ is the complex such that $C[1]^i := C^{i+1}$ and whose differential $d^i_{C[1]}\colon C[1]^i \to C[1]^{i+1}$ is $-d^{i+1}_C\colon C^{i+1} \to C^{i+2}$.

together with an R-homomorphism $M \to E^0_M$ such that the sequence

$$0 \to M \to E^0_M \to E^1_M \to E^2_M \to \cdots$$

is exact. (For definiteness one can take the canonical resolution of [Brb, p. 52, §3.4].) Then define the *local cohomology modules*

$$\mathrm{H}^i_I M := \mathrm{H}^i(\Gamma_I E^\bullet_M) \qquad (i \in \mathbb{Z}).$$

Each H^i_I can be made in a natural way into a functor from $\mathcal{M}(R)$ to $\mathcal{M}(R)$, sometimes referred to as a *higher derived functor of* Γ_I. Of course $\mathrm{H}^i_I = 0$ if $i < 0$; and since Γ_I is left-exact there is an isomorphism of functors $\mathrm{H}^0_I \cong \Gamma_I$.

To each "short" exact sequence of R-modules

$$(\sigma): \quad 0 \to M' \to M \to M'' \to 0$$

there are naturally associated *connecting R-homomorphisms*

$$\delta^i_I(\sigma)\colon \mathrm{H}^i_I M'' \to \mathrm{H}^{i+1}_I M' \qquad (i \in \mathbb{Z}),$$

varying functorially (in the obvious sense) with the sequence (σ), and such that the resulting "long" cohomology sequence

$$\cdots \to \mathrm{H}^i_I M' \to \mathrm{H}^i_I M \to \mathrm{H}^i_I M'' \to \mathrm{H}^{i+1}_I M' \to \mathrm{H}^{i+1}_I M \to \cdots$$

is exact.

A sequence of functors $(\mathrm{H}^i_*)_{i \ge 0}$, in which H^0_* is left-exact, together with connecting maps δ^i_* taking short exact sequences functorially to long exact sequences, as above, is called a *cohomological functor.* Among cohomological functors, local cohomology is characterized up to canonical isomorphism as being a *universal cohomological extension of* Γ_I—there is a functorial isomorphism $\mathrm{H}^0_I \cong \Gamma_I$, and for any cohomological functor $(\mathrm{H}^i_*, \delta^i_*)$, every functorial map $\phi^0\colon \mathrm{H}^0_I \to \mathrm{H}^0_*$ has a unique extension to a family of functorial maps $(\phi^i\colon \mathrm{H}^i_I \to \mathrm{H}^i_*)$ such that for any (σ) as above,

$$\begin{array}{ccc}
\mathrm{H}^i_I(M'') & \xrightarrow{\delta^i_I(\sigma)} & \mathrm{H}^{i+1}_I(M') \\
{\scriptstyle \phi^i(M'')}\big\downarrow & & \big\downarrow{\scriptstyle \phi^{i+1}(M')} \\
\mathrm{H}^i_*(M'') & \xrightarrow[\delta^i_*(\sigma)]{} & \mathrm{H}^{i+1}_*(M')
\end{array}$$

commutes for all $i \ge 0$.

Like considerations apply to any left-exact functor on $\mathcal{M}(R)$, cf. [Gr1, pp. 139*ff*]. For example, for a fixed R-module N the functors

$$\mathrm{Ext}^i_R(N, M) := \mathrm{H}^i\mathrm{Hom}_R(N, E^\bullet_M) \qquad (i \ge 0)$$

with their standard connecting homomorphisms form a universal cohomological extension of $\mathrm{Hom}_R(N, -)$.

From $\Gamma_I E^\bullet_M = \varinjlim_{s>0} \mathrm{Hom}_R(R/I^s, E^\bullet_M)$ one gets the canonical identification of cohomological functors

(1.1.1) $$\mathrm{H}^i_I M = \varinjlim_{s>0} \mathrm{Ext}^i_R(R/I^s, M).$$

1.2. **Generalization to complexes.** Recall that a *map of R-complexes*

$$\psi\colon (C^\bullet, d^\bullet) \to (C_*^\bullet, d_*^\bullet)$$

is a family of R-homomorphisms $(\psi^i\colon C^i \to C_*^i)_{i\in\mathbb{Z}}$ such that $d_*^i\psi^i = \psi^{i+1}d^i$ for all i. Such a map induces R-homomorphisms $\mathrm{H}^iC^\bullet \to \mathrm{H}^iC_*^\bullet$. We say that ψ is a *quasi-isomorphism* if every one of these induced homology maps is an isomorphism.

A *homotopy* between R-complex maps $\psi_1\colon C^\bullet \to C_*^\bullet$ and $\psi_2\colon C^\bullet \to C_*^\bullet$ is a family of R-homomorphisms $(h^i\colon C^i \to C_*^{i-1})$ such that

$$\psi_1^i - \psi_2^i = d_*^{i-1}h^i + h^{i+1}d^i \qquad (i \in \mathbb{Z}).$$

If such a homotopy exists we say that ψ_1 and ψ_2 are homotopic. Being homotopic is an equivalence relation, preserved by addition and composition of maps; and it follows that the R-complexes are the objects of an additive category $\mathbf{K}(R)$ whose morphisms are the homotopy-equivalence classes.

Homotopic maps induce identical maps on homology. So it is clear what a quasi-isomorphism in $\mathbf{K}(R)$ is. Moreover, H^i can be thought of as a functor from $\mathbf{K}(R)$ to $\mathcal{M}(R)$, taking quasi-isomorphisms to isomorphisms.

An R-complex $C^\bullet$ is *q-injective*[2] if any quasi-isomorphism $\psi\colon C^\bullet \to C_*^\bullet$ has a left homotopy-inverse, i.e., there exists an R-map $\psi_*\colon C_*^\bullet \to C^\bullet$ such that $\psi_*\psi$ is homotopic to the identity map of $C^\bullet$. Numerous equivalent conditions can be found in [Spn, p. 129, Prop. 1.5] and in [Lp3, §2.3]. One such is

(#): for any $\mathbf{K}(R)$-diagram $C_*^\bullet \xleftarrow{\psi} X^\bullet \xrightarrow{\phi} C^\bullet$ with ψ a quasi-isomorphism, there exists a *unique* $\mathbf{K}(R)$-map $\phi_*\colon C_*^\bullet \to C^\bullet$ such that $\phi_*\psi = \phi$.

For example, any bounded-below injective complex $C^\bullet$ (i.e., C^i is an injective R-module for all i, and $C^i = 0$ for $i \ll 0$) is q-injective [Ha1, p. 41, Lemma 4.5]. And if $C^\bullet$ vanishes in all degrees except one, say $C^j \neq 0$, then $C^\bullet$ is q-injective iff this C^j is an injective R-module [Spn, p. 128, Prop. 1.2].

A *q-injective resolution* of an R-complex $C^\bullet$ is a q-injective complex $E^\bullet$ equipped with a quasi-isomorphism $C^\bullet \to E^\bullet$. Such exists for any $C^\bullet$, with $E^\bullet$ the total complex of an injective Cartan-Eilenberg resolution of $C^\bullet$ [EG3, p. 32, (11.4.2)].[3]

An injective resolution of a single R-module M can be regarded as a q-injective resolution of the complex $M^\bullet$ such that $M^0 = M$ and $M^i = 0$ for all $i \neq 0$.

[2]K-injective in the terminology of [Spn]. ("q" connotes "quasi-isomorphism.")

[3]It has been shown only recently that a q-injective resolution exists for any complex in an arbitrary Grothendieck category, i.e., an abelian category with exact direct limits and having a generator [AJS, p. 243, Thm. 5.4]. Injective Cartan-Eilenberg resolutions always exist in Grothendieck categories; and their totalizations—which generally require countable direct products—give q-injective resolutions when such products of epimorphisms are epimorphisms (a condition which fails, e.g., in categories of sheaves on most topological spaces), see [Wb2, p. 1661].

After choosing for each R-complex $C^\bullet$ a specific q-injective resolution $C^\bullet \to E^\bullet_C$, we can define the *local cohomology modules of* $C^\bullet$ by:

$$\mathrm{H}^i_I C^\bullet := \mathrm{H}^i(\Gamma_I E^\bullet_C) \qquad (i \in \mathbb{Z}). \tag{1.2.1}$$

It results from (#) that for any $\mathbf{K}(R)$-diagram

$$\begin{array}{ccc} C^\bullet_1 & \xrightarrow{\ \psi_1\ } & E^\bullet_{C_1} \\ {\scriptstyle\phi}\big\downarrow & & \\ C^\bullet_2 & \xrightarrow[\ \psi_2\ ]{} & E^\bullet_{C_2} \end{array}$$

with ψ_1 and ψ_2 q-injective resolutions, there is a unique $\phi_*\colon E^\bullet_{C_1} \to E^\bullet_{C_2}$ such that $\phi_*\psi_1 = \psi_2\phi$. From this follows that the H^i_I can be viewed as functors from $\mathbf{K}(R)$ to $\mathcal{M}(R)$, independent (up to canonical isomorphism) of the choices of $E^\bullet_C$, and taking quasi-isomorphisms to isomorphisms.

It will be explained in §1.4, in the context of derived categories, how a short exact sequence of complexes in $\mathcal{M}(R)$—a sequence $C^\bullet_1 \to C^\bullet \to C^\bullet_2$ with $0 \to C^i_1 \to C^i \to C^i_2 \to 0$ exact for every i—gives rise functorially to a long exact cohomology sequence

$$\cdots \to \mathrm{H}^i_I C^\bullet_1 \to \mathrm{H}^i_I C^\bullet \to \mathrm{H}^i_I C^\bullet_2 \to \mathrm{H}^{i+1}_I C^\bullet_1 \to \mathrm{H}^{i+1}_I C^\bullet \to \cdots$$

Similar considerations lead to the definition of Ext functors of complexes:

$$\mathrm{Ext}^i_R(D^\bullet, C^\bullet) := \mathrm{H}^i\mathrm{Hom}^\bullet_R(D^\bullet, E^\bullet_C) \qquad (i \in \mathbb{Z}) \tag{1.2.2}$$

where for two R-complexes $(X^\bullet, d^\bullet_X)$, $(Y^\bullet, d^\bullet_Y)$, the complex $\mathrm{Hom}^\bullet_R(X^\bullet, Y^\bullet)$ is given in degree n by

$$\mathrm{Hom}^n_R(X^\bullet, Y^\bullet) := \big\{\text{families of } R\text{-homomorphisms } f = (f_j\colon X^j \to Y^{j+n})_{j\in\mathbb{Z}}\big\}$$

with differential $d^n\colon \mathrm{Hom}^n_R(X^\bullet, Y^\bullet) \to \mathrm{Hom}^{n+1}_R(X^\bullet, Y^\bullet)$ specified by

$$d^n f := \Big(d^{j+n}_Y \circ f_j - (-1)^n f_{j+1} \circ d^j_X\Big)_{j\in\mathbb{Z}}.$$

There is a functorial identification, compatible with connecting maps,

$$\mathrm{H}^i_I C^\bullet = \varinjlim_{s>0} \mathrm{Ext}^i_R(R/I^s, C^\bullet) \tag{1.2.3}$$

where R/I^s is thought of as a complex vanishing outside degree 0.

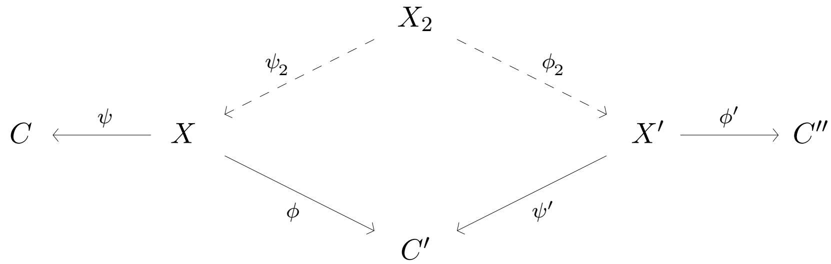
1.3. **The derived category.** An efficacious strategy in studying the behavior of and relations among various homology groups is to regard them as shadows of an underlying play among complexes, and to focus on this more fundamental reality. From such a point of view arises the notion of the *derived category* $\mathbf{D}(R)$ *of* $\mathcal{M}(R)$.

When our basic interest is in homology, we needn't distinguish between homotopic maps of complexes, so we start with the homotopy category $\mathbf{K}(R)$. Here we would like to regard the source and target of a quasi-isomorphism ψ as isomorphic objects because they have isomorphic homology. So we formally adjoin to $\mathbf{K}(R)$ an inverse for each such ψ. This *localization* procedure produces the category $\mathbf{D}(R)$, described roughly as follows. (Details can be found, e.g., in [Wb1, Chap. 10].)

The objects of $\mathbf{D}(R)$ are simply the R-complexes.[4] A $\mathbf{D}(R)$-morphism $C \to C'$ is an equivalence class ϕ/ψ of $\mathbf{K}(R)$-diagrams $C \xleftarrow{\psi} X \xrightarrow{\phi} C'$ with ψ a quasi-isomorphism, the equivalence relation being the least such that $\phi/\psi = \phi\psi_1/\psi\psi_1$ for all such ϕ, ψ and quasi-isomorphisms $\psi_1\colon X_1 \to X$. The composition of the classes of $C' \xleftarrow{\psi'} X' \xrightarrow{\phi'} C''$ and $C \xleftarrow{\psi} X \xrightarrow{\phi} C'$ is given by

$$\left(\phi'/\psi'\right)\left(\phi/\psi\right) = \phi'\phi_2/\psi\psi_2$$

where $(\phi_2\colon X_2 \to X',\ \psi_2\colon X_2 \to X')$ is any pair with ψ_2 a quasi-isomorphism and $\psi'\phi_2 = \phi\psi_2$. (Such pairs exist.)



There is a canonical functor $Q\colon \mathbf{K}(R) \to \mathbf{D}(R)$ taking any complex to itself, and taking the $\mathbf{K}(R)$-map $\phi\colon C \to C'$ to the $\mathbf{D}(R)$-map $\phi/1_C$ (where 1_C is the identity map of C). This Q takes quasi-isomorphisms to isomorphisms: if ϕ is a quasi-isomorphism then the inverse of $\phi/1_C$ is $1_C/\phi$.

The pair $(\mathbf{D}(R), Q)$ is characterized up to isomorphism by the following property:

(1.3.1) *For any category* $\mathbf{L}$, *composition with* Q *is an isomorphism of the category of functors from* $\mathbf{D}(R)$ *to* $\mathbf{L}$ (*morphisms being functorial maps*) *onto the category of those functors from* $\mathbf{K}(R)$ *to* $\mathbf{L}$ *taking quasi-isomorphisms to isomorphisms.*

(If $F\colon \mathbf{K}(R) \to \mathbf{L}$ takes quasi-isomorphisms to isomorphisms then the corresponding functor $F_D\colon \mathbf{D}(R) \to \mathbf{L}$ satisfies $F_D(\phi/\psi) = F(\phi) \circ F(\psi)^{-1}$.)

$\mathbf{D}(R)$ has a unique additive-category structure such that Q is an additive functor. For instance, to add two maps ϕ_1/ψ_1, ϕ_2/ψ_2 with the same source and target, rewrite them with a common denominator—which is always possible, because of [Ha1, pp. 35–36, proof of (FR2)]—and then just add the numerators. The characterization (1.3.1) of $(\mathbf{D}(R), Q)$ remains valid when restricted to additive functors into additive categories.

The homology functors H^i are then additive functors from $\mathbf{D}(R)$ to $\mathcal{M}(R)$. One shows easily that—in accordance with the initial motivation—*a* $\mathbf{D}(R)$-*map* α *is an isomorphism if and only if the homology maps* $\mathrm{H}^i(\alpha)$ $(i \in \mathbb{Z})$ *are all isomorphisms.*

[4]As a rule we will no longer use $^\bullet$ in denoting complexes. But the degree-n differential of a complex C will still be denoted by $d^n\colon C^n \to C^{n+1}$.

Example. When R is a field, any R-complex $(C^\bullet, d^\bullet)$ splits (non-canonically) into a direct sum of the complexes $\mathrm{im}(d^{i-1}) \hookrightarrow \ker(d^i)$ (concentrated in degrees $i-1$ and i), whence (exercise) C is *canonically* $\mathbf{D}(R)$-isomorphic to the complex

$$\cdots \xrightarrow{0} \mathrm{H}^{i-1}C \xrightarrow{0} \mathrm{H}^{i}C \xrightarrow{0} \mathrm{H}^{i+1}C \xrightarrow{0} \cdots$$

Consequently, *the functor* $C \mapsto \oplus_{i\in\mathbb{Z}} \mathrm{H}^i C$ *from* $\mathbf{D}(R)$ *to graded* R*-vector spaces is an equivalence of categories.*

Example. A common technique for comparing the homology of two bounded-below complexes C and C' is to map them of them into a first-quadrant double complex as (respectively) the vertical and horizontal zero-cycles. Thus if Y is the totalized double complex, then we have $\mathbf{K}(R)$-maps $\xi\colon C \to Y$, $\xi'\colon C' \to Y$. If the appropriate spectral sequence of the double complex degenerates then ξ' is a quasi-isomorphism, and so one has the $\mathbf{D}(R)$-map $(1_{C'}/\xi')\circ(\xi/1_C)\colon C \to C'$, from which one gets homology maps $\mathrm{H}^iC \to \mathrm{H}^iC'$. In some sense, the role of the spectral sequence is taken over here by the conceptually simpler $\mathbf{D}(R)$-map. The real advantage of the latter becomes more apparent when one has to work with a sequence of comparisons involving a variety of homological constructions—as will happen later in these lectures.

It follows at once from definitions that for any R-complexes D, E,

$$\mathrm{H}^0\mathrm{Hom}^\bullet_R(D,E) = \mathrm{Hom}_{\mathbf{K}(R)}(D,E).$$

Furthermore, `(#)` in §1.2 implies that for q-injective E the natural map $\mathrm{Hom}_{\mathbf{K}(R)}(D,E) \to \mathrm{Hom}_{\mathbf{D}(R)}(D,E)$ is bijective. Hence, with $C \to E := E_C$ the previously used q-injective resolution and $[i]$ denoting i-times-iterated translation (see footnote in §1.1),

$$\begin{aligned} \mathrm{Ext}^i_R(D,C) &= \mathrm{H}^i\mathrm{Hom}^\bullet_R(D,E) \\ &= \mathrm{H}^0\mathrm{Hom}^\bullet_R(D,E[i]) \qquad (1.3.2) \\ &= \mathrm{Hom}_{\mathbf{D}(R)}(D,E[i]) \cong \mathrm{Hom}_{\mathbf{D}(R)}(D,C[i]). \end{aligned}$$

The following illustrative Proposition will be useful. For any R-module M and any $m \in \mathbb{Z}$, $M[-m]$ is the R-complex which is M in degree m and vanishes elsewhere.

Proposition 1.3.3. *If* C *is an* R*-complex such that* $\mathrm{H}^iC = 0$ *for all* $i > m$ *then for any* R*-module* M*, the homology functor* H^m *induces an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(R)}(C, M[-m]) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{H}^mC, \mathrm{H}^m(M[-m])) = \mathrm{Hom}_R(\mathrm{H}^mC, M).$$

If, moreover, $\mathrm{H}^iC = 0$ *for all* $i < m$*, then the* $\mathbf{D}(R)$*-map corresponding in this way to the identity map of* H^mC *is an isomorphism*

$$C \xrightarrow{\sim} (\mathrm{H}^mC)[-m].$$

Proof. Let $C_{\le m} \subset C$ be the "truncated" complex

$$\cdots \to C^{m-2} \xrightarrow{d^{m-2}} C^{m-1} \xrightarrow{d^{m-1}} \ker(C^m \xrightarrow{d^m} C^{m+1}) \to 0 \to 0 \to \cdots$$

The inclusion $C_{\le m} \hookrightarrow C$ is a quasi-isomorphism, so we can replace C by $C_{\le m}$, i.e., we may assume that $C^n = 0$ for $n > m$. Then for any injective resolution $0 \to M \to I^0 \to I^1 \to \cdots$ we have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbf{D}(R)}(C, M[-m]) &\xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(R)}(C, I^\bullet[-m]) \\ &\xleftarrow{\sim} \operatorname{Hom}_{\mathbf{K}(R)}(C, I^\bullet[-m]) \xrightarrow{\sim} \operatorname{Hom}_R(\mathrm{H}^m C, M).\end{aligned}$$

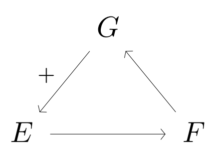
(Bijectivity of the second map follows, as above, from (#) in §1.2. Showing bijectivity of the third map—induced by H^m—is a simple exercise.) The first assertion follows. The second results from the above characterization of $\mathbf{D}(R)$-isomorphisms via their induced homology maps. (More explicitly, the $\mathbf{D}(R)$-map in question is represented by the natural diagram of quasi-isomorphisms $C \hookleftarrow C_{\le m} \twoheadrightarrow (\mathrm{H}^m C_{\le m})[-m]$.) □

Corollary 1.3.4. *The functor taking any R-module M to the R-complex which is M in degree zero and 0 elsewhere, and doing the obvious thing to R-module maps, is an equivalence of the category $\mathcal{M}(R)$ with the full subcategory of $\mathbf{D}(R)$ whose objects are the complexes with homology vanishing in all nonzero degrees. A quasi-inverse for this equivalence is given by the functor* H^0.

For a final example, we note that as the above-defined local cohomology functors $\mathrm{H}^i_I\colon \mathbf{K}(R) \to \mathcal{M}(R)$ $(i \in \mathbb{Z})$ take quasi-isomorphisms to isomorphisms, they may be regarded as functors from $\mathbf{D}(R)$ to $\mathcal{M}(R)$. In view of (1.3.2), (1.2.3) yields an interpretation of these functors in terms of $\mathbf{D}(R)$-maps, viz. a functorial isomorphism

$$\mathrm{H}^i_I C \cong \varinjlim_{s>0} \operatorname{Hom}_{\mathbf{D}(R)}(R/I^s, C[i]) \qquad (C \in \mathbf{D}(R)).$$

1.4. **Triangles.** As we have seen, exact sequences of complexes play an important role in the discussion of derived functors. But $\mathbf{D}(R)$ is not an abelian category, so it does not support a notion of exactness. Instead, $\mathbf{D}(R)$ carries a supplementary structure given by certain diagrams of the form $E \to F \to G \to E[1]$, called *triangles,* and occasionally represented in the typographically inconvenient form



Specifically, the triangles are those diagrams which are isomorphic (in the obvious sense) to diagrams of the form

$$X \xrightarrow{\alpha} Y \hookrightarrow C_\alpha \twoheadrightarrow X[1] \tag{1.4.1}$$

where α is an ordinary map of R-complexes and C_α is the *mapping cone* of α:

as a graded group, $C_\alpha := Y \oplus X[1]$, and the differential $C^n_\alpha \to C^{n+1}_\alpha$ is the sum of the differentials of d^n_Y and $d^n_{X[1]}$ *plus* the map $\alpha^{n+1}\colon X^{n+1} \to Y^{n+1}$, as depicted:

$$\begin{array}{ccccc} C^{n+1}_\alpha & = & Y^{n+1} & \oplus & X^{n+2} \\ {\scriptstyle d_{C_\alpha}}\Big\uparrow & & {\scriptstyle d_Y}\Big\uparrow & \nwarrow{\scriptstyle \alpha} & \Big\uparrow{\scriptstyle -d_X} \\ C^n_\alpha & = & Y^n & \oplus & X^{n+1} \end{array}$$

For any exact sequence

$$(\tau) \qquad 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

of R-complexes, the composite map of graded groups $C_\alpha \twoheadrightarrow Y \xrightarrow{\beta} Z$ turns out to be a quasi-isomorphism of complexes, and so becomes an isomorphism in $\mathbf{D}(R)$. Thus we get a triangle

$$X \to Y \to Z \to X[1];$$

and *up to isomorphism, these are all the triangles in* $\mathbf{D}(R)$. (See e.g., [Lp3, Example (1.4.4)].)

The operation $E \mapsto E[1]$ extends naturally to a functor on R-complexes, which preserves homotopy and quasi-isomorphisms, and hence gives rise to a functor $T\colon \mathbf{D}(R) \to \mathbf{D}(R)$, called *translation,* an automorphism of the category $\mathbf{D}(R)$.

Applying the i-fold translations T^i $(i \in \mathbb{Z})$ to a triangle

$$\triangle\colon E \to F \to G \to E[1]$$

and then taking homology, one gets a long homology sequence

$$(1.4.2) \qquad \cdots \to \mathrm{H}^i E \to \mathrm{H}^i F \to \mathrm{H}^i G \to \mathrm{H}^i E[1] = \mathrm{H}^{i+1} E \to \cdots$$

This sequence is *exact*, as one need only verify for triangles of the form (1.4.1).

If $\triangle$ is the triangle coming from the exact sequence (τ), then this homology sequence is, after multiplication of the connecting maps $\mathrm{H}^i G \to \mathrm{H}^{i+1} E$ by -1, precisely the usual long exact sequence associated to (τ).

This is why one can replace short exact sequences of R-complexes by triangles in $\mathbf{D}(R)$. And it strongly suggests that when considering functors between derived categories one should concentrate on those which respect triangles, as specified in the following definition.

Let $\mathcal{A}_1$, $\mathcal{A}_2$ be abelian categories. In the same way that one constructs the triangulated category $\mathbf{D}(R)$ from $\mathcal{M}(R)$, one gets triangulated derived categories $\mathbf{D}(\mathcal{A}_1)$, $\mathbf{D}(\mathcal{A}_2)$.[5] Denote the respective translation functors by T_1, T_2.

Definition 1.4.3. A Δ-*functor* $\Phi\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ is an additive functor which "preserves translation and triangles," in the following sense:

[5] modulo some set-theoretic conditions which we ignore here. (See [Wb1, p. 379, 10.3.3].)

Φ comes equipped with a functorial isomorphism

$$\theta\colon \Phi T_1 \xrightarrow{\sim} T_2\Phi$$

such that for any triangle

$$E \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{w} E[1] = T_1E$$

in $\mathbf{D}(\mathcal{A}_1)$, the corresponding diagram

$$\Phi E \xrightarrow{\Phi u} \Phi F \xrightarrow{\Phi v} \Phi G \xrightarrow{\theta\circ\Phi w} (\Phi E)[1] = T_2\Phi E$$

is a triangle in $\mathbf{D}(\mathcal{A}_2)$. These Δ-functors are the objects of a category whose maps, called Δ-*functorial,* are those functorial maps which commute (in the obvious sense) with the supplementary structure.

In what follows, those functors between derived categories which appear can always be equipped in some natural way with a θ making them into Δ-functors; and any noteworthy maps between such functors are Δ-functorial. For our expository purposes, however, it will not be necessary to fuss over explicit descriptions, and θ will usually be omitted from the notation.

In summary: if $\Phi\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ is a Δ-functor, then to any short exact sequence of complexes in $\mathcal{A}_1$

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0 \qquad (\tau_1)$$

there is naturally associated a long exact homology sequence in $\mathcal{A}_2$

$$\cdots \to \mathrm{H}^i(\Phi X) \to \mathrm{H}^i(\Phi Y) \to \mathrm{H}^i(\Phi Z) \to \mathrm{H}^{i+1}(\Phi X) \to \cdots,$$

that is, the homology sequence of the triangle in $\mathbf{D}(\mathcal{A}_2)$ gotten by applying Φ to the triangle given by (τ_1).

We will also need the notion of triangles in the homotopy category $\mathbf{K}(R)$. These are diagrams isomorphic in $\mathbf{K}(R)$ to diagrams of the form (1.4.1). Up to isomorphism, $\mathbf{K}(R)$-triangles come from short exact sequences of complexes which split in each degree as R-module sequences: for such sequences, the quasi-isomorphism following (τ) (above) is a $\mathbf{K}(R)$-isomorphism, see e.g, [Lp3, Example (1.4.3)]. (One might also think here about the common use of such a sequence of complexes to resolve an exact sequence of modules—see e.g., [Wb1, p. 37)] for the "dual" case of projective resolutions). The canonical functor $Q\colon \mathbf{K}(R) \to \mathbf{D}(R)$ is a Δ-functor: it commutes with translation and takes $\mathbf{K}(R)$-triangles to $\mathbf{D}(R)$-triangles. Any additive functor from $\mathcal{M}(R)$ into an additive category extends in an obvious sense to a Δ-functor between the corresponding homotopy categories.

1.5. **Right-derived functors. RHom and Ext.** Here is how in dealing with higher derived functors we lift our focus from homology to complexes.

The q-injective resolutions $q_C\colon C \to E_C$ being as in §1.2, set

$$\mathbf{R}\Gamma_I C := \Gamma_I E_C. \qquad (1.5.1)$$

Then by Definition 1.2.1, $\mathrm{H}^i_I C = \mathrm{H}^i\mathbf{R}\Gamma_I C$.

The point is that $\mathbf{R}\Gamma_I$ *can be made into a* Δ*-functor from* $\mathbf{D}(R)$ *to* $\mathbf{D}(R)$.

For, the characterization (#) (§1.2) of q-injectivity implies that any quasi-isomorphism between q-injective complexes is an isomorphism, and then that any $\mathbf{K}(R)$-diagram $C \xleftarrow{\psi} X \xrightarrow{\phi} C'$ with ψ a quasi-isomorphism embeds uniquely into a commutative $\mathbf{K}(R)$-diagram, with Ψ—and hence $\Gamma_I\Psi$—an isomorphism:

$$\begin{array}{ccccc} C & \xleftarrow{\psi} & X & \xrightarrow{\phi} & C' \\ {\scriptstyle q_C}\downarrow & & {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_{C'}} \\ E_C & \xleftarrow{\Psi} & E_X & \xrightarrow{\Phi} & E_{C'} \end{array}$$

Furthermore, the equivalence class (see §1.3) of the $\mathbf{K}(R)$-diagram

$$\Gamma_I E_C \xleftarrow{\Gamma_I\Psi} \Gamma_I E_X \xrightarrow{\Gamma_I\Phi} \Gamma_I E_{C'}$$

depends only on that of $C \xleftarrow{\psi} X \xrightarrow{\phi} C'$. Thus we can associate to the $\mathbf{D}(R)$-map $\phi/\psi\colon C \to C'$ the map $\Gamma_I\Phi/\Gamma_I\Psi\colon \mathbf{R}\Gamma_I C \to \mathbf{R}\Gamma_I C'$. This association respects identities and composition, making $\mathbf{R}\Gamma_I$ into a functor. And with $Q\colon \mathbf{K}(R) \to \mathbf{D}(R)$ as before, a Δ-structure on $\mathbf{R}\Gamma_I$ is given by the functorial isomorphism

$$\theta(C)\colon \mathbf{R}\Gamma_I(C[1]) \xrightarrow{\sim} (\mathbf{R}\Gamma_I C)[1]$$

obtained by applying $Q\Gamma_I$ to the unique isomorphism $\phi\colon E_{C[1]} \xrightarrow{\sim} E_C[1]$ such that $\phi\circ q_{C[1]} = (q_C)[1]$. (For details, cf. [Lp3, Prop. (2.2.3)]).

There is a functorial map $\zeta\colon Q\Gamma_I \to \mathbf{R}\Gamma_I Q$ such that for each C, $\zeta(C)$ is the obvious map $\Gamma_I C \to \Gamma_I E_C$. The pair $(\mathbf{R}\Gamma_I, \zeta)$ is a *right-derived functor of* Γ_I, characterized up to canonical isomorphism by the property that ζ is an initial object in the category of all functorial maps $Q\Gamma_I \to \Gamma$ where Γ ranges over the category of functors from $\mathbf{K}(R)$ to $\mathbf{D}(R)$ which take quasi-isomorphisms to isomorphisms. In other words, for each such Γ composition with ζ maps the set $[\mathbf{R}\Gamma_I Q, \Gamma]$ of functorial maps from $\mathbf{R}\Gamma_I Q$ to Γ bijectively onto the set $[Q\Gamma_I, \Gamma]$. Moreover, (1.3.1) gives a unique factorization $\Gamma = \boldsymbol{\Gamma} Q$ for some $\boldsymbol{\Gamma}\colon \mathbf{D}(R) \to \mathbf{D}(R)$, and a bijection $[\mathbf{R}\Gamma_I, \boldsymbol{\Gamma}] \xrightarrow{\sim} [\mathbf{R}\Gamma_I Q, \Gamma]$.

Similarly, one has via q-injective resolutions a right-derived Δ-functor $\mathbf{R}\Gamma$ of any Δ-functor Γ on $\mathbf{K}(R)$. The characteristic initial-object property holds with "Δ-functor" in place of "functor." Such Γ arise most often as extensions of additive functors from $\mathcal{M}(R)$ to some abelian category (see end of §1.4).

For example, for any R-complex D one has the functor $\mathbf{R}\mathrm{Hom}^\bullet_R(D,-)$ with

$$\mathbf{R}\mathrm{Hom}^\bullet_R(D,C) = \mathrm{Hom}^\bullet_R(D,E_C)^6$$

and then, as in Definition 1.2.2,

$$\mathrm{Ext}^i_R(D,C) = \mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet_R(D,C). \tag{1.5.2}$$

[6] which, with some caution regarding signs, can also be made into a contravariant Δ-functor in the first variable, see e.g., [Lp3, (1.5.3)].

To illustrate further, let us lift the homology relation (1.2.3) to a relation among complexes in $\mathbf{D}(R)$. A first guess might be that $\mathbf{R}\Gamma_I C = \varinjlim_{s>0} \mathbf{R}\mathrm{Hom}(R/I^s, C)$; but that doesn't make sense, because the $\varinjlim$ of a sequence of complexes in $\mathbf{D}(R)$ doesn't always exist. It is however possible to replace $\varinjlim$—thought of naively as the cokernel of an endomorphism of an infinite direct sum—by the summit of a triangle based on such an endomorphism, thereby expressing $\mathbf{R}\Gamma_I$ as a "homotopy colimit."

For this purpose, let $\mathbf{h}_s\colon \mathbf{D}(R) \to \mathbf{D}(R)$ be the functor described by

$$\mathbf{h}_s C := \mathbf{R}\mathrm{Hom}^\bullet_R(R/I^s, C) \qquad (s \ge 1,\ C \in \mathbf{D}(R)).$$

There are natural functorial maps $p_s\colon \mathbf{h}_s \to \mathbf{h}_{s+1}$ and $q_s\colon \mathbf{h}_s \to \mathbf{R}\Gamma_I$, satisfying $q_{s+1}p_s = q_s$. The family

$$(1, -p_m)\colon \mathbf{h}_m \to \mathbf{h}_m \oplus \mathbf{h}_{m+1} \subset \oplus_{s\ge 1}\mathbf{h}_s \qquad (m \ge 1)$$

defines a $\mathbf{D}(R)$-map

$$p\colon \oplus_{s\ge 1}\mathbf{h}_s \to \oplus_{s\ge 1}\mathbf{h}_s.$$

(Details, including the interpretation of infinite direct sums in $\mathbf{D}(R)$, are left to the reader.)

Proposition 1.5.3. *Under these circumstances,* there is a triangle

$$\oplus_{s\ge 1}\mathbf{h}_s C \xrightarrow{\ p\ } \oplus_{s\ge 1}\mathbf{h}_s C \xrightarrow{\sum q_s} \mathbf{R}\Gamma_I C \longrightarrow (\oplus_{s\ge 1}\mathbf{h}_s C)[1]$$

Proof. Replacing C by an isomorphic complex, we may assume C q-injective, so that $\mathbf{h}_s C = \mathrm{Hom}^\bullet_R(R/I^s, C)$ and $\mathbf{R}\Gamma_I C = \Gamma_I C$. Since $(\sum q_s)\circ p = 0$, it follows, with C_p the mapping cone of p, that there exists a map of R-complexes

$$\bar q\colon C_p = (\oplus_{s\ge 1}\mathbf{h}_s C) \bigoplus (\oplus_{s\ge 1}\mathbf{h}_s C)[1] \to \Gamma_I C$$

restricting to $\sum q_s$ on the first direct summand and vanishing on the second; and it suffices to show that $\bar q$ *is a quasi-isomorphism.* But from the (easily-checked) injectivity of $\mathrm{H}^i p$ and exactness of the homology sequence of the triangle (1.4.1) with α replaced by p, one finds that the homology of C_p is

$$\begin{aligned}\mathrm{H}^i_I C_p = \varinjlim_{s>0} \mathrm{H}^i\mathbf{h}_s C = \varinjlim_{s>0} \mathrm{H}^i \mathrm{Hom}^\bullet_R(R/I^s, C)& \\ = \mathrm{H}^i \varinjlim_{s>0} \mathrm{Hom}^\bullet_R(R/I^s, C) = \mathrm{H}^i\Gamma_I C,&\end{aligned}$$

whence the conclusion. □

2. Derived Hom-Tensor adjunction; Duality

2.1. **Left-derived functors. Tensor and Tor.** Dual to the notion of right-derived functor is that of left-derived functor:

Let $\gamma\colon \mathbf{K}(R) \to \mathbf{K}(R)$ be a Δ-functor. A *left-derived functor of* γ is a pair consisting of a Δ-functor $\mathbf{L}\gamma\colon \mathbf{D}(R) \to \mathbf{D}(R)$ and a functorial map $\xi\colon \mathbf{L}\gamma Q \to Q\gamma$ which is a *final object* in the category of all Δ-functorial maps $\Gamma \to Q\gamma$ where Γ ranges over the category of Δ-functors from $\mathbf{K}(R)$ to $\mathbf{D}(R)$ which take quasi-isomorphisms to isomorphisms. In other words, for each such Γ composition with ξ maps the set $[\Gamma, \mathbf{L}\gamma Q]$ of functorial maps from Γ to $\mathbf{L}\gamma Q$ bijectively onto the set $[\Gamma, Q\gamma]$. Moreover, (1.3.1) gives a unique factorization $\Gamma = \mathbf{\Gamma} Q$ for some $\mathbf{\Gamma}\colon \mathbf{D}(R) \to \mathbf{D}(R)$, and a bijection $[\mathbf{\Gamma}, \mathbf{L}\gamma] \xrightarrow{\ \sim\ } [\Gamma, \mathbf{L}\gamma Q]$.

Example. Recall that the tensor product $C \otimes_R D$ of two R-complexes is such that $(C \otimes_R D)^n = \oplus_{i+j=n} C^i \otimes_R D^j$, the differential $\delta^n\colon (C \otimes_R D)^n \to (C \otimes_R D)^{n+1}$ being determined by

$$\delta^n(x \otimes y) = d^i_C x \otimes y + (-1)^i x \otimes d^j_D y \qquad (x \in C^i,\ y \in D^j).$$

Fixing D, we get a functor $\gamma_D := \ldots \otimes_R D\colon \mathbf{K}(R) \to \mathbf{K}(R)$, which together with $\theta =$ identity is a Δ-functor. To make $\gamma'_C := C \otimes_R \ldots$ (C fixed) a Δ-functor, one uses the unique θ' ($\neq$ identity) such that the R-isomorphism $C \otimes_R D \xrightarrow{\ \sim\ } D \otimes_R C$ taking $x \otimes y$ to $(-1)^{ij} y \otimes x$ is Δ-functorial [Lp3, (1.5.4)]. One gets a left-derived functor $\ldots \underline{\otimes}_R D$ of γ_D as follows (see [Spn, p. 147, Prop. 6.5], or [Lp3, §2.5]).

An R-complex F is *q-flat* if for every exact R-complex E (i.e., $\mathrm{H}^i E = 0$ for all i), $F \otimes_R E$ is exact too. It is equivalent to say that *the functor* $F \otimes_R \ldots$ *preserves quasi-isomorphism,* because by the exactness of the homology sequence of a triangle, a map of complexes is a quasi-isomorphism if and only if its cone is exact, and tensoring with F "commutes" with forming cones.[7]

Any bounded-above flat complex is q-flat (see, e.g., [Lp3, (2.5.4)]).

Every R-complex C admits a *q-flat resolution,* i.e., there is a q-flat complex F equipped with a quasi-isomorphism $F \to C$. This can be constructed as a $\varinjlim$ of bounded flat resolutions of truncations of C (*loc. cit.*, (2.5.5)).

After choosing for each C a q-flat resolution $F_C \to C$, one shows there exists a left-derived functor, as asserted above, with

$$C \underline{\otimes}_R D = F_C \otimes_R D$$

(*loc. cit.*, (2.5.7)). Taking homology produces the (hyper)tor functors

$$\mathrm{Tor}_i(C, D) = \mathrm{H}^{-i}(C \underline{\otimes}_R D).$$

If $F_D \to D$ is a q-flat resolution, there are natural $\mathbf{D}(R)$-isomorphisms

$$C \otimes_R F_D \xleftarrow{\ \sim\ } F_C \otimes_R F_D \xrightarrow{\ \sim\ } F_C \otimes_R D,$$

[7] *Exercise:* An R-complex E is q-injective iff $\mathrm{Hom}^\bullet_R(-, E)$ preserves quasi-isomorphism.

so any of these complexes could be used to define $C \underline{\underline{\otimes}}_R D$. Using $F_C \otimes_R F_D$ one can, as before, make $C \underline{\underline{\otimes}}_R D$ into a Δ-functor of both variables C and D. As such, it has a final-object characterization as above, but with respect to two-variable functors.

2.2. **Hom-Tensor adjunction.** There is a basic duality between $\mathbf{R}\mathrm{Hom}^\bullet_R$ and $\underline{\underline{\otimes}}$, neatly encapsulating a connection between the respective homologies Ext and Tor (from which all other functorial relations between Ext and Tor seem to follow As we'll soon see, this duality underlies a simple general formulation of Local Duality.

Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let E and F be S-complexes and let G be an R-complex. There is a canonical S-isomorphism of complexes:

$$\mathrm{Hom}^\bullet_R(E \otimes_S F, G) \xrightarrow{\sim} \mathrm{Hom}^\bullet_S(E, \mathrm{Hom}^\bullet_R(F, G)), \tag{2.2.1}$$

which in degree n takes a family $\big(f_{ij}\colon E^i \otimes_S F^j \to G^{i+j+n}\big)$ to the family $\big(f_i\colon E^i \to \mathrm{Hom}^{i+n}_R(F, G)\big)$ such that for $a \in E^i$, $f_i(a)$ is the family of maps $\big(g_j\colon F_j \to G^{i+j+n}\big)$ with $g_j(b) = f_{ij}(a \otimes b)$ $(b \in F^j)$.

This relation can be upgraded to the derived-category level, as follows.

Let $\varphi_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ denote the obvious "restriction of scalars" functor. For a fixed S-complex E, the functor $\mathrm{Hom}^\bullet_R(E, G)$ from R-complexes G to S-complexes has a right-derived functor from $\mathbf{D}(R)$ to $\mathbf{D}(S)$ (gotten via q-injective resolution of G), denoted $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* E, G)$.

If we replace G in (2.2.1) by a q-injective resolution, and F by a q-flat one, then the S-complex $\mathrm{Hom}^\bullet_R(F, G)$ is easily seen to become q-injective; and consequently (2.2.1) gives a $\mathbf{D}(S)$-isomorphism

(2.2.2)

$$\alpha(E, F, G)\colon \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*(E \underline{\underline{\otimes}}_S F), G) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_S(E, \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* F, G)),$$

of which a thorough treatment (establishing canonicity, Δ-functoriality, etc.) requires some additional, rather tedious, considerations. (See [Lp3, §2.6].) Here "canonicity" signifies that α is characterized by the property that it makes the following otherwise natural $\mathbf{D}(S)$-diagram (in which $\mathrm{H}^\bullet$ stands for $\mathrm{Hom}^\bullet$) commute for all E, F and G:

$$\begin{array}{ccccc}
\mathrm{H}^\bullet_R(E \otimes F, G) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_R(\varphi_*(E \otimes F), G) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_R(\varphi_*(E \underline{\underline{\otimes}} F), G) \\
{\scriptstyle (2.2.1)}\big\downarrow \simeq & & & & \simeq \big\downarrow \alpha \\
\mathrm{H}^\bullet_S(E, \mathrm{H}^\bullet_R(F, G)) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_S(E, \mathrm{H}^\bullet_R(F, G)) & \longrightarrow & \mathbf{R}\mathrm{H}^\bullet_S(E, \mathbf{R}\mathrm{H}^\bullet_R(\varphi_* F, G))
\end{array}$$

Application of homology H^0 to (2.2.2) yields a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*(E \underline{\underline{\otimes}}_S F), G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S)}(E, \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* F, G)), \tag{2.2.3}$$

see (1.5.2) and (1.3.2). Thus *the functors* $\varphi_*(\dots \underline{\underline{\otimes}}_S F)\colon \mathbf{D}(S) \to \mathbf{D}(R)$ *and* $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* F, -)\colon \mathbf{D}(R) \to \mathbf{D}(S)$ *are adjoint.*

2.3. **Consequence: Trivial Duality.** The following proposition is a very general (and in some sense trivial) form of duality .

Proposition 2.3.1. *With* $\varphi\colon R\to S$, $\varphi_*\colon \mathbf{D}(S)\to\mathbf{D}(R)$ *as above, let* $E\in\mathbf{D}(S)$, *let* $G\in\mathbf{D}(R)$, *and let* $\Gamma\colon\mathcal{M}(S)\to\mathcal{M}(S)$ *be a functor, with right-derived functor* $\mathbf{R}\Gamma\colon\mathbf{D}(S)\to\mathbf{D}(S)$ *(see* §1.5*). Then there exists a natural functorial map*

$$E\underline{\otimes}_S\mathbf{R}\Gamma S\to\mathbf{R}\Gamma E, \tag{2.3.1a}$$

whence, via the isomorphism (2.2.2) *with* $F=\mathbf{R}\Gamma S$, *a functorial map*

$$\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma E,G)\to\mathbf{R}\mathrm{Hom}^\bullet_S(E,\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma S,G)), \tag{2.3.1b}$$

whence, upon application of the homology functor H^0, *a functorial map*

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma E,G)\to\mathrm{Hom}_{\mathbf{D}(S)}(E,\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma S,G)). \tag{2.3.1c}$$

This being so, and E being fixed,

$$\begin{aligned}(2.3.1\mathrm{a})\ \textit{is an isomorphism}&\iff(2.3.1\mathrm{b})\ \textit{is an isomorphism for all}\ \mathrm{G}\\ &\iff(2.3.1\mathrm{c})\ \textit{is an isomorphism for all}\ \mathrm{G}.\end{aligned}$$

Proof. For fixed E', the functor $\mathbf{R}\mathrm{Hom}^\bullet_S(\Gamma E',\mathbf{R}\Gamma-)\colon\mathbf{K}(S)\to\mathbf{D}(S)$ takes quasi-isomorphisms to isomorphisms. So the initial-object characterization of right-derived functors (§1.5) gives a unique functorial map $\nu_{E'}$ making the following otherwise natural $\mathbf{D}(S)$-diagram commute for all S-complexes E:

$$\begin{array}{ccccc}\mathrm{Hom}^\bullet_S(E',E) & \xrightarrow{\qquad\qquad\qquad} & & & \mathbf{R}\mathrm{Hom}^\bullet_S(E',E)\\ \big\downarrow & & & & \big\downarrow{\scriptstyle\nu_{E'}(E)}\\ \mathrm{Hom}^\bullet_S(\Gamma E',\Gamma E) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet_S(\Gamma E',\Gamma E) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet_S(\Gamma E',\mathbf{R}\Gamma E)\end{array}$$

Taking E' to be a q-injective resolution of S, one has the map

$$\nu_{E'}(E)\colon E=\mathbf{R}\mathrm{Hom}^\bullet_S(S,E)\to\mathbf{R}\mathrm{Hom}^\bullet_S(\mathbf{R}\Gamma S,\mathbf{R}\Gamma E)$$

which gives, via (2.2.3) (with $R=S$ and $\varphi=$ identity), the natural map (2.3.1a).

It is clear then that for any E, G:

$$\begin{aligned}[(2.3.1\mathrm{a})\text{ is an isomorphism}]&\implies[(2.3.1\mathrm{b})\text{ is an isomorphism}]\\ &\implies[(2.3.1\mathrm{c})\text{ is an isomorphism}].\end{aligned}$$

Conversely, if (2.3.1c) is an isomorphism for all G then using (2.2.3) one sees that (2.3.1a) induces for all G an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma E,G)\xrightarrow{\sim}\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*(E\underline{\otimes}_S\mathbf{R}\Gamma S),G),$$

whence φ_*(2.3.1a) is an isomorphism. Thus (2.3.1a) induces homology isomorphisms after, *hence before,* restriction of scalars, and this means that (2.3.1a) itself is an isomorphism (§1.3).[8] □

[8](2.3.1a) is an isomorphism iff $\mathbf{R}\Gamma$ commutes with direct sums, see Prop. 3.5.5 below.

The map (2.3.1a) is the obvious one when Γ is the identity functor $\mathbf{1}$; and it behaves well with respect to functorial maps $\Gamma \to \Gamma'$, in particular the inclusion $\Gamma_J \hookrightarrow \mathbf{1}$ with J an S-ideal. For noetherian S it follows that (2.3.1a) is identical with the *isomorphism* $\psi(S,E)$ in Corollary 3.3.1 below (with $I \subset R$ replaced by $J \subset S$), whence (2.3.1b) and (2.3.1c) are also isomorphisms. Thus:

Theorem 2.3.2 ("Trivial" Local Duality)**.** *For $\varphi\colon R \to S$ a map of commutative rings with S noetherian, J an S-ideal, and $\varphi_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ the restriction-of-scalars functor, there is a functorial $\mathbf{D}(S)$-isomorphism*

$$\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma_J E, G) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_S\big(E, \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma_J S, G)\big)$$

($E \in \mathbf{D}(S),\ G \in \mathbf{D}(R)$)*; and hence with $\varphi^\#_J\colon \mathbf{D}(R) \to \mathbf{D}(S)$ the functor*

$$\varphi^\#_J(-) := \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma_J S, -) \cong \mathbf{R}\mathrm{Hom}^\bullet_S\big(\mathbf{R}\Gamma_J S, \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* S, -)\big)$$

there is a natural adjunction isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma_J E, G) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}(E, \varphi^\#_J G).$$

Now with (S,J) and $\varphi\colon R \to S$ as above, let $\psi\colon S \to T$ be another ring-homomorphism, with T noetherian, and let $\psi_*\colon \mathbf{D}(T) \to \mathbf{D}(S)$ be the corresponding derived restriction-of-scalars functor. Let K be a T-ideal containing $\psi(J)$. Then $\psi_*\mathbf{D}_K(T) \subset \mathbf{D}_J(S)$, and therefore by Corollary 3.2.1 below, the natural map is an isomorphism $\mathbf{R}\Gamma_J\psi_*\mathbf{R}\Gamma_K \xrightarrow{\ \sim\ } \psi_*\mathbf{R}\Gamma_K$, giving rise to a functorial isomorphism

$$\varphi_*\mathbf{R}\Gamma_J\,\psi_*\mathbf{R}\Gamma_K \xrightarrow{\ \sim\ } \varphi_*\psi_*\mathbf{R}\Gamma_K = (\psi\varphi)_*\mathbf{R}\Gamma_K$$

whence a functorial isomorphism between the right adjoints (see Thm. 2.3.2):

$$(\psi\varphi)^\#_K \xrightarrow{\ \sim\ } \psi^\#_K\varphi^\#_J. \tag{2.3.3}$$

2.4. **Nontrivial dualities.** From now on, the standing assumption that all rings are *noetherian* as well as commutative is essential.

"Nontrivial" versions of Theorem 2.3.2 convey more information about $\varphi^\#_J$.

Suppose, for example, that S is module-finite over R, and let $G \in \mathbf{D}_\mathrm{c}(R)$, by which is meant that each homology module of $G \in \mathbf{D}(R)$ is finitely generated. (Here "c" connotes "coherent".) Suppose further that $\mathrm{Ext}^i(S,G)$ is a finitely-generated R-module for all $i \in \mathbb{Z}$, i.e., $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*S, G) \in \mathbf{D}_\mathrm{c}(R)$. (This holds, e.g., if $\mathrm{H}^iG = 0$ for all $i \ll 0$, cf. [Ha1, p. 92, Prop. 3.3(a)].) Then Greenlees-May duality (Corollary 4.1.1 below, with (R,I) replaced by (S,J)—so that $\hat S$ denotes J-adic completion of S—and F replaced by $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*S,G)$) gives the first of the natural isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*S,G)\otimes_S \hat S &\underset{(4.1.1)}{\xrightarrow{\ \sim\ }} \mathbf{R}\mathrm{Hom}^\bullet_S\big(\mathbf{R}\Gamma_J S, \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*S,G)\big)\\ &\underset{(2.2.2)}{\xrightarrow{\ \sim\ }} \mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma_J S, G) = \varphi^\#_J G.\end{aligned} \tag{2.4.1}$$

More particularly, for $S=R$ and $\varphi=\mathrm{id}$ (the identity map) we get

$$\mathrm{id}_J^{\#}G = G\otimes_R \hat R \qquad (G\in \mathbf{D}_{\mathrm c}(R)).$$

Specialize further to where R is local, $\varphi=\mathrm{id}$, $J=\mathfrak m$, the maximal ideal of R, and $G\in\mathbf{D}_{\mathrm c}(R)$ is a *normalized dualizing complex,*[9] so that in $\mathbf{D}(R)$, $\mathbf{R}\Gamma_{\mathfrak m}G\cong\mathcal I$ with $\mathcal I$ an R-injective hull of the residue field $R/\mathfrak m$ [Ha1, p. 276, Prop. 6.1]. Then there are natural isomorphisms

$$\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak m}E,G) \underset{(3.2.2)}{\cong} \mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak m}E,\mathbf{R}\Gamma_{\mathfrak m}G)\cong \mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak m}E,\mathcal I)$$

Substitution into Theorem 2.3.2 gives then a natural isomorphism

$$\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak m}E,\mathcal I)\xrightarrow{\ \sim\ }\mathbf{R}\mathrm{Hom}^{\bullet}_R(E,G\otimes_R\hat R)\qquad (E\in\mathbf{D}(R)). \tag{2.4.2}$$

For $E\in\mathbf{D}_{\mathrm c}(R)$ this is just *classical local duality* [Ha1, p. 278], modulo Matlis dualization.[10]

Applying homology $\mathrm H^{-i}$ we get the duality isomorphism

$$\mathrm{Hom}_R(\mathrm H^i_{\mathfrak m}E,\mathcal I)\xrightarrow{\ \sim\ }\mathrm{Ext}^{-i}_R(E,G\otimes_R\hat R). \tag{2.4.3}$$

If R is Cohen-Macaulay, i.e., there is an $\mathfrak m$-primary ideal generated by an R-regular sequence of length $d:=\dim(R)$, then by Cor. 3.1.4, $\mathrm H^i_{\mathfrak m}R=0$ for $i>d$; and in view of [Gr2, p. 31, Prop. 2.4], (1.1.1) gives $\mathrm H^i_{\mathfrak m}R=0$ for $i<d$. (Or, see [BS, p. 110, Cor. 6.2.9].) Since $\hat R$ is R-flat, (2.4.3) now yields

$$0=\mathrm{Ext}^{-i}_R(R,G\otimes_R\hat R)=\mathrm H^{-i}(G\otimes_R\hat R)=(\mathrm H^{-i}G)\otimes_R\hat R\qquad (i\ne d).$$

Hence the homology of G vanishes outside degree $-d$, so by Proposition 1.3.3 there is a derived-category isomorphism $G\cong\omega[d]$ where $\omega:=\mathrm H^{-d}G$ (a *canonical module* of R). In conclusion, (2.4.3) takes the form

$$\mathrm{Hom}_R(\mathrm H^i_{\mathfrak m}E,\mathcal I)\xrightarrow{\ \sim\ }\mathrm{Ext}^{d-i}_R(E,\hat\omega).$$

Another situation in which $\varphi^{\#}_J$ can be described concretely is when S is a power-series ring over R, see §5.1 below.

For more along these lines, see [AJL, pp. 7–9, (c)] and [DFS, §2.1].

3. Koszul complexes and local cohomology

Throughout, R is a commutative noetherian ring and $\mathbf t=(t_1,\dots,t_m)$ is a sequence in R, generating the ideal $I:=\mathbf tR$. The symbol $\otimes$ without a subscript denotes $\otimes_R$, and similarly for $\underline{\underline{\otimes}}$.

[9] which exists if R is a homomorphic image of a Gorenstein local ring [Ha1, p. 299].

[10] which is explained e.g., in [BS, Chapter 10]. For more details, see [AJL, p. 8].

3.1. **$\mathbf{R}\Gamma_I$ = stable Koszul homology.** Before proceeding with our exploration of local cohomology, we must equip ourselves with Koszul complexes. They provide, via Čech cohomology, a link between the algebraic theory and the topological theory on Spec(R)—a link which will remain implicit here. (See [Gr2, Exposé II].)

For $t \in R$, let $\mathcal{K}(t)$ be the complex $\cdots \to 0 \to R \xrightarrow{t} R_t \to 0 \to \cdots$ which in degrees 0 and 1 is the natural map from $R =: \mathcal{K}^0(t)$ to its localization $R_t =: \mathcal{K}^1(t)$ by powers of t, and which vanishes elsewhere.

For any R-complex C, define the "stable" Koszul complexes

$$\mathcal{K}(\mathbf{t}) := \mathcal{K}(t_1) \otimes \cdots \otimes \mathcal{K}(t_m), \qquad \mathcal{K}(\mathbf{t}, C) := \mathcal{K}(\mathbf{t}) \otimes C.$$

Since the complex $\mathcal{K}(\mathbf{t})$ is flat and bounded, the functor of complexes $\mathcal{K}(\mathbf{t}, -)$ takes quasi-isomorphisms to quasi-isomorphisms (apply [Ha1, p. 93, Lemma 4.1, b2] to the mapping cone of a quasi-isomorphism), and so may—and will—be regarded as a functor from $\mathbf{D}(R)$ to $\mathbf{D}(R)$.

Given a q-injective resolution $C \to E_C$ (§1.2) we have for $E = E_C^j$ ($j \in \mathbb{Z}$),

$$\Gamma_I E = \ker\big(\mathcal{K}^0(\mathbf{t}, E) = E \to \oplus_{i=1}^m E_{t_i} = \mathcal{K}^1(\mathbf{t}, E)\big),$$

whence a $\mathbf{D}(R)$-map

$$\delta(C)\colon \mathbf{R}\Gamma_I C \underset{(1.5.1)}{=} \Gamma_I E_C \hookrightarrow \mathcal{K}(\mathbf{t}, E_C) \cong \mathcal{K}(\mathbf{t}, C),$$

easily seen to be functorial in C, making the following diagram commute:

$$\begin{array}{ccc} \mathbf{R}\Gamma_I C & \xrightarrow{\delta(C)} & \mathcal{K}(\mathbf{t}, C) = \mathcal{K}(\mathbf{t}) \otimes C \\ {\scriptstyle\text{natural}}\downarrow & & \downarrow{\scriptstyle \pi(C)} \\ C & \xrightarrow{\quad\sim\quad} & R \otimes C \end{array} \tag{3.1.1}$$

where $\pi(C)$ is obtained by tensoring the projection $\mathcal{K}(\mathbf{t}) \twoheadrightarrow \mathcal{K}^0(\mathbf{t}) = R$ (which is a map of complexes) with the identity map of C.

The key to the store of properties of local cohomology in this section is:[11]

Proposition 3.1.2. *The $\mathbf{D}(R)$-map $\delta(C)$ is a functorial* isomorphism

$$\mathbf{R}\Gamma_I C \xrightarrow{\sim} \mathcal{K}(\mathbf{t}, C).$$

Proof. (Indication.) We can choose E_C to be injective as well as q-injective (see footnote in §1.2), and replace C by E_C; thus we need only show that if C is injective then the inclusion map $\Gamma_{\mathbf{t}R} C \hookrightarrow \mathcal{K}(\mathbf{t}, C)$ is a quasi-isomorphism. Elementary "staircase" diagram-chasing (or a standard spectral-sequence argument) allows us to replace C by each C^i ($i \in \mathbb{Z}$), reducing the problem to where C is a single injective R-module. In this case the classical proof can be found in [Gr2, pp. 23–26] or [Wb1, p. 118, Cor. 4.6.7] (with arrows in the two lines preceding Cor. 4.6.7 reversed).

There is another approach when C is a bounded-below injective complex (applying in particular when C is a single injective module). Every injective

[11] But see §3.5 for a Koszul-free, more general, approach.

R-module is a direct sum of injective hulls of R-modules of the form R/P with $P \subset R$ a prime ideal, and in such a hull every element is annihilated by a power of P [Mt1]. It follows that *for every $t \in R$, the localization map $C \to C_t$ is surjective,*[12] so that the inclusion $\Gamma_{tR}C \hookrightarrow \mathcal{K}((t), C)$ is a quasi-isomorphism; and that *the complex $\Gamma_{tR}C$ is injective,* whence $\Gamma_{\mathbf{t}R}C$ is bounded-below and injective, therefore q-injective (§1.2).

Moreover, $\mathcal{K}(\mathbf{t}, C)$ is bounded-below and injective, hence q-injective, since for any flat R-module F and injective R-module E, the functor

$$\mathrm{Hom}_R(M, F \otimes E) \cong F \otimes \mathrm{Hom}_R(M, E)$$

of finitely-generated R-modules M is exact, i.e., $F \otimes E$ is injective.

One shows now, by induction on $m \geq 2$, that with $\mathbf{t}' := (t_2, \dots, t_m)$, the top row of

$$\begin{array}{ccccccc} \Gamma_{\mathbf{t}R}C & = & \Gamma_{t_1R}\Gamma_{\mathbf{t}'R}C & \longrightarrow & \Gamma_{t_1R}\mathcal{K}(\mathbf{t}', C) & \longrightarrow & \mathcal{K}\big((t_1), \mathcal{K}(\mathbf{t}', C)\big) \\ {\scriptstyle\simeq}\big\downarrow & & {\scriptstyle\simeq}\big\downarrow & & {\scriptstyle\simeq}\big\downarrow & & \big\| \\ \mathbf{R}\Gamma_I C & & \mathbf{R}\Gamma_{t_1R}\mathbf{R}\Gamma_{\mathbf{t}'R}C & & \mathbf{R}\Gamma_{t_1R}\mathcal{K}(\mathbf{t}', C) & & \mathcal{K}(\mathbf{t}, C) \end{array}$$

is a $\mathbf{D}(R)$-isomorphism. $\square$

For R-ideals I and I' there is, according to the initial-object characterization of right-derived functors (§1.5), a unique functorial map χ making the following otherwise natural $\mathbf{D}(R)$-diagram commute

$$\begin{array}{ccc} \Gamma_{I+I'} = \Gamma_I\Gamma_{I'} & \longrightarrow & \mathbf{R}\Gamma_I\Gamma_{I'} \\ \big\downarrow & & \big\downarrow \\ \mathbf{R}\Gamma_{I+I'} & \underset{\chi}{\longrightarrow} & \mathbf{R}\Gamma_I\mathbf{R}\Gamma_{I'} \end{array}$$

Corollary 3.1.3. *The preceding natural functorial map is an isomorphism*

$$\chi\colon \mathbf{R}\Gamma_{I+I'} \xrightarrow{\sim} \mathbf{R}\Gamma_I\mathbf{R}\Gamma_{I'}.$$

Proof. Let $I = \mathbf{t}R$ $\big(\mathbf{t} := (t_1, \dots, t_m)\big)$ and $I' = \mathbf{t}'R$ $\big(\mathbf{t}' := (t'_1, \dots, t'_n)\big)$, so that $I + I' = (\mathbf{t} \vee \mathbf{t}')R$ $\big(\mathbf{t} \vee \mathbf{t}' := (t_1, \dots, t_m, t'_1, \dots, t'_n)\big)$. It is a routine exercise to deduce from Proposition 3.1.2 an identification of $\chi(C)$ with the natural isomorphism $\mathcal{K}(\mathbf{t} \vee \mathbf{t}', C) \xrightarrow{\sim} \mathcal{K}\big(\mathbf{t}, \mathcal{K}(\mathbf{t}', C)\big)$. $\square$

We see next that the functor $\mathbf{R}\Gamma_I$ is "bounded"—a property of considerable importance in matters involving unbounded complexes.[13]

Corollary 3.1.4. *Let C be an R-complex such that* $\mathrm{H}^iC = 0$ *for all* $i > i_1$ (resp. $i < i_0$). *Then* $\mathrm{H}^i\mathbf{R}\Gamma_I C = 0$ *for all* $i > i_1 + m$ (resp. $i < i_0$).

[12] There are easier ways to prove this.

[13] This boundedness property, called "way-out in both directions" in [Ha1], often enters via the "way-out" lemmas [*loc. cit.,* p. 69, (iii) and p. 74, (iii)]. See, for instance, the proof of Corollary 3.2.1 below.

Proof. If $\mathrm{H}^i C = 0$ for all $i > i_1$, then replacing C^i by 0 for all $i > i_1$ and C^{i_1} by the kernel of $C^{i_1} \to C^{i_1+1}$ produces a quasi-isomorphic subcomplex $C_1 \subset C$ vanishing in all degrees above i_1. There are then isomorphisms

$$\mathbf{R}\Gamma_I C \xleftarrow{\sim} \mathbf{R}\Gamma_I C_1 \underset{(3.1.2)}{\xrightarrow{\sim}} \mathcal{K}(\mathbf{t}, C_1),$$

and $\mathrm{H}^i\mathcal{K}(\mathbf{t}, C_1)$ (indeed, $\mathcal{K}(\mathbf{t}, C_1)$ itself) vanishes in all degrees above $i_1 + m$.

A dual argument applies to the case where $\mathrm{H}^i C = 0$ for all $i < i_0$. (More generally, without Prop. 3.1.2 there is in this case a surjective quasi-isomorphism $C \twoheadrightarrow C_0$ with C_0 vanishing in all degrees below i_0, and a quasi-isomorphism $C_0 \to E_0$ into an injective E_0 vanishing likewise [Ha1, p. 43]; and so $\mathrm{H}^i\mathbf{R}\Gamma_I C \cong \mathrm{H}^i\Gamma_I E_0$ vanishes for all $i < i_0$.) □

3.2. **The derived torsion category.** We will say that an R-module M is *I-power torsion* if $\Gamma_I M = M$, or equivalently, for any prime R-ideal $P \not\supset I$ the localization $M_P = 0$. (Geometrically, this means the corresponding sheaf on $\mathrm{Spec}(R)$ is supported inside the subscheme $\mathrm{Spec}(R/I)$.) For any R-module M, $\Gamma_I M$ is I-power torsion.

Let $\mathbf{D}_I(R) \subset \mathbf{D}(R)$ be the full subcategory with objects those complexes C whose homology modules are all I-power torsion, i.e., the localization C_P is exact for any prime R-ideal $P \not\supset I$. For any R-complex C, (1.5.1) implies that $\mathbf{R}\Gamma_I C \in \mathbf{D}_I(R)$.

The subcategory $\mathbf{D}_I(R)$ is stable under translation, and for any $\mathbf{D}(R)$-triangle with two vertices in $\mathbf{D}_I(R)$ the third must be in $\mathbf{D}_I(R)$ too, as follows from exactness of the homology sequence (1.4.2).

Corollary 3.2.1. *The complex C is in $\mathbf{D}_I(R)$ if and only if the natural map $\iota(C)\colon \mathbf{R}\Gamma_I C \to C$ is a $\mathbf{D}(R)$-isomorphism.*

Proof. ($\Leftarrow$) Clear, since $\mathbf{R}\Gamma_I C \in \mathbf{D}_I(R)$.

($\Rightarrow$) The boundedness of $\mathbf{R}\Gamma_I$ (3.1.4) allows us to apply [Ha1, p. 74, (iii)] to reduce to the case where C is a single I-power-torsion module. But then $\mathcal{K}(t_i) \otimes C = C$ for $i = 1, \dots, m$, whence (by induction on m) $\mathcal{K}(\mathbf{t}, C) = C$, and so by Proposition 3.1.2 and the commutativity of (3.1.1), $\iota(C)$ is an isomorphism. □

We show next that $\mathbf{R}\Gamma_I$ is right-adjoint to the inclusion $\mathbf{D}_I(R) \hookrightarrow \mathbf{D}(R)$.

Proposition 3.2.2. *The map $\iota(G)\colon \mathbf{R}\Gamma_I G \to G$ induces an isomorphism*

$$\mathbf{R}\mathrm{Hom}^\bullet(F, \mathbf{R}\Gamma_I G) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet(F, G) \qquad (F \in \mathbf{D}_I(R),\ G \in \mathbf{D}(R)),$$

whence, upon application of homology H^0, *an adjunction isomorphism*

$$\varrho(F,G)\colon \mathrm{Hom}_{\mathbf{D}_I(R)}(F, \mathbf{R}\Gamma_I G) = \mathrm{Hom}_{\mathbf{D}(R)}(F, \mathbf{R}\Gamma_I G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(R)}(F, G).$$

Proof. Since $\mathbf{D}(R)$-isomorphism means homology-isomorphism (§1.3), and since (see (1.3.2))

$$\mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet(F', G') = \mathrm{Hom}_{\mathbf{D}(R)}(F', G'[i]) \qquad \big(F', G' \in \mathbf{D}(R)\big),$$

we need only show that $\varrho(F,G)$ is an isomorphism for all $F\in\mathbf{D}_I(R)$ and $G\in\mathbf{D}(R)$. Referring then to

$$\operatorname{Hom}_{\mathbf{D}(R)}(F,G)\xrightarrow[\nu]{}\operatorname{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_I F,\mathbf{R}\Gamma_I G)\xleftarrow[\rho]{\ \sim\ }\operatorname{Hom}_{\mathbf{D}(R)}(F,\mathbf{R}\Gamma_I G)$$

where ν is the natural map and where ρ is induced by the isomorphism $\iota(F)\colon\mathbf{R}\Gamma_I F\xrightarrow{\ \sim\ }F$ (Corollary 3.2.1), let us show that $\rho^{-1}\nu$ is inverse to ϱ.

That $\varrho\rho^{-1}\nu(\alpha)=\alpha$ for any $\alpha\in\operatorname{Hom}_{\mathbf{D}(R)}(F,G)$ amounts to the (obvious) commutativity of the diagram

$$\begin{array}{ccc}
\mathbf{R}\Gamma_I F & \xrightarrow{\mathbf{R}\Gamma_I\alpha} & \mathbf{R}\Gamma_I G\\
{\scriptstyle\iota(F)}\big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\iota(G)}\\
F & \xrightarrow[\alpha]{} & G
\end{array}$$

That $\rho^{-1}\nu\varrho(\beta)=\beta$ for $\beta\in\operatorname{Hom}_{\mathbf{D}(R)}(F,\mathbf{R}\Gamma_I G)$ amounts to commutativity of

$$\begin{array}{ccc}
\mathbf{R}\Gamma_I F & \xrightarrow{\mathbf{R}\Gamma_I\beta} & \mathbf{R}\Gamma_I\mathbf{R}\Gamma_I G\\
{\scriptstyle\iota(F)}\big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\mathbf{R}\Gamma_I\iota(G)}\\
F & \xrightarrow[\beta]{} & \mathbf{R}\Gamma_I G
\end{array}$$

and so (since ι is functorial) it suffices to show that $\mathbf{R}\Gamma_I\iota(G)=\iota(\mathbf{R}\Gamma_I G)$. We may assume that G is injective and q-injective, and then the second paragraph in the proof of Prop. 3.1.2 shows that $\Gamma_I G$ is injective and that

$$\Gamma_I\Gamma_I G\hookrightarrow\mathcal{K}(\mathbf{t},\Gamma_I G)\cong\mathbf{R}\Gamma_I\mathbf{R}\Gamma_I G$$

is a $\mathbf{D}(R)$-isomorphism. It follows that $\mathbf{R}\Gamma_I\iota(G)$ and $\iota(\mathbf{R}\Gamma_I G)$ are both canonically isomorphic to the identity map $\Gamma_I\Gamma_I G\hookrightarrow\Gamma_I G$, so that they are indeed equal. □

3.3. **Local cohomology and tensor product.**

Corollary 3.3.1. *There is a unique bifunctorial isomorphism*

$$\psi(C,C')\colon\mathbf{R}\Gamma_I C\underset{=}{\otimes}C'\xrightarrow{\ \sim\ }\mathbf{R}\Gamma_I(C\underset{=}{\otimes}C')\qquad(C,\ C'\in\mathbf{D}(R))$$

whose composition with the natural map $\mathbf{R}\Gamma_I(C\underset{=}{\otimes}C')\to C\underset{=}{\otimes}C'$ *is the natural map* $\mathbf{R}\Gamma_I C\underset{=}{\otimes}C'\to C\underset{=}{\otimes}C'$.

Proof. Replacing C and C' by q-flat resolutions, we may assume that C and C' are themselves q-flat. Existence and bifunctoriality of the isomorphism ψ are given then, via Prop. 3.1.2 and commutativity of (3.1.1), by the natural isomorphism

$$\mathcal{K}(\mathbf{t},C)\otimes C'=(\mathcal{K}(\mathbf{t})\otimes C)\otimes C'\xrightarrow{\ \sim\ }\mathcal{K}(\mathbf{t})\otimes(C\otimes C')=\mathcal{K}(\mathbf{t},C\otimes C').$$

It follows in particular that $\mathbf{R}\Gamma_I C\underset{=}{\otimes}C'\in\mathbf{D}_I(R)$,[14] and so uniqueness of ψ results from Proposition 3.2.2. □

[14]This is easily shown without using $\mathcal{K}$.

Here is a homological consequence. (Proof left to the reader.)

Corollary 3.3.2. *For any R-complex C and flat R-module M there are natural isomorphisms*

$$\mathrm{H}^i_I(C)\otimes M \xrightarrow{\sim} \mathrm{H}^i_I(C\otimes M) \qquad (i\in\mathbb{Z}).$$

Here is an interpretation of some basic properties of the functor $\mathbf{R}\Gamma_I$ in terms of the complex $\mathbf{R}\Gamma_I R \cong \mathcal{K}(\mathbf{t})$. (Proof left to the reader.)

Corollary 3.3.3. *Via the isomorphism $\psi(R,-)$ of the functor $\mathbf{R}\Gamma_I R \underline{\underline{\otimes}} (-)$ with $\mathbf{R}\Gamma_I(-)$ the natural map $\mathbf{R}\Gamma_I C' \to C'$ corresponds to the map*

$$\iota(R)\underline{\underline{\otimes}}1\colon \mathbf{R}\Gamma_I R\underline{\underline{\otimes}} C' \to R\underline{\underline{\otimes}} C' = C',$$

and the above map $\psi(C,C')$ corresponds to the associativity isomorphism[15]

$$(\mathbf{R}\Gamma_I R\underline{\underline{\otimes}} C)\underline{\underline{\otimes}} C' \xrightarrow{\sim} \mathbf{R}\Gamma_I R\underline{\underline{\otimes}}(C\underline{\underline{\otimes}} C').$$

3.4. **Change of rings.** Let $\varphi\colon R\to S$ be a homomorphism of noetherian rings. The functor "restriction of scalars" from S-complexes to R-complexes preserves quasi-isomorphisms, so it extends to a functor $\varphi_*\colon \mathbf{D}(S)\to\mathbf{D}(R)$.

As in §2.1, we find that the functor $M\mapsto M\otimes_R S$ from R-modules to S-modules has a left-derived functor $\varphi^*\colon \mathbf{D}(R)\to\mathbf{D}(S)$ such that after choosing for each R-complex C a q-flat resolution $F_C\to C$ we have $\varphi^*C = F_C\otimes_R S$. If S is R-flat, then the natural map is an isomorphism $\varphi^*C \xrightarrow{\sim} C\otimes_R S$.

There are natural functorial isomorphisms

$$B\underline{\underline{\otimes}}_R\varphi_*D \xrightarrow{\sim} \varphi_*(\varphi^*B\underline{\underline{\otimes}}_S D) \qquad \big(B\in\mathbf{D}(R),\ D\in\mathbf{D}(S)\big), \tag{3.4.1}$$

$$\varphi^*(B\underline{\underline{\otimes}}_R C) \xrightarrow{\sim} \varphi^*B\underline{\underline{\otimes}}_S\varphi^*C \qquad \big(B,\ C\in\mathbf{D}(R)\big). \tag{3.4.2}$$

Proofs are left to the reader. (In view of [Lp3, (2.6.5)] one may assume that all the complexes involved are q-flat, in which case $\underline{\underline{\otimes}}$ becomes $\otimes$, and then the isomorphisms are the obvious ones.)

For example, there are natural isomorphisms (self-explanatory notation):

$$\varphi^*\mathcal{K}_R(\mathbf{t})\cong\mathcal{K}_R(\mathbf{t})\otimes_R S\cong\mathcal{K}_S(\varphi\mathbf{t}).$$

So putting $B=\mathcal{K}(\mathbf{t})$ in the isomorphisms (3.4.1) and (3.4.2) we obtain, via Propositions 3.1.2 and 3.2.2, and commutativity of (3.1.1), the following two corollaries.

Corollary 3.4.3. *There is a unique $\mathbf{D}(R)$-isomorphism*

$$\varphi_*\mathbf{R}\Gamma_{IS}D \xrightarrow{\sim} \mathbf{R}\Gamma_I\varphi_*D \qquad (D\in\mathbf{D}(S))$$

whose composition with the natural map $\mathbf{R}\Gamma_I\varphi_*D\to\varphi_*D$ is the natural map $\varphi_*\mathbf{R}\Gamma_{IS}D\to\varphi_*D$. Thus there are natural R-isomorphisms

$$\varphi_*\mathrm{H}^i_{IS}D \xrightarrow{\sim} \mathrm{H}^i_I\varphi_*D \qquad (i\in\mathbb{Z}).$$

---

[15] derived from associativity for tensor product of R-complexes as in, e.g., [Lp3, (2.6.5)].

Corollary 3.4.4. *There is a unique* $\mathbf{D}(S)$*-isomorphism*

$$\varphi^*\mathbf{R}\Gamma_I C \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_{IS}\varphi^* C \qquad (C \in \mathbf{D}(R))$$

whose composition with the natural map $\mathbf{R}\Gamma_{IS}\varphi^*C \to \varphi^*C$ *is the natural map* $\varphi^*\mathbf{R}\Gamma_I C \to \varphi^*C$*. Consequently, if* S *is* R-flat *then there are natural* S*-isomorphisms*

$$\mathrm{H}^i_I C \otimes S \xrightarrow{\ \sim\ } \mathrm{H}^i_{IS}(C \otimes S) \qquad (i \in \mathbb{Z}).$$

If M is an I-power-torsion R-module, for example, $M = \mathrm{H}^i_I C$ (see §3.2), and $\hat{R}$ is the I-adic completion of R, then the canonical map $\gamma\colon M \to M \otimes \hat{R}$ is bijective: indeed, since this map commutes with $\varinjlim$ we may assume that M is finitely generated, in which case for large n the natural map

$$M \otimes \hat{R} \to M \otimes (\hat{R}/I^n\hat{R}) = M \otimes (R/I^n)$$

as well as its composition with γ is bijective, so that γ is too. Thus putting $S = \hat{R}$ in the preceding Corollary we get:

Corollary 3.4.5. *For* $C \in \mathbf{D}(R)$ *the local cohomology modules* $\mathrm{H}^i_I C$ $(i \in \mathbb{Z})$ *depend only on the topological ring* $\hat{R}$ *and* $C \otimes \hat{R}$*, in that for any defining ideal* J $\big($*i.e.,* $\sqrt{J} = \sqrt{I\hat{R}}\,\big)$ *there are natural isomorphisms*

$$\mathrm{H}^i_I C \xrightarrow{\ \sim\ } \mathrm{H}^i_{\hat{I}}(C \otimes \hat{R}) = \mathrm{H}^i_J(C \otimes \hat{R}).$$

Remark. For $(\hat{R}, J)$ as in 3.4.5, the functor $\Gamma_J = \mathrm{H}^0_J$ on $\hat{R}$-modules M depends only on the topological ring $\hat{R}$: $\Gamma_J M$ consists of those $m \in M$ which are annihilated by some open $\hat{R}$-ideal.

Exercise. (a) Let F be a q-injective resolution of the S-complex D. Show that applying $\mathrm{H}^i\Gamma_I$ to a q-injective R-resolution $\varphi_*F \to G$ produces the homology maps in Corollary 3.4.3.

(b) Suppose that S is R-flat. Let $C \to E$ be a q-injective resolution of the R-complex C and $\eta\colon E \otimes S \to F$ a q-injective S-resolution. Show that the homology maps in Corollary 3.4.4 factor naturally as

$$\mathrm{H}^i_I C \otimes S \cong \mathrm{H}^i\Gamma_I E \otimes S \xrightarrow{\ \sim\ } \mathrm{H}^i(\Gamma_I E \otimes S) \xrightarrow{\ \sim\ } \mathrm{H}^i\Gamma_{IS}(E \otimes S)$$
$$\xrightarrow{\mathrm{H}^i\Gamma_{IS}\eta} \mathrm{H}^i\Gamma_{IS}F \cong \mathrm{H}^i_{IS}(C \otimes S).$$

3.5. **Appendix: Generalization.** In this appendix, we sketch a more general version (not needed elsewhere) of local cohomology, and its connection with the theory of "localization of categories." In establishing the corresponding generalizations of the properties of local cohomology developed above, we make use of the structure of injective modules over a noetherian ring together with some results of Neeman about derived categories of noetherian rings, rather than of Koszul complexes.

At the end, these local cohomology functors are characterized as being *all those idempotent* Δ*-functors from* $\mathbf{D}(R)$ *to itself which respect direct sums.*

Let R be a noetherian topological ring. The topology $\mathfrak{U}$ on R is *linear* if there is a neighborhood basis $\mathcal{N}$ of 0 consisting of ideals. An ideal is open iff it contains a member of $\mathcal{N}$.

We assume further that the square of any open ideal is open. Then $\mathfrak{U}$ is determined by the set O of its open prime ideals: an ideal is open iff it contains a power product of finitely many members of O. Thus endowing R with such a topology is equivalent to giving a set O of prime ideals such that for any prime ideals $p \subset p'$, $p \in O \Rightarrow p' \in O$. The case we have been studying, where $\mathcal{N}$ consists of the powers of a single ideal I, is essentially that in which O has finitely many minimal members (namely the minimal prime ideals of I, whose product can replace I).

Let $\varGamma' = \varGamma'_{\mathfrak{U}}$ be the *left-exact subfunctor of the identity functor on* $\mathcal{M}(R)$ such that for any R-module M,

$$\varGamma' M = \{\, x \in M \mid \text{for some open ideal } I,\ Ix = 0\,\}.$$

The functor $\varGamma'$ *commutes with direct sums.* If p is a prime R-ideal and I_p is the injective hull of R/p, then $\varGamma' I_p = I_p$ if p is open (because every element of I_p is annihilated by a power of p), and $\varGamma' I_p = 0$ otherwise. Thus $\varGamma'$ determines the set of open primes, and hence determines the topology $\mathfrak{U}$. Moreover, $\varGamma'$ *preserves injectivity of modules,* since every injective S-module is a direct sum of I_p's, and any such direct sum is injective.

Conversely, every left-exact subfunctor $\varGamma$ of the identity which commutes with direct sums and preserves injectivity is of the form $\varGamma'_{\mathfrak{U}}$. Indeed, since I_p is an indecomposable injective, the injective module $\varGamma(I_p)$ must be I_p or 0. If $p \subset p'$, then by left-exactness, $\varGamma(I_p) \subset \varGamma(I_{p'})$; and hence the set of p such that $\varGamma(I_p) = I_p$ is the set of open primes for a topology $\mathfrak{U}$. One checks then that $\varGamma = \varGamma'_{\mathfrak{U}}$ by applying both functors to representations of modules as kernels of maps between injectives.

Lemma 3.5.1. *If F is an injective complex, then the natural* $\mathbf{D}(R)$*-map is an isomorphism* $\iota(C)\colon \varGamma' F \xrightarrow{\sim} \mathbf{R}\varGamma' F$.

Proof. The mapping cone C of a q-injective resolution $F \to E_F$ is injective and exact, and as $\mathbf{R}\varGamma' F = \varGamma' E_F$, it suffices to show that $\varGamma' C$ is exact. To this end, consider for any ideal $I = (t_1, \dots, t_n)R$ the topology $\mathfrak{U}_I$ for which the powers of I form a neighborhood basis of 0, so that with previous notation, $\varGamma'_{\mathfrak{U}_I} = \varGamma'_I$. Then

$$\varGamma' = \varGamma'_{\mathfrak{U}} = \varinjlim_{I \text{ open}} \varGamma'_I,$$

which reduces the problem to where $\mathfrak{U} = \mathfrak{U}_I$; and $\varGamma'_I = \varGamma'_{t_1R}\varGamma'_{t_2R}\cdots\varGamma'_{t_nR}$ gives a further reduction to where $I = tR$ ($t \in R$). Finally, exactness of the complex C and of its localization C_t in the exact sequence $0 \to \varGamma'_{tR}C \to C \to C_t \to 0$ (see proof of Proposition 3.1.2) imply that $\varGamma'_{tR}C$ is exact. □

Since any direct sum of q-injective resolutions is an injective resolution, and since $\varGamma'$ commutes with direct sums, one has:

Corollary 3.5.2. *For any small family* (E_α) *in* $\mathbf{D}(R)$, *the natural map is an isomorphism*

$$\oplus_\alpha \mathbf{R}\varGamma' E_\alpha \xrightarrow{\sim} \mathbf{R}\varGamma'(\oplus_\alpha E_\alpha).$$

From Lemma 3.5.1, and the fact that $\varGamma'$ preserves injectivity of complexes, one readily deduces the ("colocalizing") idempotence of $\mathbf{R}\varGamma'$:

Proposition 3.5.3. (i) *For an R-complex E, with q-injective resolution $E \to I_E$, the maps $\iota(\mathbf{R}\varGamma' E)$ and $\mathbf{R}\varGamma'\iota(E)$ from $\mathbf{R}\varGamma'\mathbf{R}\varGamma' E$ to $\mathbf{R}\varGamma' E$ are both inverse to the isomorphism $\mathbf{R}\varGamma' E \xrightarrow{\sim} \mathbf{R}\varGamma'\mathbf{R}\varGamma' E$ given by the identity map of $\varGamma' I_E = \varGamma'\varGamma' I_E$, and so are* equal isomorphisms.

(ii) *For* $E,\ F\in\mathbf{D}(R)$ *the map* $\iota(F)\colon \mathbf{R}\mathit{\Gamma}'F\to F$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\mathit{\Gamma}'E,\mathbf{R}\mathit{\Gamma}'F)\ \xrightarrow{\ \sim\ }\ \mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\mathit{\Gamma}'E,F),$$

with inverse

$$\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\mathit{\Gamma}'E,F)\xrightarrow{\text{natural}}\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\mathit{\Gamma}'\mathbf{R}\mathit{\Gamma}'E,\mathbf{R}\mathit{\Gamma}'F)\ \underset{\text{(i)}}{\xrightarrow{\ \sim\ }}\ \mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\mathit{\Gamma}'E,\mathbf{R}\mathit{\Gamma}'F).$$

The properties given in Corollary 3.5.2 and Proposition 3.5.3 (i) *characterize* functors of the form $\mathbf{R}\mathit{\Gamma}'_{\mathfrak{U}}$ among Δ-functors from $\mathbf{D}(R)$ to itself. This will be shown at the end of this appendix (Proposition 3.5.7).

Next we generalize §3.2. Let $\mathcal{M}_{\mathfrak{U}}(R)=\mathit{\Gamma}'_{\mathfrak{U}}\mathcal{M}(R)$ be the full abelian subcategory of $\mathcal{M}(R)$ whose objects are the $\mathfrak{U}$*-torsion* R*-modules*—those R-modules M such that $\mathit{\Gamma}'M=M$, i.e., the localization $M_p=0$ for every non-open prime R-ideal p. The subcategory $\mathcal{M}_{\mathfrak{U}}(R)\subset\mathcal{M}(R)$ is *plump,* i.e., if $M_1\to M_2\to M\to M_3\to M_4$ is an exact sequence of R-modules such that $M_i\in\mathcal{M}_{\mathfrak{U}}(R)$ for $i=1,2,3,4$, then also $M\in\mathcal{M}_{\mathfrak{U}}(R)$. (To see this one reduces to the case where $M_1=M_4=0$, and uses that the product of two open ideals is open.) One can think of $\mathit{\Gamma}'$ as a functor from $\mathcal{M}(R)$ to $\mathcal{M}_{\mathfrak{U}}(R)$, right-adjoint to the inclusion functor $\mathcal{M}_{\mathfrak{U}}(R)\hookrightarrow\mathcal{M}(R)$.

Upgrading to the derived level, let $\mathbf{D}_{\mathfrak{U}}(R)\subset\mathbf{D}(R)$ be the full subcategory with objects those complexes C whose homology modules are all in $\mathcal{M}_{\mathfrak{U}}(R)$, i.e., the localization C_p is exact for every non-open prime R-ideal p. The exact homology sequence (1.4.2) of a triangle, together with plumpness of $\mathcal{M}_{\mathfrak{U}}(R)$, entails that $\mathbf{D}_{\mathfrak{U}}(R)$ is a *triangulated subcategory* of $\mathbf{D}(R)$, that is, if two vertices of a $\mathbf{D}(R)$-triangle lie in $\mathbf{D}_{\mathfrak{U}}(R)$ then so does the third. In fact $\mathbf{D}_{\mathfrak{U}}(R)$ is a *localizing subcategory* of $\mathbf{D}(R)$ (= full triangulated subcategory closed under arbitrary $\mathbf{D}(R)$-direct sums).

If $C\to E_C$ is a q-injective resolution then $\mathbf{R}\mathit{\Gamma}'C=\mathit{\Gamma}'E_C\in\mathbf{D}_{\mathfrak{U}}(R)$, and so $\mathbf{R}\mathit{\Gamma}'\mathbf{D}(R)\subset\mathbf{D}_{\mathfrak{U}}(R)$. Thus (i) in the following Proposition implies that $\mathbf{D}_{\mathfrak{U}}$ *is the essential image of the functor* $\mathbf{R}\mathit{\Gamma}'$ (i.e., the full subcategory whose objects are the complexes isomorphic to one of the form $\mathbf{R}\mathit{\Gamma}'C$); and (ii) says that $\mathbf{R}\mathit{\Gamma}'$ *can be thought of as being right-adjoint to the inclusion functor* $\mathbf{D}_{\mathfrak{U}}(R)\hookrightarrow\mathbf{D}(R)$.

Proposition 3.5.4. (i) *An* R*-complex* C *is in* $\mathbf{D}_{\mathfrak{U}}(R)$ *if and only if the natural map* $\iota(C)\colon\mathbf{R}\mathit{\Gamma}'C\to C$ *is an isomorphism.*

(ii) *For all* $E\in\mathbf{D}_{\mathfrak{U}}(R)$ *and* $F\in\mathbf{D}(R)$ *the natural map* $\iota(F)\colon\mathbf{R}\mathit{\Gamma}'F\to F$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(R)}(E,\mathbf{R}\mathit{\Gamma}'F)\ \xrightarrow{\ \sim\ }\ \mathrm{Hom}_{\mathbf{D}(R)}(E,F).$$

Proof. (i) "If" is clear since, as noted above, $\mathbf{R}\mathit{\Gamma}'C\in\mathbf{D}_{\mathfrak{U}}(R)$.

As for "only if," by Corollary 3.5.2 those $E\in\mathbf{D}_{\mathfrak{U}}(R)$ for which $\iota(E)$ is an isomorphism are the objects of a localizing subcategory $\mathbf{L}\subset\mathbf{D}_{\mathfrak{U}}(R)$. Now [Nm1, p. 528, Thm. 3.3] says that any localizing subcategory $\mathbf{L}'\subset\mathbf{D}(R)$ is completely determined by the set of prime R-ideals p such that the fraction field κ_p of R/p is in $\mathbf{L}'$. As $\kappa_p\in\mathbf{D}_{\mathfrak{U}}(R)\Leftrightarrow\kappa_p$ is $\mathfrak{U}$-torsion $\Leftrightarrow p$ is open, it follows that $\mathbf{L}=\mathbf{D}_{\mathfrak{U}}(R)$ if only $\iota(\kappa_p)$ is an isomorphism for any such p, which in fact it is because κ_p admits a quasi-isomorphism into a bounded-below complex of $\mathfrak{U}$-torsion R-injective modules, as follows easily from the fact that if an $\mathfrak{U}$-torsion module M is contained in an injective R-module J then M is contained in the $\mathfrak{U}$-torsion injective module $\mathit{\Gamma}'J$.

(ii) In view of (i), the assertion results from Proposition 3.5.3 (ii). □

To generalize the results of §3.3—details left to the reader—one can use the next Proposition (cf. Brown representability [Nm2, p. 223, Thm. 4.1].)

Proposition 3.5.5. *Let* $\Gamma\colon \mathbf{K}(R)\to\mathbf{K}(R)$ *be a* Δ*-functor, with right-derived functor* $\mathbf{R}\Gamma\colon \mathbf{D}(R)\to\mathbf{D}(R)$*. Then the following conditions are equivalent.*

(i) $\mathbf{R}\Gamma$ *commutes with direct sums, i.e., for any small family* (E_α) *in* $\mathbf{D}(R)$*, the natural map is an isomorphism*

$$\oplus_\alpha \mathbf{R}\Gamma E_\alpha \xrightarrow{\;\sim\;} \mathbf{R}\Gamma(\oplus_\alpha E_\alpha).$$

(ii) *For any* $E\in\mathbf{D}(R)$ *the natural map* (2.3.1a) *is an isomorphism*

$$E \underset{=}{\otimes} \mathbf{R}\Gamma R \xrightarrow{\;\sim\;} \mathbf{R}\Gamma E.$$

(iii) $\mathbf{R}\Gamma$ *has a right adjoint.*

Proof. One verifies that the map (2.3.1a) respects triangles and direct sums. Hence if (i) holds then the E for which (ii) holds are the objects of a localizing subcategory $\mathbf{E}\subset\mathbf{D}(R)$. Since $R\in\mathbf{E}$ (easy check), therefore by [Nm2, p. 222, Lemma 3.2], $\mathbf{E}=\mathbf{D}(R)$. Thus (i) $\Rightarrow$ (ii).

Derived adjoint associativity ((2.2.3), with φ the identity map of R) gives a bifunctorial isomorphism, for $E,F\in\mathbf{D}(R)$,

$$\mathrm{Hom}_{\mathbf{D}(R)}\big(E\underset{=}{\otimes}\mathbf{R}\Gamma R, F\big) \xrightarrow{\;\sim\;} \mathrm{Hom}_{\mathbf{D}(R)}\big(E, \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma R, F)\big).$$

Hence (ii) $\Rightarrow$ (iii); and the implication (iii) $\Rightarrow$ (i) is straightforward. □

We conclude this appendix with a remarkably simple characterization of derived local cohomology (Proposition 3.5.7), of which a more general form—for noetherian separated schemes—can be found in [Sou, §4.3].

Definition 3.5.6. An *R-colocalizing pair* is a pair $(\boldsymbol{\Gamma},\boldsymbol{\iota})$ with $\boldsymbol{\Gamma}$ a Δ-functor from $\mathbf{D}(R)$ to $\mathbf{D}(R)$ respecting direct sums and $\boldsymbol{\iota}\colon\boldsymbol{\Gamma}\to\mathbf{1}$ a Δ-functorial isomorphism (Def. 1.4.3) which is "symmetrically idempotent," i.e., the two maps $\boldsymbol{\Gamma\iota}$ and $\boldsymbol{\iota}(\boldsymbol{\Gamma})$ are *equal isomorphisms* from $\boldsymbol{\Gamma\Gamma}$ to $\boldsymbol{\Gamma}=\boldsymbol{\Gamma}\mathbf{1}=\mathbf{1}\boldsymbol{\Gamma}$.

For example, if $\boldsymbol{\iota}_{\mathfrak{U}}\colon \mathbf{R}\varGamma'_{\mathfrak{U}}\to\mathbf{1}$ is the natural map, then $(\mathbf{R}\varGamma'_{\mathfrak{U}},\boldsymbol{\iota}_{\mathfrak{U}})$ is a colocalizing pair (see Corollary 3.5.2 and Proposition 3.5.3 (i)).

This is essentially the *only* example:

Proposition 3.5.7. *Every R-colocalizing pair* $(\boldsymbol{\Gamma},\boldsymbol{\iota})$ *is canonically isomorphic to one of the form* $(\mathbf{R}\varGamma'_{\mathfrak{U}},\boldsymbol{\iota}_{\mathfrak{U}})$ *for exactly one topology* $\mathfrak{U}=\mathfrak{U}_{\boldsymbol{\Gamma}}$*. More precisely,* $\boldsymbol{\iota}$ *factors (uniquely, by Proposition* 3.5.4 (ii)*) as* $\boldsymbol{\iota}_{\mathfrak{U}}\boldsymbol{i}_{\boldsymbol{\Gamma}}$ *where* $\boldsymbol{i}_{\boldsymbol{\Gamma}}\colon\boldsymbol{\Gamma}\xrightarrow{\;\sim\;}\mathbf{R}\varGamma'_{\mathfrak{U}}$ *is a* Δ*-functorial isomorphism.*

Remarks. The set of topologies on R is ordered by inclusion, so may be regarded as a category in which $\mathrm{Hom}(\mathfrak{U},\mathfrak{V})$ has one member if $\mathfrak{U}\subset\mathfrak{V}$ and is empty otherwise. The colocalizing pairs form a category too, a morphism $(\boldsymbol{\Gamma},\boldsymbol{\iota})\to(\boldsymbol{\Gamma}',\boldsymbol{\iota}')$ being a functorial map $\psi\colon\boldsymbol{\Gamma}\to\boldsymbol{\Gamma}'$ such that $\boldsymbol{\iota}'\psi=\boldsymbol{\iota}$. Proposition 3.5.7 can be amplified slightly to state that the functor taking $\mathfrak{U}$ to $(\mathbf{R}\varGamma'_{\mathfrak{U}},\boldsymbol{\iota}_{\mathfrak{U}})$ is an *equivalence of categories.*

It follows from Propositions 3.5.7 and 3.5.5 that by associating to a colocalizing pair $(\boldsymbol{\Gamma},\boldsymbol{\iota})$ the pair $(\boldsymbol{\Gamma}(R),\boldsymbol{\iota}(R))$ one gets another equivalence of categories, between colocalizing pairs and pairs (A,ι) with $A\in\mathbf{D}(R)$ and $\iota\colon A\to R$ a $\mathbf{D}(R)$-map such that $1\underset{=}{\otimes}\iota$ and $\iota\underset{=}{\otimes}1$ are equal isomorphisms from $A\underset{=}{\otimes}A$ to A. The quasi-inverse association takes (A,ι) to the functor $\boldsymbol{\Gamma}(-):=-\underset{=}{\otimes}A$ together with the functorial map $\boldsymbol{\iota}:=1\underset{=}{\otimes}\iota$.

Proof of Proposition 3.5.7. There is at most one $\mathfrak{U}_{\boldsymbol{\Gamma}}$, since a prime R-ideal p is $\mathfrak{U}$-open iff with I_p the R-injective hull of the fraction field κ_p of R/p, $\mathbf{R}\varGamma'_{\mathfrak{U}} I_p \neq 0$.

Let us first construct $\mathfrak{U}_{\boldsymbol{\Gamma}}$. Since $\boldsymbol{\Gamma}$ is a Δ-functor commuting with direct sums and $\boldsymbol{\iota}$ is Δ-functorial, therefore the complexes E for which $\boldsymbol{\Gamma}E = 0$ are the objects of a localizing subcategory $\mathbf{L}_0 \subset \mathbf{D}(R)$ and the complexes F for which $\boldsymbol{\iota}(F)$ is an isomorphism are the objects of a localizing subcategory $\mathbf{L}_1 \subset \mathbf{D}(R)$.

If $\boldsymbol{\Gamma}\kappa_p \neq 0$ then $\boldsymbol{\iota}(\kappa_p) \neq 0$, since $\boldsymbol{\Gamma}\iota(\kappa_p)\colon \boldsymbol{\Gamma}\boldsymbol{\Gamma}\kappa_p \to \boldsymbol{\Gamma}\kappa_p$ is an isomorphism; and so the natural commutative diagram, with bottom row the identity map of κ_p,

$$
\begin{array}{ccccc}
\boldsymbol{\Gamma}\kappa_p = \boldsymbol{\Gamma}\kappa_p \underset{=}{\otimes} R & \longrightarrow & \boldsymbol{\Gamma}\kappa_p \underset{=}{\otimes} \kappa_p & & \\
{\scriptstyle \boldsymbol{\iota}(\kappa_p)}\big\downarrow & & \big\downarrow{\scriptstyle \boldsymbol{\iota}(\kappa_p)\underset{=}{\otimes} 1} & & \\
\kappa_p = \kappa_p \underset{=}{\otimes} R & \longrightarrow & \kappa_p \underset{=}{\otimes} \kappa_p & \longrightarrow & \kappa_p \otimes \kappa_p \; = \kappa_p\,,
\end{array}
$$

shows that $\boldsymbol{\Gamma}\kappa_p \underset{=}{\otimes} \kappa_p \neq 0$. Idempotence of $\boldsymbol{\iota}$ gives that $\boldsymbol{\Gamma}\kappa_p \in \mathbf{L}_1$, whence, as in the proof of [Nm1, p. 528, (1)] (with $X = \boldsymbol{\Gamma}\kappa_p$), $\kappa_p \in \mathbf{L}_1$. But $\kappa_p \in \mathbf{L}_1$ (resp. $\mathbf{L}_0$) implies the same for I_p ([Nm1, p. 526, Lemma 2.9]). So we have

$$
(*) \qquad [\boldsymbol{\Gamma}\kappa_p \neq 0] \implies [\kappa_p \in \mathbf{L}_1] \implies [I_p \in \mathbf{L}_1] \implies [\boldsymbol{\Gamma} I_p \neq 0] \implies [\boldsymbol{\Gamma}\kappa_p \neq 0].
$$

If $p \subset p'$ are prime ideals and $\boldsymbol{\Gamma} I_p \neq 0$ (so that $I_p \in \mathbf{L}_1$), the natural surjection $R/p \twoheadrightarrow R/p'$ extends to a non-zero map $\nu\colon I_p \to I_{p'}$, and the commutative diagram

$$
\begin{array}{ccc}
\boldsymbol{\Gamma} I_p & \xrightarrow{\ \boldsymbol{\Gamma}\nu\ } & \boldsymbol{\Gamma} I_{p'} \\
{\scriptstyle \iota(I_p)}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \iota(I_{p'})} \\
I_p & \xrightarrow[\ \nu\ ]{} & I_{p'}
\end{array}
$$

shows that $\boldsymbol{\Gamma} I_{p'} \neq 0$. Thus those p satisfying the equivalent conditions in $(*)$ are the open prime ideals for a topology $\mathfrak{U} = \mathfrak{U}_{\boldsymbol{\Gamma}}$ on R.

Now, keeping in mind that every injective R-module is a direct sum of I_p's, one sees that for any injective complex E, the I_p's appearing as direct summands (in any degree) of the injective complex $\varGamma'_{\mathfrak{U}}E$ correspond to open p's—so that by [Nm1, p. 527, Lemma 2.10], $\varGamma'_{\mathfrak{U}}E \in \mathbf{L}_1$; and that the I_p's appearing as direct summands of $E/\varGamma'_{\mathfrak{U}}E$ correspond to non-open p's, i.e., p's such that $\kappa_p \in \mathbf{L}_0$—so that by *loc. cit.* again, $E/\varGamma'_{\mathfrak{U}}E \in \mathbf{L}_0$. From this follows that the maps $\boldsymbol{\iota}(\varGamma'_{\mathfrak{U}}E)\colon \boldsymbol{\Gamma}\varGamma'_{\mathfrak{U}}E \to \varGamma'_{\mathfrak{U}}E$ and $\boldsymbol{\Gamma}\boldsymbol{\iota}_{\mathfrak{U}}(E)\colon \boldsymbol{\Gamma}\varGamma'_{\mathfrak{U}}E \to \boldsymbol{\Gamma}E$ are both isomorphisms.

Thus $\boldsymbol{\iota}(E)$ factors in $\mathbf{D}(R)$ as $\boldsymbol{\Gamma}E \underset{\boldsymbol{i}(E)}{\xrightarrow{\sim}} \varGamma'_{\mathfrak{U}}E \underset{\boldsymbol{i}_{\mathfrak{U}}(E)}{\hookrightarrow} E$, with $\boldsymbol{i}(E)$ functorial to the extent that if $\nu\colon E \to F$ is a homomorphism of injective q-injective complexes then the following $\mathbf{D}(R)$-diagram commutes:

$$
\begin{array}{ccccc}
\boldsymbol{\Gamma}E & \xrightarrow[\boldsymbol{i}(E)]{\sim} & \varGamma'_{\mathfrak{U}}E \cong \mathbf{R}\varGamma'_{\mathfrak{U}}E & \longrightarrow & E \\
{\scriptstyle \boldsymbol{\Gamma}\nu}\big\downarrow & & {\scriptstyle \varGamma'_{\mathfrak{U}}\nu}\big\downarrow & & \big\downarrow{\scriptstyle \nu} \\
\boldsymbol{\Gamma}F & \xrightarrow[\boldsymbol{i}(F)]{\sim} & \varGamma'_{\mathfrak{U}}F \cong \mathbf{R}\varGamma'_{\mathfrak{U}}F & \longrightarrow & F
\end{array}
$$

(For the right square and for the outer border, commutativity is clear; and then Proposition 3.5.4 (ii) gives it for the left square.) One finds then that the q-injective resolutions $q_C\colon C \to E_C$ of §1.2 give rise to the desired Δ-functorial isomorphism

$$
\boldsymbol{i}_{\boldsymbol{\Gamma}}(C)\colon \boldsymbol{\Gamma}C \xrightarrow[\boldsymbol{\Gamma}q_C]{\sim} \boldsymbol{\Gamma}E_C \xrightarrow[\boldsymbol{i}(E_C)]{\sim} \varGamma'_{\mathfrak{U}}E_C = \mathbf{R}\varGamma'_{\mathfrak{U}}C \qquad \big(C \in \mathbf{D}(R)\big). \qquad \square
$$

4. Greenlees-May duality; applications

This section revolves about a far-reaching generalization of local duality, first formulated in the 1970s by Strebel [Str, pp. 94–95, 5.9] and Matlis [Mt2, p. 89, Thm. 20] for ideals generated by regular sequences, then proved for arbitrary ideals in noetherian rings—and somewhat more generally than that—by Greenlees and May in 1992 [GM1]. While we approach this topic from the point of view of commutative algebra and its geometric globalizations, it should be noted that Greenlees and May came to it motivated primarily by topological applications, see [GM2].

The main result globalizes (nontrivially) to formal schemes [DGM], where it is important for the duality theory for complexes with coherent homology. Brief mention of such applications is made in Sections 5.4 and 5.6 below.

Here we confine ourselves to the case of a noetherian commutative ring R and an ideal $I \subset R$, to which as before we associate Γ_I, the I-power-torsion subfunctor of the identity functor on R-modules M, such that

$$\Gamma_I M = \varinjlim_{s>0} \mathrm{Hom}_R(R/I^s, M).$$

Dually, the *I-completion functor* is such that

$$\Lambda_I M = \varprojlim_{s>0} \big(M \otimes_R (R/I^s)\big).$$

These functors extend to Δ-functors from $\mathbf{K}(R)$ to itself. With $\mathbf{1}$ the identity functor, there are natural Δ-functorial maps $\Gamma_I \to \mathbf{1} \to \Lambda_I$.

The basic result is that Λ_I has a left-derived functor (§2.1) which is naturally right-adjoint to the local cohomology functor $\mathbf{R}\Gamma_I$. In brief: *left-derived completion is canonically right-adjoint to right-derived power-torsion.*

We know from Prop. 2.3.1 (with $S = R$ and φ the identity map) that $\mathbf{R}\Gamma_I$ has the right adjoint $\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_I R, -)$, which Greenlees and May call the "local homology" functor. So *local homology = left-derived completion.*

Throughout §4, $\mathrm{Hom}^\bullet$ (resp. $\otimes$) with no subscript means $\mathrm{Hom}^\bullet_R$ (resp. $\otimes_R$).

Theorem 4.1. *With $Q\colon \mathbf{K}(R) \to \mathbf{D}(R)$ as usual, there exists a unique Δ-functorial map*

$$\zeta(F)\colon \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R, QF) \to Q\Lambda_I F \qquad (F \in \mathbf{K}(R))$$

such that

(i) *the pair $(\mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R, -), \zeta)$ is a left-derived functor of Λ_I, and*

(ii) *for any R-complex F the $\mathbf{D}(R)$-composition*

$$F = \mathrm{Hom}^\bullet(R, F) \xrightarrow[\text{via } \mathbf{R}\Gamma_I \to \mathbf{1}]{\rho(F)} \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R, F) \xrightarrow{\zeta(F)} \Lambda_I F$$

is the canonical completion map $F \to \Lambda_I F$.

Moreover, $\zeta(F)$ is an isomorphism whenever F is a q-flat complex.

For a complete *proof*—which plays no role elsewhere in these lectures—see [AJL]. (The generalization to formal schemes is in [DGM].) The mildly curious reader can find a few brief indications at the end of this subsection.

Duality statements in which inverse limits play some role are often consequences of the following Corollary of Thm. 4.1. Two such consequences, Local Duality and Affine Duality, are discussed in succeeding subsections. (For more, see [AJL, §5].)

We write $\mathbf{D}$ for $\mathbf{D}(R)$ and let $\mathbf{D}_{\mathrm{c}} \subset \mathbf{D}$ be the full subcategory whose objects are those R-complexes all of whose homology modules are finitely generated. (Here "c" signifies "coherent.") The I-adic completion $\hat{R}$ of R being R-flat, we can identify the derived tensor product $F \underline{\underline{\otimes}} \hat{R}$ (§2.1) with the ordinary tensor product $F \otimes \hat{R}$.

Corollary 4.1.1. (i) *There exists a unique functorial map*

$$\theta(F)\colon F \otimes \hat{R} \to \mathbf{R}\mathrm{Hom}^{\bullet}(\mathbf{R}\Gamma_I R, F) \underset{(3.2.2)}{\cong} \mathbf{R}\mathrm{Hom}^{\bullet}(\mathbf{R}\Gamma_I R, \mathbf{R}\Gamma_I F) \qquad (F \in \mathbf{D})$$

whose composition with the natural map $\kappa(F)\colon \mathbf{R}\mathrm{Hom}^{\bullet}(R, F) = F \to F \otimes \hat{R}$ *is the map* $\rho(F)$ *induced by the natural map* $\mathbf{R}\Gamma_I R \to R$.

(ii) *If* $F \in \mathbf{D}_{\mathrm{c}}$ *then* $\theta(F)$ *is an* isomorphism.

Proof. (i) Extension of scalars gives a functorial $\hat{R}$-map $\hat{\kappa}(F)\colon F \otimes \hat{R} \to \Lambda_I F$ such that $\hat{\kappa}(F)\kappa(F)$ is the completion map $\lambda_F\colon F \to \Lambda_I F$. Since $\hat{R}$ is R-flat, the functor $\_\_ \otimes \hat{R}$ takes quasi-isomorphisms to quasi-isomorpisms, so by Theorem 4.1(i) and the definition of left-derived functors there exists a unique functorial map $\theta(F)\colon F \otimes \hat{R} \to \mathbf{R}\mathrm{Hom}^{\bullet}(\mathbf{R}\Gamma_I R, F)$ such that in $\mathbf{D}$, $\hat{\kappa}(F) = \zeta(F)\theta(F)$. Then

$$\zeta(F)\theta(F)\kappa(F) = \hat{\kappa}(F)\kappa(F) = \lambda_F \underset{4.1(\mathrm{ii})}{=} \zeta(F)\rho(F),$$

and therefore—by the definition of left-derived functors—$\theta(F)\kappa(F) = \rho(F)$.

For uniqueness, note that $\kappa(F)$ induces an *isomorphism*

$$\mathbf{R}\Gamma_I R \underline{\underline{\otimes}} F \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_I R \underline{\underline{\otimes}} (F \otimes \hat{R}).$$

(apply the isomorphism $\psi(R, -)$ of Cor. 3.3.1, and then use Cor. 3.4.5 or just combine the remarks preceding it with Prop. 3.1.2), whence the top row of the following commutative diagram must be an isomorphism:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{D}}(F \otimes \hat{R}, \mathbf{R}\mathrm{Hom}^{\bullet}(\mathbf{R}\Gamma_I R, F)) & \xrightarrow{\text{via } \kappa} & \mathrm{Hom}_{\mathbf{D}}(F, \mathbf{R}\mathrm{Hom}^{\bullet}(\mathbf{R}\Gamma_I R, F)) \\
{\scriptstyle (2.2.3)}\big\downarrow \simeq & & \simeq \big\downarrow {\scriptstyle (2.2.3)} \\
\mathrm{Hom}_{\mathbf{D}}((F \otimes \hat{R}) \underline{\underline{\otimes}} \mathbf{R}\Gamma_I R, F) & \underset{\text{via } \kappa}{\xrightarrow{\ \sim\ }} & \mathrm{Hom}_{\mathbf{D}}(F \underline{\underline{\otimes}} \mathbf{R}\Gamma_I R, F)
\end{array}$$

(ii) To show that $\theta(F)$ is an isomorphism whenever $F \in \mathbf{D}_{\mathrm{c}}$, use the fact (nontrivial, cf. [AJL, Lemma (4.3)]) that the functor $\mathbf{R}\mathrm{Hom}^{\bullet}(\mathbf{R}\Gamma_I R, -)$ is bounded to get a reduction to the case where F is a single finite-rank free R-module [Ha1, p. 68, Prop. 7.1]. In this case $\hat{\kappa}(F) = \zeta(F)\theta(F)$ is an isomorphism, whence, by the last statement in Theorem 4.1, so is $\theta(F)$. $\square$

Here is an outline of the proof of Theorem 4.1. For details, see [AJL, §4].

Uniqueness of ζ. Set $\mathbf{\Lambda}_I F := \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R, F)$. If $\zeta'\colon \mathbf{\Lambda}_I Q \to Q\Lambda_I$ is such that $(\mathbf{\Lambda}_I, \zeta')$ is a left-derived functor of Λ_I then by definition (§2.1) there is a functorial map $\vartheta\colon \mathbf{\Lambda}_I \to \mathbf{\Lambda}_I$ inducing $\vartheta_Q\colon \mathbf{\Lambda}_I Q \to \mathbf{\Lambda}_I Q$ such that $\zeta'\vartheta_Q = \zeta$; and if ζ' also satisfies (ii), so that $\zeta'\rho = \zeta\rho = \zeta'\vartheta_Q\rho$, then $\rho = \vartheta_Q\rho$. But $\rho(F)\colon F \to \mathbf{\Lambda}_I F$ induces a bijection from $\mathrm{Hom}_{\mathbf{D}}(\mathbf{\Lambda}_I F, \mathbf{\Lambda}_I F)$ to $\mathrm{Hom}_{\mathbf{D}}(F, \mathbf{\Lambda}_I F)$. (This, and other relations involving $\mathbf{R}\Gamma_I$ and $\mathbf{\Lambda}_I$, all following formally from adjointness and from "idempotence" of $\mathbf{R}\Gamma_I$, are given in [DFS, §6.3].) Thus $\vartheta_Q = \text{identity}$ and $\zeta' = \zeta$.

As for the *existence* of ζ, one first establishes that Λ_I has a left-derived functor $\mathbf{L}\Lambda_I$ such that for any R-complex C, with q-flat resolution $F_C \to C$ as in §2.1,

$$\mathbf{L}\Lambda_I(C) = \Lambda_I(F_C).$$

This is given by [Ha1, p. 53, Thm. 5.1], for if F is q-flat and exact then so is $\Lambda_I(F)$, the $\varprojlim$ of the surjective system of exact complexes $F \otimes (R/I^s)$, see [EG3, p. 66, (13.2.3)]. (If $F_s \to R/I^s$ is a q-flat resolution then $F \otimes F_s$ is quasi-isomorphic to $F \otimes R/I^s$ and exact.)

Now we may assume that F is q-flat. With $R \to G$ an injective resolution (so that in $\mathbf{D}$, $F \otimes G \cong F$) and $s > 0$, the natural map

$$(F \otimes R/I^s) \otimes \mathrm{Hom}^\bullet(R/I^s,\, G) \cong F \otimes \big(R/I^s \otimes \mathrm{Hom}^\bullet(R/I^s,\, G)\big) \to F \otimes G$$

corresponds under Hom–$\otimes$ adjunction to a functorial map

$$F \otimes R/I^s \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(R/I^s, G),\, F \otimes G).$$

So there is a natural composition, call it :

$$\begin{aligned}
\mathbf{L}\Lambda_I F \xrightarrow{\ \sim\ } \Lambda_I F &= \varprojlim_{s>0}\,(F \otimes R/I^s)\\
&\to \varprojlim_{s>0}\,\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(R/I^s, G),\, F \otimes G)\\
&\cong \mathrm{Hom}^\bullet(\varinjlim_{s>0}\,\mathrm{Hom}^\bullet(R/I^s, G),\, F \otimes G)\\
&\cong \mathrm{Hom}^\bullet(\Gamma_I G, F \otimes G)\\
&\xrightarrow[\zeta]{} \mathbf{R}\mathrm{Hom}^\bullet(\Gamma_I G, F \otimes G) \cong \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I G, F).
\end{aligned}$$

The essential problem is to show that $\Phi(F)$ is an isomorphism.

The next step is to apply "way-out" reasoning (a kind of induction, [Ha1, p. 69, (iii)]) to reduce the problem to where F is a single flat R-module. A nontrivial prerequisite is *boundedness* (cf. 3.1.4) of the functors $\mathbf{L}\Lambda_I$ and $\mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I G, -)$.

Then $F \to F \otimes G$ is an injective resolution (so that ζ is an isomorphism). With $\mathbf{t} = (t_1, \dots, t_m)$ such that $I = \mathbf{t}R$, one uses that $\mathcal{K}(\mathbf{t}) = \varinjlim_{s>0}$ of the ordinary Koszul complexes $K(\mathbf{t}^s) = K(t_1^s, \dots, t_m^s)$ (defined by replacing $R \to R_t$ in §3.1 with $R \xrightarrow{t^s} R$, the maps $K(\mathbf{t}^u) \to K(\mathbf{t}^v)$ $(v \ge u)$ being derived from the maps of complexes $K(t^u) \to K(t^v)$ which are identity in degree 0 and multiplication by t^{v-u} in degree 1) to turn the basic problem into showing for all i that the natural map is an *isomorphism*

$$\mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I G, F) \underset{3.1.2}{\cong} \mathrm{H}^i \varprojlim_{s>0} \mathrm{Hom}^\bullet(K(\mathbf{t}^s), F \otimes G) \xrightarrow{\ \sim\ } \varprojlim_{s>0} \mathrm{H}^i\,\mathrm{Hom}^\bullet(K(\mathbf{t}^s), F \otimes G).$$

(This is used to show that a certain map $\Psi(\mathbf{t}, F)\colon \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I G, F) \to \mathbf{L}\Lambda_I F$ depending a priori on $\mathbf{t}$ is an isomorphism. One must also show that $\Phi = \Psi(\mathbf{t}, F)^{-1}$.)

Treating such questions about the interchange of homology and inverse limits requires some nontrivial "Mittag-Leffler conditions," see [EG3, p. 66, (13.2.3)].

4.2. **Application: local duality, again.** In §2.4, Greenlees-May duality was used to relate a form of classical local duality (2.4.2) to "Trivial" local duality (2.3.2). More directly (and more generally), for $E \in \mathbf{D}(R)$ and $F \in \mathbf{D}_{\mathrm{c}}(R)$, and with $\hat{R}$ the I-adic completion of R, apply the functor $\mathbf{R}\mathrm{Hom}^\bullet(E, -)$ to the isomorphism in Corollary 4.1.1, and then use the isomorphisms (2.2.2) (with $R = S$, $\varphi =$ identity) and (3.3.1) to get a natural isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet(E, F \otimes \hat{R}) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I E, F) \underset{(3.2.2)}{\cong} \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I E, \mathbf{R}\Gamma_I F).$$

4.3. **Application: affine duality.** For any R-complexes F and G there is a natural $\mathbf{D}(R)$-map

$$\sigma(F,G)\colon F \to \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mathrm{Hom}^\bullet(F,G), G)$$

corresponding via (2.2.3) to the natural composition

$$F \underset{=}{\otimes} \mathbf{R}\mathrm{Hom}^\bullet(F,G) \xrightarrow{\ \tau\ } \mathbf{R}\mathrm{Hom}^\bullet(F,G) \underset{=}{\otimes} F \xrightarrow{\ \eta\ } G$$

where η corresponds via (2.2.3) to the identity map of $\mathbf{R}\mathrm{Hom}^\bullet(F,G)$, and τ is the map (clearly an isomorphism) determined by the following property: replacing F by a q-flat resolution and G by a q-injective resolution, one can change $\underset{=}{\otimes}$ to $\otimes$ and drop the $\mathbf{R}$'s, and then for $x \in F^i$ and $\phi \in \mathrm{Hom}^j(F,G)$, $\tau(x \otimes \phi) = (-1)^{ij}(\phi \otimes x)$. (Proving the existence of such a τ—by means, e.g., of the general technique for constructing functorial maps in derived categories given in [Lp3, Prop. (2.6.4)]—is left as an exercise.)

With $\phi = (\phi_n\colon F^n \to G^{n+j})_{n\in\mathbb{Z}}$, we have then

$$[\sigma(F,G)(x)](\phi) = (-1)^{ij}\phi_i(x) \in G^{i+j}.$$

Let D be a bounded injective R-complex such that for any $F \in \mathbf{D}_{\mathrm{c}}(R)$, $\sigma(F,D)$ is an isomorphism. For example, D could be a *dualizing complex* ([Ha1, pp. 257–258]), which exists if[16] R is a homomorphic image of a finite-dimensional Gorenstein ring [Ha1, p. 299]. Define the *I-dualizing functor* $\mathcal{D}_I$ by

$$\mathcal{D}_I(F) := \mathbf{R}\mathrm{Hom}^\bullet(F, \mathbf{R}\Gamma_I D) \qquad (F \in \mathbf{D}(R)).$$

The following result "double-dual=completion" is called Affine Duality. ([Ha2, p. 152, Thm. 4.2]; see also [DFS, p. 28, Prop. 2.5.8] for a formal-scheme-theoretic version).

Theorem 4.3.1. *Let $\hat{R}$ be the I-adic completion of R. Then there is a functorial isomorphism*

$$F \otimes \hat{R} \xrightarrow{\ \sim\ } \mathcal{D}_I\mathcal{D}_I F \qquad (F \in \mathbf{D}_{\mathrm{c}}(R))$$

whose composition with the natural map $F \to F \otimes_R \hat{R}$ is $\sigma(F, \mathbf{R}\Gamma_I D)$.

[16] and only if—[Kwk, Cor. 1.4].

Example. When R is local with maximal ideal I and D is a normalized dualizing complex of R then $\mathbf{R}\Gamma_I D$ is an R-injective hull of the residue field R/I (see §2.4), and Theorem 4.3.1 is a well-known component of Matlis Duality [BS, p. 194, Thm. 10.2.19(ii)].

Proof of Theorem 4.3.1*.* One checks (see below) that $\sigma(F, \mathbf{R}\Gamma_I D)$ is the following natural composition:

$$\begin{aligned} F \to F \otimes_R \hat{R} &\underset{(4.1.1)}{\xrightarrow{\sim}} \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R, F) \\ &\underset{\text{via } \sigma}{\xrightarrow{\sim}} \mathbf{R}\mathrm{Hom}^\bullet\big(\mathbf{R}\Gamma_I R, \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mathrm{Hom}^\bullet(F, D), D)\big) \\ &\underset{(2.2.2)}{\xrightarrow{\sim}} \mathbf{R}\mathrm{Hom}^\bullet\big(\mathbf{R}\Gamma_I R \underline{\otimes} \mathbf{R}\mathrm{Hom}^\bullet(F, D), D)\big) \\ &\underset{(3.3.1)}{\xrightarrow{\sim}} \mathbf{R}\mathrm{Hom}^\bullet\big(\mathbf{R}\Gamma_I \mathbf{R}\mathrm{Hom}^\bullet(F, D), D)\big) \\ &\underset{(3.2.2)}{\xrightarrow{\sim}} \mathbf{R}\mathrm{Hom}^\bullet\big(\mathbf{R}\Gamma_I \mathbf{R}\mathrm{Hom}^\bullet(F, D), \mathbf{R}\Gamma_I D)\big) \\ &\underset{\text{via } \nu}{\xrightarrow{\sim}} \mathbf{R}\mathrm{Hom}^\bullet\big(\mathbf{R}\mathrm{Hom}^\bullet(F, \mathbf{R}\Gamma_I D), \mathbf{R}\Gamma_I D)\big) = \mathcal{D}_I\mathcal{D}_I F \end{aligned}$$

where ν is the isomorphism given by:

Lemma 4.3.2. *There is a unique map*

$$\nu\colon \mathbf{R}\mathrm{Hom}^\bullet(F, \mathbf{R}\Gamma_I D) \to \mathbf{R}\Gamma_I \mathbf{R}\mathrm{Hom}^\bullet(F, D)$$

whose composition with the natural map $\mathbf{R}\Gamma_I \mathbf{R}\mathrm{Hom}^\bullet(F, D) \to \mathbf{R}\mathrm{Hom}^\bullet(F, D)$ *is the map induced by the natural map* $\mathbf{R}\Gamma_I D \to D$*; and this* ν *is an isomorphism.*

Proof. By Prop. 3.1.2, $\mathbf{R}\Gamma_I D$ is $\mathbf{D}(R)$-isomorphic to a complex $\mathcal{K}(\mathbf{t}) \otimes D$, which is bounded and injective; and hence

$$\mathbf{R}\mathrm{Hom}^\bullet(F, \mathbf{R}\Gamma_I D) \cong \mathrm{Hom}^\bullet(F, \mathcal{K}(\mathbf{t}) \otimes D) \in \mathbf{D}_I(R), \tag{4.3.3}$$

as one sees by "way-out" reduction to the simple case where F is a finite-rank free R-module [Ha1, pp. 73–74, Prop. 7.3]. Then Prop. 3.2.2 ensures the existence of ν.

For ν to be an isomorphism it suffices that for an arbitrary $A \in \mathbf{D}_I(R)$, the image of ν under application of the functor $\mathrm{Hom}_{\mathbf{D}(R)}(A, -)$ be an isomorphism. By (2.2.3) and Prop. 3.2.2, this amounts to the natural map

$$\mathrm{Hom}_{\mathbf{D}(R)}(A \underline{\otimes} F, \mathbf{R}\Gamma_I D) \to \mathrm{Hom}_{\mathbf{D}(R)}(A \underline{\otimes} F, D)$$

being an isomorphism, so, by Prop. 3.2.2, it suffices that $A \underline{\otimes} F \in \mathbf{D}_I(R)$, i.e., (Cor. 3.2.1) that the natural map $\mathbf{R}\Gamma_I(A \underline{\otimes} F) \to A \underline{\otimes} F$ be an isomorphism, which it is, by Cor. 3.3.1, since $\mathbf{R}\Gamma_I A \cong A$ (Cor. 3.2.1, again). □

The patient reader may apprehend more of the functorial flavor of our overall approach by perusing the following details of the check mentioned at the outset of the proof of Theorem 4.3.1.

Consider the following natural diagram, in which $\mathcal{D}$ is the dualizing functor $\mathbf{R}\mathrm{Hom}^\bullet(-, D)$ and the functorial map $\mathcal{D}_I \to \mathcal{D}$ is induced by the canonical map $\mathbf{R}\Gamma_I \to \mathbf{1}$, as are the horizontal arrows preceding the right column, which along with the top row is as in the sequence of maps near the beginning of the proof of Theorem 4.3.1.

$$
\begin{array}{ccccc}
F & \xrightarrow{\ \sim\ } & \mathbf{R}\mathrm{Hom}^\bullet(R, F) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R, F) \\
\big\downarrow{\scriptstyle\sigma} & & \big\downarrow & & \big\downarrow \\
\mathcal{D}\mathcal{D}F & \xrightarrow{\ \sim\ } & \mathbf{R}\mathrm{Hom}^\bullet(R, \mathcal{D}\mathcal{D}F) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R, \mathcal{D}\mathcal{D}F) \\
\Big\| & & \big\downarrow & & \big\downarrow \\
\Big\| & \text{(A)} & \mathbf{R}\mathrm{Hom}^\bullet(R \underline{\underline{\otimes}} \mathcal{D}F, D) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I R \underline{\underline{\otimes}} \mathcal{D}F, D) \\
\Big\| & & \big\downarrow & \text{(B)} & \big\downarrow \\
\mathcal{D}\mathcal{D}F & = & \mathbf{R}\mathrm{Hom}^\bullet(\mathcal{D}F, D) & \longrightarrow & \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\Gamma_I \mathcal{D}F, D) \\
\big\downarrow & & \big\downarrow & \text{(C)} & \big\downarrow \\
\mathcal{D}\mathcal{D}_I F & = & \mathbf{R}\mathrm{Hom}^\bullet(\mathcal{D}_I F, D) & \xleftarrow[\varrho]{} & \mathbf{R}\mathrm{Hom}^\bullet(\mathcal{D}_I F, \mathbf{R}\Gamma_I D)
\end{array}
$$

The unlabeled squares obviously commute. To verify commutativity of subdiagram (A) one checks (exercise) that the isomorphism (2.2.2) for $E = S = R$ and $\varphi =$ identity is naturally isomorphic to the identity map of $\mathbf{R}\mathrm{Hom}^\bullet_R(F, G)$. Commutativity of (B) follows from Corollary 3.3.3. Commutativity of (C) follows from Lemma 4.3.2, as one sees by drawing the arrow induced by ν from the upper right to the lower left corner. Thus the whole diagram commutes.

Since $\mathcal{D}_I F \in \mathbf{D}_I(R)$ (see (4.3.3), Proposition 3.2.2 gives that the map ϱ in the diagram is an *isomorphism.* It remains only to show that the left column followed by ϱ^{-1} is $\sigma(F,\, \mathbf{R}\Gamma_I D)$, and this is straightforward.

5. Residues and Duality

This section begins with a concrete interpretation of the duality functor $\varphi_J^\#$ of Theorem 2.3.2, for φ the inclusion of a noetherian commutative ring R into a power-series ring $S := R[[\mathbf{t}]] := R[[t_1, \dots, t_m]]$ and J the ideal $\mathbf{t}S = (t_1, \dots, t_m)S$. The resulting concrete versions of Local Duality lead to an introductory discussion of the *residue map,* its expression through the *fundamental class* of a map of formal schemes, and hence to canonical versions of, and relations between, local and global duality—at least for smooth residually separable maps.

Henceforth we omit "φ_*" from the notation for derived functors when the context makes the meaning clear. For example, for $G \in \mathbf{D}(R)$ we write $\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_J S, G)$ in place of $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_*\mathbf{R}\Gamma_J S, G)$, and $G \underline{\underline{\otimes}}_R\, \omega_{\mathbf{t}}[m]$ in place of $G \underline{\underline{\otimes}}_R\, \varphi_*\omega_{\mathbf{t}}[m]$.

5.1. **The duality functor for power series rings.** $\varphi\colon R \hookrightarrow S = R[[\mathbf{t}]]$ and $J = \mathbf{t}S$ are as above. We first give some concrete representations of the duality functor $\varphi_J^{\#}\colon \mathbf{D}(R) \to \mathbf{D}(S)$ (see Theorem 2.3.2).

Using the definition of the stable Koszul S-complex $\mathcal{K}(\mathbf{t})$ (§3.1), one finds that

$$\nu_{\mathbf{t}}^S := \mathrm{H}^m\mathcal{K}(\mathbf{t}) = \mathrm{coker}\big[\mathcal{K}^{m-1}(\mathbf{t}) = \oplus_{i=1}^m S_{t_1t_2\cdots\hat{t}_i\cdots t_m} \to S_{t_1t_2\cdots t_m} = \mathcal{K}^m(\mathbf{t})\big]$$

is a free R-module with basis $\{\, t_1^{-n_1}\cdots t_m^{-n_m} \mid n_1 > 0, \dots, n_m > 0\,\}$, and an S-submodule of $S_{t_1t_2\cdots t_m}/S$. Since the sequence $\mathbf{t}$ is regular, $\mathcal{K}(\mathbf{t})$ is exact except in degree m [EG3, p. 83, (1.1.4)]. Hence by Propositions 3.1.2 and 1.3.3 there are natural $\mathbf{D}(S)$-isomorphisms

$$\mathbf{R}\Gamma_J S \xrightarrow{\ \sim\ } \mathcal{K}(\mathbf{t}) \xrightarrow{\ \sim\ } \nu_{\mathbf{t}}[-m]; \tag{5.1.1}$$

and so there is a functorial $\mathbf{D}(S)$-isomorphism
(5.1.2)

$$\varphi_J^{\#}G = \mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, G) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}[-m], G) \qquad (G \in \mathbf{D}(R)).$$

Since $\nu_{\mathbf{t}}$ is R-free the functor $\mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}[-m], -)$ preserves exactness, and so takes quasi-isomorphisms to quasi-isomorphisms (as quasi-isomorphisms in $\mathbf{K}(R)$ are just those maps whose cones are exact), so that it may be regarded as a functor from $\mathbf{D}(R)$ to $\mathbf{D}(S)$. Replacing G in (5.1.2) by a quasi-isomorphic q-injective complex, we see then that the canonical map is a functorial $\mathbf{D}(S)$-isomorphism

$$\mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}[-m], G) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}[-m], G). \tag{5.1.3}$$

Thus we have a *functorial* $\mathbf{D}(S)$*-isomorphism*

$$\varphi_J^{\#}G \xrightarrow{\ \sim\ } \mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}[-m], G) \qquad (G \in \mathbf{D}(R)). \tag{5.1.4}$$

Here is another interpretation of $\varphi_J^{\#}G$, for $G \in \mathbf{D}_{\mathrm{c}}(R)$ (i.e., the homology modules of G are all finitely-generated). Set

$$\omega_{\mathbf{t}} = \omega_{\mathbf{t}}^{\varphi} := \mathrm{Hom}_R(\nu_{\mathbf{t}}^S, R), \tag{5.1.5}$$

a "relative canonical module." This $\omega_{\mathbf{t}}$ is a free rank-one S-module generated by the R-homomorphism $\gamma_{\mathbf{t}}\colon \nu_{\mathbf{t}} \to R$ such that

$$\gamma_{\mathbf{t}}(t_1^{-n_1}\cdots t_m^{-n_m}) = \begin{cases} 1 & \text{if } n_1 = \cdots = n_m = 1,\\ 0 & \text{otherwise.}\end{cases} \tag{5.1.6}$$

That's because the map $\big(\sum_{n_i>0} r_{n_1\dots n_m} t_1^{n_1-1}\cdots t_m^{n_m-1}\big)\gamma_{\mathbf{t}}$ takes $t_1^{-n_1}\cdots t_m^{-n_m}$ to $r_{n_1\dots n_m}$.

For any R-complex G there is a unique map of S-complexes

$$\chi_m(G)\colon G \otimes_R \omega_{\mathbf{t}}[m] \to \mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}[-m], G),$$

whose degree-n component χ_m^n satisfies

$$\chi_m^n(g \otimes w)(v) = w(v)g \qquad (g \in G^{n+m},\ w \in \omega_{\mathbf{t}},\ v \in \nu_{\mathbf{t}}).$$

Since $\omega_{\mathbf{t}}$ is S-flat, the functor $\dots \otimes_R \omega_{\mathbf{t}}$ takes quasi-isomorphisms to quasi-isomorphisms, so may be viewed as a functor from $\mathbf{D}(R)$ to $\mathbf{D}(S)$, and then $\chi_m(G)$ is a functorial $\mathbf{D}(S)$-map. "Way-out" reduction to the trivial case where G is a finite-rank free R-module ([Ha1, p. 68, 7.1(dualized)], with $A' \subset A := \mathcal{M}(R)$ the category of finitely-generated R-modules), shows that *for* $G \in \mathbf{D}_{\mathrm{c}}(R)$, $\chi_m(G)$ *is a* $\mathbf{D}(S)$*-isomorphism.*

In conclusion, for $G \in \mathbf{D}_{\mathrm{c}}(R)$ we can represent $\varphi_J^{\#}G$ concretely via the functorial $\mathbf{D}(S)$-isomorphisms

$$\varphi_J^{\#}G \xrightarrow[(5.1.4)]{\sim} \mathrm{Hom}_R^{\bullet}(\nu_{\mathbf{t}}[-m], G) \xrightarrow[\chi_m(G)^{-1}]{\sim} G \otimes_R \omega_{\mathbf{t}}[m]. \tag{5.1.7}$$

5.2. **Functors represented via relative canonical modules.** We continue with a nontrivial instantiation of Trivial Local Duality (2.3.2).

Set, as above, $\omega_{\mathbf{t}} := \mathrm{Hom}_R(\nu_{\mathbf{t}}, R)$, so that there is an "evaluation" map

$$\mathrm{ev}\colon \omega_{\mathbf{t}} \otimes_S \nu_{\mathbf{t}} \to R.$$

Moreover, $\nu_{\mathbf{t}}$ being R-free, if F is a finitely-generated R-module then the natural map is an isomorphism (see also above)

$$\chi_0(F)\colon F \otimes_R \omega_{\mathbf{t}} = F \otimes_R \mathrm{Hom}_R(\nu_{\mathbf{t}}, R) \xrightarrow{\sim} \mathrm{Hom}_R(\nu_{\mathbf{t}}, F). \tag{5.2.1}$$

The local cohomology functor H_J^m on the category $\mathcal{M}(S)$ of S-modules can be realized through the functorial S-isomorphism

$$\varepsilon_{\mathbf{t}}(E)\colon \mathrm{H}_J^m E \xrightarrow{\sim} E \otimes_S \nu_{\mathbf{t}} \qquad (E \in \mathcal{M}(S)), \tag{5.2.2}$$

defined to be the composition

$$\mathrm{H}_J^m E = \mathrm{H}^m \mathbf{R}\Gamma_J E \xrightarrow[(3.3.1)]{\sim} \mathrm{H}^m(E \underline{\underline{\otimes}}_S \mathbf{R}\Gamma_J S) \xrightarrow[(5.1.1)]{\sim} \mathrm{H}^m(E \underline{\underline{\otimes}}_S \nu_{\mathbf{t}}[-m]) = E \otimes_S \nu_{\mathbf{t}}.$$

Via (5.2.1) and (5.2.2), the natural isomorphism

$$\mathrm{Hom}_R(E \otimes_R \nu_{\mathbf{t}}, F) \xrightarrow{\sim} \mathrm{Hom}_S\big(E, \mathrm{Hom}_R(\nu_{\mathbf{t}}, F)\big)$$

(see (2.2.1)) gets transformed into the following down-to-earth duality, whose substance comes then from Proposition 3.1.2 and the structure of $\mathrm{H}^m\mathcal{K}(\mathbf{t})$. (Details are left to the reader.) Insofar as this duality involves a choice of power-series variables $\mathbf{t}$ it lacks canonicity, a deficiency to be remedied in Theorem 5.3.3.

Proposition 5.2.3. *For any finitely-generated R-module F there is a functorial isomorphism*

$$\mathrm{Hom}_R(\mathrm{H}_J^m E, F) \xrightarrow{\sim} \mathrm{Hom}_S(E, F \otimes_R \omega_{\mathbf{t}}) \qquad \big(E \in \mathcal{M}(S)\big)$$

which for $E = F \otimes_R \omega_{\mathbf{t}}$ takes the composite map

$$\eta_{\mathbf{t}}(F)\colon \mathrm{H}_J^m(F \otimes_R \omega_{\mathbf{t}}) \xrightarrow[\varepsilon_{\mathbf{t}}(F \otimes_R \omega_{\mathbf{t}})]{\sim} F \otimes_R \omega_{\mathbf{t}} \otimes_S \nu_{\mathbf{t}} \xrightarrow[\mathbf{1}\otimes \mathrm{ev}]{} F$$

to the identity map of $F \otimes_R \omega_{\mathbf{t}}$. In other words, the functor $\mathrm{Hom}_R(\mathrm{H}_J^m E, F)$ *of S-modules E is represented by the pair $(F \otimes_R \omega_{\mathbf{t}}, \eta_{\mathbf{t}}(F))$.*

Complement. By means of 3.4.5 and 3.4.3, Proposition 5.2.3 extends as follows (exercise). Let T be an R-algebra, $\mathbf{u} := (u_1, \dots, u_m)$ a sequence in T, $I := \mathbf{u}T$, $\hat{T}$ the I-adic completion of T, and $\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_m)$ the image of $\mathbf{u}$ in $\hat{T}$. $\hat{T}$ is an $S(= R[[\mathbf{t}]])$-algebra via the continuous R-homomorphism taking t_i to $\hat{u}_i$ for all i. As above, set $J := \mathbf{t}S$, so that for any $\hat{T}$-module E considered as a T-module and S-module, respectively, $\mathrm{H}^m_I E = \mathrm{H}^m_J E$.

Let $\mathrm{ev}'\colon \mathrm{Hom}_S(\hat{T}, F \otimes_R \omega_{\mathbf{t}}) \to F \otimes_R \omega_{\mathbf{t}}$ be the S-homomorphism "evaluation at 1." Then *for any finitely-generated R-module F, the functor* $\mathrm{Hom}_R(\mathrm{H}^m_I E, F)$ *of T-modules E is represented by the pair* $\big(\mathrm{Hom}_S(\hat{T}, F \otimes_R \omega_{\mathbf{t}}), \eta_{\mathbf{t}}(F) \circ \mathrm{H}^m_J(\mathrm{ev}')\big)$.

The next Proposition provides a canonical identification of the duality isomorphism of Proposition 5.2.3 with the one coming out of Theorem 2.3.2, namely

$$\mathrm{Hom}_R(\mathrm{H}^m_J E, F) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}\big(E, \varphi_J^{\#} F[-m]\big).$$

Proposition 5.2.4. *For any S-module E and any R-module F the following sequence of natural isomorphisms composes to the map given by* (2.2.1):

$$\begin{aligned}
\mathrm{Hom}_R(E \otimes_S \nu_{\mathbf{t}}, F) &\xrightarrow[(5.2.2)]{\ \sim\ } \mathrm{Hom}_R(\mathrm{H}^m_J E, F)\\
&\xrightarrow[(1.3.3)]{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(R)}\big(\mathbf{R}\Gamma_J E, F[-m]\big) \quad \text{(see Cor. 3.1.4)}\\
&\xrightarrow[(2.3.2)]{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}\big(E, \varphi_J^{\#} F[-m]\big)\\
&\xrightarrow[(5.1.4)]{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}\big(E, \mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}[-m], F[-m])\big)\\
&\xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}\big(E, \mathrm{Hom}^{\bullet}_R(\nu_{\mathbf{t}}, F)\big)\\
&\xrightarrow[(1.3.3)]{\ \sim\ } \mathrm{Hom}_S\big(E, \mathrm{Hom}_R(\nu_{\mathbf{t}}, F)\big).
\end{aligned}$$

Proof. The proof, left to the reader as an exercise in patience, is a matter of reformulating the assertion as the commutativity of a certain diagram, which can be verified by decomposing the maps involved into their elementary constituents, as given by their definitions, thereby expanding the diagram in question into a patchwork of simple diagrams all of whose commutativities are obvious. □

5.3. **Differentials, residues, canonical local duality.** Let $\hat{\Omega}_{S/R}$ be an S-module equipped with an R-derivation $d\colon S \to \hat{\Omega}_{S/R}$ such that $(dt_1, \dots, dt_m)$ is a free S-basis of $\hat{\Omega}_{S/R}$. Then for any $\mathbf{u} = (u_1, u_2, \dots, u_m)$ such that $S = R[[\mathbf{u}]]$, it holds that $(du_1, \dots, du_m)$ is a free basis of $\hat{\Omega}_{S/R}$. This follows e.g., from the fact that the pair $(\hat{\Omega}_{S/R}, d)$ has a universal property which characterizes it up to canonical isomorphism: for any finitely-generated S-module M and R-derivation $D\colon S \to M$ there is a unique S-linear map $\delta\colon \hat{\Omega}_{S/R} \to M$ such that $D = \delta d$.

Let $\hat{\Omega}^m$ $(m > 0)$ be the m-th exterior power of $\hat{\Omega}_{S/R}$, a free rank-one S-module with basis $dt_1 \wedge dt_2 \cdots \wedge dt_m$. Let $\phi_{\mathbf{t}}\colon \hat{\Omega}^m \xrightarrow{\ \sim\ } \omega_{\mathbf{t}}$ be the isomorphism which takes $dt_1 \wedge dt_2 \cdots \wedge dt_m$ to the generator $\gamma_{\mathbf{t}}$ of $\omega_{\mathbf{t}}$ (see (5.1.6)).

Let $\mathrm{res}_{\mathbf{t}}$ be the composition

$$\mathrm{H}^m_{\mathbf{t}S}\hat{\Omega}^m \xrightarrow{\text{via }\phi_{\mathbf{t}}} \mathrm{H}^m_{\mathbf{t}S}\omega_{\mathbf{t}} \xrightarrow[(5.2.3)]{\eta_{\mathbf{t}}(R)} R. \tag{5.3.1}$$

For any $\mathbf{u}$ as above, $\mathrm{res}_{\mathbf{u}}$ is similarly defined. Moreover, if θ is a *bicontinuous* R-automorphism of S ($\mathbf{t}$-adically topologized) and $\mathbf{u} = \theta\mathbf{t}$, then $\mathrm{H}^m_{\mathbf{t}S} = \mathrm{H}^m_{\mathbf{u}S}$ (see remark following Corollary 3.4.5).

Proposition 5.3.2. *The R-linear map $\mathrm{res}_{\mathbf{t}}\colon \mathrm{H}^m_{\mathbf{t}S}\hat{\Omega}^m \to R$ depends only on the R-algebra $S = R[[\mathbf{t}]]$ and its $\mathbf{t}$-adic topology: if a bicontinuous R-automorphism of S takes $\mathbf{t}$ to $\mathbf{u}$ (so that $S = R[[\mathbf{u}]]$, the $\mathbf{t}$-adic and $\mathbf{u}$-adic topologies on S coincide, and $\mathrm{H}^m_{\mathbf{t}S} = \mathrm{H}^m_{\mathbf{u}S}$) then $\mathrm{res}_{\mathbf{t}} = \mathrm{res}_{\mathbf{u}}$.*

The *proof* of this key fact will be discussed below.

In summary, there is given a complete topological R-algebra S having an ideal J such that:
(i) The topology on S is the J-adic topology, and
(ii) J generated by an S-regular sequence $\mathbf{t} = (t_1, \dots, t_m)$, and
(iii) the natural map is an isomorphism $R \xrightarrow{\sim} S/J$.

It follows that the continuous R-algebra homomorphism from the power-series ring $R[[T_1, \dots, T_m]]$ to S taking T_i to t_i ($1 \le i \le m$) is an isomorphism. Then the S-module $\hat{\Omega}^m$ and the local cohomology functor H^m_J depend only on the R-algebra S and its topology, as does the R-linear *residue map*

$$\mathrm{res}_{S/R} := \mathrm{res}_{\mathbf{t}}\colon \mathrm{H}^m_J\hat{\Omega}^m \to R.$$

This being so, and by the definition (5.3.1) of $\mathrm{res}_{\mathbf{t}}$, Cor. 5.2.3 gives the following **canonical version of local duality for power-series algebras:**

Theorem 5.3.3. *In the preceding situation, the functor $\mathrm{Hom}_R(\mathrm{H}^m_J E, R)$ of S-modules E is represented by the pair $(\hat{\Omega}^m, \mathrm{res}_{S/R})$.*

Remark. Again, $J = \mathbf{t}S$. Recall that the stable Koszul S-complex $\mathcal{K}(\mathbf{t})$ is the direct limit of ordinary Koszul complexes $K(t_1^{n_1}, \dots, t_m^{n_m})$ (cf. paragraph immediately preceding §4.2). So we can specify any element of

$$\mathrm{H}^m_J\hat{\Omega}^m \overset{(3.1.2)}{=} \varinjlim_{n_1,\dots,n_m} \mathrm{H}^m K(t_1^{n_1}, \dots, t_m^{n_m}, \hat{\Omega}^m)$$

by a symbol (non-unique) of the form

$$\begin{bmatrix} \nu \\ t_1^{n_1}, \dots, t_m^{n_m} \end{bmatrix} := \kappa_{n_1,\dots,n_m}\pi_{n_1,\dots,n_m}\nu$$

for suitable $\nu \in \hat{\Omega}^m$ and positive integers $n_1, \dots, n_m$, with π and κ the natural maps

$$\begin{aligned} \pi_{n_1,\dots,n_m}&\colon \hat{\Omega}^m = K^m(t_1^{n_1}, \dots, t_m^{n_m}, \hat{\Omega}^m) \twoheadrightarrow \mathrm{H}^m K(t_1^{n_1}, \dots, t_m^{n_m}, \hat{\Omega}^m), \\ \kappa_{n_1,\dots,n_m}&\colon \mathrm{H}^m K(t_1^{n_1}, \dots, t_m^{n_m}, \hat{\Omega}^m) \to \mathrm{H}^m_J\hat{\Omega}^m. \end{aligned} \tag{5.3.4}$$

Then, recalling that $\phi_{\mathbf{t}}\nu \in \omega_{\mathbf{t}} = \mathrm{Hom}_R(\nu_{\mathbf{t}}, R)$ and that $t_1^{-n_1} \cdots t_m^{-n_m} \in \nu_{\mathbf{t}}$, we get

$$\mathrm{res}_{S/R}\begin{bmatrix} \nu \\ t_1^{n_1}, \dots, t_m^{n_m} \end{bmatrix} = (\phi_{\mathbf{t}}\nu)\bigl(t_1^{-n_1} \cdots t_m^{-n_m}\bigr).$$

In particular, since $\phi_{\mathbf t}dt_1\cdots dt_m=\gamma_{\mathbf t}$ we have

$$\operatorname{res}_{S/R}\begin{bmatrix} dt_1\cdots dt_m\\ t_1^{n_1},\dots,t_m^{n_m}\end{bmatrix}=\begin{cases}1 & \text{if } n_1=\cdots=n_m=1,\\ 0 & \text{otherwise.}\end{cases} \tag{5.3.5}$$

When $m=1$, $H^1_J\hat\Omega^1$ is the cokernel of the canonical map $\hat\Omega^1\to\hat\Omega^1_t$ (localization w.r.t. the powers of $t:=t_1$), and $\left[\begin{smallmatrix}dt\\ t^n\end{smallmatrix}\right]=\pi(dt/t^n)$ with $\pi\colon\hat\Omega^1_t\twoheadrightarrow H^1_J\hat\Omega^1$ the natural map. Then (5.3.5) yields the formula $\operatorname{res}_{R[[t]]/R}\pi\big((\sum_{i\ge0}r_it^i)dt/t^n\big)=r_{n-1}$, which has an obvious relation to the classical formula for residues of one-variable meromorphic functions.

Exercise. (i) Using Prop. 3.1.2, or otherwise, establish for R-modules F and S-modules G a bifunctorial isomorphism

$$\xi(F,G)\colon F\otimes_R\mathrm H^m_J(G)\ \xrightarrow{\ \sim\ }\ \mathrm H^m_J(F\otimes_R G)$$

such that, with notation as in Proposition 5.2.3,

$$\varepsilon_{\mathbf t}(F\otimes_R\omega_{\mathbf t})\circ\xi(F,\omega_{\mathbf t})=\mathbf 1\otimes_R\varepsilon_{\mathbf t}(\omega_{\mathbf t})\colon F\otimes_R\mathrm H^m_J(\omega_{\mathbf t})\to F\otimes_R\omega_{\mathbf t}\otimes_S\nu_{\mathbf t}.$$

(ii) Show that for any finitely-generated R-module F, the functor $\operatorname{Hom}_R(\mathrm H^m_J E,F)$ of S-modules E is represented by the pair $\big(F\otimes_R\hat\Omega^m,(\mathbf 1\otimes\operatorname{res}_{S/R})\circ\xi(F,\hat\Omega^{m-1})\big)$.

Next, let $\varphi\colon R\to S$ be any *flat* (hence injective) local homomorphism of complete noetherian local rings with respective maximal ideals $\mathfrak m$ and $\mathfrak M$, such that $S/\mathfrak mS$ is a Cohen-Macaulay local ring with residue field $S/\mathfrak M$ finite over $R/\mathfrak m$. Then any sequence $\mathbf t:=(t_1,\dots,t_m)$ in S whose image in $S/\mathfrak mS$ is a system of parameters is S-regular, and $P:=S/\mathbf tS$ is a finitely-generated projective R-module. (See [EG4, p. 18, Prop. (15.1.16)]) and [ZS, p. 259, Cor. 2].) After $\varphi(R)$ is identified with R, it follows that the R-homomorphism from the formal power-series ring $R[[T_1,\dots,T_m]]$ to S taking T_i to t_i is an isomorphism onto $R[[\mathbf t]]\subset S$, and that S is $R[[\mathbf t]]$-module-isomorphic to $P\otimes_R R[[\mathbf t]]$ (see [Lp2, §3]).

To such a φ there is associated a finitely-generated S-module $\hat\Omega_\varphi$ together with an R-derivation $d\colon S\to\hat\Omega_\varphi$ which has the universal property that for any finitely-generated S-module M, composition with d maps $\operatorname{Hom}_S(\hat\Omega_\varphi,M)$ bijectively onto the S-module of R-derivations from S into M (see [SS, §1]). There is also a *trace map*

$$\tau\colon\Lambda^m_S\hat\Omega_\varphi=:\hat\Omega^m_\varphi\to\hat\Omega^m_{R[[\mathbf t]]/R},$$

see [Knz, §16], [Hü, §4]. The definition of this map is somewhat subtle. However, in the special case when $\mathfrak M=\mathfrak mS+\mathbf tS$ and in addition $S/\mathfrak M$ is a finite *separable* field extension of $R/\mathfrak m$ (so that S is *formally smooth* over R [EG4, p. 102, (19.6.4) and p. 104, (19.7.1)]), and P is a finite flat unramified (= étale) $R-$algebra, it follows e.g., from [EG4, p. 148, (20.7.6)] that

$$\hat\Omega_\varphi\cong S\otimes_{R[[\mathbf t]]}\hat\Omega_{R[[\mathbf t]]/R}\cong P\otimes_R\hat\Omega_{R[[\mathbf t]]/R}, \tag{5.3.6}$$

a free S-module with basis $(dt_1,\dots,dt_m)$. (In other words every R-derivation of $R[[\mathbf t]]$ into a finitely-generated S-module extends uniquely to S.)

So $\hat\Omega^m_\varphi \cong P\otimes_R \hat\Omega^m_{R[[\mathbf t]]/R}$, and correspondingly τ becomes the map induced by the usual trace map $\mathrm{tr}\colon P\to R$.

Now define $\mathrm{Res}_{\mathbf t}\colon H^m_{\mathfrak M}\hat\Omega^m_\varphi \to R$ to be the composite map

$$H^m_{\mathfrak M}\hat\Omega^m_\varphi \xrightarrow{\text{natural}} H^m_{\mathbf t S}\hat\Omega^m_\varphi \underset{(3.4.3)}{=} H^m_{\mathbf t R[[\mathbf t]]}\hat\Omega^m_\varphi \xrightarrow{\text{via }\tau} H^m_{\mathbf t R[[\mathbf t]]}\hat\Omega^m_{R[[\mathbf t]]/R} \xrightarrow[(5.3.1)]{\mathrm{res}_{\mathbf t}} R.$$

Proposition 5.3.2′. *This map* $\mathrm{Res}_{\mathbf t}$ *does not depend on the choice of* $\mathbf t$.

Thus we have a residue map

$$\mathrm{Res}_\varphi\colon H^m_{\mathfrak M}\hat\Omega^m_\varphi \to R.$$

There are several approaches to the proofs of Propositions 5.3.2 and 5.3.2′. For 5.3.2, the most elementary one, brute-force calculation, is rather tedious (cf. e.g., [Lp1, pp. 64–67]), and not particularly illuminating.

It is more satisfying first to find an a priori intrinsic definition of the residue map, and then to show that it agrees with the above one. For example, such a definition via Hochschild homology is the foundation of [Lp2]. (See [*ibid.,* §4.7], or [Hü, §7], for the connection between residues and traces.)

Another, richly-textured, intrinsic approach is undertaken in [HüK]. In fact Hübl and Kunz prove Theorem 5.3.3 in a more general situation, for maps $R\to S$ factoring as $R\to R[[t_1,\dots,t_m]] \xrightarrow{f} S$ with f a finite generic complete intersection. In such a situation, it is easy to generalize Corollary 5.2.3, with the representing object $\omega_{\mathbf t}$ replaced by $\mathrm{Hom}_{R[[\mathbf t]]}(S,\omega_{\mathbf t})$; but the trick is to find a *canonical* representing object, not depending on $\mathbf t$. For this Hübl and Kunz use the module of "regular differential forms," constructed via the theory of traces of differential forms.

For example, if $\varphi\colon R\to S$ as above makes S formally smooth and residually separable over R then the trace map $\mathrm{tr}\colon P\to R$ gives rise, via (5.3.6), to an $R[[\mathbf t]]$-isomorphism $\hat\Omega^m_\varphi \xrightarrow{\sim} \mathrm{Hom}_{R[[\mathbf t]]}(S,\hat\Omega^m_{R[[\mathbf t]]/R})$. In the non-separable case the same isomorphism obtains by means of the general trace map for differential forms. There results a canonical local duality theorem for formally smooth local algebras:

Theorem 5.3.3′. *If* $\varphi\colon (R,\mathfrak m)\to (S,\mathfrak M)$ *is a formally smooth local homomorphism of complete noetherian local rings making* $S/\mathfrak M$ *finite over* $R/\mathfrak m$, *and* $m := \dim S/\mathfrak m S$, *then the functor* $\mathrm{Hom}_R(\mathrm H^m_{\mathfrak M}E, R)$ *of* S*-modules* E *is represented by* $(\hat\Omega^m_\varphi, \mathrm{Res}_\varphi)$.

We will now outline yet another approach to residues, which is perhaps the "least elementary," but has the advantage of connecting immediately with the global theory of duality on formal schemes [DFS], through the *fundamental class* of certain flat maps of formal schemes. There result canonical realizations of, and relations between, local and global duality, summarized by the *Residue Theorem.* The introductory discussion here will be confined to smooth maps.

5.4. **Flat base change.** Our definition of the fundamental class makes use of a basic property of duality, having to do with its behavior under flat base change, (Proposition 5.4.2).

Henceforth ring homomorphisms will be continuous maps between noetherian topological rings, mostly *adic.* That is, we work in the category of pairs (R, I) with R a commutative noetherian ring and I an R-ideal such that R is complete and separated with respect to the I-adic topology, morphisms $\varphi\colon (R, I) \to (S, J)$ being ring homomorphisms $\varphi\colon R \to S$ such that $\varphi(I) \subset \sqrt{J}$. (Pairs (R, I_1) and (R, I_2) are considered identical if I_1 and I_2 define the same topology, i.e., $\sqrt{I_1} = \sqrt{I_2}$.) For such a φ, we simply write $\varphi^{\#}$ for the functor $\varphi^{\#}_J$ of Theorem 2.3.2, because it depends only on the J-adic topology, which is a part of (the target of) φ.

Consider then a coproduct square in this category, i.e., a commutative diagram of morphisms

$$\begin{array}{ccc} (R,I) & \xrightarrow{\ \varphi\ } & (S,J) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ (U,L) & \xrightarrow[\ \xi\ ]{} & (V,M) \end{array}$$

such that the resulting map into V from the complete tensor product $S\,\hat{\otimes}_R\,U$ (the completion of $V_0 := S \otimes_R U$ with respect to $M_0 := LV_0 + JV_0$) is an *isomorphism,* and where $M := LV + JV$.

(For simplicity we proceed as if V_0 were noetherian. Usually this is not so, and a more complicated approach is needed, cf. [DFS, p. 76, Definition 7.3; p. 86, Theorem 8.1].)

Let $\kappa\colon V_0 \to V$, $\xi_0\colon U \to V_0$, and $\nu_0\colon S \to V_0$ be the natural maps, so that $\xi = \kappa\xi_0$ and $\nu = \kappa\nu_0$. Suppose μ, hence ν_0 and ν, to be *flat.* Then the functor $\dots \otimes_R U$ from R-modules to U-modules is exact, so takes quasi-isomorphisms to quasi-isomorphisms, and consequently extends to a functor $\mu^*\colon \mathbf{D}(R) \to \mathbf{D}(U)$ (cf. §3.4). Similarly we have $\nu_0^*\colon \mathbf{D}(S) \to \mathbf{D}(V_0)$ and $\nu^* = \kappa^*\nu_0^*\colon \mathbf{D}(S) \to \mathbf{D}(V)$. For any Δ-functor $\Gamma\colon \mathbf{K}(R) \to \mathbf{K}(S)$, and $\mathbf{K}(R)$-quasi-isomorphism $C \to E_C$ with E_C q-injective, there is an isomorphism $\nu_0^*\mathbf{R}\Gamma(C) \cong \nu_0^*\Gamma(E_C)$; hence $\nu_0^*\mathbf{R}\Gamma\colon \mathbf{D}(R) \to \mathbf{D}(V_0)$ is a right-derived functor of $\Gamma(-) \otimes_R U\colon \mathbf{K}(R) \to \mathbf{K}(V_0)$ (see §1.5).

The *base-change map* $\beta\colon \nu^*\varphi^{\#} \to \xi^{\#}\mu^*$, that is, the functorial map

$$\beta(G)\colon \nu^*\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, G) \to \mathbf{R}\mathrm{Hom}^{\bullet}_U(\mathbf{R}\Gamma_M V,\, \mu^*G) \qquad (G \in (\mathbf{D}(R)),$$

is defined as follows.

First, as noted above, $\nu_0^*\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, -)$ is a right-derived functor of $\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, -) \otimes_R U$; so by the characteristic universal property of right-derived functors (§1.5), there exists a unique functorial map $\beta'(G)$ making the following otherwise natural $\mathbf{D}(V_0)$-diagram commute:

$$\begin{array}{ccc} \mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, G)\otimes_R U & \longrightarrow & \mathrm{Hom}^{\bullet}_U(\mathbf{R}\Gamma_J S\otimes_R U, G\otimes_R U) \\ \Big\downarrow & & \Big\downarrow \\ \nu_0^*\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, G) & \underset{\beta'(G)}{\longrightarrow} & \mathbf{R}\mathrm{Hom}^{\bullet}_U(\nu_0^*\mathbf{R}\Gamma_J S,\, \mu^*G); \end{array}$$

and the natural composition

$$\mathbf{R}\Gamma_{M_0}V_0 \longrightarrow \mathbf{R}\Gamma_{JV_0}V_0 \underset{(3.4.4)}{\xrightarrow{\sim}} \nu_0^*\mathbf{R}\Gamma_J S$$

combines with $\beta'(G)$ to give a functorial map

$$\beta_0(G)\colon \nu_0^*\varphi^{\#}G = \nu_0^*\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, G) \to \mathbf{R}\mathrm{Hom}^{\bullet}_U(\mathbf{R}\Gamma_{M_0}V_0,\, \mu^*G) = \xi_0^{\#}\mu^*G.$$

Second, for any $F \in \mathbf{D}(V_0)$, we have a natural isomorphism

$$\kappa_*\mathbf{R}\Gamma_M\kappa^*F \underset{(3.4.4)}{\xrightarrow{\sim}} \kappa_*\kappa^*\mathbf{R}\Gamma_{M_0}F.$$

Also, the natural map is an isomorphism $\mathbf{R}\Gamma_{M_0}F \xrightarrow{\sim} \kappa_*\kappa^*\mathbf{R}\Gamma_{M_0}F$: to verify this, since the functors κ_* and κ^* are both exact and isomorphism means "homology isomorphism" (§1.3), we can replace $\mathbf{R}\Gamma_{M_0}F$ by its homology, and then the assertion follows because the homology is M_0-power torsion (see §3.2). The resulting composition $\kappa_*\mathbf{R}\Gamma_M\kappa^*F \xrightarrow{\sim} \mathbf{R}\Gamma_{M_0}F \to F$ is dual to a map (see 2.3.2)

$$\iota(F)\colon \kappa^*F \to \kappa^{\#}F. \tag{5.4.1}$$

Finally, $\beta(G)$ is defined to be the composite map

$$\nu^*\varphi^{\#}G = \kappa^*\nu_0^*\varphi^{\#}G \xrightarrow[\kappa^*(\beta_0(G))]{} \kappa^*\xi_0^{\#}\mu^*G \xrightarrow[\iota(\xi_0^{\#}\mu^*G)]{} \kappa^{\#}\xi_0^{\#}\mu^*G \underset{(2.3.3)}{\cong} \xi^{\#}\mu^*G.$$

Let $\mathbf{D}^+(R)$ (resp. $\mathbf{D}^-(R)$) be the full subcategory of $\mathbf{D}(R)$ with objects those complexes G whose homology H^iG vanishes for $i \ll 0$ (resp. $i \gg 0$). The full subcategories $\mathbf{D}_{\mathrm{c}}^+(R)$ and $\mathbf{D}_{\mathrm{c}}^-(R)$ of $\mathbf{D}_{\mathrm{c}}(R)$ are defined similarly. (As before, $\mathbf{D}_{\mathrm{c}}(R) \subset \mathbf{D}(R)$ is the full subcategory whose objects are complexes having finitely-generated homology modules.)

Theorem 5.4.2 (Flat Base-Change)**.** *In the preceding situation, if S/J is (via φ) a finite R-module and $G \in \mathbf{D}_{\mathrm{c}}^+(R)$ then $\beta(G)$ is an isomorphism.*

Proof. (Outline.) The finiteness of S/J over R means that $\varphi = \varphi_2\varphi_1$ where $\varphi_1\colon R \to R[[\mathbf{t}]] := R[[t_1, \dots, t_m]]$ is the natural map of R into a power-series ring, which is complete for the $I' := (I, \mathbf{t})R[[\mathbf{t}]]$-topology, and $\varphi_2\colon R[[\mathbf{t}]] \to S$ is such that $J = I'S$, so that φ_2 makes S into a finite $R[[\mathbf{t}]]$-module having the I'-adic topology ([ZS, p. 259, Cor. 2]). A readily-established transitivity property of the base-change map β then reduces the problem to the two cases $\varphi = \varphi_1$ and $\varphi = \varphi_2$.

When $\varphi = \varphi_1$ then ξ is the natural map $U \to U[[\mathbf{t}]]$, and so (5.1.7) reduces the problem to identifying $\beta(G)$ with the natural isomorphism (notation as in (5.1.5))

$$(G \otimes_R \omega_{\mathbf{t}}^{\varphi}) \otimes_{R[[\mathbf{t}]]} U[[\mathbf{t}]] \xrightarrow{\ \sim\ } (G \otimes_R U) \otimes_U \omega_{\mathbf{t}}^{\xi},$$

an exercise in unraveling definitions.

When $\varphi = \varphi_2$, i.e., S is a finite R-module and $J = IS$, then $V_0 = S \otimes_R U$ is a finite U-module with L-adic topology, and so is complete, i.e., $V = V_0$. Now Greenlees-May duality enters crucially to yield, via (2.4.1), an identification of $\beta(G)$ with the natural map

$$\beta^\flat(G)\colon \mathbf{R}\mathrm{Hom}_R^\bullet(S,G) \otimes_R U \to \mathbf{R}\mathrm{Hom}_U^\bullet(V, G \otimes_R U) = \mathbf{R}\mathrm{Hom}_U^\bullet(S \otimes_R U, G \otimes_R U)$$

whose existence is shown similarly to that of β' (see above). That $\beta^\flat(G)$ is an *isomorphism* for any $G \in \mathbf{D}^+(R)$ becomes clear upon replacement of S by a (finite-rank) R-projective resolution P, in view of the simple fact that the functor $\mathrm{Hom}_R^\bullet(P, -)$ takes quasi-isomorphisms to quasi-isomorphisms, a fact whose application to an injective resolution of G shows that

$$\mathbf{R}\mathrm{Hom}_R^\bullet(S,G) \cong \mathbf{R}\mathrm{Hom}_R^\bullet(P,G) \cong \mathrm{Hom}_R^\bullet(P,G),$$

and similarly (since U is R-flat)

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_U^\bullet(S \otimes_R U, G \otimes_R U) &\cong \mathbf{R}\mathrm{Hom}_U^\bullet(P \otimes_R U, G \otimes_R U) \\ &\cong \mathrm{Hom}_U^\bullet(P \otimes_R U, G \otimes_R U). \end{aligned}$$

□

5.5. **Residues via the fundamental class.** Specialize now to a coproduct square

$$\begin{array}{ccc} (R,I) & \xrightarrow{\ \varphi\ } & (S,J) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\ (S,J) & \xrightarrow[\ \xi\ ]{} & (V,M) \end{array}$$

with φ *flat,* and S/J finite over R—so that Theorem 5.4.2 is applicable. Let $\delta\colon V \cong S \hat{\otimes}_R S \to S$ be the continuous extension of the map $S \otimes_R S \to S$ taking $s_1 \otimes s_2$ to $s_1 s_2$. Let $\kappa\colon S \otimes_R S \to V$ be the completion map, so that $\nu(s) = \kappa(s \otimes 1)$ and $\xi(s) = \kappa(1 \otimes s)$ $(s \in S)$, and let L be the V-ideal (closed, since V is assumed noetherian)[17] generated by all the elements $\nu(s) - \xi(s)$. For any $f \in V$, $f - \xi\delta f \in L$ (check this first with V replaced by its dense subring $\kappa(S \otimes_R S)$, then pass to the limit); and it follows that L is the kernel of δ.

One shows then that the S-module $\hat{\Omega}_\varphi := L/L^2$ together with the R-derivation $d\colon S \to \hat{\Omega}_\varphi$ such that $d(s) = \nu(s) - \xi(s) \pmod{L^2}$ for all $s \in S$ is universal (cf. §5.3) for R-derivations of S into finitely-generated S-modules.

[17]In fact the noetherianness of V follows from that of R and S plus the R-finiteness of S/J [GD, p. 414, (10.6.4)].

Let m be the least non-negative integer such that $\mathrm{H}^i_J S = \mathrm{H}^i\mathbf{R}\Gamma_J S = 0$ for all $i > m$ (see Corollary 3.1.4). Then

$$\mathrm{H}^{-i}\varphi^{\#}R = \mathrm{H}^{-i}\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_J S, R) = 0 \qquad (i > m).$$

Set $\hat\Omega^m = \hat\Omega^m_\varphi := \Lambda^m_S\hat\Omega_\varphi$. The *fundamental class* of φ is a canonical S-linear map

$$\mathfrak{f}_\varphi\colon \hat\Omega^m \to \omega_\varphi := \mathrm{H}^{-m}\varphi^{\#}R, \tag{5.5.2}$$

defined with the assistance of flat base-change, as follows.

Since $\delta\xi = \delta\nu = \mathbf{1}_S$, we have, clearly, $\xi_*\delta_* = \mathbf{1}_{\mathbf{D}(S)}$; and with δ^* as in §3.4, there is a natural isomorphism $\delta^*\nu^* \cong \mathbf{1}_{\mathbf{D}(S)}$. There results a natural $\mathbf{D}(V)$-composition

$$\begin{aligned}\delta_*S \underset{(2.3.2)}{\longrightarrow} \xi^{\#}\xi_*\mathbf{R}\Gamma_M\delta_*S \underset{(3.4.3)}{\xrightarrow{\ \sim\ }} \xi^{\#}\xi_*\delta_*\mathbf{R}\Gamma_J S = \xi^{\#}\mathbf{R}\Gamma_J S\\ \to \xi^{\#}S \xrightarrow{\ \sim\ } \xi^{\#}\varphi^*R \underset{(5.4.2)}{\xrightarrow{\ \sim\ }} \nu^*\varphi^{\#}R,\end{aligned}$$

to which application of δ^* gives a natural $\mathbf{D}(S)$-map

$$\delta^*\delta_*S \to \delta^*\nu^*\varphi^{\#}R \cong \varphi^{\#}R,$$

whence a natural map

$$\mathrm{Tor}^V_m(S,S) = \mathrm{H}^{-m}\delta^*\delta_*S \to \mathrm{H}^{-m}\varphi^{\#}R = \omega_\varphi. \tag{5.5.2}$$

Now with $L = \ker(\delta)$ as above, there is a natural isomorphism

$$\hat\Omega_\varphi = L/L^2 \cong \mathrm{Tor}^V_1(S,S).$$

Moreover, $\oplus_{i\ge 0}\mathrm{Tor}^V_i(S,S)$ has a canonical alternating graded-algebra structure (for which the product arises from the natural maps

$$\mathrm{H}^i(S \underline{\underline{\otimes}}_V S) \otimes_V \mathrm{H}^j(S \underline{\underline{\otimes}}_V S) \to \mathrm{H}^{i+j}\big((S \underline{\underline{\otimes}}_V S) \underline{\underline{\otimes}}_V (S \underline{\underline{\otimes}}_V S)\big) \xrightarrow{\ p\ } \mathrm{H}^{i+j}(S \underline{\underline{\otimes}}_V S)$$

where p is induced by two copies of the composition of the natural maps $S \underline{\underline{\otimes}}_V S \to S \otimes_V S \to S$). The universal property of exterior algebras gives then a canonical map

$$\hat\Omega^m \to \mathrm{Tor}^V_m(S,S). \tag{5.5.3}$$

The fundamental class $\mathfrak{f}_\varphi\colon \hat\Omega^m \to \omega_\varphi$ is the composition of (5.5.2) and (5.5.3).

We can now define the R-linear *formal residue map* $\rho_\varphi\colon \mathrm{H}^m_J\hat\Omega^m \to R$ to be the canonical composition (where the unlabeled map comes from a dual form of Proposition 1.3.3):

$$\rho_\varphi\colon \mathrm{H}^m_J\hat\Omega^m = \mathrm{H}^0\mathbf{R}\Gamma_J\hat\Omega^m[m] \underset{\text{via } \mathfrak{f}}{\longrightarrow} \mathrm{H}^0\mathbf{R}\Gamma_J\omega_\varphi[m] \to \mathrm{H}^0\mathbf{R}\Gamma_J\varphi^{\#}R \underset{(2.3.1)}{\longrightarrow} \mathrm{H}^0R = R.$$

The **local Residue Theorem** states that under the conditions considered in §5.3, *the formal residue map is the same as the residue maps defined there.*

As the formal residue depends only on φ, Theorems 5.3.3 and 5.3.3′ result.

A complete *proof* of the local Residue Theorem will appear elsewhere. For the case when $S = R[[\mathbf{t}]]$ is a power-series R-algebra all the necessary definitions have been spelled out, so no further new ideas are needed, just painstaking work.

For example, for connecting the "abstract" formal residue ρ_φ with the "concrete" residue $\mathrm{res}_{\mathbf{t}}$, one needs commutativity of the diagram

$$\begin{array}{ccccc} \mathrm{H}^0\mathbf{R}\Gamma_J\omega_{\mathbf{t}}[m] & \xrightarrow{\text{natural}} & \mathrm{H}^0\mathbf{R}\Gamma_J\mathrm{Hom}^\bullet_R(\nu_{\mathbf{t}}[-m],R) & \xrightarrow{(5.1.4)} & \mathrm{H}^0\mathbf{R}\Gamma_J\varphi^\# R \\ {\scriptstyle (5.2.2)}\big\downarrow & & & & \big\downarrow{\scriptstyle (2.3.1)} \\ \omega_{\mathbf{t}}\otimes_S\nu_{\mathbf{t}} & & \xrightarrow[\text{evaluation}]{\hspace{8em}} & & R, \end{array}$$

which can be seen by detailed consideration of Proposition 5.2.4 with $E = \omega_{\mathbf{t}}$ and $F = R$.

A full treatment involves more about the relation between fundamental classes and traces of differential forms. Consider, for example, a pair of continuous maps

$$R \xrightarrow{\varphi} S = R[[\mathbf{t}]] \xrightarrow{\psi} T$$

with φ the canonical map, and T a finite $R[[\mathbf{t}]]$-module (via ψ). From (2.4.1) we find that the integer m used to define $\mathfrak{f}_{\psi\varphi}$ is the same as that used for $\mathfrak{f}_\varphi$ (namely, the number of variables in $\mathbf{t}$). There is then, by the above-mentioned dual form of Proposition 1.3.3, a natural map

$$\omega_{\psi\varphi} := \mathrm{H}^{-m}(\psi\varphi)^\# R \to (\psi\varphi)^\# R \underset{(2.3.3)}{=} \psi^\#\varphi^\# R;$$

and as part of the proof of the local Residue Theorem one needs:

Theorem 5.5.4. *The fundamental class $\mathfrak{f}_\varphi$ is the composite isomorphism*

$$\hat{\Omega}^m_\varphi \underset{\phi_{\mathbf{t}}}{\xrightarrow{\sim}} \omega_{\mathbf{t}} \underset{(5.1.7)}{\xrightarrow{\sim}} \mathrm{H}^{-m}\varphi^\# R =: \omega_\varphi.$$

So there is a unique S-linear map τ making the following $\mathbf{D}(S)$-diagram commute:

$$\begin{array}{ccccc} \psi_*\hat{\Omega}^m_{\psi\varphi} & \xrightarrow{\psi_*\mathfrak{f}_{\psi\varphi}} & \psi_*\omega_{\psi\varphi} & \longrightarrow & \psi_*\psi^\#\varphi^\# R[-m] \\ {\scriptstyle\tau}\big\downarrow & & & & {\scriptstyle (2.3.2)}\big\downarrow{\scriptstyle (2.4.1)} \\ \hat{\Omega}^m_\varphi & \underset{\mathfrak{f}_\varphi}{\xrightarrow{\sim}} & \omega_\varphi & \xrightarrow{\sim} & \varphi^\# R[-m]; \end{array}$$

and this τ coincides with the trace map for differential forms.

5.6. **Global duality; the Residue Theorem.** This culminating section introduces the connections between residues and global duality theory on noetherian formal schemes. A key advantage of working in the category of formal schemes—rather than its subcategory of ordinary schemes—is that local and global duality then become two aspects of a single theory.

We first set up some notation and briefly review necessary background material. (The prerequisite basics on formal schemes are in [GD, Chap. I, §10].)

Let $\mathcal{X} = (|\mathcal{X}|, \mathcal{O}_{\mathcal{X}})$ be a noetherian formal scheme, with ideal of definition $\mathcal{J}$. ($|\mathcal{X}|$ is a topological space and $\mathcal{O}_{\mathcal{X}}$ is a sheaf of topological rings.) Let $\mathcal{A}(\mathcal{X})$ be the abelian category of $\mathcal{O}_{\mathcal{X}}$-modules, and $\mathbf{D}(\mathcal{X})$ the derived category of $\mathcal{A}(\mathcal{X})$. Let $\mathbf{D}_{\mathrm{qc}}(\mathcal{X}) \subset \mathbf{D}(\mathcal{X})$ (resp. $\mathbf{D}_{\mathrm{c}}(\mathcal{X}) \subset \mathbf{D}(\mathcal{X})$) be the full subcategory with objects those $\mathcal{A}(\mathcal{X})$-complexes whose homology sheaves are quasi-coherent (resp. coherent), i.e., locally cokernels of maps of free (resp. free, finite-rank) $\mathcal{O}_{\mathcal{X}}$-modules. In $\mathbf{D}_{\dots}(\mathcal{X})$ the homologically bounded-below complexes—those $\mathcal{E}$ whose homology sheaves $H^i\mathcal{E}$ vanish for $i \ll 0$—are the objects of a full subcategory denoted by $\mathbf{D}^+_{\dots}(\mathcal{X})$.

The torsion subfunctor $\varGamma'_{\mathcal{X}}$ of the identity functor on $\mathcal{A}(\mathcal{X})$ is given by

$$\varGamma'_{\mathcal{X}}\mathcal{E} = \varinjlim_{s>0} \mathcal{H}om^{\bullet}_{\mathcal{X}}(\mathcal{O}_X/\mathcal{J}^s, \mathcal{E}) \qquad (\mathcal{E} \in \mathcal{A}(\mathcal{X})).$$

$\varGamma'_{\mathcal{X}}$ depends only on $\mathcal{X}$, not $\mathcal{J}$. It has a derived functor $\mathbf{R}\varGamma'_{\mathcal{X}}\colon \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X})$, which satisfies $\mathbf{R}\varGamma'_{\mathcal{X}}\mathbf{D}_{\mathrm{qc}}(\mathcal{X}) \subset \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ [DFS, p. 49, Prop. 5.2.1(b)].

To any noetherian *adic* ring R—i.e., R is a complete noetherian topological ring with topology defined by the powers of some ideal I—is associated an *affine* formal scheme $\mathrm{Spf}(R)$, whose underlying space is the same as that of the ordinary scheme $\mathrm{Spec}(R/I)$. Any noetherian formal scheme $\mathcal{X}$ has a finite open covering by affine formal schemes, with structure sheaves obtained by restricting $\mathcal{O}_{\mathcal{X}}$. Continuous maps $\varphi\colon R \to S$ of noetherian adic rings correspond bijectively to formal-scheme maps $\widetilde{\varphi}\colon \mathrm{Spf}(S) \to \mathrm{Spf}(R)$; and any map $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian formal schemes is locally of this form. The direct image functor $f_*\colon \mathcal{A}(\mathcal{X}) \to \mathcal{A}(\mathcal{Y})$ has a right-derived functor $\mathbf{R}f_*\colon \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y})$; and the inverse image functor $f^*\colon \mathcal{A}(\mathcal{Y}) \to \mathcal{A}(\mathcal{X})$ has a left-derived functor $\mathbf{L}f^*\colon \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$.

For any such $f\colon \mathcal{X} \to \mathcal{Y}$, there are ideals of definition $\mathcal{I} \subset \mathcal{O}_{\mathcal{Y}}$ and $\mathcal{J} \subset \mathcal{O}_{\mathcal{X}}$ such that $\mathcal{I}\mathcal{O}_{\mathcal{X}} \subset \mathcal{J}$ [GD, p. 416,(10.6.10)]; and correspondingly there is a map of ordinary schemes $f_0\colon (|\mathcal{X}|, \mathcal{O}_{\mathcal{X}}/\mathcal{J}) \to (|\mathcal{Y}|, \mathcal{O}_{\mathcal{Y}}/\mathcal{I})$ [GD, p. 410, (10.5.6)]. We say f is *separated* (resp. *pseudo-proper*) if f_0 is separated (resp. proper), a condition independent of the choice of $(\mathcal{I}, \mathcal{J})$. For example, a map $\widetilde{\varphi}$ as above is pseudo-proper iff S/J is, via φ, a finite R-module for some (hence any) S-ideal J defining the topology of S. We say f is *proper* if f is pseudo-proper and for some (hence any) $\mathcal{I}$, $\mathcal{I}\mathcal{O}_{\mathcal{X}}$ is an ideal of definition of $\mathcal{X}$.

We say f is *flat* if it is locally $\widetilde{\varphi}$ for some φ as above making S a flat R-module. For flat f the functor $f^*\colon \mathcal{A}(\mathcal{Y}) \to \mathcal{A}(\mathcal{X})$ is *exact* (see [DFS, p. 72, Lemma 7.1.1]), so may be thought of as a functor from $\mathbf{D}(\mathcal{Y})$ to $\mathbf{D}(\mathcal{X})$, naturally isomorphic to $\mathbf{L}f^*$.

One has then the following *globalizations* of Local Duality (Theorem 2.3.2) and Flat Base-Change (Theorem 5.4.2). (Despite the obvious formal similarities, however, fully elucidating the connection between the global and local versions requires more than a little work.) We note in passing that for proper maps Greenlees-May duality plays a basic role in the proofs.

Theorem 5.6.1 ([DFS, p. 64, Cor. 6.1.4; p. 89, Thm. 8.4]). *If $f\colon \mathcal{X}\to\mathcal{Y}$ is a separated map of noetherian formal schemes then the functor*

$$\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}}\colon \mathbf{R}\varGamma'^{\,-1}_{\mathcal{X}}(\mathbf{D}_{\mathrm{qc}}(\mathcal{X}))\to\mathbf{D}(\mathcal{Y})$$

has a right adjoint $f^{\#}$. Moreover, if f is proper (*hence separated*) *then this $f^{\#}$ induces a right adjoint for $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{c}}(\mathcal{X})\to\mathbf{D}^+_{\mathrm{c}}(\mathcal{Y})$.*

Theorem 5.6.2 ([DFS, p. 89, Cor. 8.3.3]). *Let there be given a commutative diagram of noetherian formal schemes*

$$\begin{array}{ccc}\mathcal{V} & \xrightarrow{\ v\ } & \mathcal{X}\\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f}\\ \mathcal{U} & \xrightarrow{\ u\ } & \mathcal{Y}\end{array}$$

with the induced map $\mathcal{V}\to\mathcal{U}\times_{\mathcal{Y}}\mathcal{X}$ an isomorphism, f (hence g) pseudo-proper (*hence separated*), *and u (hence v)* flat (see [DFS, p. 71, Prop. 7.1]). *Then there exists a functorial* base-change isomorphism

$$\beta(\mathcal{F})\colon v^*f^{\#}\mathcal{F}\ \xrightarrow{\ \sim\ }\ g^{\#}u^*\mathcal{F}\qquad (\mathcal{F}\in\mathbf{D}^+_{\mathrm{c}}(\mathcal{Y})).$$

The *fundamental class* $\mathfrak{f}_f$ of any flat pseudo-proper map f can now be defined, as follows: with respect to the diagram

$$\begin{array}{ccccc}\mathcal{X} & \xrightarrow{\ \delta\ } & \mathcal{X}\times_{\mathcal{Y}}\mathcal{X} & \xrightarrow{\ \pi_1\ } & \mathcal{X}\\ & & {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle f}\\ & & \mathcal{X} & \xrightarrow[\ f\ ]{} & \mathcal{Y}\end{array}$$

where δ is the diagonal map and π_1, π_2 the canonical projections (so that $\pi_1\delta=\mathbf{1}_{\mathcal{X}}$ and $\pi_2\delta=\mathbf{1}_{\mathcal{X}}$), there is a sequence of natural $\mathbf{D}(\mathcal{X})$-maps

$$\begin{aligned}\delta_*\mathcal{O}_{\mathcal{X}}\underset{(5.6.1)}{\longrightarrow}\pi_2^{\#}\mathbf{R}\pi_{2*}\mathbf{R}\varGamma'_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}}\delta_*\mathcal{O}_{\mathcal{X}}&\longrightarrow\pi_2^{\#}\mathbf{R}\pi_{2*}\delta_*\mathcal{O}_{\mathcal{X}}\\ &\xrightarrow{\ \sim\ }\pi_2^{\#}\mathcal{O}_{\mathcal{X}}\ =\pi_2^{\#}f^*\mathcal{O}_{\mathcal{Y}}\underset{(5.6.2)}{\xrightarrow{\ \sim\ }}\pi_1^*f^{\#}\mathcal{O}_{\mathcal{Y}},\end{aligned}$$

to which application of the left-derived functor $\mathbf{L}\delta^*$ produces

$$\mathfrak{f}_f\colon\mathbf{L}\delta^*\delta_*\mathcal{O}_{\mathcal{X}}\longrightarrow\mathbf{L}\delta^*\pi_1^*f^{\#}\mathcal{O}_{\mathcal{Y}}\ \xrightarrow{\ \sim\ }\ f^{\#}\mathcal{O}_{\mathcal{Y}}.$$

Let $\mathcal{L}$ be the kernel of the canonical map $\mathcal{O}_{\mathcal{X}\otimes_{\mathcal{Y}}\mathcal{X}}\to\delta_*\mathcal{O}_{\mathcal{X}}$, and let $\hat\Omega_f$ be the coherent $\mathcal{O}_{\mathcal{X}}$-module $\delta^*\mathcal{L}$, i.e., after identification of $\delta|\mathcal{X}|$ with $|\mathcal{X}|$, $\hat\Omega_f=(\mathcal{L}/\mathcal{L}^2)|_{|\mathcal{X}|}$. This $\hat\Omega_f$ is closely related to the universal finite differential modules $\hat\Omega_\varphi$ of §5.5, thus: if f looks locally like $\widetilde{\varphi}$ with $\varphi\colon R\to S$ as above, then $\Gamma(\mathrm{Spf}(S),\hat\Omega_f)=\hat\Omega_\varphi$.

As in §5.5, $\mathfrak{f}_f$ determines for each integer m a map

$$\mathfrak{f}_f^m\colon\hat\Omega_f^m\to H^{-m}f^{\#}\mathcal{O}_{\mathcal{Y}}.$$

When φ is the inclusion of R into a power-series ring $R[[\mathbf{t}]]$ ($\mathbf{t} := (t_1, \dots, t_m)$) and $f = \widetilde{\varphi}$, one shows (with some effort) that modulo the standard correspondence between modules and sheaves, $\mathfrak{f}_f^m$ agrees with the fundamental class defined in §5.5. More generally, global fundamental classes "restrict" to local ones, as we shall now illustrate—without proof—for *formally smooth* pseudo-proper maps of *relative dimension* m.

For a pseudo-proper map f to have these properties means that for any closed point $y \in f(\mathcal{X})$ and any closed point $x \in f^{-1}y$, the corresponding map of completed local rings $\hat{\mathcal{O}}_{\mathcal{Y},y} =: R \xrightarrow{\varphi} S := \hat{\mathcal{O}}_{\mathcal{X},x}$ is formally smooth and if $\mathfrak{m}$ is the maximal ideal of R then the local ring $S/\mathfrak{m}S$ has dimension m. For simplicity, one may assume in what follows that, furthermore, S is residually separable over R.

For any such x, y, there is a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Spf}(S) & \xrightarrow{\kappa_x} & \mathcal{X} \\ {\scriptstyle\widetilde{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathrm{Spf}(R) & \xrightarrow[\kappa_y]{} & \mathcal{Y} \end{array} \tag{5.6.3}$$

The maps κ_x and κ_y are flat, and both $\widetilde{\varphi}$ and $g := \kappa_y\widetilde{\varphi} = f\kappa_x$ are pseudo-proper. As the topological space $|\mathrm{Spf}(R)|$ consists of the single point $\mathfrak{m}$, the category $\mathcal{A}(\mathrm{Spf}(R))$ can be identified with the category of R-modules, and in particular $\mathcal{O}_{\mathrm{Spf}(R)} = R$. It is similar for $\mathrm{Spf}(S)$. One verifies that $\widetilde{\varphi}^{\#} = \varphi^{\#}$, and that $\kappa_x^*\hat{\Omega}_f = \hat{\Omega}_\varphi$—so that $\hat{\Omega}_f$ is locally free of rank m (see (5.3.6)).

It is a consequence of Greenlees-May duality that for any $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(\mathcal{X})$, the map in (5.4.1) is an *isomorphism*

$$\iota(F)\colon \kappa_x^*\mathcal{F} \xrightarrow{\ \sim\ } \kappa_x^{\#}\mathcal{F},$$

and similarly for κ_y. Thus (and cf. (2.3.3)) there are natural isomorphisms

$$\varphi^{\#}R = \widetilde{\varphi}^{\#}\kappa_y^*\mathcal{O}_{\mathcal{Y}} \cong \widetilde{\varphi}^{\#}\kappa_y^{\#}\mathcal{O}_{\mathcal{Y}} \cong g^{\#}\mathcal{O}_{\mathcal{Y}} \cong \kappa_x^{\#}f^{\#}\mathcal{O}_{\mathcal{Y}} \cong \kappa_x^*f^{\#}\mathcal{O}_{\mathcal{Y}}. \tag{5.6.4}$$

So, κ_x^* being exact, we have for each closed $x \in \mathcal{X}$ the map

$$\hat{\Omega}_\varphi^m = \kappa_x^*\hat{\Omega}_f^m \xrightarrow{\kappa_x^*\mathfrak{f}_f^m} \kappa_x^*H^{-m}f^{\#}\mathcal{O}_{\mathcal{Y}} \cong H^{-m}\kappa_x^*f^{\#}\mathcal{O}_{\mathcal{Y}} \cong H^{-m}\varphi^{\#}(R);$$

The assertion relating global to local fundamental classes is:

Lemma 5.6.5. *The preceding composite map is* $\mathfrak{f}_\varphi$ (*see* (5.5.2)).

From this we see, first, that $\mathfrak{f}_f^m\colon \hat{\Omega}_f^m \to H^{-m}f^{\#}\mathcal{O}_{\mathcal{Y}}$ *is an isomorphism.* Indeed, Lemma 5.6.5 localizes the problem to showing that $\mathfrak{f}_\varphi$ is an isomorphism (since then the kernel and cokernel of $\mathfrak{f}_f^m$ would each have at every closed point a stalk whose completion vanishes, and hence they would both vanish). When $\varphi = \varphi_{\mathbf{t}}$ is the inclusion of R into a power-series ring $R[[\mathbf{t}]]$ ($\mathbf{t} := (t_1, \dots, t_m)$), the local assertion is given by the first part of Theorem 5.5.4. In the general case write $\varphi = \psi\varphi_{\mathbf{t}}$ with $\psi\colon R[[\mathbf{t}]] \to S$ étale (see remarks preceding (5.3.6)), and use the following diagram, whose top row comes from the trace (remarks preceding Theorem 5.3.3′), whose bottom row comes from (2.4.1) applied to ψ, and which, as a corollary of Theorem 5.5.4, commutes:

$$\begin{array}{ccc} \hat\Omega^m_{\psi\varphi_{\mathbf t}} & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{R[[\mathbf t]]}(S, \hat\Omega^m_{\varphi_{\mathbf t}}) \\ {\scriptstyle \mathfrak f_\varphi}\big\downarrow & & \simeq\big\downarrow{\scriptstyle \text{via } \mathfrak f_{\varphi_{\mathbf t}}} \\ H^{-m}\psi^{\#}\varphi^{\#}_{\mathbf t}R & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{R[[\mathbf t]]}(S, H^{-m}\varphi^{\#}_{\mathbf t}R) \end{array}$$

Second, note that there is a natural isomorphism

$$(H^{-m}f^{\#}\mathcal O_{\mathcal Y})[m] \xrightarrow{\ \sim\ } f^{\#}\mathcal O_{\mathcal Y} \tag{5.6.6}$$

resulting via Proposition 1.3.3 from the vanishing of $H^j f^{\#}\mathcal O_{\mathcal Y}$ for all $j \ne -m$: since κ_x^* is exact for all x, the isomorphisms (5.6.4) reduce verification of this vanishing to the corresponding vanishing for $\varphi^{\#}R$, which holds by (5.1.7) when S is a power-series R-algebra, and then follows via (2.4.1) in the general case when S is an étale extension of a power-series algebra (see remarks preceding (5.3.6)). Using (5.1.7) one shows the same true with any coherent $\mathcal O_{\mathcal Y}$-module $\mathcal G$ in place of $\mathcal O_{\mathcal Y}$.

So we have the $\mathbf D(\mathcal X)$-isomorphisms

$$\Omega^m_f[m] \underset{\mathfrak f^m_f[m]}{\xrightarrow{\ \sim\ }} (H^{-m}f^{\#}\mathcal O_{\mathcal Y})[m] \xrightarrow{\ \sim\ } f^{\#}\mathcal O_{\mathcal Y}.$$

Hence, by Thm. 5.6.1, $\hat\Omega^m_f$ represents the functor $\mathrm{Hom}_{\mathbf D(\mathcal Y)}(\mathbf Rf_*\mathbf R\varGamma'_{\mathcal X}\mathcal E[m], \mathcal O_{\mathcal Y})$ of quasi-coherent $\mathcal O_{\mathcal X}$-modules $\mathcal E$; and when f is *proper,* $\hat\Omega^m_f$ represents the functor $\mathrm{Hom}_{\mathbf D(\mathcal Y)}(\mathbf Rf_*\mathcal F[m], \mathcal O_{\mathcal Y})$ of coherent $\mathcal O_{\mathcal X}$-modules $\mathcal F$.

For such $\mathcal F$, there is an n such that $R^jf_*\mathcal F := H^j\mathbf Rf_*\mathcal F = 0$ for all $j > n$ [DFS, p. 39, Prop. 3.4.3(b)]. Then with $\mathcal G$ the $\mathcal O_{\mathcal Y}$-module $H^n\mathbf Rf_*\mathcal F$, which is coherent [DFS, p. 40, Prop. 3.5.2)], there are isomorphisms

$$\mathrm{Hom}_{\mathbf D(\mathcal X)}(\mathcal F, f^{\#}\mathcal G[-n]) \underset{5.6.1}{\cong} \mathrm{Hom}_{\mathbf D(\mathcal Y)}(\mathbf Rf_*\mathcal F, \mathcal G[-n]) \underset{1.3.3}{\cong} \mathrm{Hom}_{\mathcal O_Y}(\mathcal G, \mathcal G).$$

But as noted above, $H^jf^{\#}\mathcal G[-n] = H^{j-n}f^{\#}\mathcal G = 0$ if $j - n < -m$, i.e., if $j < n - m$; and hence if $n > m$ then $\mathrm{Hom}_{\mathcal O_Y}(\mathcal G, \mathcal G) = 0$, i.e., $\mathcal G = 0$. We conclude that $R^jf_*\mathcal F = 0$ for all $j > m$, and therefore, by Proposition 1.3.3,

$$\mathrm{Hom}_{\mathbf D(\mathcal Y)}(\mathbf Rf_*\mathcal F[m], \mathcal O_{\mathcal Y}) \cong \mathrm{Hom}_{\mathcal O_{\mathcal Y}}(R^mf_*\mathcal F, \mathcal O_{\mathcal Y}). \tag{5.6.7}$$

In summary:

Theorem 5.6.8. *Let $f\colon \mathcal X \to \mathcal Y$ be a formally smooth pseudo-proper map of noetherian formal schemes, of relative dimension m. Then $\hat\Omega^m_f$ represents the functor $\mathrm{Hom}_{\mathbf D(\mathcal Y)}(\mathbf Rf_*\mathbf R\varGamma'_{\mathcal X}\mathcal E[m], \mathcal O_{\mathcal Y})$ of quasi-coherent $\mathcal O_{\mathcal X}$-modules $\mathcal E$. If this f is* proper, *then $\hat\Omega^m_f$ represents the functor $\mathrm{Hom}_{\mathcal O_{\mathcal Y}}(R^mf_*\mathcal F, \mathcal O_{\mathcal Y})$ of coherent $\mathcal O_{\mathcal X}$-modules $\mathcal F$.*

To complete the discussion, we review how the map $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}\hat{\Omega}^m_f[m] \to \mathcal{O}_{\mathscr{Y}}$ (resp., when f is proper, the map $R^m f_* \hat{\Omega}^m_f \to \mathcal{O}_{\mathscr{Y}}$) implicit in the proof of Theorem 5.6.8 is *uniquely determined by residues.* We need only look at the first of these maps, since in the proper case, they correspond under the composite isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(\mathscr{Y})}(\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}\hat{\Omega}^m_f[m], \mathcal{O}_{\mathscr{Y}}) &\underset{(5.6.1)}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\hat{\Omega}^m_f[m], f^{\#}\mathcal{O}_{\mathscr{Y}})\\ &\underset{(5.6.1)}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbf{D}(\mathscr{Y})}(\mathbf{R}f_*\hat{\Omega}^m_f[m], \mathcal{O}_{\mathscr{Y}})\\ &\underset{(5.6.7)}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{O}_{\mathscr{Y}}}(R^m f_*\hat{\Omega}^m_f, \mathcal{O}_{\mathscr{Y}}).\end{aligned}$$

That first map corresponds by duality to the fundamental class

$$\mathfrak{f}_f\colon \hat{\Omega}_f[m] \to f^{\#}\mathcal{O}_{\mathscr{Y}} \underset{(5.6.6)}{\cong} (H^{-m}f^{\#}\mathcal{O}_{\mathscr{Y}})[m],$$

and so is determined by $\mathfrak{f}^m_f\colon \hat{\Omega}_f \to H^{-m}f^{\#}\mathcal{O}_{\mathscr{Y}}$, which is in turn uniquely determined by its completions $\kappa^*_x\mathfrak{f}^m_f$ at all closed points x; and Lemma 5.6.5 implies that $\kappa^*_x\mathfrak{f}^m_f$ is dual to the formal residue map $\rho_\varphi\colon \mathrm{H}^m_{\mathfrak{M}}\hat{\Omega}^m_\varphi \to R$ of §5.5.

***

The foregoing provides for formally smooth pseudo-proper maps a *canonical* version of abstractly defined (by Theorem 5.6.1, but only up to isomorphism!) global duality, a version which pastes together all the canonical local dualities—via residues—associated to closed points of $\mathscr{X}$.

When $\mathscr{Y}$ is a perfect field and $\mathscr{X}$ is an ordinary variety, not necessarily smooth, this is essentially the principal result in [Lp1], Theorem (0.6) on p. 24. (See *loc. cit.,* §11 for the smooth case, and for a deduction via traces of differential forms of the main theorem.) A more general relative version, Theorem (10.2), involving a formal completion, starts there on p. 87. Another generalization, to certain maps of noetherian schemes, is given by Hübl and Sastry in [HüS, p. 752, (iii) and p. 785(iii)].

These results should all turn out to be special cases of one Residue Theorem for arbitrary pseudo-proper maps of noetherian formal schemes, for which the constructions sketched in this section provide a foundation. (Work in progress at the time of this writing.)

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