A NUMERICAL CRITERION FOR SIMULTANEOUS NORMALIZATION

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Abstract. We investigate conditions for *simultaneous normalizability* of a family of reduced schemes, i.e., the normalization of the total space normalizes, fiber by fiber, each member of the family. The main result (under more general conditions) is that a flat family of reduced equidimensional projective $\mathbb{C}$-varieties $(X_y)_{y\in Y}$ with normal parameter space Y—algebraic or analytic—admits a simultaneous normalization if and only if the Hilbert polynomial of the integral closure $\overline{\mathcal{O}_{X_y}}$ is locally independent of y. When the X_y are curves projectivity is not needed, and the statement reduces to the well known δ-constant criterion of Teissier. Proofs are basically algebraic, analytic results being related via standard techniques (Stein compacta, etc.) to more abstract algebraic ones.

Introduction

By default, rings are commutative and all schemes and rings are noetherian.

Let $f\colon X \to Y$ be a scheme map which is *reduced* (flat, with all nonempty fibers geometrically reduced). A *simultaneous normalization of* f is a finite map $\nu\colon Z \to X$ such that $\bar f := f\circ\nu$ is *normal* (flat, with all nonempty fibers geometrically normal), and such that for each $y \in f(X)$ the induced map of fibers $\nu_y\colon \bar f^{-1}y \to f^{-1}y$ is a normalization map.[1] For any base change $Y_1 \to Y$ these properties pass over to the projections $f_1\colon X_1 = X\times_Y Y_1 \to Y_1$ and $\nu_1\colon Z_1 = Z\times_X X_1 \to X_1$ (even if Y_1 and X_1 are not noetherian). If Y is normal then any simultaneous normalization of f is itself a normalization map (Theorem 2.3). When a normalization of X is a simultaneous normalization of f, we say that f is *equinormalizable*.

We will consider families (of fibers) given by scheme maps $f\colon X\to Y$ subjected to the following fairly mild conditions (see §1 for information about some of the terms used):

Definition 0.1. A scheme map $f\colon X\to Y$ satisfies (♣) if the following all hold:

- f is reduced.
- For every $y\in f(X)$ the local ring $\mathcal{O}_{Y,y}$ is normal and has geometrically normal formal fibers.

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[1] A finite map $\mu\colon Z'\to Z$ of noetherian schemes is *birational* if it induces an isomorphism of an open dense subscheme of Z' to an open dense subscheme of Z (a condition preserved by flat base change); and it is a *normalization map* if in addition, Z is reduced and Z' is normal.

- X is a formally equidimensional (= locally quasi-unmixed) Nagata scheme. In other words (see 1.6.1), X is a universally catenary Nagata scheme,[2] and for every closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is equidimensional.

Under (♣) the flatness condition on $\bar{f}$ in the definition of simultaneous normalization is superfluous: a normalization $\nu\colon \overline{X} \to X$ is a simultaneous normalization if and only if for each $y \in f(X)$, $\bar{f}^{-1}y$ is geometrically normal (Corollary 2.3.1).

This paper is concerned mainly with numerical conditions on the fibers of a map f satisfying (♣) that characterize equinormalizability.

The culminating results, Theorem 4.2 and its analytic avatar Theorem 5.8, state roughly that when $f\colon X \to Y$ (satisfying (♣)) is projective the sought-after condition is that *the Hilbert polynomial $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}})$ of the normalization of the structure sheaf on the fiber X_y ($y \in Y$) is locally independent of y.* Since flatness implies that $\mathcal{H}_y(\mathcal{O}_{X_y})$ is locally independent of y, one can replace $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}})$ here by $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}}/\mathcal{O}_{X_y})$.

When the fibers of f are one-dimensional, the polynomial $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}}/\mathcal{O}_{X_y})$ is a constant, namely, with $\kappa(y)$ the residue field of $\mathcal{O}_{Y,y}$, it is the sum

$$\sum_{x\in X_y} \delta_y(\mathcal{O}_{X_y,x}) := \sum_{x\in X_y} \dim_{\kappa(y)}(\overline{\mathcal{O}_{X_y,x}}/\mathcal{O}_{X_y,x}).$$

In this case, the projectivity assumption on f is not needed, and the result reduces to the δ-constant criterion of of Teissier and Raynaud [T2, p. 73, Théorème 1].[3] That result has largely inspired the present paper; and though there are gaps in our understanding of the details of the argument presented in *loc. cit.,* nevertheless in broad outline our proof runs along similar lines.

In summary, after recalling some Commutative Algebra in §1, we do the basics for equinormalization in §2, give in §3 some numerical criteria which, together with a supplementary condition (automatically satisfied when the parameter space Y is one-dimensional), characterize equinormalizability, then in §4 use Quot schemes to eliminate the supplementary condition, thereby obtaining, algebraically, the main results. In §5 these results are used to prove the corresponding ones for complex spaces.

Actually, the projective case and the one-dimensional case are treated separately. It would be better were we able to prove a theorem for projective maps of formal schemes (resp. formal complex spaces), which would yield the results for maps $f\colon X \to Y$ of schemes (resp. complex spaces) after completion along the closed subspace $N \subset X$ consisting of points which are not normal on their fibers. (Cf. 2.3.2.) Then the assumption on f could have been weakened to projectivity of the restriction of f to N, a condition

[2] That is, for every affine open subscheme $\mathsf{Spec}\, A \subset X$, A is a universally catenary Nagata ring; or equivalently, X is covered by affine schemes $\mathsf{Spec}\, A_i$ with each A_i a universally catenary Nagata ring, see 1.4.3 and 1.5.2 below. The reader who so prefers may simply assume—with minor loss in generality—that $f\colon X \to Y$ is a reduced map of *excellent* schemes, with X equidimensional and Y normal (cf. 5.1).

[3] That criterion is due to Teissier when Y is the spectrum of a discrete valuation ring—or, in the analytic case, an open disc in $\mathbb{C}$ (cf. 3.3.1 below). On hearing this, Raynaud quickly generalized it to the case of arbitrary normal Y. (Some insight into the background of Raynaud's argument might be gleaned from the introduction to [GrS].) Teissier's result is generalized to deformations of possibly nonreduced curves in [BG, Korollar 3.2.1].

satisfied both when f itself is projective and when the fibers of f are reduced curves (indeed—locally—when $\dim N = 0$).

A second unresolved issue raised by the main results is why the *global* object $\mathcal{H}_y$ should be involved in the characterization of what is really a *local* phenomenon. To wit: it follows from 1.3.1, 1.5.2, 1.4.1, 1.6.1 and 1.1.1 that if (♣) holds for f then for all $x \in X$, (♣) holds for the map $f_x \colon \mathsf{Spec}\,\mathcal{O}_{X,x} \to \mathsf{Spec}\,\mathcal{O}_{Y,f(x)}$ induced by f; and by a straightforward globalization of 1.4.3 the converse holds iff every integral scheme finite over X has a nonempty open normal subscheme; also, with $\nu_X \colon \overline{X} \to X$ a normalization, $f \circ \nu_X$ is normal iff with f_x as above and $\nu_{X,x}$ the normalization of $\mathsf{Spec}\,\mathcal{O}_{X,x}$, $f_x \circ \nu_{X,x}$ is normal for all $x \in X$. In fact, the existence of a simultaneous normalization depends, ordinarily, only on the completions of the local rings involved, see 2.3.2.

The question is, *are there local invariants, somehow related to $\mathcal{H}_y$, that characterize equinormalizability at a point?*

(For one-dimensional fibers, the δ-invariant provides an affirmative answer.)

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Placing our results in a broader context we ask, what sort of information about "singularity type" does normalization convey? In slightly less vague terms, we'd like to know more about the relation of equinormalization to various versions of equisingularity. Even for curves equinormalization is weaker, an example being the equinormalizable—but not equisingular—family $y^2 = tx^2 + x^3$ parametrized by $t \in \mathbb{C}$. (Here $\delta_t \equiv 1$.)

The question is, *which equisingularity conditions imply equinormalizability?*

For example, suppose $f \colon X \to Y$ is a map of $\mathbb{Q}$-schemes that satisfies (♣) and admits a *weak simultaneous resolution,* that is, there exists a proper birational map $\pi \colon \tilde{X} \to X$ such that $f\pi$ is smooth and and for each $y \in f(X)$ the induced map of fibers $\pi_y \colon \tilde{X}_y \to X_y$ is birational. As indicated above, the question of equinormalizability is local, so let us assume, for simplicity, that X and Y are affine. We have the Stein factorization through the normalization $\tilde{X} \to \overline{X} = \mathsf{Spec}\,\mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \xrightarrow{\nu} X$. That ν be a simultaneous normalization of f means, by 2.3.1 below, that "Stein factorization commutes with passage to the fibers," i.e., that with $\kappa(y)$ the residue field of $\mathcal{O}_{Y,y}$,

$$\mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \otimes_R \kappa(y) \cong \mathrm{H}^0(\tilde{X}_y, \mathcal{O}_{\tilde{X}_y}) \quad (y \in Y).$$

For this to hold *it suffices that* $\mathrm{H}^q(\tilde{X}, \mathcal{O}_{\tilde{X}})$ *be R-flat for all* $q \geq 0$. For, these H^q can be calculated by the R-flat Čech complex $C^\bullet$ associated to a finite affine covering of $\tilde{X}$; and for any R-module M and $p > 0$, flatness of H^q gives $\mathsf{Tor}_p(\mathrm{H}^q(C^\bullet), M) = 0$, so that the well-known spectral sequences [EGIII, (6.3.2)] yield the second of the natural isomorphisms

$$\mathrm{H}^q(\tilde{X}, \mathcal{O}_{\tilde{X}}) \otimes_R M \cong \mathrm{H}^q(C^\bullet) \otimes_R M \cong \mathrm{H}^q(C^\bullet \otimes_R M) \qquad (q \geq 0)$$

(where the last term $\cong \mathrm{H}^0(\tilde{X}_y, \mathcal{O}_{\tilde{X}_y})$ when $q = 0$ and $M = \kappa(y)$).

This suggests that *in considering the existence of a simultaneous resolution $\pi \colon \tilde{X} \to X$ as a condition of equisingularity of the family $f \colon X \to Y$ one may wish to impose the additional condition that all the higher direct images $R^q\pi_*\mathcal{O}_{\tilde{X}}$ are Y-flat.*

In this connection Jonathan Wahl has shown us an example of a one-parameter family of integral two-dimensional quasihomogeneous singularities with an "equitopological" simultaneous resolution but no simultaneous normalization. In fact the total space of this family is normal, but one of the members is not normal. Here $R^1\pi_*\mathcal{O}_{\tilde{X}}$ is not flat over the base. For more in this vein, see [Wa, p. 341, Thm. 4.6(i)].

Equinormalizability also holds for *equisaturated* families of singularities, in characteristic zero, see [Z2, pp. 1017–1018]. The arguments given there for hypersurfaces apply somewhat more generally. Consider a power-series ring $P := \mathbb{C}[[x_1, \dots, x_r, t_1, \dots, t_s]]$ and a finite module-free P-algebra with $A/(t_1, \dots, t_s)A$ reduced. Let $\overline{A}$ be the integral closure of A. Let $\nu\colon \mathsf{Spec}\,\overline{A} \to \mathsf{Spec}\,A$ and $f\colon \mathsf{Spec}\,A \to \mathsf{Spec}\,\mathbb{C}[[t_1, \dots, t_s]]$ be the obvious maps. Suppose *the discriminant of the P-algebra A (a principal P-ideal) is generated by a power series independent of the t_i*. Then by a simple extension of [Z1, p. 527, Thm. 5] (see [Lp, p. 882, Cor. 1]), $\overline{A}$ is of the form $A_0[[t_1, \dots, t_s]]$, so the closed fiber $\mathsf{Spec}\,A_0$ of $f\nu$ is normal, and by 1.7.1, and 2.3.1 below, ν is a simultaneous normalization of f.

What about *Whitney equisingularity* and equinormalization? For families of plane curve singularities Whitney equisingularity is equivalent to equisaturation, so it does imply equinormalizability. For certain families of nonplanar curve singularities, the implication still holds [BGG, p. 27, Thm. III.8]. It would be nice if there were some relation in higher-dimensional situations; but at present we have no information to offer.

1. Preliminaries.

In this section we go over briefly some subsequently-used notation, terminology, and basic (long-known) results from commutative algebra.

The total quotient ring of a commutative ring A will be denoted by K_A, and the integral closure of A in K_A by $\overline{A}$. For a prime ideal $\mathfrak{p} \in \mathsf{Spec}\,A$, the residue field of the local ring $A_\mathfrak{p}$ (= field of fractions of $A/\mathfrak{p}$) will be denoted by $\kappa(\mathfrak{p})$. A noetherian local ring A with maximal ideal $\mathfrak{m}$ will be denoted by $(A, \mathfrak{m})$. When we need to refer to the residue field $k_A := \kappa(\mathfrak{m})$, we may also write $(A, \mathfrak{m}, k_A)$.

Definition 1.1. Let k be a field. A noetherian k-algebra A is *geometrically reduced* (resp. *geometrically normal*) if the ring $A \otimes_k k'$ is reduced (resp. normal)[4] for every field extension $k' \supset k$. (It suffices that this be so for every finite, purely inseparable $k' \supset k$, see [EGA, (6.7.7), (4.6.1), (6.14.1)]. Thus if k is perfect or of characteristic 0, then [A reduced] $\Rightarrow$ [A geometrically reduced], and similarly for "normal.")

1.1.1. These geometric properties are local: they hold for A if and only if they hold for $A_\mathfrak{p}$ for all $\mathfrak{p} \in \mathsf{Spec}\,A$.

Definition 1.2. A ring homomorphism $\psi\colon A \to B$ is *reduced* (resp. *normal*) if ψ is flat and for every $\mathfrak{p} \in \mathsf{Spec}\,A$ such that $\mathfrak{p}B \neq B$ the corresponding $\kappa(\mathfrak{p})$-algebra $B \otimes_A \kappa(\mathfrak{p})$ is geometrically reduced (resp. geometrically normal).

1.2.1. These properties hold for ψ iff they hold for the induced maps $\psi_{\mathfrak{P},\mathfrak{p}}\colon A_\mathfrak{p} \to B_\mathfrak{P}$ for all $\mathfrak{P} \in \mathsf{Spec}\,B$ and $\mathfrak{p} = \psi^{-1}\mathfrak{P}$.

Definition 1.3. A noetherian ring A satisfies $\mathrm{ff}_{\mathrm{red}}$ (resp. $\mathrm{ff}_{\mathrm{nor}}$) if for every maximal A-ideal $\mathfrak{m}$ the canonical map $A_\mathfrak{m} \to \widehat{A_\mathfrak{m}}$ from the local ring $A_\mathfrak{m}$ to its completion is reduced (resp. normal). [Here "ff" is meant to suggest "formal fibers."]

[4] "A is normal" means, by definition, that $A_\mathfrak{p}$ is an integrally closed domain for all $\mathfrak{p} \in \mathsf{Spec}\,A$.

1.3.1. If A has either of these ff_ properties then so does any A-algebra essentially of finite type (i.e., any localization of a finitely-generated A-algebra), see [EGA, (7.4.5), (7.3.8), (7.3.13)].

Definition 1.4. A *Nagata ring*[5] is a noetherian ring A such that for every $\mathfrak{p} \in \operatorname{Spec} A$ and every finite field extension $L \supset \kappa(\mathfrak{p})$, the integral closure of $A/\mathfrak{p}$ in L is module-finite over $A/\mathfrak{p}$.

1.4.1. If A is a Nagata ring, then so is any A-algebra essentially of finite type ([Ng, p. 131, (36.1) and p. 132, (36.5)].

1.4.2. If A is a reduced Nagata ring, then $\overline{A}$ is module-finite over A.

Proof. Being noetherian, A has only finitely many minimal prime ideals, say $\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_t$; and being reduced, A is naturally identifiable with a subring of $A' := \prod_{i=1}^t A/\mathfrak{p}_i$. Then any regular element in A is outside $\mathfrak{p}_i$ for all i and hence is A'-regular. Thus we may write $A \subset K_A \subset K_{A'} = \prod_{i=1}^t \kappa(\mathfrak{p}_i)$, and conclude that $\overline{A} \subset \overline{A'} = \prod_{i=1}^t \overline{A/\mathfrak{p}_i}$. (Actually, $\overline{A} = \overline{A'}$.) Since A is a Nagata ring therefore for each i, $\overline{A/\mathfrak{p}_i}$ is module-finite over $A/\mathfrak{p}_i$, hence over A, and consequently $\overline{A} \subset \prod_{i=1}^t \overline{A/\mathfrak{p}_i}$ is module-finite over A. $\square$

1.4.3. A is a Nagata ring iff A satisfies ff_{red} and for every reduced finitely generated A-algebra A' the set of normal points is open and dense in $\operatorname{Spec} A'$, this last condition being implied by ff_{red} when A is semi-local. (It suffices that for all module-finite A-algebras $\psi\colon A \to A'$ with A' a domain whose fraction field is finite and purely inseparable over that of $A/\operatorname{Ker}(\psi)$, $\operatorname{Spec} A'$ has a nonempty normal open subscheme, see [EGA, (6.13.2), (7.6.4), and (7.7.2)].)

Definition 1.5. A noetherian ring A is *catenary* if for any two prime A-ideals $\mathfrak{p} \supset \mathfrak{q}$ it holds that any two saturated chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ have the same length ([Ma2, p. 31]). A is *universally catenary* if every finite-type A-algebra B is catenary.

1.5.1. Let $B \supset A$ be noetherian integral domains with B a finite-type A-algebra. We say "the dimension formula holds between A and B" if for all $\mathfrak{P} \in \operatorname{Spec} B$ and $\mathfrak{p} = \mathfrak{P} \cap A$,

$$\dim B_{\mathfrak{P}} + \text{tr.deg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{P}) = \dim A_{\mathfrak{p}} + \text{tr.deg}_{K_A} K_B.$$

A noetherian ring A is universally catenary iff the dimension formula holds between $A/\mathfrak{p}$ and B for every $\mathfrak{p} \in \operatorname{Spec} A$ and every integral domain $B \supset A/\mathfrak{p}$ of finite type over $A/\mathfrak{p}$, see [Ma2, p. 119, Theorem 15.6]. (It suffices that this be so for every *minimal* $\mathfrak{q} \in \operatorname{Spec} A$, see [R1, p. 511, (2.8) and p. 513, Prop. 2.13].)

1.5.2. If the noetherian ring A is universally catenary then so is any A-algebra essentially of finite type. Conversely, if $A_{\mathfrak{m}}$ is universally catenary for every maximal A-ideal $\mathfrak{m}$ then A is universally catenary. (See [EGA, Remarque (5.6.3)].)

Definition 1.6. A noetherian ring A is *equidimensional* if $\dim(A/\mathfrak{p}) = \dim(A) < \infty$ for all minimal prime A-ideals $\mathfrak{p}$; and *biequidimensional* if *any* maximal chain of (distinct) prime ideals in A has length $\dim A < \infty$. A is *formally equidimensional* (or "locally quasi-unmixed") if the $\mathfrak{p}$-adic completion of $A_{\mathfrak{p}}$ is equidimensional for every $\mathfrak{p} \in \operatorname{Spec} A$. (It suffices that this be so for every *maximal* $\mathfrak{p} \in \operatorname{Spec} A$, see [Ma2, p. 251, Thm. 31.6]).

[5]Also known as *pseudo-geometric ring* or *universally Japanese ring.*

1.6.1. A noetherian ring A is formally equidimensional if and only if A is universally catenary and $A_{\mathfrak{m}}$ is equidimensional for each maximal ideal $\mathfrak{m}$. (This is the equivalence of (1) and (2) in [R2, p. 104, Corollary 2.7].) So if A is formally equidimensional then for every $\mathfrak{p} \in \mathsf{Spec}\, A$ the local ring $A_{\mathfrak{p}}$, being formally equidimensional, is equidimensional. It also follows that if A has only one minimal prime ideal, or if A is biequidimensional, then A is formally equidimensional if and only if A is universally catenary.

1.6.2. If A is reduced and equidimensional, and B is a noetherian A-algebra integral over A and torsion-free as an A-module, then B is equidimensional. Indeed, torsion-freeness implies that the inverse image p in A of a minimal prime B-ideal P is a minimal prime A-ideal, and B/P is integral over A/p, so $\dim B/P = \dim A/p = \dim A$.

1.6.3. Suppose A reduced. If A is formally equidimensional (resp. biequidimensional and formally equidimensional), then so is any module-finite torsion-free A-algebra B. Indeed, B is universally catenary by 1.5.2, so it suffices, because of 1.6.1, to show that $B_{\mathfrak{P}}$ is equidimensional (resp. equidimensional of dimension $\dim A$) for any maximal B-ideal $\mathfrak{P}$. Let $\mathfrak{p}$ be the inverse image of $\mathfrak{P}$ in A, a maximal A-ideal, and let $P \subset \mathfrak{P}$ and $p \subset \mathfrak{p}$ be minimal primes as in 1.6.2. Since A is universally catenary, the dimension formula holds between A/p and its finite extension ring B/P, so that

$$\dim B_{\mathfrak{P}}/PB_{\mathfrak{P}} = \dim(B/P)_{\mathfrak{P}} + \mathrm{tr.deg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{P}) = \dim(A/p)_{\mathfrak{p}} + \mathrm{tr.deg}_{\kappa(p)}\kappa(P) = \dim A_{\mathfrak{p}},$$

whence the conclusion.

1.7. The following results (see [Ni, p. 157, (2.4)]), deducing properties of general fibers of certain flat maps from those of closed fibers, will be very useful.

Theorem 1.7.1. (Nishimura, André) *If $\phi\colon (R, \mathfrak{m}, k_R) \to (A, \mathfrak{M}, k_A)$ is a local homomorphism such that*

(i) *R satisfies* $\mathrm{ff}_{\mathrm{red}}$ (*resp.* $\mathrm{ff}_{\mathrm{nor}}$),
(ii) *the k_R-algebra $A/\mathfrak{m}A$ is geometrically reduced* (*resp. geometrically normal*), *and*
(iii) *ϕ is flat,*

then the map ϕ is reduced (*resp. normal*).

2. Equinormalization.

The main results of this section, Theorem 2.3 and its corollaries, characterize equinormalizability of a map $f\colon X \to Y$ satisfying (♣) (see Introduction) in terms of a normalization of X, and show that equinormalizability is a strictly local property.

2.1. We begin with a few facts relating condition (♣) and base change. Terminology remains as above. A generic point of irreducible component of a scheme W is simply called a *generic point of W*.

Proposition 2.1.1. *Let $f\colon X \to Y$ satisfy* (♣). *Let $\mu\colon Z \to X$ be a finite birational map*[6] *with Z reduced. Let Y_1 be a reduced, essentially-finite-type Y-scheme,*[7] *and for any Y-scheme W set $W_1 := W \times_Y Y_1$. Then:*

[6]See Introduction. Birationality also means μ induces a bijection from generic points ζ of Z to generic points of X such that for each ζ, $\mathcal{O}_{X,\mu(\zeta)} \to \mathcal{O}_{Z,\zeta}$ is an isomorphism [EGI, p. 312, (6.6.4)(ii)].

[7]In other words, Y has a covering by affine open subsets $\mathsf{Spec}\, A_i$ whose inverse image in Y_1 is covered by affine open subsets $\mathsf{Spec}\, A'_{ij}$ with each A'_{ij} a localization of a finitely-generated A_i-algebra.

(i) *X_1 is a reduced universally catenary Nagata scheme, for which any normalization map ν_1 factors uniquely as*

$$\overline{X_1} \xrightarrow{\alpha_{Z,Y_1}} Z_1 \xrightarrow{\mu_1:=\mu\times 1} X_1.$$

(ii) *The inverse image of the nonnormal locus of X is a nowhere dense closed subset of X_1.*

(iii) *If Y_1 is normal then the projection $f_1\colon X_1 \to Y_1$ satisfies (♣).*

(iv) *For all $x \in X$ and $y := f(x)$, if $\hat{f}_x\colon \mathsf{Spec}\,\widehat{\mathcal{O}_{X,x}} \to \mathsf{Spec}\,\widehat{\mathcal{O}_{Y,y}}$ is the map induced by f (where $\widehat{\ }$ denotes completion) then $\hat{f}_x$ satisfies (♣). More generally, let $\phi\colon R \to A$ be a ring homomorphism such that the corresponding scheme map $f\colon \mathsf{Spec}\,A \to \mathsf{Spec}\,R$ satisfies (♣). Let I be an R-ideal such that R/IR is a finite-dimensional Nagata ring, and let $J \supset IA$ be an A-ideal. Let $\hat{R}$ (resp. $\hat{A}$) be the I-adic completion of R (resp. J-adic completion of A). Then the map $\hat{f}\colon \mathsf{Spec}\,\hat{A} \to \mathsf{Spec}\,\hat{R}$ induced by f satisfies (♣).*

Proof. That X_1 is reduced is given by [Ma2, p. 184, Cor. (ii)] or by [EGA, (6.15.10)];[8] that X_1 is universally catenary, by 1.5.2; and that X_1 is a Nagata scheme, by 1.4.1.

In particular, X is a reduced Nagata scheme, so its nonnormal locus is closed (1.4.3). Hence, to prove (ii) we need only show that *if $x \in X$ is the image of a generic point x_1 of X_1 then X is normal at x.*

If $y := f(x)$, then the genericity of x_1 in X_1—hence in its fiber over Y_1—implies that x is a generic point of the fiber $f^{-1}y$. Since $f^{-1}y$ is geometrically reduced, one sees that the closed fiber of the flat map $f_x\colon \mathsf{Spec}\,\mathcal{O}_{X,x} \to \mathsf{Spec}\,\mathcal{O}_{Y,y}$ induced by f (namely, Spec of the residue field $\kappa(x)$) is geometrically normal. So by 1.7.1 the map f_x is normal; and by [Ma2, p. 184, Cor. (ii)] $\mathcal{O}_{X,x}$ is normal, proving (ii).

For the rest of (i), let $V_1 = \mathsf{Spec}\,A \subset X_1$ be an affine open subscheme. A is a reduced noetherian ring, and $\nu_1^{-1}V_1 = \mathsf{Spec}\,\overline{A}$. Since μ_1 is finite, $\mu_1^{-1}V_1 = \mathsf{Spec}\,B$ with B a module-finite A-algebra. Again, the normal points of X form an open dense subscheme U; and μ induces an isomorphism from $\mu^{-1}U$ to U, hence from $\mu_1^{-1}U_1$ to U_1. Furthermore, as we have just seen, U_1 contains every generic point of X_1. Thus μ_1 is an isomorphism over a neighborhood of any generic point of X_1, so for every minimal prime A-ideal $\mathfrak{p}$ there is a unique minimal prime B-ideal $\mathfrak{P}$ with inverse image $\mathfrak{p} \subset A$, and a natural isomorphism $A_\mathfrak{p} \cong B_\mathfrak{P}$. In particular, the natural map $A \to B$ is injective. Since A is reduced we have $K_A = \prod_\mathfrak{p} A_\mathfrak{p} \cong \prod_\mathfrak{P} B_\mathfrak{P}$. Let $B \xrightarrow{\alpha} K_A$ be the natural map, so that $A \subset \alpha(B) \subset K_A$. Since B is finite over A, therefore $\alpha(B) \subset \overline{A}$, so we have a factorization of the normalization map $A \hookrightarrow \overline{A}$.

This factorization is unique. Indeed, since $A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ is an isomorphism for each minimal prime A-ideal $\mathfrak{p}$ therefore the annihilator of the A-module B/A is contained in no such $\mathfrak{p}$, and so it contains a regular element $h \in A$ such that $h\alpha(B) \subset A$; and if $A \to B \xrightarrow{\alpha'} \overline{A}$ is another factorization of $A \hookrightarrow \overline{A}$, and $b \in B$, then with this h we have

$$h\alpha(b) = \alpha(hb) = \alpha'(hb) = h\alpha'(b)$$

whence, h being a unit in K_A, $\alpha(b) = \alpha'(b)$.

[8] Replace the map f (resp. g) in *loc. cit.* by our $Y_1 \to Y$ (resp. our $f\colon X \to Y$); and note that in the first paragraph of the proof there, g need only be *reduced*.

Finally, a simple pasting argument gives the existence and uniqueness of the asserted global factorization of ν_1.

(iii) From 1.3.1 it follows that all the local rings of points on Y_1 satisfy $\mathrm{ff}_{\mathrm{nor}}$. It is clear that f_1 is flat and has geometrically reduced fibers (any such fiber being obtained from a fiber of f by base change to a finitely-generated field extension). As in (i), X_1 is a universally catenary Nagata scheme. For the equidimensionality condition (local on X_1), we may assume that $Y = \mathsf{Spec}\, R$ with R a normal domain, and that $Y_1 = \mathsf{Spec}\, R_1$ where R_1 is a homomorphic image of a localization at a prime ideal of a polynomial ring $R[T_1, T_2, \dots, T_n]$. As [Ma2, p. 250, 31.5] takes care of the "homomorphic image" part, we reduce inductively to the case where $Y_1 = \mathsf{Spec}\, R_1$ with R_1 the localization at a prime ideal of the one-variable polynomial ring $R[T]$.

So, in view of 1.6.1, we need only show that if A is a noetherian ring all of whose localizations are equidimensional then so is $B := A[T]$—in other words, if $\mathfrak{P}$ is a prime B-ideal then $B_{\mathfrak{P}}$ is equidimensional. We can replace A by its localization at $\mathfrak{p} := \mathfrak{P} \cap A$, so we may assume that A is local, with maximal ideal $\mathfrak{p}$. Then $B/\mathfrak{p}B = (A/\mathfrak{p})[T]$, and so either (a): $\mathfrak{P} = \mathfrak{p}B$ or (b): $\dim B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}} = 1$.

Let $\mathfrak{Q} \subset \mathfrak{P}$ be a minimal prime $A[T]$-ideal, necessarily equal to $\mathfrak{q}A[T]$ with $\mathfrak{q} := \mathfrak{Q} \cap A$ a minimal prime A-ideal. Then by [Ma2, p. 116, 15.1], $\dim B_{\mathfrak{P}}/\mathfrak{Q} = \dim A/\mathfrak{q}$ in case (a), and $\dim B_{\mathfrak{P}}/\mathfrak{Q} = \dim A/\mathfrak{q} + 1$ in case (b). In either case, $B_{\mathfrak{P}}$ is equidimensional.

(iv) Any maximal $\hat{A}$-ideal is of the form $\mathfrak{m}\hat{A}$ for some maximal A-ideal $\mathfrak{m}$, and the $\hat{\mathfrak{m}}$-adic completion of $\hat{A}$ is the $\mathfrak{m}$-adic completion of A, so is equidimensional. Thus $\hat{A}$ is formally equidimensional (see 1.6), hence universally catenary (see 1.6.1). Moreover, by [Mr, p. 107, 2.3], $\hat{A}$ is a Nagata ring.

By [EGA, (7.4.6)] the fibers of the canonical map $\mathsf{Spec}\, \hat{R} \to \mathsf{Spec}\, R$ are geometrically normal, whence by [Ma2, p. 184, Cor. (ii)], $\hat{R}$ is normal. If $\mathfrak{q} \in \mathsf{Spec}\, \hat{R}$ is the $\hat{f}$-image of $\mathfrak{p} \in \mathsf{Spec}\, \hat{A}$ then $\hat{A}_{\mathfrak{p}}$ is a faithfully flat $\hat{R}_{\mathfrak{q}}$-algebra ([Bo, p. 103]), whence the polynomial ring $\hat{A}_{\mathfrak{p}}[T]$ is a faithfully flat $\hat{R}_{\mathfrak{q}}[T]$-algebra. As in the proof of (iii), since $\hat{A}$ is formally equidimensional, $\hat{A}_{\mathfrak{p}}[T]$ is locally equidimensional and universally catenary (see 1.6.1). By [Ma2, p. 252, 31.7] and [Ma2, p. 250, 31.5], it follows that $\hat{R}_{\mathfrak{q}}$ is universally catenary. Since $\hat{R}/I\hat{R} \cong R/IR$ is finite dimensional and $\hat{R}$ is a catenary normal ring all of whose maximal ideals contain $I\hat{R}$, therefore $\hat{R}$ is finite dimensional; and moreover the formal fibers of the Nagata ring $\hat{R}/I\hat{R}$ (i.e., of R/IR) are geometrically normal. Now [BR, Satz 2] gives that the formal fibers of $\hat{R}$ are geometrically normal.

It remains to show that $\hat{f}$ is reduced. By [Bo, p. 103], $\hat{A}$ is a flat $\hat{R}$-algebra. Whether every fiber $\hat{f}^{-1}\mathfrak{q}$ ($\mathfrak{q} \in \mathsf{Spec}\, \hat{R}$) is geometrically reduced is a local question (see 1.1.1), so we need only show for each $\mathfrak{p}$ as above that $\hat{A}_{\mathfrak{p}}/\mathfrak{q}\hat{A}_{\mathfrak{p}}$ is geometrically reduced. There is a maximal $\hat{A}$-ideal $\mathfrak{p}'$ containing $\mathfrak{p}$, whose $\hat{f}$-image is a prime $\hat{R}$-ideal $\mathfrak{q}' \supset I\hat{R}$; and it will suffice then to show that all the fibers of the map $\mathsf{Spec}\, \hat{A}_{\mathfrak{p}'} \to \mathsf{Spec}\, \hat{R}_{\mathfrak{q}'}$ induced by $\hat{f}$ are geometrically reduced. In view of 1.7.1, it will even suffice to show that $\hat{f}^{-1}\{\mathfrak{q}'\}$ is geometrically reduced.

For this, let $\mathfrak{q}''$ be the inverse image in R of $\mathfrak{q}'$. Since $\hat{R}/I\hat{R} \cong R/IR$ and $\mathfrak{q}' \supset I\hat{R}$ therefore $\mathfrak{q}' = \mathfrak{q}''\hat{R} + I\hat{R}$; and as $I\hat{R}$ is the Jacobson radical of $\hat{R}$, this means that $\mathfrak{q}' = \mathfrak{q}''\hat{R}$, whence $\hat{A}/\mathfrak{q}'\hat{A}$ is the J-adic completion of $A/\mathfrak{q}''A$.

By (i), $A/\mathfrak{q}''A$ is a reduced Nagata ring, so $A/\mathfrak{q}''A$ satisfies $\mathrm{ff}_{\mathrm{red}}$ (see 1.4.3). Consequently, the fibers of the natural map $\mathsf{Spec}\,(\hat{A}/\mathfrak{q}'\hat{A}) \to \mathsf{Spec}\,(A/\mathfrak{q}''A)$ are geometrically reduced [EGA, (7.4.6)], whence $\hat{A}/\mathfrak{q}'\hat{A}$ is reduced [Ma2, p. 184, Cor. (ii)]. Thus $\hat{f}^{-1}\{\mathfrak{q}'\}$—the Spec of a localization of $\hat{A}/\mathfrak{q}'\hat{A}$—is indeed reduced.

Similar considerations apply if $\hat{A}/\mathfrak{q}'\hat{A}$ and $A/\mathfrak{q}''A$ are replaced by their respective tensor products with a finite integral $\hat{R}/\mathfrak{q}'(\cong R/\mathfrak{q}'')$-algebra whose fraction field is purely inseparable over that of $R/\mathfrak{q}''$. It follows that $\hat{f}^{-1}\{\mathfrak{q}'\}$ is *geometrically* reduced, as desired. $\square$

2.2. Here are some conditions under which the map $\alpha := \alpha_{Z,Y_1}$ in 2.1.1 is *schematically dominant,* i.e., the associated map $\mathcal{O}_{Z_1} \to \alpha_*\mathcal{O}_{\overline{X_1}}$ is injective ([EGI, p. 284 (5.4.2)]). This condition is easily seen to be equivalent to "Z_1 reduced and μ_1 birational."

For any $y \in Y$ and any scheme map $g\colon W \to Y$, $\kappa(y)$ denotes the residue field of the local ring $\mathcal{O}_{Y,y}$ and W_y denotes the fiber $g^{-1}y := W \times_Y \mathsf{Spec}\,\kappa(y)$. Recall that an *associated point* of a scheme W is one in whose local ring every nonunit is a zerodivisor.

Proposition 2.2.1. *With notation and assumptions as in* 2.1.1, *the following conditions are equivalent.*

(i) *For all* Y_1, Z_1 *is reduced and* μ_1 *is birational.*

(ii) *For all* $y \in Y$, $\mu \times 1$ *maps any associated point of* Z_y *to a generic point of* X_y.

(ii)$'$ *Condition* (ii) *holds for all* $y = f(x)$ *with* x *a closed point of* X.

(iii) *For all* $z \in Z$, *with* $x := \mu(z)$ *and* $y := f(x)$ *the local ring* $\mathcal{O}_{Z_y,z}$ *is reduced and equidimensional of dimension* $\dim \mathcal{O}_{X_y,x}$.

(iii)$'$ *For all* $y = f(x)$ *with* x *closed in* X, *every associated point of* Z_y *is generic.*

(iv) *The map* $f\mu\colon Z \to Y$ *satisfies* (♣).

(iv)$'$ *The map* $f\mu\colon Z \to Y$ *is reduced.*

When these conditions hold, the cokernel of the natural map $\mathcal{O}_X \to \mu_*\mathcal{O}_Z$ *is* Y*-flat.*

Proof. It is clear that (i) for $Y_1 = \mathsf{Spec}\,\kappa(y)$ implies (ii), and trivial that (ii) $\Rightarrow$ (ii)$'$, (iii) $\Rightarrow$ (iii)$'$ and (iv) $\Rightarrow$ (iv)$'$ $\Rightarrow$ (iii)$'$. To establish all the equivalences it suffices then to show that (ii) $\Rightarrow$ (iii), (iii)$'$ $\Rightarrow$ (ii)$'$, (ii)$'$ $\Rightarrow$ (i), and (ii)$'$ $\Rightarrow$ (iv). Along the way, in Lemma 2.2.3, we will show that the last assertion in the Proposition follows from (ii)$'$.

(ii) $\Rightarrow$ (iii). Let $R := \mathcal{O}_{Y,y}$, with maximal ideal $\mathfrak{m}$, and let $A := \mathcal{O}_{X,x}$. Since μ is finite and birational, and Z is reduced, therefore $Z \times_X \mathsf{Spec}\,A = \mathsf{Spec}\,B$ where $A \subset B \subset \overline{A}$. Set $A_1 := A/\mathfrak{m}A$, $B_1 := B/\mathfrak{m}B$. By (♣), A_1 is reduced, i.e., $\mathfrak{m}A$ is an intersection of prime ideals. Since B is a finite extension of A, it follows that $\mathfrak{m}A = \mathfrak{m}B \cap A$, i.e., the natural map $A_1 \to B_1$ is injective, making B_1 a finite A_1-module. Hence $\dim B_1 = \dim A_1$.

If (ii) holds then B_1 is a torsion-free A_1-module, because any regular element $\bar{h} \in A_1$ lies in no minimal prime of A_1, hence in no associated prime of (0) in B_1, so that $\bar{h}$ is B_1-regular. That B_1 is reduced follows easily from the existence, given by Lemma 2.2.2 below, of an A_1-regular—hence B_1-regular—element $h \in A$ such that $hB \subset A$ (whence $hB_1 \subset A_1$). By (♣) and 1.6.1, A is universally catenary, equidimensional, and flat over R; so by [Ma2, p. 250, 31.5], A_1 is equidimensional, and it is also universally catenary (since A is), hence biequidimensional. Then 1.6.3 shows that B_1 is biequidimensional, and since $\dim B_1 = \dim A_1$, (iii) follows.

[Conversely, assuming (iii) we have in the above situation that B_1 is reduced and equidimensional, and if $\mathfrak{P}$ is a minimal prime of B_1 and $\mathfrak{p} := \mathfrak{P} \cap A_1$ then, since B_1 is an integral extension of A_1 and $B_1/\mathfrak{P}$ is an integral extension of $A_1/\mathfrak{p}$,

$$\dim B_1 = \dim B_1/\mathfrak{P} = \dim A_1/\mathfrak{p} \le \dim A_1 = \dim B_1,$$

whence $\mathfrak{p}$ is a minimal prime of A_1; and (ii) follows.]

Lemma 2.2.2. *Under the preceding circumstances, there exists $h \in A$ such that*:
(a) $0 :_A h = 0$.
(b) $\mathfrak{m}A :_A h = \mathfrak{m}A$.
(c) $h\overline{A} \subset A$.
(d) $h\overline{A_1} \subset A_1$.

Proof. Let $\mathfrak{C} = A :_A \overline{A}$ be the conductor of A, let $\mathfrak{C}_1$ be the conductor of A_1, and with $\pi\colon A \to A_1$ the natural map let $\mathfrak{C}' := \pi^{-1}\mathfrak{C}_1$. Since both A and A_1 are reduced (see 2.1.1(i)), it suffices that there exist $h \in \mathfrak{C} \cap \mathfrak{C}'$ such that h belongs to no minimal prime divisor $\mathfrak{P}$ of (0) or of $\mathfrak{m}A$, or equivalently, that $\mathfrak{C} \cap \mathfrak{C}'$ is not contained in any such $\mathfrak{P}$, i.e., neither $\mathfrak{C}$ nor $\mathfrak{C}'$ is contained in any such $\mathfrak{P}$.

If $\mathfrak{P}$ is a minimal prime divisor of (0) then the field $A_\mathfrak{P}$ is normal; and if $\mathfrak{P}$ is a minimal prime divisor of $\mathfrak{m}A$ then $A_\mathfrak{P}$ is normal, by 2.1.1(ii) for $Y_1 = \mathsf{Spec}\,(R/\mathfrak{m})$. In either case, since $\overline{A}$ is module-finite over the Nagata ring A (see 1.4.2) therefore $\mathfrak{C}_\mathfrak{P}$ is the conductor of $A_\mathfrak{P}$, so $\mathfrak{C} \not\subset \mathfrak{P}$.

As $\overline{A_1}$ is module-finite over the reduced Nagata ring A_1, $\mathfrak{C}'$ cannot be contained in any minimal prime divisor of $\mathfrak{m}A$. Finally, if $\mathfrak{P}$ is a minimal prime divisor of (0) then $\mathfrak{C}' \not\subset \mathfrak{P}$ because the natural map $\phi\colon R \to A$ is flat, so $\phi^{-1}\mathfrak{P} = (0)$ and

$$\mathfrak{C}' \subset \mathfrak{P} \implies \mathfrak{m}A \subset \mathfrak{P} \implies \mathfrak{m} = \phi^{-1}(\mathfrak{m}A) \subset \phi^{-1}\mathfrak{P} = (0) \implies A_1 = A \implies \mathfrak{C}' = \mathfrak{C} \not\subset \mathfrak{P}.$$

□

(iii)$'$ ⇒ (ii)$'$. It suffices to show that if z is a generic point of Z_y then $x' := \mu(z)$ is a generic point of X_y. Let $(R, \mathfrak{m}) := \mathcal{O}_{Y,y}$, with maximal ideal $\mathfrak{m}$, $A' := \mathcal{O}_{X,x'}$, $B' := \mathcal{O}_{Z,z}$. It follows from 1.6.3 that $\dim B' = \dim A'$. Then [Ma2, p. 116, 15.1] gives

$$0 = \dim(B'/\mathfrak{m}B') \ge \dim B' - \dim R = \dim A' - \dim R = \dim(A'/\mathfrak{m}A'),$$

which shows that x' is indeed a generic point of X_y.

For the remaining implications we will need:

Lemma 2.2.3. *If* 2.2.1(ii)$'$ *holds then $f\mu$ is flat and the cokernel of $\mathcal{O}_X \to \mu_*\mathcal{O}_Z$ is Y-flat.*

Proof. Let $x \in X$, $y := f(x)$, and let $R \to A \subset B$ be as above. It suffices to show that both B and B/A are R-flat. The point x specializes to a closed point of X, whose f-image is a specialization of y; and for proving flatness we can replace x and y by these specializations. Thus we may assume that x is a closed point, to which (ii)$'$ applies.

Since X is reduced (by 2.1.1(i)), therefore so is A. The assumed birationality of μ makes the natural map $A \to B$ injective, and with h as in Lemma 2.2.2, $B \subset h^{-1}A \subset K_A$.

It follows from (ii)′ that h is B_1-regular; and since $hB_1 \subset A_1$, we deduce that the map $\iota_1\colon B_1 \to h^{-1}A_1 \cong h^{-1}A \otimes_R k$, derived from the inclusion $\iota\colon B \hookrightarrow h^{-1}A$ by tensoring over R with $k := R/\mathfrak{m}$, is injective. Since $h^{-1}A \cong A$ is R-flat, the natural exact sequence

$$0 = \mathsf{Tor}_1^R(h^{-1}A, k) \to \mathsf{Tor}_1^R(h^{-1}A/B, k) \to B \otimes_R k \xrightarrow{\iota_1} h^{-1}A \otimes_R k \tag{2.2.2.1}$$

gives $\mathsf{Tor}_1^R(h^{-1}A/B, k) = 0$, i.e., the finite A-module $h^{-1}A/B$ is R-flat (by the local criterion of flatness [Ma2, p. 174, 22.3, (3′)⇒ (1)]); and consequently B is R-flat. Moreover, since (as above) $A_1 \subset B_1$, therefore the natural exact sequence

$$0 = \mathsf{Tor}^1_R(B, k) \to \mathsf{Tor}^1_R(B/A, k) \to A_1 \to B_1$$

gives $\mathsf{Tor}^1_R(B/A, k) = 0$, so B/A is R-flat. □

(ii)′ ⇒ (i). There is an easy reduction to the case where, with $R \to A \to B$ as above, $Y = \mathsf{Spec}\, R$, $X = \mathsf{Spec}\, A$ and $Z = \mathsf{Spec}\, B$ (f and μ being the obvious maps), and $Y_1 = \mathsf{Spec}\, S$ with S a local R-algebra.

There exists an A-regular element h such that $hB \subset A$ and $h^{-1}A/B$ is R-flat. These assertions were deduced from (ii)′ in the proof of 2.2.3, so we have them in the case where x is closed in X, a case to which, however, the general case reduces (as in the proof of 2.2.3) by specialization.

Now from 2.1.1(i) it follows that $A \otimes_R S$ is reduced. Since, by 2.2.3, B/A is R-flat, therefore $\mathsf{Tor}_1^R(B/A, S) = 0$, and so the natural map $A \otimes_R S \to B \otimes_R S$ is injective. Moreover, $A/hA \cong h^{-1}A/A$ is R-flat (take $B = A$ above, i.e., use (ii)′ for $Z = X$), whence

$$\mathsf{Tor}_1^R(A/hA, S) = \mathsf{Tor}_1^R(h^{-1}A/A, S) = 0,$$

so that multiplication by $h \otimes 1$ is an injective endomorphism of $A \otimes_R S$ (resp. $h^{-1}A \otimes_R S$). It follows that $h^{-1}A \otimes_R S$ is isomorphic to an S-submodule of the quotient ring $K_{A\otimes_R S}$. But since $h^{-1}A/B$ is R-flat, the natural map $B \otimes_R S \to h^{-1}A \otimes_R S$ is injective, and so $B \otimes_R S$ is isomorphic to a subring of $K_{A\otimes_R S}$. This gives (i).

(ii)′ ⇒ (iv). By 2.2.3, $f\mu$ is flat; and for any $y \in Y$, taking S in the preceding paragraph to be a finite purely inseparable extension of $\kappa(y)$ we see that Z_y is geometrically reduced. So the map $f\mu$ is reduced, and hence, by 1.6.3, satisfies (♣). □

Here is one situation where the conditions in 2.2.1 are satisfied.

Proposition 2.2.4. *Suppose Y is a d-dimensional irreducible regular scheme and that $f\colon X \to Y$ satisfies* (♣). *If Z as in* 2.1.1 *satisfies the Serre condition* (S_{d+1}), (*for example if Z is Cohen-Macaulay*) *then the conditions in* 2.2.1 *hold. In particular, if $d = 1$ and Z is normal then those conditions hold.*

Proof. We reduce as in the proof of 2.2.1 to considering a regular local ring $(R, \mathfrak{m})$ of dimension, say, $n \le d$, a flat equidimensional local R-algebra A, and a finite torsion-free extension $B \supset A$ satisfying (S_{n+1}). It is to be shown that any associated prime ideal $\mathfrak{P}$ of $\mathfrak{m}B$ intersects A in a minimal associated prime of $\mathfrak{m}A$ (cf. 2.2.1(ii)).

Set $\mathfrak{p} := \mathfrak{P} \cap A$. Since B is integral over A and $B/\mathfrak{P}$ is integral over $A/\mathfrak{p}$, and since A and B are biequidimensional (see 1.6.3), it holds that

$$\dim B_{\mathfrak{P}} \le \dim A_{\mathfrak{p}} = \dim A - \dim A/\mathfrak{p} = \dim B - \dim B/\mathfrak{P} = \dim B_{\mathfrak{P}},$$

so that $(*)\colon\ \dim B_{\mathfrak{P}} = \dim A_{\mathfrak{p}}$.

Let $(r_1, r_2, \dots, r_n)$ generate $\mathfrak{m}$. Any prime A-ideal $\mathfrak{q}$ has height at least that of its inverse image in R [Ma2, p. 68, 9.5]. Hence for $m \le n$, if $\mathfrak{q} \supset (r_1, \dots, r_m)A$ then $\dim A_{\mathfrak{q}} \ge m$. Since B satisfies (S_{n+1}), it follows from $(*)$ and from [EGA, (5.7.5)], by induction on m, that the sequence $(r_1, \dots, r_m)$ is B-regular and every associated prime ideal $\mathfrak{Q}$ of $(r_1, \dots, r_m)B$ has height m. Thus $n = \dim B_{\mathfrak{P}} = \dim A_{\mathfrak{p}}$, and the conclusion follows. $\square$

Theorem 2.3. *Let $f\colon X \to Y$ be a reduced scheme map with Y normal, and $\mu\colon Z \to X$ a finite map. If μ is a simultaneous normalization of f then μ is a normalization and the nonempty fibers of $f\mu$ are geometrically normal. The converse holds whenever f satisfies* (♣).

Proof. Suppose μ is a simultaneous normalization of f. The fiber Z_y is geometrically normal by definition. That X is reduced and Z is normal follow from [Ma2, p. 184, Cor. (ii)]; so to show that μ is a normalization, we need only prove that μ is birational, in other words that μ induces a bijection from the set of generic points $z \in Z$ to the set of generic points of X such that for each such z, the corresponding local homomorphism $\mathcal{O}_{X,\mu(z)} \to \mathcal{O}_{Z,z}$ is an isomorphism. Flatness of f and of $f\mu$ implies that every generic point of X and every generic point of Z maps to a generic point of Y. We may therefore localize at a generic point of Y, i.e., assume that Y is the spectrum of a field. Then by definition μ is a normalization map, thus birational.

For the converse implication, with $\alpha = \alpha_{Z,\kappa(y)}\colon \overline{X_y} \to Z_y$ $(y \in f(X))$ as in 2.1.1(i), 2.2.1(iii)$'$ gives that $f\mu$ satisfies (♣), so is flat, and that the natural map $\mathcal{O}_{Z_y} \to \alpha_*\mathcal{O}_{\overline{X_y}}$ is injective. Normality of Z_y implies then that this map is surjective too, whence α, being finite, is an isomorphism. Thus the map $Z_y \to X_y$ induced by μ is a normalization (see 2.1.1(i)). $\square$

Corollary 2.3.1. *A map $f\colon X \to Y$ satisfying* (♣) *is equinormalizable iff for one* (*hence any*) *normalization map $\nu\colon \overline{X} \to X$ and all $y \in f(X)$, $\overline{X}_y$ is geometrically normal.*

Corollary 2.3.2. *Let $f\colon X \to Y$ satisfy* (♣). *Assume that for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ satisfies* $\mathrm{ff}_{\mathrm{nor}}$. *Then f is equinormalizable iff for each $x \in X$ the map $\hat{f}_x$ in* 2.1.1(iv) *is equinormalizable.*

Proof. Let k be a field, let $(A, \mathfrak{m})$ be a reduced local k-algebra satisfying $\mathrm{ff}_{\mathrm{nor}}$, and let $\overline{A}$ be its integral closure. By [EGA, (7.6.1)], $\overline{A}$ is a finite A-module, the completion $\hat{A}$ is reduced, and $\overline{\hat{A}}$ is the $\mathfrak{m}$-adic completion of $\overline{A}$. Moreover, A *is geometrically normal over k if and only if $\hat{A}$ is.* For if the local ring $A' := A \otimes_k k'$ is normal for every finite purely inseparable field extension k' of k, then so is its completion $\hat{A} \otimes_k k'$ (since A' satisfies $\mathrm{ff}_{\mathrm{nor}}$, see 1.3.1); and conversely, if $\hat{A} \otimes_k k'$ is normal then so is $A' = (\hat{A} \otimes_k k') \cap K_{A'}$. Similar considerations hold if A is semilocal.

Now suppose that f is equinormalizable. Let $C := \mathcal{O}_{X,x}$ and $D := \overline{C}$. As above, $\mathsf{Spec}\,\hat{D} \to \mathsf{Spec}\,\hat{C}$ is a normalization map. With $y := f(x)$ and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{Y,y}$, we have that $\hat{D}/\mathfrak{m}\hat{D}$ is the completion of the semilocal ring $D/\mathfrak{m}D$; and since, by assumption, the latter is geometrically normal, therefore so is the former. Thus by 2.3.1, $\hat{f}_x$ is equinormalizable.

Conversely, if $\hat{f}_x$ is equinormalizable then $\hat{D}/\mathfrak{m}\hat{D}$ is geometrically normal, whence so is $D/\mathfrak{m}D$. It follows therefore from 2.3.1 that if f satisfies (♣) and $\hat{f}_x$ is equinormalizable for every $x \in X$ then f is equinormalizable. □

3. Partially numerical criteria for equinormalization.

Under suitable conditions, Proposition 3.3 and Corollary 3.3.1 give a criterion for equinormalizability of $f\colon X \to Y$ in terms of constancy of a numerical invariant δ associated to each fiber $X_y := f^{-1}y$ $(y \in Y)$, namely, with $\kappa(y)$ the residue field of $\mathcal{O}_{Y,y}$, $f_y\colon X_y \to \operatorname{Spec}\kappa(y)$ the obvious map, and $\overline{\mathcal{O}_{X_y}}$ the integral closure of $\mathcal{O}_{X_y}$,

$$\delta_y := \dim_{\kappa(y)} f_{y*}(\overline{\mathcal{O}_{X_y}}/\mathcal{O}_{X_y}).$$

It is assumed here that X_y has isolated nonnormal points, with residue fields finite over $\kappa(y)$, so that this δ is finite.

When f is a flat projective map, we can associate to each fiber X_y the Hilbert polynomial of $\overline{\mathcal{O}_{X_y}}/\mathcal{O}_{X_y}$; and show under suitable conditions that equinormalizability is equivalent to the local constancy of this function of y (see Proposition 3.4, noting that the Hilbert polynomial of $\mathcal{O}_{X_y}$ is locally independent of y).[9]

It may be noted that when a fiber of a projective map has isolated nonnormal points, the above Hilbert polynomial is just the constant δ.

These results will be improved in §4—the above-mentioned "suitable conditions" will be weakened to where they refer *solely to the fibers themselves.*

Definition 3.1. Let k be a field and let $g : X \to \operatorname{Spec} k$ be a scheme map with X a reduced Nagata scheme. Let $\mathfrak{C} \subset \mathcal{O}_X$ be the conductor of the normalization $\nu\colon \overline{X} \to X$, (a finite map, see 1.4.2), i.e., the annihilator of the $\mathcal{O}_X$-module $\nu_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X$; and assume that the closed subscheme $X_{\mathfrak{C}} \subset X$ corresponding to the coherent $\mathcal{O}_X$-ideal $\mathfrak{C}$ is *finite* over k. When these conditions hold we say "$\delta_k(X)$ is finite" and set

$$\delta_k(X) := \dim_k g_*(\nu_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X) = \sum_{x\in X} \dim_k(\overline{\mathcal{O}_{X,x}}/\mathcal{O}_{X,x}) < \infty.$$

If X is affine, say $X = \operatorname{Spec} A$, we write $\delta_k(A)$ in place of $\delta_k(X)$.

Definition 3.2. A ring homomorphism $\phi\colon R \to A$ satisfies (♠) if:

(1) $(R, \mathfrak{m}, k)$ is a normal local ring satisfying $\mathrm{ff}_{\mathrm{nor}}$ and such that the residue field k is either of characteristic 0 or of characteristic > 0 and perfect.
(2) A is a formally equidimensional Nagata ring.
(3) The map ϕ is flat, $\mathfrak{m}A$ is contained in every maximal A-ideal, $A/\mathfrak{m}A$ is reduced and $\delta_k(A/\mathfrak{m}A)$ is finite.
(4) $\overline{A}/A$ is a finite R-module.

[9]There should be something interesting to be said about this criterion being *global* on the fibers whereas equinormalizability is a *local* condition (see 2.3.2)—but we don't know what that might be.

Remarks 3.2.1. (a) Suppose that R is a complete local ring, or that R is henselian and A is a localization of a finitely generated R-algebra, or that R and A are both analytic local rings, i.e., homomorphic images of convergent power series rings over a complete nondiscrete valued field. Then conditions (1), (2) and (3) in 3.2 imply (4).[10]

Proof. If $\mathfrak{P}$ is a prime A-ideal containing $\mathfrak{m}A$ and such that $A_{\mathfrak{P}}/\mathfrak{m}A_{\mathfrak{P}}$ is normal, hence geometrically normal over k (since k is perfect or of characteristic 0), then by 1.7.1 the homomorphism $R \to A_{\mathfrak{P}}$ is normal, so $A_{\mathfrak{P}}$ is normal [Ma2, p. 184, Cor. (ii)].

Finiteness of $\delta_k(A/\mathfrak{m}A)$ means that if $\mathfrak{M}$ is a prime A-ideal containing $\mathfrak{m}A$ such that $A_{\mathfrak{M}}/\mathfrak{m}A_{\mathfrak{M}}$ is *not* normal, then $\mathfrak{M}$ is a maximal ideal and $[A/\mathfrak{M} : k] < \infty$; and moreover, there are only finitely many such $\mathfrak{M}$.

The ring A is reduced (see 2.1.1(i)), so the integral closure $\overline{A}$ is a finite A-module (see 1.4.2). Let $\mathfrak{C}$ be the A-conductor, i.e., the annihilator of the A-module $\overline{A}/A$. Then $\mathfrak{C}_{\mathfrak{P}} = A_{\mathfrak{P}}$ for any $\mathfrak{P}$ as above, whence, by the preceding paragraph, $(A/\mathfrak{C}) \otimes_R k$ is a finite-dimensional k-vector space. Hence $A/\mathfrak{C}$ is a finite R-module:

- if R is complete, by [Ma2, p. 58, 8.4], since $\mathfrak{m}A$ is contained in the Jacobson radical of A, so $A/\mathfrak{C}$ is $\mathfrak{m}$-adically separated;
- if R is henselian and A is a finitely generated R-algebra, by [EGA, 18.5.11 c′)] (since $A/\mathfrak{C}$ has only finitely many maximal ideals, all of which contract in R to $\mathfrak{m}$);
- if R and A are analytic local rings, [C, p. 18-01, Thm. 1].

In any of these cases the finite $A/\mathfrak{C}$-module $\overline{A}/A$ is also finite over R. □

(b) If $\phi\colon (R, \mathfrak{m}, k) \to A$ satisfies (♠) then $\mathsf{Spec}\,\phi\colon \mathsf{Spec}\,A \to \mathsf{Spec}\,R$ satisfies (♣), and the $\mathfrak{m}$-adic completion $\hat{\phi}\colon \hat{R} \to \hat{A}$ satisfies (♠). If, in addition, A satisfies $\mathrm{ff}_{\mathrm{nor}}$ and $A/\mathfrak{m}A$ has finite Krull dimension, then $\hat{A}$ satisfies $\mathrm{ff}_{\mathrm{nor}}$.

Proof. Since k is perfect or of characteristic 0, the k-algebra $A/\mathfrak{m}A$ is geometrically reduced (see 1.1). If $\mathfrak{M}$ is any maximal A-ideal then the composition $R \to A \to A_{\mathfrak{M}}$ is reduced (see 1.7.1). It follows that $\mathsf{Spec}\,\phi$ is reduced (since being reduced is a local property, and every prime A-ideal is contained in some $\mathfrak{M}$). It is now immediate that $\mathsf{Spec}\,\phi$ satisfies (♣).

Since R is normal and satisfies $\mathrm{ff}_{\mathrm{nor}}$, therefore $\hat{R}$ is normal [Ma2, p. 184, Cor. (ii)], and of course $\hat{R}$ satisfies $\mathrm{ff}_{\mathrm{nor}}$. That $\mathsf{Spec}\,\hat{\phi}$ satisfies (♣) is given by 2.1.1(iv) with $I = \mathfrak{m}$ and $J = \mathfrak{m}A$. Since $\hat{A}/\mathfrak{m}\hat{A} = A/\mathfrak{m}A$ and $\hat{R}$ is complete, it follows from (a) that $\hat{\phi}$ satisfies (♠).

In particular, $\hat{A}$ is universally catenary; and consequently, since $\mathfrak{m}\hat{A}$ is contained in every maximal ideal of $\hat{A}$ and $\hat{A}/\mathfrak{m}\hat{A} = A/\mathfrak{m}A$ has finite Krull dimension, therefore $\hat{A}$ has finite Krull dimension. Then Satz 2 of [BR] shows that $\hat{A}$ satisfies $\mathrm{ff}_{\mathrm{nor}}$. □

Proposition 3.3. *Let $\phi\colon (R, \mathfrak{m}, k) \to A$ satisfy (♠). Set $K := K_R$, $A_0 := A \otimes_R K$, $A_1 := A/\mathfrak{m}A$, $B := \overline{A}$, $B_0 := B \otimes_R K$, $B_1 := B/\mathfrak{m}B$. For $\mathfrak{p} \in \mathsf{Spec}\,R$ set $A_{\langle\mathfrak{p}\rangle} := A \otimes_R \kappa(\mathfrak{p})$ and $B_{\langle\mathfrak{p}\rangle} := B \otimes_R \kappa(\mathfrak{p})$. Then:*

[10] Complete local rings and analytic local rings are all universally catenary Nagata rings which satisfy $\mathrm{ff}_{\mathrm{nor}}$—in fact they are *excellent* [EGA, 7.8.3, 5.6.4]. For various proofs see [EGA, Chap. 0, (22.3.2)] plus [Ng, p. 193, (45.5)], [Ma1, p. 291, Remark], [BKKN, p. 96, Satz 3.3.3], [Ku, p. 1001, Thm. 2.5], or the last sentence in the introduction to [Ki].

(i) *If* $f := \mathsf{Spec}\,\phi$ *is equinormalizable then* $\delta_{\kappa(\mathfrak{p})}(A_{\langle\mathfrak{p}\rangle})$ *is independent of* $\mathfrak{p}$.

Assume further that the map $\alpha\colon B_1 \to \overline{A_1}$ *arising from* 2.1.1(i) *is injective. Then*:

(ii) *If* $\delta_{\kappa(\mathfrak{p})}(A_{\langle\mathfrak{p}\rangle}) < \infty$ *then*

$$\delta_{\kappa(\mathfrak{p})}(A_{\langle\mathfrak{p}\rangle}) - \delta_{\kappa(\mathfrak{p})}(B_{\langle\mathfrak{p}\rangle}) = \delta_K(A_0).$$

(iii) *If* $\delta_k(A_1) \le \delta_K(A_0)$ *then* f *is equinormalizable.*

Proof. For any simultaneous normalization $\mu\colon Z \to X := \mathsf{Spec}\,A$ of f, $Z \cong \mathsf{Spec}\,B$ and condition 2.2.1(ii) is satisfied (see Theorem 2.3), whence so is 2.2.1(i), so that $\alpha\colon B_1 \to \overline{A_1}$ is injective. Thus we may assume this injectivity throughout the proof. This assumption implies condition (ii)$'$ in Proposition 2.2.1, so by that Proposition both B and B/A are flat R-modules; and being finitely generated (by (♠)), B/A *is a finite-rank free* R*-module.*

We begin with the case $\mathfrak{p} = \mathfrak{m}$. By assumption, B_1 and its subring A_1 have the same integral closure, whence

$$\delta_k(A_1) - \delta_k(B_1) = \dim_k(B_1/A_1) = \dim_k(B/A \otimes_R k).$$

Also, it is easy to see that $B \otimes_R K = \overline{A \otimes_R K}$. Therefore,

$$\dim_k(B/A \otimes_R k) = \dim_K(B/A \otimes_R K) = \delta_K(A_0).$$

Thus (ii) holds for $\mathfrak{p} = \mathfrak{m}$.

For arbitrary $\mathfrak{p}$, the implication (ii)$'$ $\Rightarrow$ (i) in Proposition 2.2.1 shows that $B_{\langle\mathfrak{p}\rangle}$ and its subring $A_{\langle\mathfrak{p}\rangle}$ have the same integral closure. So one can localize at $\mathfrak{p}$ and argue as before to prove (ii). Moreover, if f is equinormalizable then $B_{\langle\mathfrak{p}\rangle}$ is normal, and hence

$$\delta_{\kappa(\mathfrak{p})}(A_{\langle\mathfrak{p}\rangle}) = \dim_{\kappa(\mathfrak{p})}(B_{\langle\mathfrak{p}\rangle}/A_{\langle\mathfrak{p}\rangle}) = \mathrm{rank}_R(B/A)$$

is independent of $\mathfrak{p}$, proving (i). Finally, if $\delta_k(A_1) \le \delta_K(A_0)$ then by (ii), $\delta_k(B_1) = 0$, i.e., B_1 is normal, hence geometrically normal, since k is perfect or of characteristic 0; and since any prime B-ideal is contained in a maximal ideal, which contracts to a maximal ideal in A, hence to $\mathfrak{m}$ in R, it follows from 1.7.1 that all the fibers of $\mathsf{Spec}\,B \to \mathsf{Spec}\,R$ are geometrically normal. Thus Corollary 2.3.1 gives (iii). □

Corollary 3.3.1. *Let* $\phi\colon R \to A$ *satisfy* (♠), *and assume further that* R *is a regular local ring of dimension, say,* d *and that* $\overline{A}$ *satisfies the Serre condition* (S_{d+1}) (*which it always does in case* $d = 1$). *Then with the notation of Proposition* 3.3, $\mathsf{Spec}\,\phi$ *is equinormalizable iff* $\delta_k(A_1) = \delta_K(A_0)$.

Proof. In view of Proposition 2.2.4, this results from Proposition 3.3. □

We turn now to the case where the map $f\colon X \to Y$, satisfying (♣), is *projective.*

We need some notation. For any Y-scheme W and any $y \in Y$, with residue field $\kappa(y)$, let $W_y := W \otimes_Y \mathsf{Spec}\,\kappa(y)$ be the fiber over y, and let $\iota_y^W : W_y \to W$ be the projection. For any $\mathcal{O}_W$-module $\mathcal{F}$ let $\mathcal{F}_y$ be the $\mathcal{O}_{W_y}$-module $(\iota_y^W)^*\mathcal{F}$.

Projectivity of $f\colon X \to Y$ entails the existence of invertible $\mathcal{O}_X$-modules which are very ample relative to f. Fix one such and call it $\mathcal{L}$. Then $\mathcal{L}_y$ is very ample relative to the projection $f_y\colon X_y \to \mathsf{Spec}\,\kappa(y)$ [EGII, (4.4.10)(iii)]. For any coherent $\mathcal{O}_{X_y}$-module $\mathcal{M}$ set $\mathcal{M}(n) := \mathcal{M} \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes n}$ and let $\mathcal{H}_y(\mathcal{M})$ be the polynomial function (depending on $\mathcal{L}$) which takes integers $n \gg 0$ to $\dim_{\kappa(y)} \mathrm{H}^0(X_y, \mathcal{M}(n))$. (See [EGIII, (2.5.3)].)

Proposition 3.4. *Let* $(R, \mathfrak{m}, k)$ *be a normal local ring satisfying* $\mathrm{ff}_{\mathrm{nor}}$ *and such that* k *is either of characteristic* 0 *or of characteristic* >0 *and perfect. Let* $f\colon X \to Y := \mathsf{Spec}\, R$ *be a projective map whose fibers are all geometrically reduced and which is locally equidimensional* [EGA, $\mathrm{Err}_{\mathrm{IV}}$, 35].[11] *Let* $\mathcal{L}$, $\mathcal{H}_y$ *be as above. Let* $\mu\colon Z \to X$ *be a normalization map. Let* y_0 *and* y_1 *be, respectively, the closed and generic points of* Y, *and set* $X_i := X_{y_i}$, *resp.* $Z_i := Z_{y_i}$. *Then:*

(i) *If the map* $\alpha\colon \overline{X_1} \to Z_1$ *from* 2.1.1(i) *is schematically dominant* (§2.2), *and if*

$$\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}) = \mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}})$$

then f *is equinormalizable.*

(ii) *If* f *is equinormalizable then* $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}})$ *is independent of* y.

Proof. First of all, f satisfies (♣). In fact:

Lemma 3.4.1. *Let* $(R, \mathfrak{m}, k)$ *be a normal local ring satisfying* $\mathrm{ff}_{\mathrm{nor}}$. *Let* $f\colon X \to \mathsf{Spec}\, R$ *be a finite-type scheme map whose fibers are all geometrically reduced. Then the following conditions are equivalent.*

(i) f *satisfies* (♣).

(ii) f *is locally equidimensional.*

(iii) *For each* $x \in X$, $\mathcal{O}_{X,x}$ *is equidimensional, and* f *is universally open.*

(iii)′ *For each* $x \in X$, $\mathcal{O}_{X,x}$ *is equidimensional, and* f *is open.*

(iv) *For each* $x \in X$, $\mathcal{O}_{X,x}$ *is equidimensional, and* f *is flat.*

(v) *For each* $x \in X$ *and* $y = f(x)$, $\mathcal{O}_{X,x}$ *is equidimensional, and with* $\mathfrak{m}_y$ *the maximal ideal of* $\mathcal{O}_{Y,y}$,

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}).$$

Proof. (i) ⇒ (iv). Trivial.

(iv) ⇒ (i). It suffices to show that $\mathsf{Spec}\, R$ is a universally catenary Nagata scheme, since then the same will be true of X (see 1.5.2, 1.4.1). Since $\hat{R}$ is normal [Ma2, p. 184, Cor. (ii)], therefore R is universally catenary (see 1.6, 1.6.1). Since R satisfies $\mathrm{ff}_{\mathrm{nor}}$, therefore R is a Nagata ring (see 1.4.3).

(iv) ⇒ (iii). [EGA, (2.4.6)].

(iii) ⇒ (iv). [EGA, (15.2.3)].

(iii) ⇔ (iii)′. [EGA, (14.4.3)]. (Normal schemes are geometrically unibranch [EGA, (6.15.1)].)

(iv) ⇒ (v). [EGA, (6.1.2)].

(v) ⇒ (ii). [EGA, (13.3.6)].

(ii) ⇒ (iii). [EGA, (14.4.4)]. □

Now, as in the proof of Proposition 3.3, we may assume throughout that α is schematically dominant.

The projective map f takes closed points of X to the closed point of Y. The assumed dominance of α implies, via 2.2.1, that $f\mu$ is flat, and so $\overline{\mathcal{O}_X} = \mu_*\mathcal{O}_Z$ is Y-flat. Hence by [EGIII, 7.9.13], $\mathcal{H}_y((\overline{\mathcal{O}_X})_y)$ is independent of y.

[11] Pub. Math. I.H.E.S. **32**, p. 357.

If f is equinormalizable, then $\overline{X}_y = \overline{X_y}$, and one deduces (directly, or by [EGI, p. 366, (9.3.3)]) a natural isomorphism $(\overline{\mathcal{O}_X})_y \xrightarrow{\sim} \overline{\mathcal{O}_{X_y}}$, proving (ii).

As for (i), since α is schematically dominant, therefore $\overline{\mathcal{O}_{X_1}} \supset (\overline{\mathcal{O}_X})_{y_1}$, and so

$$\begin{aligned}\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}) &= \mathcal{H}_{y_1}\big(\overline{\mathcal{O}_{X_1}}/(\overline{\mathcal{O}_X})_{y_1}\big) + \mathcal{H}_{y_1}\big((\overline{\mathcal{O}_X})_{y_1}\big)\\ &= \mathcal{H}_{y_1}\big(\overline{\mathcal{O}_{X_1}}/(\overline{\mathcal{O}_X})_{y_1}\big) + \mathcal{H}_{y_0}\big((\overline{\mathcal{O}_X})_{y_0}\big)\\ &= \mathcal{H}_{y_1}\big(\overline{\mathcal{O}_{X_1}}/(\overline{\mathcal{O}_X})_{y_1}\big) + \mathcal{H}_{y_0}\big(\overline{\mathcal{O}_{X_0}}\big).\end{aligned}$$

The equality $\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}) = \mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}})$ implies then that $\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}/(\overline{\mathcal{O}_X})_{y_1}) = 0$, whence $\overline{\mathcal{O}_{X_1}} = (\overline{\mathcal{O}_X})_{y_1}$, i.e., $(\overline{\mathcal{O}_X})_{y_1}$ *is normal.* (Indeed, for any coherent $\mathcal{O}_{X_y}$-module $\mathcal{M}$, $\mathcal{H}_y(\mathcal{M}) = 0 \Leftrightarrow \mathcal{M} = 0$ because for all $n \gg 0$, $\mathcal{M}(n)$ is generated by global sections.) That f is equinormalizable follows now just as in the proof of 3.3(iii). □

As before, Proposition 2.2.4 yields:

Corollary 3.4.2. *In* 3.4 *assume that R is a regular local ring of dimension, say, d and that Z satisfies the Serre condition* (S_{d+1}) *(which it always does in case $d = 1$). Then α has to be schematically dominant, and so f is equinormalizable iff $\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}) = \mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}})$.*

4. Main Theorems.

Here is the first main theorem, affirming that for families of curves it is not necessary in Proposition 3.3(iii) to assume that the map α is injective.

Theorem 4.1. *Let $\phi\colon (R, \mathfrak{m}, k) \to A$ satisfy (♠). Suppose also that R and A satisfy one of the conditions in Remark* 3.2.1(a) *and that $A/\mathfrak{m}A$ has Krull dimension* 1. *Set $K := K_R$, $A_0 := A \otimes_R K$, $A_1 := A \otimes_R k$. If $\delta_k(A_1) = \delta_K(A_0)$ then $f := \mathsf{Spec}\,\phi$ is equinormalizable.*

Proof. Set $B := \overline{A}$, $B_0 := B \otimes_R K$, $B_1 := B \otimes_R k$. By Proposition 3.3, it suffices to show that the map $\alpha\colon B_1 \to \overline{A_1}$ given by 2.1.1(i) is injective.

Recall that f satisfies (♣) (Remark 3.2.1(b)), so that by 2.1.1 both A and $A/\mathfrak{m}A$ are reduced, and for any minimal prime divisor $\mathfrak{P}$ of $\mathfrak{m}A$ the local ring $A_{\mathfrak{P}}$ is normal. This being so, the proof of Lemma 2.2.2 is valid in the present situation.

Lemma 4.1.1. *If h is as in Lemma* 2.2.2 *then the R-module $h^{-1}A/B$ is torsion-free.*

Proof. Multiplication by the A-regular element h is an R-isomorphism $h^{-1}A/B \cong A/hB$. Note that the A-ideal hB is the integral closure of hA. As A is formally equidimensional, [R3, p. 189, Thm. 2.12] implies that every associated prime A-ideal $\mathfrak{Q}$ of hB is minimal.

Now these $\mathfrak{Q}$ are also the minimal prime divisors of hA, because $hA \subset hB \subset \sqrt{hA}$. Since A is R-flat and multiplication by h is an injective endomorphism of $A/\mathfrak{m}A = A \otimes_R k$, therefore $\mathsf{Tor}_1^R(A/hA, k) = 0$; so by [Ma2, p. 174, 22.3], A/hA is R-flat, hence torsion-free, and therefore $\mathfrak{P} \cap R = 0$ for every associated prime $\mathfrak{P}$ of hA, in particular for $\mathfrak{P} = \mathfrak{Q}$. Hence A/hB is R-torsion-free. □

Remark 4.1.2. As just shown, $h^{-1}A/A \cong A/hA$ is R-flat; and since $A \subset \overline{A} \subset h^{-1}A$, therefore $\overline{A}/A$ *is R-torsion-free.*

Note next that the above h can be chosen so that *A/hA is a finite R-module.* Indeed, as in Remark 3.2.1(a), it suffices to arrange that $A/hA\otimes_R k$ be finite over k. Since $A/\mathfrak{m}A$ has dimension one and h is $A/\mathfrak{m}A$-regular, therefore $A/hA\otimes_R k$ is artinian, so it suffices that every maximal A-ideal $\mathfrak{M}$ containing h satisfy $[A/\mathfrak{M}:k]<\infty$; and again as in Remark 3.2.1(a), this will be so if $A_{\mathfrak{M}}/\mathfrak{m}A_{\mathfrak{M}}$ is not normal. Reviewing the proof of Lemma 2.2.2, one finds it enough to show that *if $A_{\mathfrak{M}}/\mathfrak{m}A_{\mathfrak{M}}$ is normal then neither $\mathfrak{C}'$ nor $\mathfrak{C}$ is contained in $\mathfrak{M}$.* For $\mathfrak{C}'$ this is evident; and for $\mathfrak{C}$ it follows from 1.7.1—which with [Ma2, p. 184, Cor. (ii)] shows that $A_{\mathfrak{M}}$ is normal, so that $\mathfrak{C}A_{\mathfrak{M}}$, the conductor of $A_{\mathfrak{M}}$, is the unit ideal.

As observed in the proof of 4.1.1, $h^{-1}A/A\cong A/hA$ is R-flat, and hence, being finitely generated, it is R-free, of rank, say, d. For any R-module C and any R-algebra T, set $C_T := C\otimes_R T$. The natural exact sequence

$$0=\mathsf{Tor}^1_R(A/hA,T)\to A_T\xrightarrow{h}A_T \tag{4.1.3}$$

shows that h is A_T-regular, so that there are natural isomorphisms

$$(h^{-1}A)_T\ \underset{h\otimes 1}{\xrightarrow{\sim}}\ A_T\ \underset{h^{-1}}{\xrightarrow{\sim}}\ h^{-1}A_T\subset K_{A_T}.$$

Assume henceforth that $R\subset T\subset K$ and that T is normal and essentially of finite type over R. Then with $\phi_T\colon T\to A_T$ the map induced by ϕ, 2.1.1(iii) shows that $\mathsf{Spec}\,\phi_T$ satisfies (♣). Hence, by Remark 4.1.2, $\overline{A_T}/A_T$ is T-torsion free. Consequently,

$$A_T\subset\overline{A_T}\subset h^{-1}A_T; \tag{4.1.4}$$

indeed, $A_K=A_T\otimes_T K$ and $\overline{A_K}=\overline{A_T}\otimes_T K$, and since $h\overline{A}\subset A$ therefore $h\overline{A_K}\subset A_K$, so for any $a\in\overline{A_T}$ there is a nonzero $t\in T$ such that $tha\in A_T$, whence, by torsion-freeness of $\overline{A_T}/A_T$, $ha\in A_T$. We denote the inclusion $\overline{A_T}\hookrightarrow h^{-1}A_T$ by ι_T.

With $\delta:=\delta_K(A_1)=\delta_k(A_0)$, let $g\colon \mathsf{G}\to\mathsf{Spec}\,R$ be the Grassmannian of locally free rank-$(d-\delta)$ quotients of $h^{-1}A/A$ ([EGI, p. 384, (9.7.5)]). There is then a functorial bijection between R-morphisms $\mathsf{Spec}\,T\to\mathsf{G}$ and T-submodules $L\subset h^{-1}A_T/A_T$ such that the T-module $(h^{-1}A_T/A_T)/L$ is locally free of rank $d-\delta$. The map g is projective ([EGI, p. 390, (9.8.4)]).

Let $\psi\colon\mathsf{Spec}\,K\to\mathsf{G}$ be the R-morphism corresponding to the δ-dimensional K-vector space $B_0/A_0\subset h^{-1}A_0/A_0$. Regarding ψ as a rational R-morphism $\psi_T\colon\mathsf{Spec}\,T\to\mathsf{G}$ (see [EGI, p. 345, (8.1.11)]), *suppose that the domain of definition* $\mathrm{D}(\psi_T)$ *is all of* $\mathsf{Spec}\,T$. Then ψ_T corresponds to a T-module $L_1\subset E:=h^{-1}A_T/A_T$ with E/L_1 locally free of rank $(d-\delta)$, such that $L_1\otimes_T K=B_0/A_0=L_2\otimes_T K$ where $L_2:=\overline{A_T}/A_T$. Since $\mathsf{Spec}\,A_T\to\mathsf{Spec}\,T$ satisfies (♣) (see Remark 3.2.1(b) and 2.1.1(iii)), the T-module E/L_2 is torsion-free (see 4.1.1). The following Lemma 4.1.6 shows then that $L_1=L_2$, so that $h^{-1}A_T/\overline{A_T}\cong E/L_2=E/L_1$ is locally free of rank $d-\delta$, and there is a split-exact sequence of T-modules

$$0\to\overline{A_T}\xrightarrow{\iota_T}h^{-1}A_T\to h^{-1}A_T/\overline{A_T}\to 0. \tag{4.1.5}$$

Lemma 4.1.6. *Let S be a commutative domain, E a torsion-free S-module, and L_1, L_2 S-submodules of E. If E/L_1 and E/L_2 are both torsion-free and if the natural images of $L_1\otimes_S K_S$ and $L_2\otimes_S K_S$ in $E\otimes_S K_S$ coincide then $L_1=L_2$.*

Proof. The natural map $\rho\colon E \to E\otimes_S K_S$ is injective. Since the images of $L_1\otimes_S K_S$ and $L_2\otimes_S K_S$ in $E\otimes_S K_S$ coincide, there exists for each $f\in L_1$ a nonzero $s\in S$ and a $g\in L_2$ such that $\rho(f)=\rho(g)/s$, i.e., $\rho(sf)=\rho(g)$, i.e., $sf=g$. Since E/L_2 is torsion-free, $sf=g$ implies that $f\in L_2$, and thus $L_1\subset L_2$. Similarly, $L_2\subset L_1$. $\square$

Now if $T=R$, so that $A_T=A$ and $\overline{A_T}=B$, then applying $\otimes_R k$ to the split-exact sequence (4.1.5) we get an exact sequence

$$0\to B_1 \xrightarrow{\alpha'} h^{-1}A_1 \to h^{-1}A_1/B_1 \to 0.$$

As $\overline{A_1}\subset h^{-1}A_1\subset K_{A_1}$, one sees that $\alpha'(B_1)\subset \overline{A_1}$, and that the natural composition $A_1\to B_1 \xrightarrow{\alpha'} \overline{A_1}$ is a normalization map, whence, by 2.1.1(i), $\alpha'=\iota_k\circ\alpha$. Thus injectivity of α' implies that of α.

In summary: *For α to be injective it suffices that* $\mathrm{D}(\psi_R)$ *be all of* $\mathsf{Spec}\,R$, which we will now show to be the case.

Let Z be the schematic closure of $\psi(\mathsf{Spec}\,K)$, so that g induces a birational projective map $\gamma\colon Z\to \mathsf{Spec}\,R$. According to [EGI, p. 347, (8.2.7)], ψ is defined on all of $\mathsf{Spec}\,R$ if γ is an isomorphism, for which, since R is normal, it suffices by Zariski's Main Theorem (see e.g., [EGIII, (4.4.8)]) that the closed fiber $\gamma^{-1}\{\mathfrak{m}\}$ be zero-dimensional. We need only show then that $\gamma^{-1}\{\mathfrak{m}\}$ *has a unique closed point.*

Let z be any closed point in Z. Let S be the local ring of the generic point of a component of the closed fiber of the normalization of the blowup of the maximal ideal of $\mathcal{O}_{Z,z}$. Then S is a discrete valuation ring with fraction field K, essentially of finite type over R (because R is a Nagata ring, see 1.4.3), and with residue field k_S a separable extension of k.

As above (with $T=S$), the natural map $\mathsf{Spec}\,S\to\mathsf{G}$ corresponds to $L_2:=\overline{A_S}/A_S$, which, being the kernel of the surjective map of free S-modules $h^{-1}A_S/A_S\to h^{-1}A_S/\overline{A_S}$ of respective ranks d and $d-\delta$, is free of rank δ. So

$$\dim_{k_S}\big((\overline{A_S}\otimes_S k_S)/A_{k_S}\big)=\dim_{k_S}\big((\overline{A_S}/A_S)\otimes_S k_S\big)=\delta.$$

As k_S is separable over k, $\overline{A_{k_S}}=\overline{A_1}\otimes_k k_S$ [EGA, (6.14.2)]. So

$$\dim_{k_S}\big(\overline{A_{k_S}}/A_{k_S}\big)=\dim_{k_S}\big((\overline{A_1}/A_1)\otimes_k k_S\big)=\delta_k(A_1)=\delta.$$

Moreover, as above, $\mathsf{Spec}\,A_S\to\mathsf{Spec}\,S$ satisfies (♣), so 2.2.4 gives that the natural map $\overline{A_S}\otimes_S k_S\to\overline{A_{k_S}}$ is injective, whence so is the resulting map $(\overline{A_S}\otimes_S k_S)/A_{k_S}\to\overline{A_{k_S}}/A_{k_S}$. Since the source and target of this last map have the same dimension δ, it is *bijective,* and so there are natural identifications

$$(\overline{A_S}/A_S)\otimes_S k_S=(\overline{A_S}\otimes_S k_S)/A_{k_S}=\overline{A_{k_S}}/A_{k_S}=(\overline{A_1}/A_1)\otimes_k k_S\subset h^{-1}A_{k_S}/A_{k_S}.$$

This means that if $\psi_1\colon k\to\mathsf{G}$ corresponds to $\overline{A_1}/A_1\subset h^{-1}A_1/A_1$ then the following natural diagram commutes:

$$\begin{array}{ccc}\mathsf{Spec}\,k_S & \longrightarrow & \mathsf{Spec}\,S\\ \big\downarrow & & \big\downarrow\\ \mathsf{Spec}\,k & \xrightarrow[\psi_1]{} & \mathsf{G}\end{array}$$

Hence $z=\psi_1(\mathsf{Spec}\,k)$, and so $\gamma^{-1}\{\mathfrak{m}\}$ does indeed have a unique closed point. $\square$

Theorem 4.2. *Let $(R, \mathfrak{m}, k)$ be a normal local domain satisfying* $\mathrm{ff}_{\mathrm{nor}}$ *and such that k is either of characteristic 0 or of characteristic > 0 and perfect. Let $f\colon X \to Y := \operatorname{Spec} R$ be a projective map whose fibers are all geometrically reduced and which is locally equidimensional. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is very ample for f, and for $y \in Y$ let $\mathcal{H}_y$ denote the corresponding Hilbert polynomial on the fiber X_y (see the paragraphs preceding Proposition 3.4). Let y_1 and y_0 be, respectively, the closed and generic points of Y, and set $X_i := X_{y_i}$. In this situation, one has:*

(i) *If $\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}) = \mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}})$ then f is equinormalizable.*

(ii) *If f is equinormalizable then $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}})$ is independent of y.*

Proof. The proof is analogous to that of Theorem 4.1. Recall that f satisfies (♣), (see 3.4.1). In view of Proposition 3.4, we need only prove (i), for which it suffices to show, with $Z \to X$ a normalization map and $Z_1 := Z_{y_1}$, that *if $\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}) = \mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}})$ then the map $\alpha\colon \overline{X_1} \to Z_1$ from 2.1.1(i) is schematically dominant.*

First, some notation. Let T be an R-algebra whose only idempotents are 0 and 1 (i.e., $\operatorname{Spec} T$ is connected). Set $X_T := X \otimes_R T$. Hilbert polynomials on the fibers of the projection $f_T\colon X_T \to \operatorname{Spec} T$ are defined via the very ample (relative to f_T) invertible $\mathcal{O}_{X_T}$-module $\mathcal{L}_T := \mathcal{L} \otimes_R T$ (cf. again, the paragraphs preceding Proposition 3.4). If $\mathcal{F}$ is a T-flat coherent $\mathcal{O}_{X_T}$-module then by [EGIII, (7.9.13)] the Hilbert polynomial $\mathcal{H}_t(\mathcal{F}_t)$ is the same for all $t \in \operatorname{Spec} T$. We denote that polynomial by $\mathcal{H}_T(\mathcal{F})$.

We will need some global object to take the place of the element h in Lemma 2.2.2. By [EGII, (4.4.7)] we may assume there exists a graded R-algebra $A = R \oplus A_{[1]} \oplus A_{[2]} + \cdots$ generated by a finite R-module $A_{[1]}$, such that $X = \operatorname{Proj} A$ and $\mathcal{L} = \mathcal{O}_X(1)$. We may also assume that we are not in the trivial situation where X_1—and hence X—is empty. Let $\mathcal{C}$ be the conductor of $\mathcal{O}_X$, and let $\mathfrak{C}$ be a graded A-ideal whose associated $\mathcal{O}_X$-ideal $\widetilde{\mathfrak{C}}$ is $\mathcal{C}$ (see [EGII, (2.7.11)(ii)]). With $i\colon X_1 \hookrightarrow X$ the inclusion and $\pi\colon \mathcal{O}_X \to i_*\mathcal{O}_{X_1}$ the natural map, let $\mathcal{C}_1$ be the conductor of $\mathcal{O}_{X_1}$ and let $\mathfrak{C}'$ be a graded A-ideal whose associated $\mathcal{O}_X$-ideal $\widetilde{\mathfrak{C}'}$ is $\mathcal{C}' := \pi^{-1} i_*\mathcal{C}_1$. Then *neither $\mathfrak{C}$ nor $\mathfrak{C}'$ is contained in any minimal prime divisor $\mathfrak{P}$ of (0) or of $\mathfrak{m}A$.* Indeed, such a $\mathfrak{P}$ is graded, and does not contain every element of positive degree in A because otherwise $X_1 = \operatorname{Proj} A/\mathfrak{m}A$ would be empty. For any homogeneous $a \notin \mathfrak{P}$, $\mathfrak{C}_{(a)}$ is the conductor of $A_{(a)}$ (the ring of degree-0 elements in the localization A_a), $\mathfrak{C}'_{(a)}$ is the inverse image in $A_{(a)}$ of the conductor of $A_{(a)}/\mathfrak{m}A_{(a)}$, and $\mathfrak{P}_{(a)}$ is a minimal prime divisor in $A_{(a)}$ of (0) or of $\mathfrak{m}A_{(a)}$, as the case may be; and so either $\mathfrak{C} \subset \mathfrak{P}$ or $\mathfrak{C}' \subset \mathfrak{P}$ would lead to a contradiction, as in the proof of Lemma 2.2.2 (with A replaced by $A_{(a)}$). Since $A_{(a)}$ and $\mathfrak{m}A_{(a)}$ have no embedded associated primes (see 2.1.1(i)), homogeneous prime avoidance implies then that there is a homogeneous $h \in \mathfrak{C} \cap \mathfrak{C}'$, of degree, say, $n > 0$, such that $(0 :_A h)^{\sim} = (0)$ and $(\mathfrak{m}A :_A h)^{\sim} = \mathfrak{m}\mathcal{O}_X$. Thus, if $a \in A_{[1]}$, then the pair $(A_{(a)}, h/a^n)$ has the same properties as the pair (A, h) in Lemma 2.2.2.

Globally, this h gives rise to a section of $\mathcal{C}(n)$, i.e., to a map $\tilde{h}\colon \mathcal{O}_X \to \mathcal{C}(n)$. There results a composed map

$$\mathfrak{h}\colon \overline{\mathcal{O}_X} = \overline{\mathcal{O}_X} \otimes \mathcal{O}_X \xrightarrow{1\otimes\tilde{h}} \overline{\mathcal{O}_X} \otimes \mathcal{C}(n) \xrightarrow{\text{natural}} \mathcal{O}_X(n).$$

An examination of the basic definitions involved shows that for any affine open subset $X_a := \mathsf{Spec}\, A_{(a)} \subset X$ $(a \in A_{[1]})$, $\Gamma(X_a, \mathfrak{h})$ is the composed $A_{(a)}$-homomorphism

$$\Gamma(X_a, \overline{\mathcal{O}_X}) = \overline{A_{(a)}} \xrightarrow{h/a^n} A_{(a)} \xrightarrow{\sim} \Gamma(X_a, \mathcal{O}_X(n)) \tag{4.2.1}$$

where the first map, multiplication by the regular element h/a^n lying in the conductor of $A_{(a)}$, is injective, and the second map is the isomorphism taking $1 \in A_{(a)}$ to the section defined by $a^n \in A_{[n]}$.

We are now in a position to prove properties of $\mathfrak{h}$ analogous to those of h which were needed for the proof of Theorem 4.1.

Lemma 4.2.2. *There exist $n > 0$ and an $\mathcal{O}_X$-homomorphism $\mathfrak{h}\colon \overline{\mathcal{O}_X} \to \mathcal{O}_X(n)$ such that if $R \subset T \subset K$ and T is a normal essentially-finite-type R-algebra, or if T is a finite-type field extension of k, then with $X_T := X \otimes_R T$, the map $\mathfrak{h} \otimes 1$ factors uniquely as*

$$\overline{\mathcal{O}_X} \otimes_R T \xrightarrow[2.1.1]{\alpha_\bullet} \overline{\mathcal{O}_{X_T}} \xrightarrow{\mathfrak{h}_T} \mathcal{O}_{X_T}(n) = \mathcal{O}_X(n) \otimes_R T;$$

and $\mathfrak{h}_T$ is injective, $\mathcal{O}_{X_T}(n)/\mathfrak{h}_T(\mathcal{O}_{X_T})$ is T-flat, and $\mathcal{O}_{X_T}(n)/\mathfrak{h}_T(\overline{\mathcal{O}_{X_T}})$ is T-torsion-free.

Proof. The restriction of the previously described map $\mathfrak{h}$ to $\mathcal{O}_X$ corresponds to multiplication by h in A; and for any R-algebra T at all, the map

$$\mathcal{O}_{X_T} = \mathcal{O}_X \otimes_R T \xrightarrow{\mathfrak{h}\otimes 1} \mathcal{O}_X(n) \otimes_R T = \mathcal{O}_{X_T}(n)$$

is given by multiplication by $h \otimes 1 \in A \otimes_R T$. Let $a \in A_{[1]}$, and set $S = S^a := A_{(a)} \otimes_R T$. As in (4.2.1), over $\mathsf{Spec}\, A_{(a)} \subset X$ the map $\mathfrak{h}$ is given by multiplication by the regular element h/a^n; and similarly, over $\mathsf{Spec}\, S \subset X_T$ the map $\mathfrak{h} \otimes 1$ is given by multiplication by $h_T := h/a^n \otimes 1$—which is regular in S (see line following (4.1.3)).

Let us first check uniqueness, i.e., that if $\mathfrak{h}'_T \circ \alpha_\bullet = \mathfrak{h}''_T \circ \alpha_\bullet = \mathfrak{h} \otimes 1$, then $\mathfrak{h}'_T = \mathfrak{h}''_T$. This can be done locally, say over $\mathsf{Spec}\, S$. Set $\alpha^a := \Gamma(\mathsf{Spec}\, S, \alpha_\bullet)$, $\mathfrak{h}^{*a}_T := \Gamma(\mathsf{Spec}\, S, \mathfrak{h}^*_T)$. Let $\sigma \in \overline{S}$, and let $g \in S$ be a regular element such that $g\sigma \in S$. Let $\iota\colon A_{(a)} \hookrightarrow \overline{A_{(a)}}$ be the inclusion map. From 2.1.1 we get that $\alpha^a \circ (\iota \otimes 1)$ is the inclusion $S \hookrightarrow \overline{S}$. Then

$$g\mathfrak{h}'^a_T(\sigma) = \mathfrak{h}'^a_T(g\sigma) = \mathfrak{h}'^a_T\alpha^a(\iota \otimes 1)(g\sigma) = \mathfrak{h}''^a_T\alpha^a(\iota \otimes 1)(g\sigma) = \mathfrak{h}''^a_T(g\sigma) = g\mathfrak{h}''^a_T(\sigma);$$

and so, g being regular, $\mathfrak{h}'^a_T(\sigma) = \mathfrak{h}''^a_T(\sigma)$. Thus $\mathfrak{h}'^a_T = \mathfrak{h}''^a_T$, and therefore $\mathfrak{h}'_T = \mathfrak{h}''_T$.

Given the preceding uniqueness, we need only prove existence over $\mathsf{Spec}\, S = \mathsf{Spec}\, S^a$, since then gluing gives existence in the general case. So assume $X = \mathsf{Spec}\, A_{(a)}$. When $R \subset T \subset K$, (4.1.4)—suitably interpreted—shows that multiplication by $(h/a^n)\otimes 1$ maps $\overline{S}$ into S. Call the resulting *injective* map $\mathfrak{h}^a_T$ (see (4.2.1)). The S-homomorphisms $\mathfrak{h}^a_T \circ \alpha^a \circ (\iota\otimes 1)$ and $\Gamma(\mathsf{Spec}\, S, \mathfrak{h}\otimes 1) \circ (\iota\otimes 1)$ are both given by multiplication by $(h/a^n)\otimes 1$, so the restrictions of the maps $\mathfrak{h}^a_T \circ \alpha^a$ and $\Gamma(\mathsf{Spec}\, S, \mathfrak{h}\otimes 1)$ to $(\iota\otimes 1)(S) \subset \overline{A_{(a)}}\otimes T$ coincide. Since $(h/a^n)\otimes 1$ is regular in S and $((h/a^n)\otimes 1)(\overline{A_{(a)}} \otimes T) \subset (\iota \otimes 1)(S)$, therefore $\mathfrak{h}^a_T \circ \alpha^a$ and $\Gamma(\mathsf{Spec}\, S, \mathfrak{h} \otimes 1)$ coincide on $\overline{A_{(a)}} \otimes T$, whence $(\mathsf{Spec}\, \mathfrak{h}^a_T) \circ \alpha_\bullet = \mathfrak{h} \otimes 1$, giving the asserted existence.

A similar argument holds when T is a field extension, necessarily separable, of k, except that now (4.1.4) (with S in place of A_T) holds for $T = k$ because $h \in \mathfrak{C}'$, and then for any field extension $T \supset k$ because $\overline{S} = (\overline{A_{(a)} \otimes_R k}) \otimes_k T$ [EGA, (6.14.2)].

The *flatness* assertion is local on X, and so in view of (4.2.1), it is given by the fact that $S/((h/a^n)\otimes 1)S = (A_{(a)}/(h/a^n)A_{(a)})\otimes_R T$ is T-flat (see third line before (4.1.3)). Similarly, the *torsion-freeness* assertion is given by Lemma 4.1.1. □

Noting that with n as in 4.2.2, $\mathcal{H}_{y_i}(\mathcal{O}_{X_i}(n)) = \mathcal{H}_R(\mathcal{O}_X(n))$, we redefine $g\colon \mathsf{G}\to \operatorname{\mathsf{Spec}} R$ to be the *Quot-scheme* of R-flat quotients of $\mathcal{O}_X(n)$ with Hilbert polynomial

$$\mathcal{H}(n):=\mathcal{H}_{y_1}\big(\mathcal{O}_{X_1}(n)\big)-\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}) = \mathcal{H}_{y_0}\big(\mathcal{O}_{X_0}(n)\big)-\mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}}),$$

see [G, p. 221-12, Thm. 3.2]. For R-algebras T and $X_T := X\otimes_R T$ there is a functorial bijection between R-morphisms $T\to\mathsf{G}$ and coherent $\mathcal{O}_{X_T}$-submodules L of $\mathcal{O}_{X_T}(n)$ such that $\mathcal{O}_{X_T}(n)/L$ is T-flat and $\mathcal{H}_T(\mathcal{O}_{X_T}(n)/L)=\mathcal{H}$. The map g is projective.

Assume further that $R\subset T\subset K$ and that T is normal and essentially of finite type over R. Let $\psi\colon \operatorname{\mathsf{Spec}} K\to\mathsf{G}$ be the R-morphism corresponding to $\mathfrak{h}_K(\overline{\mathcal{O}_{X_0}})\subset\mathcal{O}_{X_0}(n)$. Suppose the domain of definition $\mathrm{D}(\psi_T)$ of the corresponding rational R-morphism $\psi_T\colon T\to\mathsf{G}$ is all of $\operatorname{\mathsf{Spec}} T$. Then ψ_T determines a coherent $\mathcal{O}_{X_T}$-module $L_1\subset\mathcal{O}_{X_T}(n)$ with $\mathcal{O}_{X_T}(n)/L_1$ T-flat, such that $L_1\otimes_T K=\mathfrak{h}_K(\overline{\mathcal{O}_{X_0}}) = L_2\otimes_T K$ where $L_2:=\mathfrak{h}_T(\overline{\mathcal{O}_{X_T}})$; and by 4.2.2, $\mathcal{O}_{X_T}(n)/L_2$ is T-torsion-free. The assertion being local, Lemma 4.1.6 shows then that $L_1=L_2$, so that $\mathcal{O}_{X_T}(n)/L_2\cong\mathcal{O}_{X_T}(n)/L_1$ *is T-flat,* and there is an exact sequence of T-flat $\mathcal{O}_{X_T}$-modules

$$0\to\overline{\mathcal{O}_{X_T}}\xrightarrow{\mathfrak{h}_T}\mathcal{O}_{X_T}(n)\to\mathcal{O}_{X_T}(n)/L_2\to 0.$$

If $T=R$, so that $\mathcal{O}_{X_T}=\mathcal{O}_X$, then we deduce that $\mathfrak{h}\otimes 1\colon \overline{\mathcal{O}_X}\otimes_R k\to\mathcal{O}_{X_1}(n)$ is injective, whence, by 4.2.2, so is $\alpha_\bullet\colon\overline{\mathcal{O}_X}\otimes_R k\to\overline{\mathcal{O}_{X_1}}$—in other words, the schematic dominance mentioned in the first paragraph of the proof of Theorem 4.2 holds.

Thus it suffices to show that *if* $\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}})=\mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}})$ *then* $\mathrm{D}(\psi)$ *is all of* $\operatorname{\mathsf{Spec}} R$.

To do so we proceed in essentially the same way as in the corresponding part of the proof of Theorem 4.1. Let $(S,\mathfrak{m}_S,k_S)$ be as in that proof. The natural map $\operatorname{\mathsf{Spec}} S\to\mathsf{G}$, which localizes generically to ψ and hence is the same as the above ψ_S, corresponds to $\mathfrak{h}_S(\overline{\mathcal{O}_{X_S}})$, which is S-flat (so that $\overline{\mathcal{O}_{X_S}}$ is S-flat); and

$$\begin{aligned}\mathcal{H}_{\mathfrak{m}_S}(\overline{\mathcal{O}_{X_S}}\otimes_S k_S)=\mathcal{H}_S(\overline{\mathcal{O}_{X_S}}) &= \mathcal{H}_S\big(\mathcal{O}_S(n)\big)-\mathcal{H}_S\big(\mathcal{O}_S(n)/\mathfrak{h}_S\big(\overline{\mathcal{O}_{X_S}}\big)\big)\\ &=\mathcal{H}_{y_0}\big(\mathcal{O}_{X_0}(n)\big)-\mathcal{H}(n)=\mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_0}})=\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}).\end{aligned}$$

Since k_S is separable over k, therefore (by [EGA, (6.14.2)]),

$$\mathcal{H}_{\mathfrak{m}_S}(\overline{\mathcal{O}_{X_{k_S}}})=\mathcal{H}_{\mathfrak{m}_S}(\overline{\mathcal{O}_{X_1}\otimes_k k_S})=\mathcal{H}_{\mathfrak{m}_S}(\overline{\mathcal{O}_{X_1}}\otimes_k k_S)=\mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_1}}).$$

where the last equality holds because cohomology is compatible with flat base change [EGIII, (1.4.15)]. Moreover, by 2.2.4, the map $\overline{\mathcal{O}_{X_S}}\otimes_S k_S\to\overline{\mathcal{O}_{X_{k_S}}}$ in 4.2.2 (applied to $f\otimes 1\colon X_S\to\operatorname{\mathsf{Spec}} S$, with $T=k_S$) is injective. Since the source and target of this map have the same Hilbert polynomial, it must be an isomorphism, and so

$$\mathfrak{h}_{k_S}(\overline{\mathcal{O}_{X_{k_S}}})=(\mathfrak{h}_S\otimes 1)(\overline{\mathcal{O}_{X_S}}\otimes_S k_S)\subset\mathcal{O}_{k_S}(n).$$

This signifies that the natural composition $\operatorname{\mathsf{Spec}} k_S\hookrightarrow\operatorname{\mathsf{Spec}} S\to\mathsf{G}$ corresponds to $\mathfrak{h}_{k_S}(\overline{\mathcal{O}_{X_{k_S}}})\subset\mathcal{O}_{k_S}(n)$. But $\mathfrak{h}_{k_S}(\overline{\mathcal{O}_{X_{k_S}}})=(\mathfrak{h}_k\otimes 1)(\overline{\mathcal{O}_{X_1}}\otimes_k k_S)$.

Thus if $\psi_1\colon \mathsf{Spec}\, k \to \mathsf{G}$ corresponds to $\mathfrak{h}_k\overline{\mathcal{O}_{X_1}} \subset \mathcal{O}_{X_1}(n)$ then the following natural diagram commutes:

$$\begin{array}{ccc} \mathsf{Spec}\, k_S & \longrightarrow & \mathsf{Spec}\, S \\ \downarrow & & \downarrow \\ \mathsf{Spec}\, k & \xrightarrow[\psi_1]{} & \mathsf{G} \end{array}$$

As in the proof of Theorem 4.1, the desired conclusion results. □

5. Application to complex spaces.

In this section we translate Theorems 4.1 and 4.2 from algebraic into analytic geometry, that is, from the context of schemes to the context of complex spaces, see Theorems 5.6 and 5.8. The treatment of these theorems comes after a few pages of preliminaries on equinormalization for maps of complex spaces.

To establish terminology and notation, referring to [C, exposés 9, 10, 13] or [GR] for details we first review a few foundational facts about complex spaces, beginning with their definition. For any $n \geq 0$ and open $U \subset \mathbb{C}^n$ let $\mathcal{O}_U$ be the sheaf of holomorphic functions $U \to \mathbb{C}$ (functions given locally by convergent power series). The *local model* determined by an open $U \subset \mathbb{C}^n$ $(n \geq 0)$ and a finite-type $\mathcal{O}_U$-ideal $\mathcal{I}$, is the topological space $X_{\mathcal{I}} :=$ support of $\mathcal{O}_U/\mathcal{I}$ together with the sheaf of $\mathbb{C}$-algebras $(\mathcal{O}_U/\mathcal{I})|_{X_{\mathcal{I}}}$. A *complex space* is a $\mathbb{C}$-ringed space—a topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras—such that each point of X has an open neighborhood isomorphic as a $\mathbb{C}$-ringed space to some local model.

Any open subset U of a complex space X can, and will, be viewed as a complex space, with $\mathcal{O}_U := \mathcal{O}_X|_U$. A subset of X will be called *analytic* if it is the underlying topological space of a closed complex subspace of X, i.e., if it is the support of $\mathcal{O}_X/\mathcal{I}$ for some finite-type $\mathcal{O}_X$-ideal $\mathcal{I}$.

An *analytic* (or *holomorphic*) *map* $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ *of complex spaces* is a continuous map $f_0\colon X \to Y$ together with a homomorphism of sheaves of $\mathbb{C}$-algebras $f_1\colon \mathcal{O}_Y \to f_{0*}\mathcal{O}_X$. The sheaf $\mathcal{O}_X$ is canonically isomorphic to the sheaf of (germs of) analytic maps from $(X,\mathcal{O}_X)$ to $(\mathbb{C}^1,\mathcal{O}_{\mathbb{C}^1})$, and f_1 is given locally by composition of analytic maps. It is a consequence of (indeed, essentially equivalent to) Rückert's nullstellensatz that if X is *reduced*, i.e., $\mathcal{O}_X$ has no nonzero nilpotents, then f is uniquely determined by f_0 [C, p.19-18, Cor.5]; in particular, $\mathcal{O}_X$ is canonically isomorphic to a subsheaf of the sheaf of continuous $\mathbb{C}$-valued functions on X.

Though principally interested here in reduced complex spaces, we must allow for the possible occurrence of nonreduced ones, for instance as fibers of morphisms $f\colon X \to Y$ of reduced complex spaces. (The fiber $X_y := f^{-1}y$ $(y \in Y)$ is by definition the complex space $X \times_Y [y]$ where $[y]$ is the complex subspace $(\{y\},\mathbb{C}) \subset (Y,\mathcal{O}_Y)$.)

A *pointed complex space* is a pair (X,x) where X is a complex space and $x \in X$. A *morphism* $(X,x) \to (Y,y)$ *of pointed complex spaces,* or *pointed analytic map,* is an analytic map $f\colon X \to Y$ such that $f(x) = y$. The localization of the category of pointed complex spaces with respect to open immersions is the category AG of complex germs. Explicitly, the objects of AG—called *germs*—are the pointed complex spaces, and the morphisms $(X,x) \to (Y,y)$ in AG—called *map germs*—are equivalence classes of pointed analytic maps $(U,x) \to (Y,y)$ with domain U an open neighborhood of x in X, where two such maps are equivalent if they agree on an open neighborhood of x contained in both their domains. The composition of two map germs is defined in the obvious way. Two germs (X, x) and (Y, y) are isomorphic iff there exist open neighborhoods U and V of x and y respectively, such that the pointed spaces (U, x) and (V, y) are isomorphic.

The category AA of *analytic* $\mathbb{C}$*-algebras* has as its objects homomorphic images of rings of convergent power-series with coefficients in $\mathbb{C}$, and as its morphisms all $\mathbb{C}$-algebra homomorphisms. Analytic $\mathbb{C}$-algebras are *excellent* noetherian local rings (see footnote under Remark 3.2.1(a)), and $\mathbb{C}$-algebra homomorphisms between them are automatically local (i.e., take nonunits to nonunits).

Associating to any germ the stalk $\mathcal{O}_{X,x}$, and to any map germ $(X, x) \to (Y, y)$ the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ one gets a contravariant functor $\mathcal{O}\colon \mathsf{AG} \to \mathsf{AA}$ which is in fact an *antiequivalence of categories* [C, p. 13-2, Thm. 1.3]. This is a fundamental link between local algebra and local analytic geometry. The connection between algebra and geometry is illustrated in the following Proposition.

Definition 5.1. A map of complex spaces satisfies $(\clubsuit)_{\mathrm{an}}$ if

- f is flat (i.e., the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat for all $x \in X$) and all the fibers of f are reduced.
- Y is normal (i.e., $\mathcal{O}_{Y,y}$ is normal for all $y \in Y$).
- X is locally equidimensional (i.e., for each $x \in X$ the irreducible components of X passing through x all have the same dimension, i.e., the connected components of X are equidimensional). *Equivalently* (see [Ma2, p. 116, 15.1]), the fibers of f are locally equidimensional.

With $y := f(x)$ and $\mathfrak{m}_y$ the maximal ideal of $\mathcal{O}_{Y,y}$, the local ring of x on the analytic fiber $f^{-1}y$ is $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ [C, p. 10-07]. It follows from 1.7.1 and [Ma2, p. 184, Cor. (ii)] that *if f satisfies $(\clubsuit)_{\mathrm{an}}$ then X is reduced.*

Proposition 5.2. *The map $\mathsf{Spec}\,\phi\colon \mathsf{Spec}\,R \to \mathsf{Spec}\,A$ associated to a $\mathbb{C}$-homomorphism of analytic $\mathbb{C}$-algebras $\phi\colon R \to A$ satisfies $(\clubsuit)$ iff ϕ is isomorphic to $\mathcal{O}(F)$ where F is a map germ represented by a pointed analytic map $f\colon (X, x) \to (Y, y)$ satisfying $(\clubsuit)_{\mathrm{an}}$.*

Moreover, if $\mathsf{Spec}\,\phi$ does satisfy $(\clubsuit)$ then such an f exists with X reduced and connected, Y normal and irreducible, f surjective, and all fibers of f of pure dimension $\dim_x X - \dim_y Y$; and if, in addition, $A/\mathfrak{m}A$ is normal, then there is an f with the preceding properties and also such that X is normal and all the fibers of f are normal.

Proof. Suppose given f satisfying $(\clubsuit)_{\mathrm{an}}$ and $\phi \cong \mathcal{O}(F)$. The closed fiber of $\mathsf{Spec}\,\phi$ is isomorphic to the Spec of the local ring of x on the analytic fiber $f^{-1}y$, see [C, p. 10-07]; hence $\mathsf{Spec}\,\phi$ is flat, with geometrically reduced closed fiber, see 1.1. The local ring $R \cong \mathcal{O}_{Y,y}$ is normal, and, being excellent, satisfies $\mathrm{ff}_{\mathrm{nor}}$. So by 1.7.1, all the fibers of $\mathsf{Spec}\,\phi$ are geometrically reduced, so that $\mathsf{Spec}\,\phi$ is reduced. The local ring $A \cong \mathcal{O}_{X,x}$, being excellent, is a universally catenary Nagata ring, whose equidimensionality results from [C, p. 20-03, Cor. 2]. Thus $\mathsf{Spec}\,\phi$ satisfies $(\clubsuit)$.

Conversely, suppose that $\mathsf{Spec}\,\phi$ satisfies $(\clubsuit)$ (resp. and in addition $A/\mathfrak{m}A$ is normal). By the above-mentioned equivalence of categories, $\mathsf{Spec}\,\phi \cong \mathcal{O}(F)$ for some map germ F represented by a pointed analytic map $f\colon (X, x) \to (Y, y)$. For any open neighborhoods U of x and V of y with $f(U) \subset V$, we are free to replace f by $f|_U\colon (U, x) \to (V, y)$, a process referred to as "shrinking."

Since A is equidimensional, [GR, Chap. 5, §4.2, p. 106] shows that after shrinking X if necessary, we may assume that all components of X have dimension $d := \dim_x X$ at each of their points. Further, Frisch's theorem [Fr] allows to assume, after more shrinking, that f is flat. Still further, [Fi, p. 160, 3.22] allows us to assume that all the fibers of f are reduced (resp. normal). Then by 1.7.1 and [Ma2, p. 184, 23.9], X is reduced (resp. normal). Since y is a normal point (because $\mathcal{O}_{Y,y} \cong R$) and the nonnormal points of Y form an analytic subset [C, p. 21-09, Thm. 2], Y has a normal open neighborhood V, which may be assumed to be connected, hence irreducible and of dimension $e := \dim_y Y$ at each of its points. It follows from the dimension relation for flat local homomorphisms [Ma2, p. 116, 15.1], applied to $\mathcal{O}_{Y,f(s)} \to \mathcal{O}_{X,s}$ $(s \in f^{-1}V)$, that after Y is shrunk to V and X to $f^{-1}V$, every fiber of f has pure dimension $d - e$. Since flat maps are open, [Fi, p. 156, 3.19], therefore f maps the connected component X_0 of X containing x onto a connected open subset $Y_0 \subset Y$. Shrinking X to X_0 and Y to Y_0 finishes the proof. $\square$

Equinormalizability of a *reduced* map $f\colon X \to Y$ of analytic spaces (i.e., f is flat, with reduced fibers) means, again, the existence of a simultaneous normalization, now defined as follows. Recall first that an analytic map $\mu\colon Z' \to Z$ is *finite* if it is proper and has finite fibers; that μ is *bimeromorphic* if there are analytically rare (hence nowhere dense, and conversely when X is reduced [Fi, pp. 38–40, §0.43]) analytic subsets $W \subset Z$ and $W' = \mu^{-1}(W) \subset Z'$ such that μ induces an isomorphism from $Z' \setminus W'$ onto $Z \setminus W$; and that μ is a *normalization map,* or a *normalization of* Z, if μ is finite and bimeromorphic, and in addition Z is reduced and Z' is normal. For fixed Z, any two normalizations are isomorphic [C, p. 21-11, Cor. 3]. We will use freely properties of normalization found, e.g., in [GR, Chapter 8] or [C, Exposé 21, §4].

Definition 5.3. A *simultaneous normalization* of a reduced analytic map $f\colon X \to Y$ is a finite analytic map $\nu\colon Z \to X$ such that such that $\bar{f} := f \circ \nu$ is *normal* (flat, with all nonempty fibers normal), and such that for each $y \in f(X)$ the induced map of fibers $\nu_y\colon \bar{f}^{-1}y \to f^{-1}y$ is a normalization map.

Proposition 5.4. *Let $f\colon X \to Y$ be an analytic map satisfying $(\clubsuit)_{\mathrm{an}}$. Then:*

(i) *Any simultaneous normalization of f is a normalization of X.*

(ii) *A normalization $\nu\colon \overline{X} \to X$ is a simultaneous normalization of f if and only if the nonempty fibers of $f\nu$ are all normal.*

Proof. It suffices to prove the assertions for the restriction of f to any of the connected components of X. So we may assume that X and Y are connected. Then Y, being normal, is irreducible, and so has the same dimension, say e, at each of its points. Moreover, since X is locally equidimensional, for any $n \geq 0$ the (locally finite) union of the n-dimensional irreducible components of X is an open and closed complex subspace; and hence we may assume that X has the same dimension, say d, at each of its points.

(i) It follows from 5.2 that every fiber $f^{-1}y$ $(y \in f(X))$ has pure dimension $d - e$. For a simultaneous normalization $\nu\colon Z \to X$ the same is therefore true of $(f\nu)^{-1}y$, the normalization of $f^{-1}y$ (because a proper bimeromorphic map $\mu\colon W' \to W$ maps each irreducible component $C \subset W'$ bimeromorphically onto an irreducible component of W having the same dimension as C.) The restriction of $f\nu$ to any open subset U of Z meeting only one irreducible component C of Z is flat, so [Ma2, p. 116, 15.1] implies that $\dim C = \dim U = e + (d-e) = d$. Since ν is finite, we conclude that $\nu(C)$ *is an irreducible component of* X.

By 1.7.1 and [Ma2, p. 184, Cor. (ii)], X is reduced and Z is normal. It remains then to show that ν is bimeromorphic.

Lemma 5.4.1. *For any* $x \in X$ *the following conditions are equivalent.*

(a) x *is a normal point of* $f^{-1}f(x)$.

(b) *There is a unique* $z \in Z$ *such that* $\nu(z) = x$; *and for that* z, *the induced map germ* $(Z, z) \to (X, x)$ *is an isomorphism.*

Once the lemma is proved, then since the set of points $W \subset X$ not satisfying (a) is analytic [Fi, p. 160, Proposition], clearly nowhere dense (because f has reduced fibers), we see, by (b), that the restriction of ν maps $Z \setminus \nu^{-1}(W)$ one-to-one and locally isomorphically, hence globally isomorphically, onto $X \setminus W$; and moreover, $\nu^{-1}(W)$ is nowhere dense in Z, because ν maps components of Z onto components of X. Thus ν is bimeromorphic.[12]

As for the proof of the lemma, the implication (b) $\Rightarrow$ (a) is simple, because the map $f\nu$ has normal fibers. Conversely, if (a) holds, then, by standard properties of normalization, since ν normalizes fiberwise, there just one z with $\nu(z) = x$, and with $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{Y,f(x)}$, ν induces an isomorphism $\mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{Z,z}/\mathfrak{m}\mathcal{O}_{Z,z}$. Then, since $\mathcal{O}_{Z,z}$ is a finite $\mathcal{O}_{X,x}$-module, Nakayama's lemma shows that ν induces a surjection $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{Z,z}$, which must be an isomorphism, because $\mathcal{O}_{X,x}$ is normal (see 5.2) and, as above, $\dim \mathcal{O}_{Z,z} = \dim \mathcal{O}_{X,x} = d$.

This completes the proof of (i).

(ii) By definition, if ν is a simultaneous normalization then the nonempty fibers of $f\nu$ are normal.

Suppose, conversely, that the fibers of $f\nu$ are normal. Let $y \in Y$ and let C be an irreducible component of $\overline{X}_y := (f\nu)^{-1}y$. Recall that $\nu(C)$ can be viewed as a closed complex subspace of $X_y := f^{-1}y$ [GR, p. 65, Proposition]. We claim that $\nu(C)$ *is an irreducible component of* $X_y := f^{-1}y$. To see this, let $z \in C$ lie on no other component of $\overline{X}_y$ and set $x := \nu(z)$. Since ν is a normalization and X is equidimensional, therefore $\dim_z \overline{X} = \dim_x X$. Also, by [Ma2, p. 116, 15.1(ii)],

$$\dim_x \nu(C) = \dim_z C = \dim_z \overline{X}_y \ge \dim_z \overline{X} - \dim_y Y = \dim_x X - \dim_y Y = \dim_x X_y,$$

and the assertion results.

Next, let D be the dense open subset of normal points of the reduced analytic space X_y. By 1.7.1 and [Ma2, p. 184, Cor. (ii)], every point $w \in D$ is normal on X, and so w has an open X-neighborhood U_w such that $\nu^{-1}(U_w) \to U_w$ is an isomorphism. Every component C of $\overline{X}_y$ must meet $\nu^{-1}(U_w)$ for some w, since D meets the component $\nu(C)$ of X_y in an open dense set. It follows then from [C, p. 21-11, Cor. 3] that $\nu\colon \overline{X}_y \to X_y$ is a normalization; and by 2.2.1(iv), $f\nu$ is flat. Thus ν is a simultaneous normalization of f. $\square$

We say that an analytic map $f\colon X \to Y$ is *equinormalizable at* $x \in X$ if the induced scheme-map $\mathsf{Spec}\,\mathcal{O}_{X,x} \to \mathsf{Spec}\,\mathcal{O}_{Y,f(x)}$ is equinormalizable.

[12] One way to avoid the referenced theorem is via differentials and ramification [C, p.20-12, Corollaire].

Corollary 5.4.2. *Let $f: X \to Y$ be an analytic map satisfying $(\clubsuit)_{\mathrm{an}}$.*

(i) *If f is equinormalizable at $x \in X$ then the restriction of f to some neighborhood of x is equinormalizable.*

(ii) *f is equinormalizable iff f is equinormalizable at x for all $x \in X$.*

Proof. Let $\nu: Z \to X$ be a normalization, and let $z_1, \dots, z_n$ be the distinct points in $\nu^{-1}x$. The integral closure of $\mathcal{O}_{X,x}$ is $B := \prod_{i=1}^{n} \mathcal{O}_{Z,z_i}$, which is equidimensional since $\mathcal{O}_{X,x}$ is.

Let $(R, \mathfrak{m})$ be the local ring $\mathcal{O}_{Y,f(x)}$. By 2.3, a simultaneous normalization of schemes is a normalization. So equinormalizability of f at x implies that B is R-flat and that $B/\mathfrak{m}B$ is normal, whence, by 1.7.1, the natural map $\mathsf{Spec}\, B \to \mathsf{Spec}\, R$ satisfies $(\clubsuit)$. By 5.2, $f\nu$ restricted to some neighborhood V of $\nu^{-1}x$ has normal fibers. Since ν is finite there is a neighborhood W of x such that $\nu^{-1}W \subset V$. By 5.4(ii), the restriction of f to W is equinormalizable, proving (i).

As for (ii), if f is equinormalizable at every $x \in X$ then it results from (i) that the nonempty fibers of $f\nu$ are normal, so that f is equinormalizable.

Conversely, if f is equinormalizable, so that the nonempty fibers of $f\nu$ are normal, then by 5.2, 1.7.1 and 2.3.1, f is equinormalizable at every $x \in X$. □

Corollary 5.4.3. *If the set $S_f := \{\, x \in X \mid x \text{ not normal on } X_{f(x)} \,\}$ (which is analytic [Fi, p. 160, 3.22]) is proper over Y then there is a nowhere dense analytic subset $T \subset Y$ such that f is equinormalizable at every $x \notin f^{-1}T$.*

Proof. Let $\nu: \overline{X} \to X$ be a normalization. Set $S_{f\nu} := \{\, z \in \overline{X} \mid z \text{ not normal on } \overline{X}_{f\nu(z)} \,\}$, an analytic subset of $\overline{X}$. Then $\nu(S_{f\nu}) \subset S_f$—for if $x = \nu(z)$ is normal on $X_{f(x)}$ then x is normal on X (see 5.2), so that for some neighborhood U of x, ν induces an isomorphism $\nu^{-1}U \xrightarrow{\sim} U$, whence $z \in \nu^{-1}U$ is normal on $\overline{X}_{f\nu(z)}$. By 5.4(ii), f is equinormalizable at every point outside $\nu(S_{f\nu})$. Since S_f is proper over Y, the set $T := f\nu(S_{f\nu})$, which is analytic [GR, p. 213], and nowhere dense in Y [Mn, p. 270, Thm. (I.5)], is as asserted. □

From now on, by *curve* we mean *reduced one-dimensional complex space with no isolated points.* An *analytic family of curves* is a flat map $f: X \to Y$ of complex spaces whose fibers X_y $(y \in Y)$ are curves. If Y is normal then such an f satisfies $(\clubsuit)_{\mathrm{an}}$, and X is reduced. For families of curves with Y reduced a simultaneous normalization is the same thing as a "very weak simultaneous resolution" [T2, p. 72, Défn. 2].

The following theorem 5.6 gives a numerical criterion of equinormalizability at a point $x \in X$ for a family of curves $f: X \to Y$. Arguing as in 5.2, and with the aid of the dimension relation [Ma2, p. 116, 15.1], we see that a pointed analytic map $(X, x) \to (Y, y)$ defines a map germ represented by a family of curves $f: X \to Y$ with normal Y if and only if $(A, \mathfrak{M}) := \mathcal{O}_{X,x}$ is flat over $(R, \mathfrak{m}) := \mathcal{O}_{Y,y}$, $A/\mathfrak{m}A$ is reduced and of pure dimension 1, and R is normal. For such a family the set $\mathrm{Sing}(f)$ of points on X where the fiber $f^{-1}f(x)$ is singular is analytic [Fi, p. 160, 3.22], meeting each fiber in a discrete set; so we may, and will, assume that the induced map $\mathrm{Sing}(f) \to Y$ is *finite*, and that $\mathrm{Sing}(f) \cap f^{-1}y = \{x\}$.

The theorem is due to Teissier when $Y = \mathbb{C}^1$, and was extended to arbitrary normal Y by Raynaud (see remarks in the Introduction). Here we will deduce the theorem from its algebraic counterpart Theorem 4.1.

Definition 5.5. For a curve C and $x \in C$, set

$$\delta(C,x) := \dim_{\mathbb{C}}(\overline{\mathcal{O}_{C,x}}/\mathcal{O}_{C,x}).$$

If C has only finitely many singular points, we set

$$\delta(C) := \sum_{x\in C} \delta(C,x).$$

Theorem 5.6. *Let $f\colon (X,x_0) \to (Y,y_0)$ be a flat pointed analytic map, with Y normal, such that the nonempty fibers of f are curves. Assume that* $\mathrm{Sing}(f)$ *is finite over Y and that* $\mathrm{Sing}(f) \cap f^{-1}y_0 = \{x_0\}$. *Then f is equinormalizable at x_0 if and only if $\delta(X_y)$ is the same for all y in some neighborhood of y_0.*

Proof. As above, f satisfies $(\clubsuit)_{\mathrm{an}}$ and X is reduced. Since flat analytic maps are open [Fi, p. 156, 3.19], we may assume further that f is surjective.

Let $\nu\colon \overline{X} \to X$ be a normalization, and set $\overline{\mathcal{O}_X} := \nu_*\mathcal{O}_{\overline{X}}$. As in 5.2 we find that if $x \in X$ is a normal point of $f^{-1}f(x)$ then x is normal on X. Hence the support of the coherent sheaf $\overline{\mathcal{O}_X}/\mathcal{O}_X$—topologically the set of nonnormal points of X—is finite over Y, and the $\mathcal{O}_Y$-module $f_*(\overline{\mathcal{O}_X}/\mathcal{O}_X)$ is *coherent* [GR, p. 64, **3**]. After shrinking we may assume that Y is irreducible and that $f_*(\overline{\mathcal{O}_X}/\mathcal{O}_X)$ is the cokernel of a map $\mathcal{O}_Y^m \to \mathcal{O}_Y^n$ represented by an $n\times m$ matrix whose entries are holomorphic functions on Y. The rank r_y of this matrix when evaluated at the point $y \in Y$ is a function on Y taking its maximum value r on a dense open subset $U \subset Y$, the complement of the analytic set defined by the vanishing of the $r\times r$ subdeterminants. For any $y\in Y$ the germs at $\mathcal{O}_y := \mathcal{O}_{Y,y}$ of these $r \times r$ subdeterminants are not all zero, since every neighborhood of y meets U; while the germs of the $(r+1)\times(r+1)$ subdeterminants do vanish. Setting $K_y :=$ the fraction field of $\mathcal{O}_y$, we deduce that *the dimension of the K_y-vector space $f_*(\overline{\mathcal{O}_X}/\mathcal{O}_X)_y \otimes_{\mathcal{O}_y} K_y$ is $(n-r)$*—which does not depend on y.

Since for any $y \in Y$, $f_*(\overline{\mathcal{O}_X}/\mathcal{O}_X)_y = \oplus_{x\in f^{-1}y}(\overline{\mathcal{O}_X}/\mathcal{O}_X)_x$ (where x contributes nothing to the sum unless $x\in N$, the nonnormal locus of X), we have

$$n - r = \sum_{x\in N\cap f^{-1}y} \dim_{K_y}\big((\overline{\mathcal{O}_X}/\mathcal{O}_X)_x \otimes_{\mathcal{O}_y} K_y\big). \tag{5.6.1}$$

Lemma 5.6.2. *If the fiber $\overline{X}_y := (f\nu)^{-1}y$ is normal then $x \in X_y$ is normal iff x is normal on X. Thus,*

$$\delta(X_y) = \sum_{x\in N\cap f^{-1}y} \delta(X_y,x) = \sum_{x\in N\cap f^{-1}y} \dim_{\mathbb{C}}\big(\overline{(\mathcal{O}_{X_y})_x}/(\mathcal{O}_{X_y})_x\big).$$

Proof. That [x normal on X_y] implies [x normal on X] is shown as in 5.2. Conversely, if x is normal on X, then x has a normal neighborhood V, and the normalization ν maps $\nu^{-1}V$ isomorphically onto V; so $X_y \cap V \cong \overline{X}_y \cap \nu^{-1}V$ is normal. □

Now suppose f equinormalizable at x_0. By 5.4.2 we may assume (after shrinking X) that f is equinormalizable for all $x\in X$. Then 2.2.1(iii)$'$ holds for $\mu = \nu$ (see 2.3.1), and hence, with $k_y \cong \mathbb{C}$ the residue field of $\mathcal{O}_{Y,y}$, $(\overline{\mathcal{O}_X}/\mathcal{O}_X)_x$ is a free $\mathcal{O}_y$-module, of rank

$$\dim_{K_y}\big((\overline{\mathcal{O}_X}/\mathcal{O}_X)_x \otimes_{\mathcal{O}_y} K_y\big) = \dim_{\mathbb{C}}\big((\overline{\mathcal{O}_X}/\mathcal{O}_X)_x \otimes_{\mathcal{O}_y} k_y\big) = \dim_{\mathbb{C}}\big(\overline{(\mathcal{O}_{X_y})_x}/(\mathcal{O}_{X_y})_x\big),$$

where the last equality holds because by the definition of simultaneous normalization, $\overline{(\mathcal{O}_{X_y})_x} = \overline{\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_y} k_y$. So by (5.6.1) and 5.6.2, $\delta(X_y) = n - r$ for all y.

Assume conversely that $\delta(X_y)$ is constant near y_0. There exists a sequence $y_1, y_2, y_3, \dots$ on Y approaching y_0 such that for all i, X_{y_i} is nonempty and f is equinormalizable at each $x \in X_{y_i}$ (see 5.4.3 or [Mn, p. 271, (I.10)]). As shown above, this implies that

$$\delta(X_{y_i}) = n - r = \dim_{K_{y_0}}\big((\overline{\mathcal{O}_X}/\mathcal{O}_X)_{x_0} \otimes_{\mathcal{O}_{y_0}} K_{y_0}\big).$$

Since for $i \gg 0$, $\delta(X_{y_i}) = \delta(X_{y_0})$, we conclude that, with $\mathcal{O}_{y_0} := \mathcal{O}_{Y,y_0}$ and $\mathcal{O}_{x_0} := \mathcal{O}_{X,x_0}$,

$$\delta_{\mathbb{C}}(\mathcal{O}_{x_0} \otimes_{\mathcal{O}_{y_0}} k_{y_0}) = \delta(X_{y_0}) = \dim_{K_{y_0}}\big((\overline{\mathcal{O}_X}/\mathcal{O}_X)_{x_0} \otimes_{\mathcal{O}_{y_0}} K_{y_0}\big) = \delta_{K_{y_0}}(\mathcal{O}_{x_0} \otimes_{\mathcal{O}_{y_0}} K_{y_0}).$$

In view of Proposition 5.2, Theorem 4.1 gives that f is equinormalizable at x_0. □

5.7. Theorem 5.8 below deals with a *projective* analytic map $f\colon X \to Y$, that is, a proper map such that there exists an f-ample invertible $\mathcal{O}_X$-module [BS, p. 141]. (To avoid some trivialities we assume henceforth that X is nonempty.) The theorem concerns properties which are local on Y, so no essential loss in generality results from assuming Y connected and, with $\mathbb{P}^r$ the r-dimensional projective space over $\mathbb{C}$ for some $r > 0$, that $f = p \circ i$ where $i\colon X \to Y \times \mathbb{P}^r$ is a closed immersion and $p\colon Y \times \mathbb{P}^r \to Y$ is the projection (see [BS, p. 143]). Using Serre's "GAGA" comparison theorem (see e.g., [BS, p. 148, Thm. 2.6]) one sees that any fiber X_y ($y \in Y$) is the complex subspace associated to a closed subscheme X_y^0 of $\mathbb{P}^r$ (Chow's theorem) and that for any coherent $\mathcal{O}_{X_y}$-module $\mathcal{M}$ the definition of the Hilbert polynomial $\mathcal{H}_y(\mathcal{M})$—recalled just before Prop. 3.4, with $\mathcal{L} := i^*\mathcal{O}_{Y\times\mathbb{P}^r}(1)$—makes sense in the present context: in fact if $\gamma\colon X_y \to X_y^0$ is the canonical ringed-space map then $\mathcal{M} \cong \gamma^*\mathcal{M}^0$ for some coherent $\mathcal{O}_{X_y^0}$-module $\mathcal{M}^0$, unique up to isomorphism, and $\mathcal{H}_y(\mathcal{M})$ (analytic) coincides with $\mathcal{H}_y(\mathcal{M}^0)$ (algebraic)—the latter calculated with respect to the algebraic $\mathcal{O}(1)$ on $X_y^0 \subset \mathbb{P}^r$.

In particular, when X_y (and hence X_y^0) is reduced and $\mathcal{M}^0$ is the integral closure of $\mathcal{O}_{X_y^0}$ then by basic properties of the association of complex spaces to finite-type $\mathbb{C}$-schemes (clearly described in [SGA, Exposé XII]), $\mathcal{M} := \gamma^*\mathcal{M}^0$ is the integral closure of $\mathcal{O}_{X_y}$. (For this fact, another argument appears in the proof of Lemma 5.7.4.)

For simplicity, we assume throughout that the complex space Y is normal and irreducible. To deduce the analytic Theorem 5.8 from its algebraic counterpart Theorem 4.2 we will need some results on projective analytic maps analogous to ones given for schemes in [EGII] and [EGIII], and enabled by Grauert-Remmert's analytic version [Bi, p. 36, (5.11)] of Serre's "fundamental theorem of projective morphisms" [EGIII, Thm (2.2.1)]. In our description of these results, some details will be left implicitly to the reader.

Every finitely-presented $\mathcal{O}_Y$-algebra $\mathcal{G} = \oplus_{n\ge0}\,\mathcal{G}_n$ [C, p. 19-01, Défn. 1] determines an analytic map $g\colon \mathsf{Projan}\,\mathcal{G} \to Y$ such that for each Stein open $U \subset Y$ whose closure $\overline{U}$ is a Stein compactum,[13] there is an $s > 0$ such that the reduced space underlying $g^{-1}U$ looks like the zeros in $U \times \mathbb{P}^s$ of a set of homogeneous polynomials in $\Gamma(\overline{U}, \mathcal{O}_Y)[T_0, T_1, \dots, T_s]$

[13]That is, a semianalytic compact set having a neighborhood basis of open Stein subspaces of Y, for example, the inverse image of a small closed ball under a local embedding of Y into some $\mathbb{C}^n$.

generating the kernel of a surjection $(\mathcal{O}_Y|_{\overline{U}})[T_0, T_1, \dots, T_s] \twoheadrightarrow \mathcal{G}|_{\overline{U}}$. (Details are in [Bi, p. 36].) For instance,

$$\mathsf{Projan}\, \mathcal{O}_Y[T_0, T_1, \dots, T_s] = Y \times \mathbb{P}^s.$$

If $g\colon Y_1 \to Y$ is an analytic map, then there is a natural Y_1-isomorphism

$$\mathsf{Projan}\, g^*\mathcal{G} \xrightarrow{\ \sim\ } (\mathsf{Projan}\, \mathcal{G}) \times_Y Y_1. \tag{5.7.1}$$

To see this, restrict to a suitable Stein open subset of Y, cover $\mathsf{Projan}\, \mathcal{G}$ by open subspaces of the form $\mathsf{Specan}\, \mathcal{G}'$ (see [C, p. 19-02, Défn. 2], [Bi, p. 37, top], [EGII, (3.1.4)]) and then using that $\mathsf{Specan}\, g^*\mathcal{G}' = \mathsf{Specan}\, \mathcal{G} \times_Y Y_1$ [C, p.19-03, Prop. 2(iv)] argue by pasting, as in [C, p. 10-05, Lemme 2.3].

Set $Z := Y \times \mathbb{P}^r$ and let $p\colon Z \to Y$ be the projection. If Y is a Stein space, then the calculations leading to [EGA, (2.1.12)] carry over to the analytic context, giving a natural isomorphism

$$H^0(Y, \mathcal{O}_Y)[T_0, T_1, \dots, T_r] \xrightarrow{\ \sim\ } H^0\big(Z, \bigoplus_{n\ge 0} \mathcal{O}_Z(n)\big),$$

and hence, for general Y, a natural isomorphism of $\mathcal{O}_Y$-algebras

$$\mathcal{O}_Y[T_0, T_1, \dots, T_r] \xrightarrow{\ \sim\ } \bigoplus_{n\ge 0} p_*\big(\mathcal{O}_Z(n)\big).$$

For $f\colon X \overset{i}{\hookrightarrow} Z \xrightarrow{p} Y$ as above, since $f_*\mathcal{O}_X(n)$ is coherent, the image $\mathcal{G}$ of the natural graded homomorphism $\pi\colon \oplus_{n\ge 0}\, p_*(\mathcal{O}_Z(n)) \to \oplus_{n\ge 0}\, f_*(\mathcal{O}_X(n))$ is finitely presented, see [TL, p. 2, Prop. 1.4]. For all $n \gg 0$ the graded component π_n of π is *surjective,* because if I is the (coherent) kernel of the natural map $\mathcal{O}_Z \to i_*\mathcal{O}_X$ then we have the natural exact sequence

$$0 \to I(n) \to \mathcal{O}_Z(n) \to i_*\mathcal{O}_X(n) \to 0,$$

and for $n \gg 0$, $R^1p_*I(n) = 0$ (by the above-mentioned theorem of Grauert-Remmert). By [Bi, p. 39, (5.13)(2)] we have then a Y-isomorphism $X \xrightarrow{\ \sim\ } \mathsf{Projan}\, \mathcal{G}$.

For any $y \in Y$, the fiber X_y is $\mathsf{Projan}\, \mathcal{G}(y)$ where $\mathcal{G}(y) := \mathcal{G}_y \otimes_{\mathcal{O}_{Y,y}} \mathbb{C}$ ($\mathbb{C}$ being identified with the residue field of $\mathcal{O}_{Y,y}$), see (5.7.1). The construction of Projan in [Bi, p. 36] shows that X_y is the complex space associated to the projective $\mathbb{C}$-scheme $X_y^0 := \mathsf{Proj}\, \mathcal{G}(y)$, the closed fiber of the natural map $f_y\colon \mathsf{Proj}\, \mathcal{G}_y \to \mathsf{Spec}\, \mathcal{O}_{Y,y}$ ($\mathcal{G}_y$ being the stalk at y of $\mathcal{G}$). So if X_y is reduced and y^0 is the closed point of $\mathsf{Spec}\, \mathcal{O}_{Y,y}$ then by the above remarks the analytic and algebraic Hilbert polynomials of the integral closures of the structure sheaves on these fibers coincide:

$$\mathcal{H}_y(\overline{\mathcal{O}_{X_y}}) = \mathcal{H}_{y^0}(\overline{\mathcal{O}_{X_y^0}}). \tag{5.7.2}$$

Assume henceforth that f *satisfies* $(\clubsuit)_{\mathrm{an}}$. By 1.7.1 and [Ma2, p. 184, Cor. (ii)], X is *reduced.* Let $\eta = \eta_y$ be the generic point of $\mathsf{Spec}\, \mathcal{O}_{Y,y}$, and with K_y the fraction field of $\mathcal{O}_{Y,y}$ let X_η be the K_y-scheme $\mathsf{Proj}\, (\mathcal{G}_y \otimes_{\mathcal{O}_{Y,y}} K_y)$. The integral closure $\overline{\mathcal{O}_{X_\eta}}$ of $\mathcal{O}_{X_\eta}$ is a coherent $\mathcal{O}_{X_\eta}$-module; and as in the paragraph preceding Proposition 3.4 (with $\mathcal{L} = \mathcal{O}_{X_\eta}(1)$) we have the Hilbert polynomial $\mathcal{H}_\eta(\overline{\mathcal{O}_{X_\eta}})$.

Lemma 5.7.3. *In the preceding circumstances, $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}}) = \mathcal{H}_\eta(\overline{\mathcal{O}_{X_\eta}})$ iff there is an open neighborhood U of y such that f is equinormalizable everywhere on $f^{-1}U$.*

Proof. We will apply Theorem 4.2. To relate the algebraic and complex-analytic setups, note first that the natural ringed-space map $\varphi\colon X_y = \mathsf{Projan}\,\mathcal{G}(y) \to \mathsf{Proj}\,\mathcal{G}(y) = X_y^0$ maps the points of X_y bijectively to the closed points of X_y^0, and that for each $x \in X_y$ the induced maps are isomorphisms of completions $\hat\varphi_x\colon \hat{\mathcal{O}}_{X_y^0,\varphi(x)} \xrightarrow{\sim} \hat{\mathcal{O}}_{X_y,x}$ (see [SGA, Exposé XII, Thm. 1.1]). Since $\mathcal{O}_{X_y,x}$ is reduced (by $(\clubsuit)_{\rm an}$) and excellent, it follows that X_y^0, the closed fiber of the above map $f_y\colon \mathsf{Proj}\,\mathcal{G}_y \to \mathsf{Spec}\,\mathcal{O}_{Y,y}$, is reduced.

More generally, setting $Z_y := \mathsf{Proj}\,\mathcal{G}_y$ we claim that *the natural map $\mathcal{O}_{Z_y,\varphi(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism of completions.* This granted, it follows (since f satisfies $(\clubsuit)_{\rm an}$) that $\mathcal{O}_{Z_y,\varphi(x)}$ is equidimensional and flat over $\mathcal{O}_{Y,y}$. Since the points $\varphi(x)$ are all the closed points of Z_y, one concludes, using 1.7.1, that f_y *satisfies* $(\clubsuit)$. Then in view of (5.7.2), Theorem 4.2 gives that $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}}) = \mathcal{H}_\eta(\overline{\mathcal{O}_{X_\eta}})$ *iff f_y is equinormalizable.*

By 2.3.2, if f_y is equinormalizable then its completion $\hat f_x\colon \mathsf{Spec}\,\widehat{\mathcal{O}_{X,x}} \to \mathsf{Spec}\,\widehat{\mathcal{O}_{Y,y}}$ is equinormalizable, for all $x \in X_y$. Conversely, if $\hat f_x$ is equinormalizable then, with $C := \mathcal{O}_{Z_y,\varphi(x)}$ and $D := \overline{C}$, one has, as in the proof of 2.3.2, that $\hat D$ is $\widehat{\mathcal{O}_{Y,y}}$-flat, whence D is $\mathcal{O}_{Y,y}$-flat, and that the closed fiber of the obvious map $\bar f_x\colon \mathsf{Spec}\,D \to \mathsf{Spec}\,\mathcal{O}_{Y,y}$ is geometrically normal, whence, by 1.7.1, all the fibers of $\bar f_x$ are geometrically normal. So if $\hat f_x$ is equinormalizable for all $x \in X_y$ then it results from 2.3.1 that f_y is equinormalizable. Thus *f_y is equinormalizable iff $\hat f_x$ is equinormalizable for all $x \in X_y$.*

Now *$\hat f_x$ is equinormalizable for all $x \in X_y$ iff f is equinormalizable at every such x.* Indeed, using 1.7.1 one finds as above that since f satisfies $(\clubsuit)_{\rm an}$ therefore the induced map $\mathsf{Spec}\,\mathcal{O}_{X,x} \to \mathsf{Spec}\,\mathcal{O}_{Y,y}$ satisfies $(\clubsuit)$; and as in the preceding paragraph, we see that this map is equinormalizable—i.e., f is equinormalizable at x—if and only if its completion at x, which, by the above claim, is *the same $\hat f_x$ as before,* is equinormalizable.

In conclusion, $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}}) = \mathcal{H}_\eta(\overline{\mathcal{O}_{X_\eta}})$ *iff f is equinormalizable at every* $x \in X_y$, i.e., by 5.4.2, *iff f is equinormalizable on some neighborhood of $f^{-1}y$*, i.e., since f is proper, *iff y has a neighborhood on whose inverse image f is equinormalizable,* as asserted.

As for the the claim, let U be any Stein neighborhood of y whose closure $\overline{U}$ is a Stein compactum, set $A_U := \Gamma(U, \mathcal{O}_Y)$ and $S_U := \Gamma(\overline{U}, \mathcal{G}) \otimes_{\Gamma(\overline{U},\mathcal{O}_Y)} A_U$, so that $f^{-1}U$ is the complex space associated to the A_U-scheme $X_U := \mathsf{Proj}\,S_U$, coming with a canonical ringed-space map $\varphi_U\colon f^{-1}U \to X_U$ [Bi, p. 36]. As $\mathcal{G}_y = \varinjlim_U S_U$, it follows for any $x \in X_y$ that $\mathcal{O}_{Z_y,\varphi(x)} = \varinjlim_U \mathcal{O}_{X_U,\varphi_U(x)}$. With $\mathfrak{m}_x$ (resp. $\mathfrak{m}_U$) the maximal ideal of $\mathcal{O}_{X,x}$ (resp $\mathcal{O}_{Z_U,\varphi_U(x)}$), φ_U induces for any $n > 0$ an isomorphism $\mathcal{O}_{X_U,\varphi_U(x)}/\mathfrak{m}_U^n \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ [Bi, p. 1, (1.1)]; taking the direct limit over U gives, with $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{Z_y,\varphi(x)}$, an isomorphism $\mathcal{O}_{Z_y,\varphi(x)}/\mathfrak{m}^n \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$, whence the claim. □

Lemma 5.7.4. *The Hilbert polynomial $\mathcal{H}_\eta(\overline{\mathcal{O}_{X_\eta}})$ does not depend on y.*

Proof. Let $\overline{\mathcal{O}_X}$ be the integral closure of $\mathcal{O}_X$, a coherent $\mathcal{O}_X$-module [C, p. 21-09, Cor. 2]. For each $n > 0$ the $\mathcal{O}_Y$-module $\overline{\mathcal{G}}_n := f_*(\overline{\mathcal{O}_X}(n))$ is coherent, hence—as in the proof of Theorem 5.6—locally free on a dense open subset of Y, of rank, say r_n; and *for each $y \in Y$, and K_y the fraction field of $\mathcal{O}_{Y,y}$, the dimension of the K_y-vector space $(\overline{\mathcal{G}}_n)_y \otimes_{\mathcal{O}_y} K_y$ is r_n.*

Let us describe the (algebraically defined) polynomial $\mathcal{H}_\eta(\overline{\mathcal{O}_{X_\eta}})$ in terms of the (analytically defined) integers r_n—which do not depend on y. Let $U \subset Y$ be any Stein open set whose closure $\overline{U}$ is a Stein compactum, and let A_U, and $\varphi_U\colon f^{-1}U \to X_U$ be as in the preceding proof. The canonical very ample invertible sheaves on X_U and $f^{-1}U$ are related in the obvious way: $\varphi_U^*\mathcal{O}_{X_U}(1) = \mathcal{O}_{X_U}(1)$. It follows trivially from the definition [Bi, p. 1, Satz (1.1)] that the U-analytic space associated to $\mathsf{Spec}\, A_U$ is U itself, together with the canonical ringed-space map $i_U\colon U \to \mathsf{Spec}\, A_U$.

Let $\overline{\mathcal{O}_{X_U}}$ be the integral closure of $\mathcal{O}_{X_U}$. Then $\varphi_U^*\overline{\mathcal{O}_{X_U}} = \overline{\mathcal{O}_{f^{-1}U}}$: indeed, for $x \in f^{-1}U$ the local rings $\mathcal{O}_{X_U,\varphi_U(x)}$ and $\mathcal{O}_{X,x}$ are excellent [Bi2, p. 153], of equicharacteristic zero, and have the same completion [Bi, p. 1, Satz (1.1)], so the canonical map from the first to the second is *regular* (flat, with geometrically regular fibers) [EGA, (6.6.1)]; and [EGA, (6.14.1)] implies the assertion. (One could also argue on the basis of various "comparison" results in [Bi, §§2–3].) Hence by [Bi, p. 16, (4.2)] ("relative GAGA"), if $f_U\colon X_U \to \mathsf{Spec}\, A_U$ is the projection then for all $n \ge 0$, $i_U^* f_{U*}\big(\overline{\mathcal{O}_{X_U}}(n)\big) = f_*\big(\overline{\mathcal{O}_{f^{-1}U}}(n)\big)$.

Now $f_{U*}\big(\overline{\mathcal{O}_{X_U}}(n)\big)$ is a coherent sheaf on $\mathsf{Spec}\, A_U$, associated to the finitely-generated A_U-module $\Gamma\big(X_U, \overline{\mathcal{O}_{X_U}}(n)\big)$, so its restriction to some nonempty open subscheme is locally free, necessarily of constant rank r_n, since that is the rank of the restriction of $f_*\big(\overline{\mathcal{O}_{f^{-1}U}}(n)\big)$ to some dense open subset of U. Thus r_n *is the rank of the finitely generated A_U-module* $\Gamma\big(X_U, \overline{\mathcal{O}_{X_U}}(n)\big)$.

For any $y \in U$ this last statement remains true under base change from A_U to A_y, where A_y is the localization of A_U at the maximal ideal $i_U(y)$ of A_U (because Γ "commutes" with flat base change [EGI, p. 366, (9.3.3)], and integral closure is compatible with localization, or more generally, with flat base change having geometrically normal fibers [EGA, (6.14.4)]). Furthermore, as before, the natural local homomorphism $A_y \to \mathcal{O}_{Y,y}$ is regular; and therefore the statement remains true under the base change from A_y to $\mathcal{O}_{Y,y}$. Now, fixing $y \in Y$, let V be a neighborhood such that the finitely presented $\mathcal{O}_{Y,y}$-algebra $\mathcal{G}_y$ is specified by generators and relations which are stalks of generators and relations of the A_V-algebra $\Gamma(V, \mathcal{G})$, and choose a neighborhood U as above such that $\overline{U} \subset V$. Then $X_U \otimes_{A_U} \mathcal{O}_{Y,y}$ is the above scheme X_y, and r_n *is the rank of the $\mathcal{O}_{Y,y}$-module* $\Gamma\big(X_y, \overline{\mathcal{O}_{X_y}}(n)\big)$. Hence for all $n \gg 0$,

$$\mathcal{H}_\eta(\overline{\mathcal{O}_{X_\eta}})(n) = \dim_{K_y} \Gamma\big(X_\eta, \overline{\mathcal{O}_{X_\eta}}(n)\big) = r_n,$$

which, again, does not depend on y. □

Theorem 5.8. *Let $f\colon X \hookrightarrow Y \times \mathbb{P}^r \twoheadrightarrow Y$ be a projective analytic map satisfying $(\clubsuit)_{\mathrm{an}}$, with Y irreducible. Then f is equinormalizable if and only if $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}})$ is the same for all $y \in Y$.*

Proof. If f is equinormalizable and $\nu\colon \overline{X} \to X$ is a normalization then $f\nu$ is flat over Y, and hence $\overline{\mathcal{O}_X} = \nu_*\mathcal{O}_{\overline{X}}$ is Y-flat (its stalk at $x \in X$ being the product of the stalks of $\mathcal{O}_{\overline{X}}$ at the finitely many points in $\nu^{-1}x$). So by a theorem of Grauert [BS, p. 134, Thm. 4.12(iii)], $\mathcal{H}_y\big((\overline{\mathcal{O}_X})_y\big)$ is independent of y. Furthermore, the fibers of $f\nu$ normalize those of f, and hence one gets a natural isomorphism $(\overline{\mathcal{O}_X})_y \xrightarrow{\sim} \overline{\mathcal{O}_{X_y}}$. Thus $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}})$ is independent of y.

Suppose conversely that $\mathcal{H}_y(\overline{\mathcal{O}_{X_y}})$ does not depend on y. By Prop. 5.4.3 or [Mn, p. 271, (I.10)], there is a $y_1 \in Y$ such that f is equinormalizable at each $x \in X_{y_1}$. For any $y_0 \in Y$ Lemmas 5.7.3 and 5.7.4 give

$$\mathcal{H}_{y_0}(\overline{\mathcal{O}_{X_{y_0}}}) = \mathcal{H}_{y_1}(\overline{\mathcal{O}_{X_{y_1}}}) = \mathcal{H}_{\eta_1}(\overline{\mathcal{O}_{X_{\eta_1}}}) = \mathcal{H}_{\eta_0}(\overline{\mathcal{O}_{X_{\eta_0}}}),$$

and then Lemma 5.7.3 shows that f is equinormalizable over a neighborhood of y_0. $\square$

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