REDUCTION OF DERIVED HOCHSCHILD FUNCTORS OVER COMMUTATIVE ALGEBRAS AND SCHEMES

LUCHEZAR L. AVRAMOV, SRIKANTH B. IYENGAR, JOSEPH LIPMAN, AND SURESH NAYAK

Abstract. We study functors underlying derived Hochschild cohomology, also called Shukla cohomology, of a commutative algebra S essentially of finite type and of finite flat dimension over a commutative noetherian ring K. We construct a complex of S-modules D, and natural reduction isomorphisms $\operatorname{Ext}^*_{S\otimes^{\mathsf{L}}_K S}(S|K; M\otimes^{\mathsf{L}}_K N) \simeq \operatorname{Ext}^*_S(\mathsf{R}\operatorname{Hom}_S(M,D),N)$ for all complexes of S-modules N and all complexes M of finite flat dimension over K whose homology $\mathrm{H}(M)$ is finitely generated over S; such isomorphisms determine D up to derived isomorphism. Using Grothendieck duality theory we establish analogous isomorphisms for any essentially finite-type flat map $f\colon X\to Y$ of noetherian schemes, with $f^!\mathcal{O}_Y$ in place of D.

Contents

Introduction

We study commutative algebras essentially of finite type over some commutative noetherian ring K. Let $\sigma\colon K\to S$ denote the structure map of such an algebra. When S is projective as a K-module, for example, when K is a field, the Hochschild cohomology $\mathrm{HH}^*(S|K;-)$ allows one to investigate certain properties of the homomorphism σ in terms of properties of S, viewed as a module over the enveloping

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algebra $S^{\mathsf{e}} = S \otimes_K S$. This comes about via isomorphisms

$$\mathrm{HH}^n(S\,|\,K; L) = \mathrm{Ext}^n_{S^{\mathsf{e}}}(S, L)\,,$$

established by Cartan and Eilenberg [10] for an arbitrary S-bimodule L.

In the absence of projectivity, one can turn to a cohomology theory introduced by MacLane [21] for $K = \mathbb{Z}$, extended by Shukla [28] to all rings K, and recognized by Quillen [26] as a derived version of Hochschild cohomology; see Section 3.

A central result of this article is a reduction of the computation of derived Hochschild cohomology with coefficients in $M \otimes^{\mathsf{L}}_K N$ to a computation of iterated derived functors over the ring S itself; this is new even in the classical situation.

We write $\mathsf{D}(S)$ for the derived category of S-modules, and $\mathsf{P}(\sigma)$ for its full subcategory consisting of complexes with finite homology that are isomorphic in $\mathsf{D}(K)$ to bounded complexes of flat K-modules. As part of Theorem 4.1 we prove:

Theorem 1. *When S has finite flat dimension as a K-module there exists a unique up to isomorphism complex $D^\sigma \in \mathsf{P}(\sigma)$, such that for each $M \in \mathsf{P}(\sigma)$ and every $N \in \mathsf{D}(S)$ there is an isomorphism that is natural in M and N:*

$$\mathsf{R}\mathrm{Hom}_{S \otimes^{\mathsf{L}}_K S}(S, M \otimes^{\mathsf{L}}_K N) \simeq \mathsf{R}\mathrm{Hom}_S(\mathsf{R}\mathrm{Hom}_S(M, D^\sigma), N)\,.$$

The complex D^σ is an algebraic version of a *relative dualizing complex* used in algebraic geometry, see (6.2.1). A direct, explicit construction of D^σ is given in Section 1. When S is flat as a K-module, M and N are S-modules, and M is flat over K and finite over S, the theorem yields isomorphisms of S-modules

$$\mathrm{Ext}^n_{S^{\mathsf{e}}}(S, M \otimes_K N) \cong \mathrm{Ext}^n_S(\mathsf{R}\mathrm{Hom}_S(M, D^\sigma), N)$$

for all $n \in \mathbb{Z}$; they were originally proved in the first preprint version of [5].

Our second main result is a global version of part of Theorem 1. For a map of schemes $f\colon X \to Y$, $f_0^{-1}\mathcal{O}_Y$ is a sheaf of commutative rings on X, whose stalk at any point $x \in X$ is $\mathcal{O}_{Y,f(x)}$ (see Section 6). The derived category of (sheaves of) $f_0^{-1}\mathcal{O}_Y$-modules is denoted by $\mathsf{D}(f_0^{-1}\mathcal{O}_Y)$. Corollary 6.5 of Theorem 6.1 gives:

Theorem 2. *Let $f\colon X \to Y$ be an essentially finite-type, flat map of noetherian schemes; let $X \xleftarrow{\pi_1} X \times_Y X \xrightarrow{\pi_2} X$ be the canonical projections; let $\delta\colon X \to X \times_Y X$ be the diagonal morphism; and let M and N be complexes of $\mathcal{O}_X$-modules.*

If M has coherent cohomology and is isomorphic in $\mathsf{D}(f_0^{-1}\mathcal{O}_Y)$ to a bounded complex of $f_0^{-1}\mathcal{O}_Y$-modules that are flat over Y, and if N has bounded-above quasi-coherent homology, then one has an isomorphism

$$\delta^!(\pi_1^*M \otimes^{\mathsf{L}}_{X\times_Y X} \pi_2^*N) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(M, f^!\mathcal{O}_Y), N)\,.$$

When both schemes X and Y are affine, and f corresponds to an essentially finite-type ring homomorphism, Theorem 2 reduces to a special case of Theorem 1, namely, where the K-algebra S is flat and N is homologically bounded above. In Section 6 we also obtain global analogs of other results proved earlier in the paper for complexes over rings. A pattern emerging from these series of parallel results is that neither version of a theorem implies the other one in full generality. This intriguing discrepancy suggests the existence of stronger global results.

The proofs of Theorems 1 and 2 follow very different routes. The first one is based on isomorphisms in derived categories of differential graded algebras; background material on the topic is collected in Section 2. The second one involves fundamental results of Grothendieck duality theory, systematically developed in [15, 11, 19]; the relevant notions and theorems are reviewed in Section 5.

1. Relative dualizing complexes

In this section $\sigma\colon K \to S$ denotes a homomorphism of commutative rings.

For any K-algebra P and each $n \in \mathbb{Z}$ we write $\Omega_{P|K}$ for the P-module of Kähler differentials of P over K, and set $\Omega^n_{P|K} = \bigwedge^n_P \Omega_{P|K}$ for each $n \in \mathbb{N}$.

Recall that σ is said to be *essentially of finite type* if it can be factored as

$$K \hookrightarrow K[x_1,\dots,x_e] \to V^{-1}K[x_1,\dots,x_e] = Q \twoheadrightarrow S\,, \tag{1.0.1}$$

where $x_1,\dots,x_e$ are indeterminates, V is a multiplicatively closed set, the first two maps are canonical, the equality defines Q, and the last arrow is a surjective ring homomorphism. We fix such a factorization and set

$$D^\sigma = \Sigma^e \,\mathsf{R}\mathrm{Hom}_P(S, \Omega^e_{Q|K}) \quad \text{in} \quad \mathsf{D}(S)\,, \tag{1.0.2}$$

where $\mathsf{D}(S)$ denotes the derived category of S-modules. Any complex isomorphic to D^σ in $\mathsf{D}(S)$ is called a *relative dualizing complex* of σ. To obtain such complexes we factor σ through essentially smooth maps, see 1.3.

Theorem 1.1. *If $K \to P \to S$ is a factorization of σ, with $K \to P$ essentially smooth of relative dimension d and $P \to S$ finite, then there exists an isomorphism*

$$D^\sigma \simeq \Sigma^d \,\mathsf{R}\mathrm{Hom}_P(S, \Omega^d_{P|K}) \quad \textit{in} \quad \mathsf{D}(S)\,.$$

The isomorphism in the theorem can be chosen in a coherent way for all K-algebras essentially of finite type. To prove this statement, or even to make it precise, we need to appeal to the theory of the pseudofunctor $^!$ of Grothendieck duality theory; see [19, Ch. 4]. Canonicity is not used in this paper.

We write $\mathsf{P}(\sigma)$ for the full subcategory of $\mathsf{D}(S)$ consisting of complexes $M \in \mathsf{D}(S)$ such that $\mathrm{H}(M)$ is finite over S and M is isomorphic in $\mathsf{D}(K)$ to some bounded complex of flat K-modules.

The name given to the complex D^σ is explained by the next result.

Theorem 1.2. *When $\mathrm{fd}_K S$ is finite the complex D^σ has the following properties.*

(1) *For each M in $\mathsf{P}(\sigma)$ the complex $\mathsf{R}\mathrm{Hom}_S(M, D^\sigma)$ is in $\mathsf{P}(\sigma)$, and the biduality morphism gives a canonical isomorphism*

$$\delta^M\colon M \simeq \mathsf{R}\mathrm{Hom}_S(\mathsf{R}\mathrm{Hom}_S(M, D^\sigma), D^\sigma) \quad \textit{in} \quad \mathsf{D}(S)\,.$$

(2) *One has $D^\sigma \in \mathsf{P}(\sigma)$, and the homothety map gives a canonical isomorphism*

$$\chi^{D^\sigma}\colon S \simeq \mathsf{R}\mathrm{Hom}_S(D^\sigma, D^\sigma) \quad \textit{in} \quad \mathsf{D}(S)\,.$$

The theorems are proved at the end of the section. The arguments use various properties of (essentially) smooth homomorphisms, which we record next.

1.3. Let $\varkappa\colon K \to P$ be a homomorphism of commutative noetherian rings.

One says that $\varkappa\colon K \to P$ is (*essentially*) *smooth* if it is (essentially) of finite type, flat, and the ring $k \otimes_K P$ is regular for each homomorphism of rings $K \to k$ when k is a field; see [14, 17.5.1] for a proof that this notion of smoothness is equivalent to that defined in terms of lifting of homomorphisms.

When $\varkappa$ is essentially smooth $\Omega^1_{P|K}$ is finite projective, so for each prime ideal $\mathfrak{p}$ of P the $P_\mathfrak{p}$-module $(\Omega^1_{P|K})_\mathfrak{p}$ is free of finite rank. If this rank is equal to a fixed integer d for all $\mathfrak{p}$, then $K \to P$ is said to be of *relative dimension* d; (essentially) smooth homomorphism of relative dimension zero are called (*essentially*) *étale*.

1.3.1. Set $P^{\mathsf{e}} = P \otimes_K P$ and $I = \operatorname{Ker}(\mu\colon P^{\mathsf{e}} \to P)$, where μ is the multiplication. There exist canonical isomorphisms of P-modules

$$\Omega^1_{P|K} \cong I/I^2 \cong \operatorname{Tor}_1^{P^{\mathsf{e}}}(P,P)\,.$$

As μ is a homomorphism of commutative rings, $\operatorname{Tor}^{P^{\mathsf{e}}}(P,P)$ has a natural structure of a strictly graded-commutative P-algebra, so the composed isomorphism above extends to a homomorphism of graded P-algebras

$$\lambda^{P|K}\colon {\textstyle\bigwedge}_P \Omega^1_{P|K} \longrightarrow \operatorname{Tor}^{P^{\mathsf{e}}}(P,P)\,.$$

1.3.2. Let $X \xrightarrow{\simeq} P$ be a projective resolution over P^{e}. The morphism of complexes

$$\delta\colon X \otimes_{P^{\mathsf{e}}} P \to \operatorname{Hom}_{P^{\mathsf{e}}}(\operatorname{Hom}_{P^{\mathsf{e}}}(X,P^{\mathsf{e}}),P)$$
$$\delta(x\otimes p)(\chi) = (-1)^{(|x|+|p|)|\chi|}\chi(x)p$$

yields the first map in the composition below, where κ is a Künneth homomorphism:

$$\begin{aligned} \operatorname{H}(X\otimes_{P^{\mathsf{e}}} P) &\xrightarrow{\operatorname{H}(\delta)} \operatorname{H}(\operatorname{Hom}_{P^{\mathsf{e}}}(\operatorname{Hom}_{P^{\mathsf{e}}}(X,P^{\mathsf{e}}),P)) \\ &\xrightarrow{\ \kappa\ } \operatorname{Hom}_{P^{\mathsf{e}}}(\operatorname{H}(\operatorname{Hom}_{P^{\mathsf{e}}}(X,P^{\mathsf{e}})),P) \\ &=\!=\!= \operatorname{Hom}_{P}(\operatorname{H}(\operatorname{Hom}_{P^{\mathsf{e}}}(X,P^{\mathsf{e}})),P)\,. \end{aligned}$$

Thus, one gets a homomorphism of graded P-modules

$$\tau^{P|K}\colon \operatorname{Tor}^{P^{\mathsf{e}}}(P,P) \longrightarrow \operatorname{Hom}_P(\operatorname{Ext}_{P^{\mathsf{e}}}(P,P^{\mathsf{e}}),P)\,.$$

1.3.3. The composition below, where the first arrow is a biduality map,

$$\begin{aligned} \operatorname{Ext}_{P^{\mathsf{e}}}(P,P^{\mathsf{e}}) &\longrightarrow \operatorname{Hom}_P(\operatorname{Hom}_P(\operatorname{Ext}_{P^{\mathsf{e}}}(P,P^{\mathsf{e}}),P),P) \\ &\xrightarrow{\operatorname{Hom}_P(\tau^{P|K},P)} \operatorname{Hom}_P(\operatorname{Tor}^{P^{\mathsf{e}}}(P,P),P)\,. \end{aligned}$$

is a homomorphism of graded P-modules

$$\epsilon_{P|K}\colon \operatorname{Ext}_{P^{\mathsf{e}}}(P,P^{\mathsf{e}}) \longrightarrow \operatorname{Hom}_P(\operatorname{Tor}^{P^{\mathsf{e}}}(P,P^{\mathsf{e}}),P)\,.$$

The maps above appear in homological characterizations of smoothness:

1.3.4. Let $K \to P$ be a flat and essentially of finite type homomorphism of rings, and set $I = \operatorname{Ker}(\mu\colon P^{\mathsf{e}} \to P)$. The following conditions are equivalent.

(i) The homomorphism $K \to P$ is essentially smooth.
(ii) The ideal $I_{\mathfrak{m}}$ is generated by a regular sequence for each prime ideal $\mathfrak{m} \supseteq I$.
(iii) The P-module $\Omega^1_{P|K}$ is projective and the map $\lambda^{P|K}$ from 1.3.1 is bijective.
(iv) The projective dimension $\operatorname{pd}_{P^{\mathsf{e}}} P$ is finite.

The equivalence of the first three conditions is due to Hochschild, Kostant, and Rosenberg when K is a perfect field, and to André [1, Prop. C] in general. The implication (ii) $\Longrightarrow$ (iv) is clear, and the converse is proved by Rodicio [27, Cor. 2].

In the next lemma we use homological dimensions for complexes, as introduced in [3]. They are based on notions of semiprojective and semiflat resolutions, recalled in 2.3.1. The *projective dimension* of $M \in \mathsf{D}(P)$ in defined by the formula

$$\operatorname{pd}_P M = \inf\left\{ n\in\mathbb{Z} \;\middle|\; \begin{matrix} n \geq \sup \operatorname{H}(M) \text{ and } F \simeq M \text{ in } \mathsf{D}(P) \text{ with } F \\ \text{semiprojective and } \operatorname{Coker}(\partial^F_{n+1}) \text{ projective} \end{matrix} \right\}\,.$$

The number obtained by replacing 'semiprojective' with 'semiflat' and 'projective' with 'flat' is the *flat dimension* of M, denoted $\operatorname{fd}_P M$.

For the rest of this section we fix a factorization $K \to P \to S$ of σ, with $K \to P$ essentially smooth of relative dimension d and $P \to S$ finite.

Lemma 1.4. *For every complex M of P-modules the following inequalities hold:*

$$\operatorname{fd}_K M \leq \operatorname{fd}_P M \leq \operatorname{fd}_K M + \operatorname{pd}_{P^{\mathsf{e}}} P\,.$$

In particular, $\operatorname{fd}_P M$ and $\operatorname{fd}_K M$ are finite simultaneously.

When the S-module $\mathrm{H}(M)$ is finite one can replace $\operatorname{fd}_P M$ with $\operatorname{pd}_P M$.

Proof. The inequality on the left is a consequence of [3, 4.2(F)].

For the one on the right we may assume $\operatorname{fd}_K M = q < \infty$. Thus, if $F \to M$ is a semiflat resolution over P, then $G = \operatorname{Coker}(\partial^F_{q+1})$ is flat as a K-module. For each $n \in \mathbb{Z}$ there is a canonical isomorphism of functors of P-modules

$$\operatorname{Tor}^P_n(G, -) \cong \operatorname{Tor}^{P^{\mathsf{e}}}_n(P, G \otimes_K -)\,,$$

see [10, X.2.8], so the desired inequality holds. Since $K \to P$ is essentially smooth one has $\operatorname{pd}_{P^{\mathsf{e}}} P < \infty$, see 1.3.4, so they imply that $\operatorname{fd}_P M$ is finite if only if so is $\operatorname{fd}_K M$. In case $\mathrm{H}(M)$ is finite over P one has $\operatorname{fd}_P M = \operatorname{pd}_P M$; see [3, 2.10(F)]. □

Lemma 1.5. *The canonical homomorphisms $\lambda^{P|K}_d$, $\tau^{P|K}_d$, and $\epsilon^d_{P|K}$ defined in 1.3.1, 1.3.2, and 1.3.3, respectively, provide isomorphisms of P-modules*

(1.5.1) $$\operatorname{Ext}^n_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) = 0 \quad \textit{for} \quad n \neq d\,;$$

(1.5.2) $$\operatorname{Hom}_S(\lambda^{P|K}_d, P) \circ \epsilon^d_{P|K} \colon \operatorname{Ext}^d_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) \cong \operatorname{Hom}_P(\Omega^d_{P|K}, P)\,;$$

(1.5.3) $$\tau^{P|K}_d \circ \lambda^{P|K}_d \colon \Omega^d_{P|K} \cong \operatorname{Hom}_P(\operatorname{Ext}^d_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}), P)\,.$$

Proof. Set $I = \operatorname{Ker}(\mu)$. It suffices to prove that the maps above induce isomorphisms after localization at every $\mathfrak{n} \in \operatorname{Spec} P$. Fix one, then set $T = P_{\mathfrak{m}}$, $R = P^{\mathsf{e}}_{\mathfrak{n} \cap P^{\mathsf{e}}}$ and $J = I_{\mathfrak{n} \cap P^{\mathsf{e}}}$. The ideal J is generated by a regular sequence, see 1.3.4. Any such sequence consists of d elements: This follows from the isomorphisms of T-modules

$$J/J^2 \cong (I/I^2)_{\mathfrak{n}} \cong (\Omega^1_{P|K})_{\mathfrak{n}} \cong T^d\,.$$

The Koszul complex Y on such a sequence is a free resolution of T over R. A well known isomorphism $\operatorname{Hom}_R(Y, R) \cong \Sigma^{-d} Y$ of complexes of R-modules yields $\operatorname{Ext}^n_R(T, R) = 0$ for $n \neq d$ and $\operatorname{Ext}^d_R(T, R) \cong T$. This establishes (1.5.1) and shows that $\operatorname{Ext}^d_{P^{\mathsf{e}}}(P, P^{\mathsf{e}})$ is invertible; as a consequence, (1.5.2) follows from (1.5.3).

We analyze the maps in (1.5.3). From 1.3.4 we know that $\lambda^{P|K}_d$ is bijective. By 1.3.2 one has $\tau^{P|K}_d = \kappa_d \circ \mathrm{H}_d(\delta)$. The map $\mathrm{H}_d(\delta)$ is bijective, as it can be computed from a resolution X of P by finite projective P^{e}-modules, and then δ itself is an isomorphism. To establish the isomorphism in (1.5.3) it remains to show that $(\kappa_d)_{\mathfrak{m}}$ is bijective. This is a Künneth map, which can be computed using the Koszul complex Y above. Thus, we need to show that the natural T-linear map

$$\mathrm{H}_d(\operatorname{Hom}_R(\operatorname{Hom}_R(Y, R), T)) \longrightarrow \operatorname{Hom}_R(\mathrm{H}_{-d}(\operatorname{Hom}_R(Y, R)), T)$$

is bijective. It has been noted above that both modules involved are isomorphic to T, and an easy calculation shows that the map itself is an isomorphism. □

To continue we need a lemma from general homological algebra.

Lemma 1.6. *Let R be an associative ring and M a complex of R-modules.*

If the graded R-module $\mathrm{H}(M)$ is projective, then there exists a unique up to homotopy morphism of complexes $\mathrm{H}(M) \to M$ inducing $\mathrm{id}^{\mathrm{H}(M)}$, and a unique isomorphism $\alpha\colon \mathrm{H}(M) \to M$ in $\mathsf{D}(R)$ with $\mathrm{H}(\alpha) = \mathrm{id}^{\mathrm{H}(M)}$.

Proof. One has $\mathrm{H}(M) \cong \coprod_{i\in\mathbb{Z}} \Sigma^i \mathrm{H}_i(M)$ as complexes with zero differentials. The projectivity of the R-modules $\mathrm{H}_i(M)$ provides the second link in the chain

$$
\begin{aligned}
\mathrm{H}(\mathrm{Hom}_R(\mathrm{H}(M), M)) &\cong \mathrm{H}(\prod_{i\in\mathbb{Z}} \Sigma^{-i}\,\mathrm{Hom}_R(\mathrm{H}_i(M), M))\\
&\cong \prod_{i\in\mathbb{Z}} \Sigma^{-i}\,\mathrm{Hom}_R(\mathrm{H}_i(M), \mathrm{H}(M))\\
&\cong \mathrm{Hom}_R(\coprod_{i\in\mathbb{Z}} \Sigma^{i}\,\mathrm{H}_i(M), \mathrm{H}(M))\\
&\cong \mathrm{Hom}_R(\mathrm{H}(M), \mathrm{H}(M))
\end{aligned}
$$

of isomorphisms of graded modules. The composite map is given by $\mathrm{cls}(\alpha) \mapsto \mathrm{H}(\alpha)$. The first assertion follows because $\mathrm{H}_0(\mathrm{Hom}_R(\mathrm{H}(M), M))$ is the set of homotopy classes of morphisms $\mathrm{H}(M) \to M$. For the second, note that one has

$$\mathrm{Mor}_{\mathsf{D}(R)}(\mathrm{H}(M), M) \cong \mathrm{H}_0(\mathrm{Hom}_R(\mathrm{H}(M), M))$$

because each complex $\Sigma^i \mathrm{H}_i(M)$ is semiprojective, and hence so is $\mathrm{H}(M)$. □

Lemma 1.7. *In $\mathsf{D}(P)$ there exist canonical isomorphisms*

$$\mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) \simeq \Sigma^{-d}\,\mathrm{Hom}_P(\Omega^d_{P|K}, P)\,. \tag{1.7.1}$$

$$\mathsf{R}\mathrm{Hom}_P(\mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}), P) \simeq \Sigma^d \Omega^d_{P|K}\,. \tag{1.7.2}$$

Proof. Since $K \to P$ is essentially smooth of relative dimension d, the P-module $\Omega^d_{P|K}$ is projective of rank one, and hence so is $\mathrm{Hom}_P(\Omega^d_{P|K}, P)$. The isomorphisms (1.5.1) and (1.5.2) imply that $\mathrm{H}(\mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}))$ is an invertible graded P-module. In particular, it is projective. Now choose (1.7.1) to be the canonical isomorphism provided by Lemma 1.6, and (1.7.2) the isomorphism induced by it. □

Lemma 1.8. *When σ is finite there is a canonical isomorphism*

$$\Sigma^d\,\mathsf{R}\mathrm{Hom}_P(S, \Omega^d_{P|K}) \cong \mathsf{R}\mathrm{Hom}_K(S, K) \quad \text{in} \quad \mathsf{D}(S)\,.$$

Proof. One has a chain of canonical isomorphisms:

$$
\begin{aligned}
\Sigma^d\,\mathsf{R}\mathrm{Hom}_P(S, \Omega^d_{P|K}) &\simeq \Sigma^d\,\mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, \mathsf{R}\mathrm{Hom}_K(S, \Omega^d_{P|K}))\\
&\simeq \Sigma^d\,\mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) \otimes^{\mathsf{L}}_{P^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_K(S, \Omega^d_{P|K})\\
&\simeq \mathsf{R}\mathrm{Hom}_P(\Omega^d_{P|K}, P) \otimes^{\mathsf{L}}_{P^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_K(S, \Omega^d_{P|K})\\
&\simeq \mathsf{R}\mathrm{Hom}_P(\Omega^d_{P|K}, P) \otimes^{\mathsf{L}}_{P^{\mathsf{e}}} \big(\Omega^d_{P|K} \otimes^{\mathsf{L}}_K \mathsf{R}\mathrm{Hom}_K(S, K)\big)\\
&\simeq \mathsf{R}\mathrm{Hom}_P(\Omega^d_{P|K}, P) \otimes^{\mathsf{L}}_{P} \big(P \otimes^{\mathsf{L}}_{P^{\mathsf{e}}} \big(\Omega^d_{P|K} \otimes^{\mathsf{L}}_K \mathsf{R}\mathrm{Hom}_K(S, K)\big)\big)\\
&\simeq \mathsf{R}\mathrm{Hom}_P(\Omega^d_{P|K}, P) \otimes^{\mathsf{L}}_{P} \big(\Omega^d_{P|K} \otimes^{\mathsf{L}}_P \mathsf{R}\mathrm{Hom}_K(S, K)\big)\\
&\simeq \mathsf{R}\mathrm{Hom}_P(\Omega^d_{P|K}, \Omega^d_{P|K}) \otimes^{\mathsf{L}}_{P} \mathsf{R}\mathrm{Hom}_K(S, K)\\
&\simeq \mathsf{R}\mathrm{Hom}_K(S, K)\,.
\end{aligned}
$$

The first one holds by a classical associativity formula, see (2.1.1), the second one because $\operatorname{pd}_{P^{\mathrm{e}}} P$ is finite, see 1.3.4, the third one by (1.7.1). The last one is induced by the homothety $P \to \mathsf{R}\mathrm{Hom}_P(\Omega^d_{P|K}, \Omega^d_{P|K})$, which is bijective as $(\Omega^d_{P|K})_{\mathfrak{p}} \cong P_{\mathfrak{p}}$ holds as $P_{\mathfrak{p}}$-modules for each $\mathfrak{p} \in \operatorname{Spec} P$. The other isomorphisms are standard. □

Proof of Theorem 1.1. Let $K \to Q \to S$ be the factorization of σ given by (1.0.1), with $Q = V^{-1}K[x_1, \dots, x_e]$. The isomorphism

$$\Omega^1_{(P\otimes_K Q)|K} \cong (\Omega^1_{P|K} \otimes_K Q) \oplus (P \otimes_K \Omega^1_{Q|K})$$

induces the first isomorphism of $(P \otimes_K Q)$-modules below:

$$\begin{aligned}\Omega^{d+e}_{(P\otimes_K Q)|K} &\cong \bigoplus_{i+j=d+e} (\Omega^i_{P|K} \otimes_K Q) \otimes_{P\otimes_K Q} (P \otimes_K \Omega^j_{Q|K}) \\ &\cong \Omega^d_{P|K} \otimes_K \Omega^e_{Q|K}\,.\end{aligned}$$

The second one holds because for each $\mathfrak{p} \in \operatorname{Spec} P$ one has $(\Omega^i_{P|K})_{\mathfrak{p}} \cong \wedge^i_{P_{\mathfrak{p}}}(P^d_{\mathfrak{p}}) = 0$ for $i > d$, and similarly $(\Omega^i_{Q|K})_{\mathfrak{p}} = 0$ for $j > e$. One also has

$$\Omega^n_{(P\otimes_K Q)|Q} \cong \Omega^n_{P|K} \otimes_K Q \quad \text{for every} \quad n \in \mathbb{N}\,. \tag{1.9.1}$$

The isomorphisms above explain the first and third links in the chain

$$\begin{aligned}\mathsf{R}\mathrm{Hom}_{P\otimes_K Q}(S, \Sigma^{d+e}\Omega^{d+e}_{(P\otimes_K Q)|K}) &\simeq \mathsf{R}\mathrm{Hom}_{P\otimes_K Q}(S, \Sigma^d\Omega^d_{P|K} \otimes_K \Sigma^e\Omega^e_{Q|K}) \\ &\simeq \mathsf{R}\mathrm{Hom}_{P\otimes_K Q}(S, \Sigma^d\Omega^d_{P|K} \otimes_K Q) \otimes_Q \Sigma^e\Omega^e_{Q|K} \\ &\simeq \mathsf{R}\mathrm{Hom}_{P\otimes_K Q}(S, \Sigma^d\Omega^d_{(P\otimes_K Q)|Q}) \otimes_Q \Sigma^e\Omega^e_{Q|K} \\ &\simeq \mathsf{R}\mathrm{Hom}_Q(S, Q) \otimes_Q \Sigma^e\Omega^e_{Q|K} \\ &\simeq \mathsf{R}\mathrm{Hom}_Q(S, \Sigma^e\Omega^e_{Q|K})\end{aligned}$$

For the fourth isomorphism, apply Lemma 1.8 to the factorization $Q \to P\otimes_K Q \to S$ of the finite homomorphism $Q \to S$, where the first map is essentially smooth by [14, 17.7.4(v)] and has relative dimension d by (1.9.1). The other isomorphisms are standard. By symmetry one also obtains an isomorphism

$$\mathsf{R}\mathrm{Hom}_{P\otimes_K Q}(S, \Sigma^{d+e}\Omega^{d+e}_{(P\otimes_K Q)|K}) \simeq \mathsf{R}\mathrm{Hom}_P(S, \Sigma^d\Omega^d_{P|K})\,. \qquad \square$$

Proof of Theorem 1.2. Recall that $K \to P \to S$ is a factorization of σ with $K \to P$ essentially smooth of relative dimension d and $P \to S$ finite. Set $L = \Sigma^d\Omega^d_{P|K}$, and note that one has $D^\sigma = \mathsf{R}\mathrm{Hom}_P(S, L)$; see Theorem 1.1.

(1) Standard adjunctions give isomorphisms of functors

$$\mathsf{R}\mathrm{Hom}_S(-, D^\sigma) \cong \mathsf{R}\mathrm{Hom}_S(-, \mathsf{R}\mathrm{Hom}_P(S, L)) \cong \mathsf{R}\mathrm{Hom}_P(-, L)\,,$$

For $M \in \mathsf{P}(\sigma)$ Lemma 1.4 yields $\operatorname{pd}_P M < \infty$, so M is represented in $\mathsf{D}(P)$ by a bounded complex F of finite projective P-modules. As L is a shift of a finite projective P-module, $\mathrm{Hom}_P(F, L)$ is a bounded complex of finite projective P-modules. It represents $\mathsf{R}\mathrm{Hom}_P(M, L)$, so one sees that $\mathrm{H}(\mathsf{R}\mathrm{Hom}_P(M, L))$ is finite over P. As P acts on it through S, it is finite over S as well; furthermore, $\operatorname{fd}_K \mathsf{R}\mathrm{Hom}_P(M, L)$ is finite by Lemma 1.4.

The map δ^M in $\mathsf{D}(S)$ is represented in $\mathsf{D}(P)$ by the canonical biduality map

$$F \to \mathrm{Hom}_P(\mathrm{Hom}_P(F, L), L)\,.$$

This is a quasiisomorphism as F is finite complex of finite projectives and L is invertible. It follows that δ^M is an isomorphism.

(2) Since $\operatorname{fd}_K S$ is finite, (1) applied to $M = S$ shows that $D^\sigma = \mathsf{R}\mathrm{Hom}_S(S, D^\sigma)$ is in $\mathsf{P}(\sigma)$ and that $\delta^S\colon S \to \mathsf{R}\mathrm{Hom}_S(\mathsf{R}\mathrm{Hom}_S(S, D^\sigma), D^\sigma)$ is an isomorphism. Composing δ^S with the map induced by the isomorphism $D^\sigma \simeq \mathsf{R}\mathrm{Hom}_S(S, D^\sigma)$ one gets $\chi^{D^\sigma}\colon S \to \mathsf{R}\mathrm{Hom}_S(D^\sigma, D^\sigma)$, hence χ^{D^σ} is an isomorphism. □

2. DG derived categories

Our purpose here is to introduce background material on differential graded homological algebra needed to state and prove the results in Sections 3 and 4.

In this section K denotes a commutative ring.

2.1. DG algebras and DG modules. Our terminology and conventions generally agree with those of MacLane [22, Ch. VI]. All DG algebras are defined over K, are zero in negative degrees, and act on their DG modules from the left. When A is a DG algebra and N a DG A-module we write $A^\natural$ and $N^\natural$ for the graded algebra and graded $A^\natural$-module underlying A and N, respectively. We set

$$\inf N = \inf\{n \in \mathbb{Z} \mid N_n \neq 0\}\,;$$
$$\sup N = \sup\{n \in \mathbb{Z} \mid N_n \neq 0\}\,.$$

Every element $x \in N$ has a well defined degree, denoted $|x|$.

When B is a DG algebra the complex $A \otimes_K B$ is a DG algebra with product $(a \otimes b)\cdot(a' \otimes b') = (-1)^{|b||a'|}(aa' \otimes bb')$.

When M' is a DG B-module the complex $N \otimes_K M'$ is canonically a DG module over $A \otimes_K B$, with $(a \otimes b)\cdot(n \otimes m') = (-1)^{|b||n|} an \otimes bm'$.

The *opposite DG K-algebra* A° has the same underlying complex of K-modules as A, and product $\cdot$ given by $a \cdot b = (-1)^{|a||b|} ba$. We identify right DG A-modules with DG modules over A°, via the formula $am = (-1)^{|a||m|} ma$.

When M is a DG B-module the complex $\mathrm{Hom}_K(M, N)$ is canonically a DG $A \otimes_K B^\circ$-module, with action given by $\big((a \otimes b)(\alpha)\big)(m) = (-1)^{|b||\alpha|} a\alpha(bm)$.

We write A^{e} for the DG K-algebra $A \otimes_K A^\circ$. Any morphism $\alpha\colon A \to B$ of DG K-algebras induces a morphism $\alpha^{\mathsf{e}} = \alpha \otimes_K \alpha^\circ$ from A^{e} to B^{e}. There is a natural DG A^{e}-module structure on A given by $(a \otimes a')x = (-1)^{|a'||x|} axa'$.

For every DG $A \otimes_K B^\circ$-module L, [22, VI.(8.7)] yields a canonical isomorphism

$$\mathrm{Hom}_{A\otimes_K B^\circ}(L, \mathrm{Hom}_K(M, N)) \cong \mathrm{Hom}_A(L \otimes_B M, N)\,. \tag{2.1.1}$$

For every DG $A^\circ \otimes_K B$-module L', [22, VI.(8.3)] yields a canonical isomorphism

$$L' \otimes_{A\otimes_K B^\circ} (N \otimes_K M') \cong (L' \otimes_A N) \otimes_B M'\,. \tag{2.1.2}$$

2.2. Properties of DG modules. A DG A-module F is said to be *semiprojective* if the functor $\mathrm{Hom}_A(F, -)$ preserves surjections and quasi-isomorphisms, and *semiflat* if $(F \otimes_A -)$ preserves injections and quasi-isomorphisms. If F is semiprojective, respectively, semiflat, then $F^\natural$ is projective, respectively, flat, over $A^\natural$; the converse is true when F is bounded below. Semiprojectivity implies semiflatness.

A DG module I is *semiinjective* if $\mathrm{Hom}_A(-, I)$ transforms injections into surjections and preserves quasi-isomorphisms. If I is semiinjective, then $I^\natural$ is injective over $A^\natural$; the converse is true when I is bounded above.

2.2.1. Every quasi-isomorphisms of DG modules, both of which are either semiprojective or semiinjective, is a homotopy equivalence.

The following properties readily follow from standard adjunction formulas.

2.2.2. Let $\alpha\colon A \to B$ be a morphism of DG K-algebras, and let X and Y be DG modules over A and B, respectively. The following statements hold:

(1) If X is semiprojective, then so is the DG B-module $B \otimes_A X$.
(2) If X is semiinjective, then so is the DG B-module $\operatorname{Hom}_A(B, X)$.
(3) If B is semiprojective over A and Y is semiprojective over B, then Y is semiprojective over A.
(4) If B is semiflat over A and Y is semiinjective over B, then Y is semiinjective over A.

2.3. Resolutions of DG modules. Let M be a DG A-module.

2.3.1. A semiprojective resolution of M is a quasi-isomorphism $F \xrightarrow{\simeq} M$ with F semiprojective. Each DG A-module M admits such a resolution; [4, §1].

A semiinjective resolution of M is a quasi-isomorphism $M \xrightarrow{\simeq} I$ with I semiinjective. Every DG A-module M admits such a resolution; see [18, §3-2].

In what follows, for each DG module M over A, we fix a semiprojective resolution $\pi_A^M\colon \mathsf{p}_A(M) \to M$, and a semiinjective resolution $\iota_A^M\colon M \to \mathsf{i}_A(M)$.

Each morphism of DG modules lifts up to homotopy to a morphism of their semiprojective resolutions and extends to a morphism of their semiinjective resolutions, and such a lifting or extension is unique up to homotopy. In particular, both F and I are unique up to homotopy equivalences inducing the identity on M.

Lemma 2.3.2. *Let $\omega\colon A \to B$ be a quasi-isomorphism of DG algebras, I a semiinjective DG A-module, J a semiinjective DG B-module, and $\iota\colon J \to I$ a quasi-isomorphism of DG A-modules.*

For every DG B-module L the following map is a quasi-isomorphism:

$$\operatorname{Hom}_\omega(L, \iota)\colon \operatorname{Hom}_B(L, J) \to \operatorname{Hom}_A(L, I)\,.$$

Proof. The morphism ι factors as a composition

$$J \xrightarrow{\iota'} \operatorname{Hom}_A(B, I) \xrightarrow{\operatorname{Hom}_A(\omega, I)} \operatorname{Hom}_A(A, I) \cong I$$

of morphisms of DG A-modules, where $\iota'(x)(b) = (-1)^{|x||b|} b\iota(x)$. It follows that ι' is a quasi-isomorphism. Now J is a semiinjective DG B-module by hypothesis, $\operatorname{Hom}_A(B, J)$ is one by 2.2.2(2), so 2.2.1 yields

$$\operatorname{Hom}_B(L, J) \xrightarrow{\simeq} \operatorname{Hom}_B(L, \operatorname{Hom}_A(B, I)) \cong \operatorname{Hom}_A(L, I)\,.$$

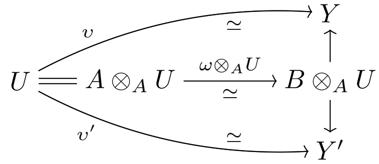
It remains to note that the composition of these maps is equal to $\operatorname{Hom}_\omega(L, \iota)$. □

Lemma 2.3.3. *Let $\omega\colon A \to B$ be a morphism of DG algebras, and let Y and Y' be DG B-modules that are quasi-isomorphic when viewed as DG A-modules.*

If ω is a quasi-isomorphism, or if there exists a morphism $\beta\colon B \to A$, such that $\omega\beta = \mathrm{id}^B$, then Y and Y' are quasi-isomorphic as DG B-modules.

Proof. By hypothesis, one has A-linear quasi-isomorphisms $Y \xleftarrow{\upsilon} U \xrightarrow{\upsilon'} Y'$.

When ω is a quasi-isomorphism, choose U semiprojective over A, using 2.3.1. With vertical arrows defined to be $b \otimes u \mapsto b\upsilon(u)$ and $b \otimes u \mapsto b\upsilon'(u)$ the diagram



commutes. The vertical maps are morphisms of DG B-modules, and $\omega \otimes_A U$ is a quasi-isomorphism because ω is one and U is semiprojective.

When ω has a right inverse β, note that the A-linear quasi-isomorphisms υ and υ' are also B-linear, and that the DG B-module structures on Y and Y' induced via β are identical with their original structures over B. □

We recall basic facts concerning DG derived categories; see Keller [18] for details.

2.4. DG derived categories. Let A be a DG algebra and M a DG A-module.

DG A-modules and their morphisms form an abelian category. The *derived category* $\mathsf{D}(A)$ is obtained by keeping the same objects and by formally inverting all quasi-isomorphisms. It has a natural triangulation, with translation functor $\mathsf{\Sigma}$ is defined on M by $(\mathsf{\Sigma}M)_i = M_{i-1}$, $\partial^{\mathsf{\Sigma}M}\varsigma(m) = -\varsigma(\partial^M(m))$, and $a\varsigma(m) = (-1)^{|a|}\varsigma(am)$, where $\varsigma\colon M \to \mathsf{\Sigma}M$ is the degree one map given by $\varsigma(m) = m$.

For any semiprojective resolution $F \to M$, and each $N \in \mathsf{D}(A)$ one has

$$\operatorname{Mor}_{\mathsf{D}(R)}(M,N) \cong \operatorname{H}_0(\operatorname{Hom}_R(F,N))\,.$$

2.4.1. For all $L \in \mathsf{D}(A^{\mathsf{o}})$ and M, N in $\mathsf{D}(A)$, the complexes of K-modules

$$L \otimes^{\mathsf{L}}_A M = L \otimes_A F \qquad \text{and} \qquad \mathsf{R}\!\operatorname{Hom}_A(M,N) = \operatorname{Hom}_A(F,N)$$

are defined uniquely up to unique isomorphisms in $\mathsf{D}(A)$. When $\omega\colon A \to B$ is a morphism of DG algebras, L', M' and N' are DG B-modules, and $\lambda\colon L \to L'$, $\mu\colon M \to M'$, and $\nu\colon N' \to N$ are ω-equivariant morphisms of DG modules, there exist uniquely defined morphisms

$$\lambda \otimes^{\mathsf{L}}_{\omega} \mu \colon L \otimes^{\mathsf{L}}_A M \to L' \otimes^{\mathsf{L}}_B M'\,,$$
$$\mathsf{R}\!\operatorname{Hom}_{\omega}(\mu,\nu)\colon\ \mathsf{R}\!\operatorname{Hom}_B(M',N') \to \mathsf{R}\!\operatorname{Hom}_A(M,N)\,.$$

that depend functorially on all three arguments, and are isomorphisms when all the morphisms involved have this property. For each $i \in \mathbb{Z}$ one sets

$$\operatorname{Tor}^A_i(L,M) = \operatorname{H}_i(L \otimes^{\mathsf{L}}_A M) \quad \text{and} \quad \operatorname{Ext}^i_A(M,N) = \operatorname{H}_{-i}(\mathsf{R}\!\operatorname{Hom}_A(M,N))\,.$$

2.4.2. Associative K-algebras are viewed as DG algebras concentrated in degree zero, in which case DG modules are simply complexes of left modules. Graded modules are complexes with zero differential, and modules are complexes concentrated in degree zero. The constructions above specialize to familiar concepts:

When $A_i = 0$ for $i \neq 0$ the derived category $\mathsf{D}(A)$ coincides with the classical unbounded derived category of the category of A_0-modules. Similarly, if M and N are DG A-modules with $M_i = 0 = N_i$ for $i \neq 0$, then for all $n \in \mathbb{Z}$ one has $\operatorname{Ext}^n_A(M,N) = \operatorname{Ext}^n_{A_0}(M_0,N_0)$ and $\operatorname{Tor}^A_n(M,N) = \operatorname{Tor}^{A_0}_n(M_0,N_0)$.

2.4.3. Let $\omega\colon A \to B$ be a morphism of DG algebras. Viewing DG B-modules as DG A-modules via restriction along ω, one gets a functor of derived categories

$$\omega^*\colon \mathsf{D}(B) \to \mathsf{D}(A)\,.$$

When ω is a quasi-isomorphism it is an equivalence, with quasi-inverse $B \otimes^{\mathsf{L}}_A -$.

3. Derived Hochschild functors

In this section we explain the left hand side of the isomorphism in Theorem 1.

Let K be a commutative ring and $\sigma\colon K \to S$ an associative K-algebra.

3.1. A *flat DG algebra resolution* of σ is a factorization $K \to A \xrightarrow{\alpha} S$ of σ as a composition of morphisms of DG algebras, where each K-module A_i is flat and α is a quasi-isomorphism; complexes of S-modules are viewed as DG A-modules via α. When $K \to B \xrightarrow{\beta} S$ is a flat DG algebra resolution of σ, we say that $\omega\colon A \to B$ is a *morphism of resolutions* if it is a morphism of DG K-algebras, satisfying $\beta\omega = \alpha$.

We set $A^{\mathsf{e}} = A \otimes_K A^{\mathsf{o}}$, note that $K \to A^{\mathsf{o}} \xrightarrow{\alpha^{\mathsf{o}}} S^{\mathsf{o}}$ is a flat DG algebra resolution of $\sigma^{\mathsf{o}}\colon K \to S^{\mathsf{o}}$, and turn S into a DG module over A^{e} by $(a \otimes a')s = \alpha(a)s\,\alpha^{\mathsf{o}}(a')$.

Flat DG algebra resolutions always exist: A resolution $K \to T \to S$, with $T^{\natural}$ the tensor algebra of some free non-negatively graded K-module, can be obtained by inductively adjoining noncommuting variables to K; see also Lemma 3.7.

Here we construct one of four functors of pairs of complexes of S-modules that can be obtained by combining $\mathsf{R}\mathrm{Hom}_{A^{\mathsf{e}}}(S, -)$ and $S \otimes^{\mathsf{L}}_{A^{\mathsf{e}}} -$ with $(- \otimes^{\mathsf{L}}_K -)$ and $\mathsf{R}\mathrm{Hom}_K(-,-)$. The other three functors are briefly discussed in 3.10 and 3.11.

The statement of the following theorem is related to results in [32, §2]. We provide a detailed proof, for reasons explained in 3.12.

Theorem 3.2. *Each flat DG algebra resolution $K \to A \to S$ of σ defines a functor*

$$\mathsf{R}\mathrm{Hom}_{A^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_K -)\colon \mathsf{D}(S) \times \mathsf{D}(S^{\mathsf{o}}) \to \mathsf{D}(S^{\mathsf{c}}),$$

where S^{c} denote the center of S, described by (3.8.1). For every flat DG algebra resolution $K \to B \to S$ of σ there is a canonical natural equivalence of functors

$$\omega^{AB}\colon\ \mathsf{R}\mathrm{Hom}_{A^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_K -) \to \mathsf{R}\mathrm{Hom}_{B^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_K -),$$

given by (3.8.2), and every flat DG algebra resolution $K \to C \to S$ of σ satisfies

$$\omega^{AC} = \omega^{BC}\omega^{AB}.$$

The theorem validates the following notation:

Remark 3.3. Fix a flat DG algebra resolution $K \to A \to S$ of σ and let

$$\mathsf{R}\mathrm{Hom}_{S \otimes^{\mathsf{L}}_K S^{\mathsf{o}}}(S, - \otimes^{\mathsf{L}}_K -)\colon \mathsf{D}(S) \times \mathsf{D}(S^{\mathsf{o}}) \to \mathsf{D}(S^{\mathsf{c}})$$

denote the functor $\mathsf{R}\mathrm{Hom}_{A \otimes_K A^{\mathsf{o}}}(S, - \otimes^{\mathsf{L}}_K -)$. For all $L \in \mathsf{D}(S)$ and $L' \in \mathsf{D}(S^{\mathsf{o}})$ it yields *derived Hochschild cohomology modules* with tensor-decomposable coefficients:

$$\mathrm{Ext}^n_{S \otimes^{\mathsf{L}}_K S^{\mathsf{o}}}(S, L \otimes^{\mathsf{L}}_K L') = \mathrm{H}^n(\mathsf{R}\mathrm{Hom}_{S \otimes^{\mathsf{L}}_K S^{\mathsf{o}}}(S, L \otimes^{\mathsf{L}}_K L')).$$

These modules are related to vintage Hochschild cohomology.

For all S-modules L and L' there are canonical natural maps

$$\mathrm{HH}^n(S\,|\,K; L \otimes_K L') \to \mathrm{Ext}^n_{S \otimes_K S^{\mathsf{o}}}(S, L \otimes_K L')$$

of S^{c}-modules, where the modules on the left are the classical ones, see 2.4.2. These are isomorphisms when S is K-projective; see [10, IX, §6]. When one of L or L' is K-flat, there exist canonical natural homomorphisms

$$\mathrm{Ext}^n_{\alpha \otimes_K \alpha^{\mathsf{o}}}(S, L \otimes_K L')\colon\ \mathrm{Ext}^n_{S \otimes_K S^{\mathsf{o}}}(S, L \otimes_K L') \to \mathrm{Ext}^n_{S \otimes^{\mathsf{L}}_K S^{\mathsf{o}}}(S, L \otimes_K L').$$

When S is K-flat the composition $K \to S \xrightarrow{=} S$ is a flat DG resolution of σ and $\alpha\colon A \to S$ is a morphism of resolutions, so the theorem shows that the maps above are isomorphisms.

Construction 3.4. Let $K \to A \xrightarrow{\alpha} S$ and $K \to A' \xrightarrow{\alpha'} S^{\mathsf{o}}$ be flat DG algebra resolutions of σ and of σ^{o}, respectively. We turn S into a DG module over $A \otimes_K A'$ by setting $(a \otimes a')s = \alpha(a)s\,\alpha'(a')$. The action of S^{c} on S commutes with that of $A \otimes_K A'$, and so confers a natural structure of complex of S^{c}-modules on

$$\operatorname{Hom}_{A\otimes_K A'}(S, \mathsf{i}_{A\otimes_K A'}(\mathsf{p}_A(L) \otimes_K \mathsf{p}_{A'}(L'))\,,$$

where p_A and $\mathsf{i}_{A\otimes_K A'}$ refer to the resolutions introduced in 2.3.1.

Let $K \to B \xrightarrow{\beta} S$ and $K \to B' \xrightarrow{\beta'} S^{\mathsf{o}}$ be DG algebra resolutions of σ and σ^{o}, respectively, and $\omega\colon A \to B$ and $\omega'\colon A' \to B'$ be morphism of resolutions. We turn DG B-modules into DG A-modules via ω, and remark that the equality $\beta\omega = \alpha$ implies that on S-modules the new action of A coincides with the old one.

Let $\lambda\colon L \to M$ be a morphism of DG S-modules and $\lambda'\colon L' \to M'$ one of DG S^{o}-modules. The lifting property of semiprojective DG modules yields diagrams

$$\text{(3.4.1)}\qquad \begin{array}{ccc} \mathsf{p}_A(L) & \xrightarrow{\widetilde{\lambda}} & \mathsf{p}_B(M) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ L & \xrightarrow{\lambda} & M \end{array} \qquad\text{and}\qquad \begin{array}{ccc} \mathsf{p}_{A'}(L') & \xrightarrow{\widetilde{\lambda}'} & \mathsf{p}_{B'}(M') \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ L' & \xrightarrow{\lambda'} & M' \end{array}$$

of DG A-modules and DG A'-modules, respectively, that commute up to homotopy. It provides the morphism in the top row of a diagram of DG $(A \otimes_K A')$-modules

$$\text{(3.4.2)}\qquad \begin{array}{ccc} \mathsf{p}_A(L) \otimes_K \mathsf{p}_{A'}(L') & \xrightarrow{\widetilde{\lambda}\otimes_K\widetilde{\lambda}'} & \mathsf{p}_B(M) \otimes_K \mathsf{p}_{B'}(M') \\ {\scriptstyle\simeq}\downarrow & & \\ \mathsf{i}_{A\otimes_K A'}(\mathsf{p}_A(L) \otimes_K \mathsf{p}_{A'}(L')) & & \downarrow{\scriptstyle\simeq} \\ \downarrow{\scriptstyle\epsilon} & & \\ \mathsf{i}_{A\otimes_K A'}\big(\mathsf{i}_{B\otimes_K B'}(\mathsf{p}_B(M) \otimes_K \mathsf{p}_{B'}(M'))\big) & \xleftarrow[\simeq]{\iota} & \mathsf{i}_{B\otimes_K B'}(\mathsf{p}_B(M) \otimes_K \mathsf{p}_{B'}(M')) \end{array}$$

that commutes up to homotopy, where ι is the chosen semiinjective resolution, and ϵ is given by the extension property of semiinjective DG module over $A \otimes_K A'$; for conciseness, we rewrite these maps as $E \xrightarrow{\epsilon} I \xleftarrow{\iota} J$. They are unique up to homotopy, as the liftings and extensions used for their construction have this property.

The hypotheses $\beta\omega = \alpha$ and $\beta'\omega' = \alpha'$ imply that ω and ω' are quasi-isomorphisms, hence so is $\omega \otimes_K \omega'$, due to the K-flatness of $A^\natural$ and $B'^\natural$. Since ι is a quasi-isomorphism, Lemma 2.3.2 shows that so is $\operatorname{Hom}_{\omega\otimes_K\omega'}(S, \iota)$; thus, the latter map defines in $\mathsf{D}(S^{\mathsf{c}})$ an isomorphism, denoted $\mathsf{R}\!\operatorname{Hom}_{\omega\otimes_K\omega'}(S, \iota)$. We set

$$\text{(3.4.3)}\qquad \begin{aligned} [\omega, \omega'](\lambda, \lambda') = \mathsf{R}\!\operatorname{Hom}_{\omega\otimes_K\omega'}(S, \iota)^{-1} \circ \mathsf{R}\!\operatorname{Hom}_{A\otimes_K A'}(S, \epsilon)\colon& \\ \mathsf{R}\!\operatorname{Hom}_{A\otimes_K A'}(S, L \otimes^{\mathsf{L}}_K L') \longrightarrow \mathsf{R}\!\operatorname{Hom}_{B\otimes_K B'}(S, M \otimes^{\mathsf{L}}_K M')& \end{aligned}$$

The first statement of the following lemma contains the existence of the functors $\mathsf{R}\!\operatorname{Hom}_{A^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_K -)$, asserted in the theorem. The second statement, concerning the uniqueness of these functors, is weaker than the desired one, because it only applies to resolutions that can be compared through a morphism $\omega\colon A \to B$. On the other hand, it allows one to compare functors defined by independently chosen resolutions of σ and σ^{o}. The extra generality is needed in the proof of Lemma 3.7.

Lemma 3.5. *In the notation of Construction 3.4, the assignment*

$$(L, L') \mapsto \operatorname{Hom}_{A\otimes_K A'}(S, \mathsf{i}_{A\otimes_K A'}(\mathsf{p}_A(L) \otimes_K \mathsf{p}_{A'}(L'))) ,$$

defines a functor

$$\mathsf{R}\mathrm{Hom}_{A\otimes_K A'}(S, - \otimes^{\mathsf{L}}_K -)\colon \mathsf{D}(S) \times \mathsf{D}(S^{\mathsf{o}}) \to \mathsf{D}(S^{\mathsf{c}}) ,$$

and the assignment

$$(\lambda, \lambda') \mapsto [\omega, \omega'](\lambda, \lambda') ,$$

given by formula (3.4.3), defines a canonical natural equivalence of functors

$$[\omega, \omega']\colon \mathsf{R}\mathrm{Hom}_{A\otimes_K A'}(S, - \otimes^{\mathsf{L}}_K -) \to \mathsf{R}\mathrm{Hom}_{B\otimes_K B'}(S, - \otimes^{\mathsf{L}}_K -) .$$

If $K \to C \xrightarrow{\gamma} S$ and $K \to C' \xrightarrow{\gamma'} S$ are flat DG algebra resolutions of σ and σ^{o}, respectively, and $\vartheta\colon B \to C$ and $\vartheta'\colon B' \to C'$ are morphism of resolutions, then

$$[\vartheta\omega, \vartheta'\omega'] = [\vartheta, \vartheta'][\omega, \omega'] .$$

Proof. Recall that the maps $E \xrightarrow{\epsilon} I \xleftarrow{\iota} J$ are unique up to homotopy. Thus, $\operatorname{Hom}_{A\otimes_K A'}(S, \epsilon)$ and $\operatorname{Hom}_{\omega\otimes_K \omega'}(S, \iota)$ are morphisms of complexes of S^{c}-modules defined uniquely up to homotopy. In view of (3.4.3), this uniqueness has the following consequences:

The morphism $[\omega, \omega'](\lambda, \lambda')$ depends only on λ and λ'; one has

$$[\mathrm{id}^A, \mathrm{id}^{A'}](\mathrm{id}^L, \mathrm{id}^{L'}) = \mathrm{id}^{\mathsf{R}\mathrm{Hom}_{A\otimes_K A'}(S, L\otimes^{\mathsf{L}}_K L')} ;$$

and for all morphism $\mu\colon M \to N$ and $\mu'\colon M' \to N'$ of complexes of S-modules and S^{o}-modules, respectively, there are equalities

$$[\vartheta\omega, \vartheta'\omega'](\mu\lambda, \mu'\lambda') = [\vartheta, \vartheta'](\mu, \mu') \circ [\omega, \omega'](\lambda, \lambda') .$$

Suitable specializations of these properties show that $\mathsf{R}\mathrm{Hom}_{A\otimes_K A'}(S, - \otimes^{\mathsf{L}}_K -)$ is a functor to $\mathsf{D}(S^{\mathsf{c}})$ from the product of the categories of complexes over S with that of complexes over S^{o}, *and* that $[\omega, \omega']$ is a natural transformation.

To prove that $[\omega, \omega']$ is an equivalence, it suffices to show that if λ and λ' are quasi-isomorphisms, then $\mathsf{R}\mathrm{Hom}_{\omega\otimes_K \omega'}(S, \lambda \otimes^{\mathsf{L}}_K \lambda')$ is an isomorphism.

By (3.4.3), it is enough to show that $\mathsf{R}\mathrm{Hom}_{A\otimes_K A'}(S, \epsilon)$ is a quasi-isomorphism. As λ and λ' are quasi-isomorphisms, the diagrams in (3.4.1) imply that so are $\widetilde{\lambda}$ and $\widetilde{\lambda}'$. Due to the K-flatness of $A^\natural$ and $B'^\natural$, their semiprojective DG modules are K-flat, hence $\widetilde{\lambda} \otimes_K \widetilde{\lambda}'$ is a quasi-isomorphism of DG modules over $A \otimes_K A'$. Now diagram (3.4.2) shows that $\epsilon\colon E \to I$ is a quasi-isomorphism. It follows that it is a homotopy equivalence, because both E and J are semiinjective DG modules over $A \otimes_K A'$. This implies that $\operatorname{Hom}_{A\otimes_K A'}(S, \epsilon)$ is a quasi-isomorphism, as desired. □

To clarify how the natural equivalence in Lemmas 3.5 depends on ω, we apply Quillen's homotopical approach in [25]. It is made available by the following result, see Baues and Pirashvili [8, A.3.1, A.3.5]:

3.6. The category of DG K-algebras has a model structure, where

- the *weak equivalences* are the quasi-isomorphisms;
- the *fibrations* are the morphisms that are surjective in positive degrees;
- any DG K-algebra, whose underlying graded algebra is the tensor algebra of a non-negatively graded projective K-module, is *cofibrant*; that is, the structure map from K is a *cofibration*.

We recall some consequences of the existence of a model structure, following [12]: For all DG K-algebras T and A, there exists a relation on the set of morphisms $T \to A$, known as *left homotopy*, see [12, 4.2]. It is an equivalence when T is cofibrant, see [12, 4.7], and then $\pi^{\ell}(T, A)$ denotes the set of equivalence classes.

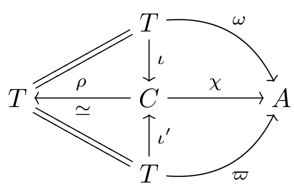
Lemma 3.7. *There is a DG algebra resolution $K \to T \to S$ of σ with T cofibrant.*

If $K \to A \xrightarrow{\alpha} S$ is a flat DG algebra resolutions of σ, then there is a morphism of resolutions $\omega\colon T \to A$. Any morphism of resolutions $\varpi\colon T \to A$ is left homotopic to ω, and the natural equivalences defined in Lemma 3.5 satisfy

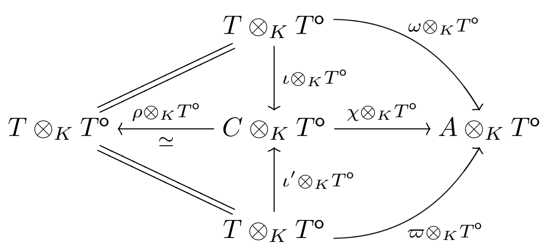
$$[\omega, \omega^{\mathsf{o}}] = [\varpi, \varpi^{\mathsf{o}}]\colon\ \mathsf{R}\mathrm{Hom}_{T^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_K -) \to \mathsf{R}\mathrm{Hom}_{A^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_K -)$$

Proof. Being both a fibration and a weak equivalence, α is, by definition, an *acyclic fibration*. The existence of ω comes from a defining property of model categories—the *left lifting property* of cofibrations with respect to acyclic fibrations; see axiom MC4(i) in [12, 3.3]. Composition with α induces a bijection $\pi^{\ell}(T, A) \to \pi^{\ell}(T, S)$, see [12, 4.9], so $\alpha\varpi = \alpha\omega$ implies that ϖ and ω are left homotopic.

By [12, 4.3, 4.4], the homotopy relation produces a commutative diagram



of DG K-algebras, with a quasi-isomorphism ρ. It induces a commutative diagram



of morphisms of DG K-algebras, where $\rho \otimes_K T^{\mathsf{o}}$ is a quasi-isomorphism because T^{o} is K-flat. The diagram above yields the following chain of equalities:

$$[\omega, \mathrm{id}^{T^{\mathsf{o}}}] = [\chi, \mathrm{id}^{T^{\mathsf{o}}}][\iota, \mathrm{id}^{T^{\mathsf{o}}}] = [\chi, \mathrm{id}^{T^{\mathsf{o}}}][\rho, \mathrm{id}^{T^{\mathsf{o}}}]^{-1} = [\chi, \mathrm{id}^{T^{\mathsf{o}}}][\iota', \mathrm{id}^{T^{\mathsf{o}}}] = [\varpi, \mathrm{id}^{T^{\mathsf{o}}}].$$

A similar argument shows that the morphisms ω^{o} and ϖ^{o} are left homotopic, and yields $[\mathrm{id}^A, \omega^{\mathsf{o}}] = [\mathrm{id}^A, \varpi^{\mathsf{o}}]$. Assembling these data, one obtains

$$[\omega, \omega^{\mathsf{o}}] = [\mathrm{id}^A, \omega^{\mathsf{o}}][\omega, \mathrm{id}^{T^{\mathsf{o}}}] = [\mathrm{id}^A, \varpi^{\mathsf{o}}][\varpi, \mathrm{id}^{T^{\mathsf{o}}}] = [\varpi, \varpi^{\mathsf{o}}]\,. \qquad \square$$

Proof of Theorem 3.2. Choose a DG algebra resolution $K \to T \to S$ of σ with T cofibrant, either by noting that the one in 3.1 has this property by 3.6, or referring to a defining property of model categories; see axiom MC5(i) in [12, 3.3].

For each flat DG algebra resolution $K \to A \to S$ of σ, form the flat DG algebra resolution $K \to A^{\mathsf{o}} \to S^{\mathsf{o}}$ of σ^{o}, and define a functor

(3.8.1) $$\mathsf{R}\mathrm{Hom}_{A^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_K -)\colon \mathsf{D}(S) \times \mathsf{D}(S^{\mathsf{o}}) \to \mathsf{D}(S^{\mathsf{c}})$$

by applying Lemma 3.5 with $A' = A^{\circ}$. As T is cofibrant, Lemma 3.7 provides a morphism of resolutions $\omega\colon T \to A$, and shows that it defines a natural equivalence

$$[\omega, \omega^{\circ}]\colon \mathsf{R}\mathrm{Hom}_{T^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_{K} -) \to \mathsf{R}\mathrm{Hom}_{A^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_{K} -)\,;$$

that does not depend on the choice of ω; set $\omega^{A}_{T} = [\omega, \omega^{\circ}]$.

When $K \to U \to S$ also is a flat DG algebra resolution of σ with U cofibrant, one gets morphisms of resolutions $\tau\colon T \to U$ and $\theta\colon U \to A$. Both $\theta\tau\colon T \to A$ and ω are morphisms of resolutions, so Lemmas 3.7 and 3.5 yield

$$\omega^{A}_{T} = [\omega, \omega^{\circ}] = [\theta\tau, \theta^{\circ}\tau^{\circ}] = [\theta, \theta^{\circ}][\tau, \tau^{\circ}] = \omega^{A}_{U}\omega^{U}_{T}\,.$$

For each flat DG algebra resolution $K \to B \to S$ of σ set

$$\omega^{AB} := \omega^{B}_{T}(\omega^{A}_{T})^{-1}\colon \mathsf{R}\mathrm{Hom}_{A^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_{K} -) \to \mathsf{R}\mathrm{Hom}_{B^{\mathsf{e}}}(S, - \otimes^{\mathsf{L}}_{K} -)\,. \tag{3.8.2}$$

One clearly has $\omega^{AC} = \omega^{BC}\omega^{AB}$, and ω^{AB} is independent of T, because

$$\omega^{B}_{T}(\omega^{A}_{T})^{-1} = \omega^{B}_{U}\omega^{U}_{T}(\omega^{A}_{U}\omega^{U}_{T})^{-1} = \omega^{B}_{U}\omega^{U}_{T}(\omega^{U}_{T})^{-1}(\omega^{A}_{U})^{-1} = \omega^{B}_{U}(\omega^{A}_{U})^{-1}\,.$$

It follows that ω^{AB} is the desired canonical natural equivalence. □

We proceed with a short discussion of other derived Hochschild functors. The proof of the next result is omitted, as it parallels that of Theorem 3.2.

Theorem 3.9. *Any flat DG algebra resolution $K \to A \to S$ of σ defines a functor*

$$A \otimes_{A^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{K}(-,-)\colon \mathsf{D}(S)^{\mathsf{op}} \times \mathsf{D}(S) \to \mathsf{D}(S^{\mathsf{c}})\,.$$

For each flat DG algebra resolution $K \to B \to S$ of σ one has a canonical equivalence

$$\omega_{BA}\colon B \otimes_{B^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{K}(-,-) \xrightarrow{\simeq} A \otimes_{A^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{K}(-,-)$$

of functors, and every flat DG algebra resolution $K \to C \to S$ of σ satisfies

$$\omega_{CA} = \omega_{BA}\omega_{CB}\,.$$ □

Remark 3.10. We fix a DG algebra resolution $K \to A \to S$ of σ and let

$$S \otimes^{\mathsf{L}}_{S\otimes^{\mathsf{L}}_{K}S^{\circ}} \mathsf{R}\mathrm{Hom}_{K}(-,-)\colon \mathsf{D}(S)^{\mathsf{op}} \times \mathsf{D}(S) \to \mathsf{D}(S^{\mathsf{c}})$$

denote the functor $A \otimes_{A\otimes_{K}A} \mathsf{R}\mathrm{Hom}_{K}(-,-)$: The preceding theorem shows that it is independent of the choice of A. For all $M, N \in \mathsf{D}(S)$ it defines *derived Hochschild homology modules* of the K-algebra S with Hom-decomposable coefficients:

$$\mathrm{Tor}^{S\otimes^{\mathsf{L}}_{K}S^{\circ}}_{n}(S, \mathsf{R}\mathrm{Hom}_{K}(M,N)) = \mathrm{H}^{n}(S \otimes^{\mathsf{L}}_{S\otimes^{\mathsf{L}}_{K}S^{\circ}} \mathsf{R}\mathrm{Hom}_{K}(M,N))\,.$$

These modules are related to classical Hochschild homology:

For all S-modules M and N there are canonical natural maps

$$\mathrm{Tor}^{S\otimes_{K}S^{\circ}}_{n}(S, \mathrm{Hom}_{K}(M,N)) \to \mathrm{HH}_{n}(S\,|\,K; \mathrm{Hom}_{K}(M,N))$$

of S^{c}-modules, where the modules on the left are the classical ones, see 2.4.2. They are isomorphisms when S is K-flat; see [10, IX, §6]. When M is K-projective there exist natural homomorphisms

$$\mathrm{Tor}^{\alpha\otimes_{K}\alpha^{\circ}}_{n}(S, \mathsf{R}\mathrm{Hom}_{K}(M,N))\colon$$
$$\mathrm{Tor}^{S\otimes^{\mathsf{L}}_{K}S^{\circ}}_{n}(S, \mathsf{R}\mathrm{Hom}_{K}(M,N)) \to \mathrm{Tor}^{S\otimes_{K}S^{\circ}}_{n}(S, \mathrm{Hom}_{K}(M,N))$$

When S is K-flat the composition $K \to S \xrightarrow{=} S$ is a flat DG resolution of σ and $\alpha\colon A \to S$ is a morphism of resolutions, so the theorem shows that the maps above are isomorphisms.

The remaining two composed functors collapse in a predictable way.

Remark 3.11. Similarly to Theorems 3.2 and 3.9, one can define functors

$$\mathsf{R}\mathrm{Hom}_{S\otimes^{\mathsf{L}}_K S^\circ}(S, \mathsf{R}\mathrm{Hom}_K(-,-))\colon \mathsf{D}(S)^{\mathsf{op}} \times \mathsf{D}(S) \to \mathsf{D}(S^{\mathsf{c}})\,,$$
$$S \otimes_{S\otimes^{\mathsf{L}}_K S^\circ} (- \otimes^{\mathsf{L}}_K -)\colon \mathsf{D}(S) \times \mathsf{D}(S) \to \mathsf{D}(S^{\mathsf{c}})\,,$$

that do not depend on the choice of the DG algebra resolution A. However, this is not necessary, as for all $M, N \in \mathsf{D}(S)$ there exit canonical isomorphisms

$$\mathsf{R}\mathrm{Hom}_{S\otimes^{\mathsf{L}}_K S^\circ}(S, \mathsf{R}\mathrm{Hom}_K(M,N)) \simeq \mathsf{R}\mathrm{Hom}_S(M,N)\,, \tag{3.11.1}$$
$$S \otimes_{S\otimes^{\mathsf{L}}_K S^\circ} (M \otimes^{\mathsf{L}}_K N) \simeq M \otimes^{\mathsf{L}}_S N\,. \tag{3.11.2}$$

They are derived extensions of classical reduction results, [10, IX.2.8, IX.2.8a].

We finish with a comparison of the content of this section and that of [32, §2].

Remark 3.12. When $M = N$ the statement of Theorem 3.2 bears a close resemblance to results of Yekutieli and Zhang, see [32, 2.2, 2.3]. One might ask whether their proof can be adapted to handle the general case.

Unfortunately, even in the special case above the argument for [32, Theorem 2.2] is deficient. It utilizes the mapping cylinder of morphisms $\phi_0, \phi_1\colon \tilde{M} \to M$ of DG modules over a DG algebra, $\tilde{B}$. On page 3225, line 11, they are described as "the two $\tilde{B}'$-linear quasi-isomorphisms ϕ_0 and ϕ_1" where $\tilde{B}'$ is a DG algebra equipped with *two* homomorphisms of DG algebras $u_0, u_1\colon \tilde{B}' \to \tilde{B}$; with this, an implicit choice is being made between u_0 and u_1. Such a choice compromises the argument, whose goal is to establish an equality $\chi_0 = \chi_1$ between morphism of complexes χ_i, which have already been constructed by using ϕ_i and u_i for $i = 0, 1$.

The basic problem is that the relation between various choices of comparison *morphisms of DG algebra resolutions* is not registered in the additive environment of derived categories. In the proof of Theorem 3.2 it is solved by using the homotopy equivalence provided by a model structure on the category of DG algebras.

4. Reduction of derived Hochschild functors over algebras

Let $\sigma\colon K \to S$ be a homomorphism of *commutative* rings.

Recall that σ is said to be *essentially of finite type* if it can be factored as

$$K \hookrightarrow K[x_1, \dots, x_d] \to V^{-1}K[x_1, \dots, x_d] \twoheadrightarrow S\,,$$

where $x_1, \dots, x_d$ are indeterminates, V is a multiplicatively closed subset, the first two maps are canonical, and the third one is a surjective ring homomorphism.

The following theorem, which is the main algebraic result in the paper, involves the relative dualizing complex D^σ described in (1.0.2).

Theorem 4.1. *If* $\mathrm{fd}_K S$ *is finite, then in* $\mathsf{D}(S)$ *there are isomorphisms*

$$\mathsf{R}\mathrm{Hom}_{S\otimes^{\mathsf{L}}_K S}(S, M \otimes^{\mathsf{L}}_K N) \simeq \mathsf{R}\mathrm{Hom}_S(\mathsf{R}\mathrm{Hom}_S(M, D^\sigma), N) \tag{4.1.1}$$
$$\mathsf{R}\mathrm{Hom}_{S\otimes^{\mathsf{L}}_K S}(S, \mathsf{R}\mathrm{Hom}_S(M, D^\sigma) \otimes^{\mathsf{L}}_K N) \simeq \mathsf{R}\mathrm{Hom}_S(M,N) \tag{4.1.2}$$

for all $M \in \mathsf{P}(\sigma)$ *and* $N \in \mathsf{D}(S)$*; these morphisms are natural in* M *and* N*.*

We record a useful special case, obtained by combining Theorems 4.1 and 1.1:

Corollary 4.2. *Assume that σ is flat, and let $K \to P \to S$ be a factorization of σ with $K \to P$ essentially smooth of relative dimension d and $P \to S$ finite.*

If M is a finite S-module that is flat over K, and N is an S-module, then for each $n \in \mathbb{Z}$ there is an isomorphism of S-modules

$$\operatorname{Ext}^n_{S\otimes_K S}(S, M \otimes_K N) \cong \operatorname{Ext}^{n-d}_S(\mathsf{R}\mathrm{Hom}_P(M, \Omega^d_{P|K}), N)\,. \qquad \square$$

Before the proof of Theorem 4.1 we make a couple of remarks.

4.3. For all complexes of P-modules L, X, and J there is a natural morphism

$$\mathrm{Hom}_P(L,P) \otimes_P X \otimes_P J \longrightarrow \mathrm{Hom}_P(\mathrm{Hom}_P(X,L),J)$$

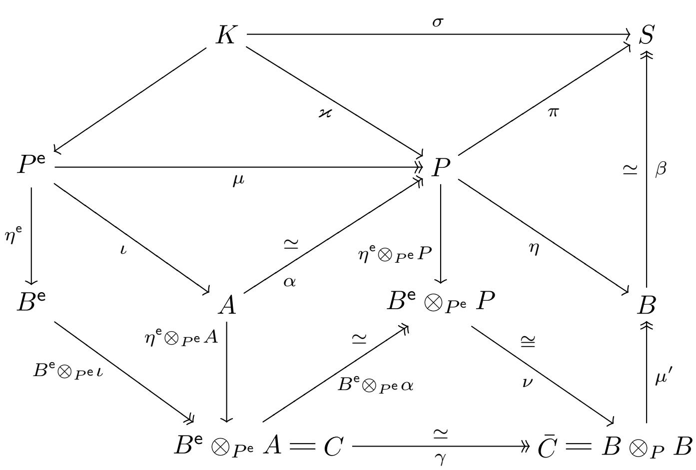
defined by the assignment $\lambda\otimes x\otimes j \mapsto \big(\chi \mapsto (-1)^{(|x|+|j|)|\lambda|}\lambda\chi(x)j\big)$. This morphism is bijective when $L^\natural$ and $X^\natural$ are finite projective: This is clear when L and X are shifts of P. The case when they are shifts of projective modules follows, as the functors involved commute with finite direct sums. The general case is obtained by induction on the number of the degrees in which L and X are not zero.

4.4. A DG algebra A is called *graded-commutative* if $ab = (-1)^{|a||b|}ba$ holds for all $a, b \in A$. The identity map $A^\circ \to A$ then is a morphism of DG algebras, so each DG A-module is canonically a DG module over A°, and for all A-modules M and N the complexes $\mathsf{R}\mathrm{Hom}_A(M,N)$ and $M \otimes^{\mathsf{L}}_A N$ are canonically DG A-modules.

When A and B are graded-commutative DG algebras, then so is $A\otimes_K B$, and the canonical isomorphisms in (2.1.1) and (2.1.2) represent morphisms in $\mathsf{D}(A \otimes_K B)$.

Proof of Theorem 4.1. The argument proceeds in several steps, with notation introduced as needed. It uses chains of quasi-isomorphisms that involve a number of auxiliary DG algebras and DG modules. We start with the DG algebras.

Step 1. There exists a commutative diagram of morphisms of DG K-algebras



where $\simeq$ flags quasi-isomorphisms and $\twoheadrightarrow$ tips surjections. The morphisms appearing in the diagram are constructed in the following sequence:

Fix a factorization $K \xrightarrow{\varkappa} P \xrightarrow{\pi} S$ of σ, with $\varkappa$ essentially smooth of relative dimension d and π finite.

Set $P^{\mathsf{e}} = P \otimes_K P$ and let $\mu\colon P^{\mathsf{e}} \to P$ denote the multiplication map, and note that the projective dimension $\mathrm{pd}_{P^{\mathsf{e}}} P$ is finite by 1.3.4.

Choose a graded-commutative DG algebra resolution $P^{\mathsf{e}} \xrightarrow{\iota} A \xrightarrow{\alpha} P$ of μ with $A_0 = P^{\mathsf{e}}$, each A_i a finite projective P^{e}-module, and $\sup A = \operatorname{pd}_{P^{\mathsf{e}}} P$; see [2, 2.2.8].

Choose a graded-commutative DG algebra resolution $P \xrightarrow{\eta} B \xrightarrow{\pi} S$ of σ, with B_0 a finite free P-module and each B_i a finite free P-module; again, see [2, 2.2.8].

Set $B^{\mathsf{e}} = B \otimes_K B$ and let $\mu' \colon B \otimes_P B \to B$ be the multiplication map.

Let $\nu \colon B^{\mathsf{e}} \otimes_{P^{\mathsf{e}}} P \to B \otimes_P B$ be the map $b \otimes b' \otimes p \mapsto (b \otimes b')p$.

Let $\gamma \colon B^{\mathsf{e}} \otimes_{P^{\mathsf{e}}} A \to B \otimes_P B$ be the map $b \otimes b' \otimes a \mapsto (b \otimes b')\alpha(a)$.

The diagram commutes by construction. The map ν is an isomorphism by (2.1.2), and $B^{\mathsf{e}} \otimes_{P^{\mathsf{e}}} \alpha$ is a quasi-isomorphism because α is one and B^{e} is a bounded below complex of flat P^{e}-modules.

We always specify the DG algebra operating on any newly introduced DG module. On DG modules of homomorphisms and tensor products the operations are those induced from the arguments of these functors; see 4.4.

Notation. Let $P^{\mathsf{e}} \xrightarrow{\simeq} U$ be a semiinjective resolution over P^{e}.

Set $H = \mathrm{H}(\operatorname{Hom}_{P^{\mathsf{e}}}(P, U))$.

Step 2. There exists a unique isomorphism inducing id^H in homology:

$$H \simeq \mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) \quad \text{in } \mathsf{D}(P)\,.$$

Proof. The isomorphism $H \cong \operatorname{Ext}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}})$ of graded P-modules and (1.7.1) show that H is projective, so Lemma 1.6 applies. □

Notation. Set $L = \operatorname{Hom}_P(H, P)$.

Let $L \xrightarrow{\simeq} I$ be a semiinjective resolution over P.

Step 3. There exists an isomorphism $D^\sigma \simeq \mathsf{R}\mathrm{Hom}_P(S, I)$ in $\mathsf{D}(S)$.

Proof. Theorem 1.1 provides the first isomorphism in the chain

$$\begin{aligned} D^\sigma &\simeq \mathsf{R}\mathrm{Hom}_P(S, \Sigma^d \Omega^d_{P|K}) \\ &\simeq \mathsf{R}\mathrm{Hom}_P(S, \mathsf{R}\mathrm{Hom}_P(\mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}), P)) \\ &\simeq \mathsf{R}\mathrm{Hom}_P(S, \mathsf{R}\mathrm{Hom}_P(H, P)) \\ &\simeq \mathsf{R}\mathrm{Hom}_P(S, I)\,. \end{aligned}$$

The remaining ones come from (1.7.2), Step 2, and the resolution $L \simeq I$. □

Notation. Let $X' \xrightarrow{\simeq} M$ be a semiprojective resolution over B, with X'_i a finite projective P-module for each i and $\inf X' = \inf \mathrm{H}(M)$, see 2.3.1; set $q = \operatorname{pd}_P M$, and note that q is finite by Lemma 1.4.

Set $X = X'/X''$, where $X''_i = X'_i$ for $i > q$, $X''_q = \partial(X_{q+1})$, and $X''_i = 0$ for $i < q$. It is easy to see that X'' is a DG submodule of X', so the canonical map $X' \to X$ is a surjective quasi-isomorphism of DG B-modules. Since $X' \xrightarrow{\simeq} M$ is a semiprojective resolution over P, each P-module X_i is projective; see [3, 2.4.P].

Let $G \xrightarrow{\simeq} \operatorname{Hom}_P(X, L)$ be a semiprojective resolution over B.

Let $N \xrightarrow{\simeq} J$ be a semiinjective resolution over B.

Set $\underline{J} = \operatorname{Hom}_B(S, J)$.

Step 4. There exists an isomorphism

$$\mathsf{R}\mathrm{Hom}_S(\mathsf{R}\mathrm{Hom}_S(M, D^\sigma), N) \simeq \mathsf{R}\mathrm{Hom}_B(\mathsf{R}\mathrm{Hom}_P(M, L), N) \quad \text{in } \mathsf{D}(B)\,.$$

Proof. The map $N \xrightarrow{\simeq} J$ induces the vertical arrows in the commutative diagram

$$\begin{array}{ccccc}
N & = & \operatorname{Hom}_B(S,N) & \xrightarrow[\operatorname{Hom}_B(\beta,N)]{\cong} & \operatorname{Hom}_B(B,N) \\
\vdots & & \downarrow & & \downarrow \simeq \\
\underline{J} & = & \operatorname{Hom}_B(S,J) & \xrightarrow[\operatorname{Hom}_B(\beta,J)]{\simeq} & \operatorname{Hom}_B(B,J)
\end{array}$$

Note that B acts on N through β, which is surjective, so $\operatorname{Hom}_B(\beta,N)$ is bijective. The map $\operatorname{Hom}_B(\beta,J)$ is a quasi-isomorphism because β is one and J is semiinjective. By 2.2.2(2), $\underline{J}$ is semiinjective, so $N \to \underline{J}$ is a semiinjective resolution over S. In the following chain of morphisms of DG B-modules the isomorphisms are adjunctions:

$$\begin{aligned}
\operatorname{Hom}_S(\operatorname{Hom}_S(M,\operatorname{Hom}_P(S,I)),\underline{J}) &\cong \operatorname{Hom}_S(\operatorname{Hom}_P(M,I),\underline{J}) \\
&= \operatorname{Hom}_S(\operatorname{Hom}_P(M,I),\operatorname{Hom}_B(S,J)) \\
&\cong \operatorname{Hom}_B(S\otimes_S \operatorname{Hom}_P(M,I),J) \\
&\cong \operatorname{Hom}_B(\operatorname{Hom}_P(M,I),J) \\
&\simeq \operatorname{Hom}_B(\operatorname{Hom}_P(X',I),J) \\
&\simeq \operatorname{Hom}_B(\operatorname{Hom}_P(X,I),J) \\
&\simeq \operatorname{Hom}_B(\operatorname{Hom}_P(X,L),J) \\
&\simeq \operatorname{Hom}_B(G,J)
\end{aligned}$$

The quasi-isomorphisms are induced by $M \xleftarrow{\simeq} X' \xrightarrow{\simeq} X$, $L \xrightarrow{\simeq} I$, and $G \xrightarrow{\simeq} X$, because I is semiinjective over P, J is semiinjective over B, and X is semiprojective over P. The chain yields the desired isomorphism in $\mathsf{D}(B)$ as $\underline{J}$ is semiinjective over S, G is semiprojective over B, and Step 3 gives $\operatorname{Hom}_P(S,I) \simeq D^\sigma$. ☐

Notation. Let $F \xrightarrow{\simeq} B$ be a semiprojective resolution over C.

Step 5. There exists an isomorphism

$$\mathsf{R}\mathrm{Hom}_B(\mathsf{R}\mathrm{Hom}_P(M,L),N) \simeq \mathsf{R}\mathrm{Hom}_C(B,\mathsf{R}\mathrm{Hom}_P(\mathsf{R}\mathrm{Hom}_P(M,L),N)) \quad \text{in } \mathsf{D}(C)\,.$$

Proof. The DG $\bar{C}$-module $\bar{C}\otimes_C F$ is semiprojective by 2.2.2(1). The map $F \xrightarrow{\simeq} B$ induces the vertical arrows in the commutative diagram of DG C-modules

$$\begin{array}{ccccc}
F & \xrightarrow{\cong} & C\otimes_C F & \xrightarrow[\gamma\otimes_C F]{\simeq} & \bar{C}\otimes_C F \\
\simeq\downarrow & & \simeq\downarrow & & \downarrow \\
B & \xrightarrow{\cong} & C\otimes_C B & \xrightarrow[\gamma\otimes_C B]{\cong} & \bar{C}\otimes_C B
\end{array}$$

where $\gamma\otimes_C B$ is an isomorphism because γ is surjective and C acts on B through γ, and $\gamma\otimes_C F$ is a quasi-isomorphism because γ is one and F is semiprojective.

The resulting quasi-isomorphism $\bar{C}\otimes_C F \xrightarrow{\simeq} B$ induces the quasi-isomorphism in the following chain, because $\operatorname{Hom}_P(G,J)$ is semiinjective over $\bar{C}$ by 2.2.2(2):

$$\begin{aligned}
\operatorname{Hom}_B(G,J) &\cong \operatorname{Hom}_{\bar{C}}(B,\operatorname{Hom}_P(G,J)) \\
&\simeq \operatorname{Hom}_{\bar{C}}(\bar{C}\otimes_C F,\operatorname{Hom}_P(G,J)) \\
&\cong \operatorname{Hom}_C(F,\operatorname{Hom}_P(G,J))\,.
\end{aligned}$$

The first isomorphism reflects the action of $\bar{C} = B \otimes_P B$ on $\operatorname{Hom}_P(G, J)$, the second one holds by adjunction. The chain represents the desired isomorphism because $\operatorname{Hom}_P(G, J)$ is semiinjective over C; see 4.4. □

Notation. Let $Y \xrightarrow{\simeq} N$ be a semiprojective resolution over B.

Step 6. There exists an isomorphism

$$\mathsf{RHom}_P(\mathsf{RHom}_P(M, L), N) \simeq \mathsf{RHom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) \otimes^{\mathsf{L}}_{P^{\mathsf{e}}} (M \otimes^{\mathsf{L}}_K N) \quad \text{in } \mathsf{D}(B^{\mathsf{e}}).$$

Proof. From $G \xrightarrow{\simeq} \operatorname{Hom}_P(X, L)$ one gets the first link in the chain

$$\begin{aligned}
\operatorname{Hom}_P(G, J) &\simeq \operatorname{Hom}_P(\operatorname{Hom}_P(X, L), J)\\
&\cong \operatorname{Hom}_P(L, P) \otimes_P X \otimes_P J\\
&= H \otimes_P X \otimes_P J\\
&\simeq H \otimes_P X \otimes_P Y\\
&\cong H \otimes_{P^{\mathsf{e}}} (X \otimes_K Y)\\
&\simeq \operatorname{Hom}_{P^{\mathsf{e}}}(P, U) \otimes_{P^{\mathsf{e}}} (X \otimes_K Y)
\end{aligned}$$

of morphisms of DG $\bar{C}$-modules; it is a quasi-isomorphism because the semiinjective DG B-module J is semiinjective over P, see 2.2.2(4).

The equality reflects the definition of L.

The composition $Y \xrightarrow{\simeq} N \xrightarrow{\simeq} J$ induces the third link; which is a quasi-isomorphism because H and X are semiprojective over P.

The second isomorphism holds by associativity of tensor products; see 2.1.2.

The quasi-isomorphism $H \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, P^{\mathsf{e}})$ from Step 2 induces the last link, which is a quasi-isomorphism because $X \otimes_K Y$ is semiflat over P^{e}.

Finally, the semiinjectivity of $\bar{J}$ and the semiflatness of $X \otimes_K Y$ imply that the chain above represents the desired isomorphism in $\mathsf{D}(B^{\mathsf{e}})$. □

Step 7. There exists an isomorphism

$$\mathsf{RHom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) \otimes^{\mathsf{L}}_{P^{\mathsf{e}}} (M \otimes^{\mathsf{L}}_K N) \simeq \mathsf{RHom}_{P^{\mathsf{e}}}(P, M \otimes^{\mathsf{L}}_K N) \quad \text{in } \mathsf{D}(B^{\mathsf{e}}).$$

Proof. The resolutions $A \xrightarrow{\simeq} P \xrightarrow{\simeq} U$ over P^{e} induce quasi-isomorphisms

$$\operatorname{Hom}_{P^{\mathsf{e}}}(P, U) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, U) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, P^{\mathsf{e}})$$

of complexes of P^{e}-modules, which in turn induce a quasi-isomorphism

$$\operatorname{Hom}_{P^{\mathsf{e}}}(P, U) \otimes_{P^{\mathsf{e}}} (X \otimes_K Y) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, P^{\mathsf{e}}) \otimes_{P^{\mathsf{e}}} (X \otimes_K Y)$$

of DG B^{e}-modules. To wrap things up, we use the canonical evaluation morphism

$$\operatorname{Hom}_{P^{\mathsf{e}}}(A, P^{\mathsf{e}}) \otimes_{P^{\mathsf{e}}} (X \otimes_K Y) \to \operatorname{Hom}_{P^{\mathsf{e}}}(A, X \otimes_K Y)$$

given by $\lambda \otimes x \otimes y \mapsto \big(a \mapsto (-1)^{(|x|+|y|)|a|}\lambda(a)(x \otimes y)\big)$; it is bijective, because the DG algebra A is a bounded complex of finite projective P^{e}-modules. □

Notation. Let $X \otimes_K Y \xrightarrow{\simeq} V$ be a semiinjective resolution over B^{e}.

Step 8. There exists an isomorphism

$$\mathsf{RHom}_C(B, \mathsf{RHom}_{P^{\mathsf{e}}}(P, M \otimes^{\mathsf{L}}_K N)) \simeq \mathsf{RHom}_{B^{\mathsf{e}}}(S, M \otimes^{\mathsf{L}}_K N) \quad \text{in } \mathsf{D}(B^{\mathsf{e}}).$$

Proof. The isomorphisms below come from adjunction formulas, see (2.1.1):

$$\begin{aligned}\operatorname{Hom}_C(F, \operatorname{Hom}_{P^{\mathsf{e}}}(A, X \otimes_K Y)) &\cong \operatorname{Hom}_{B^{\mathsf{e}}}(F \otimes_A A, X \otimes_K Y)\\ &\cong \operatorname{Hom}_{B^{\mathsf{e}}}(F, X \otimes_K Y)\\ &\simeq \operatorname{Hom}_{B^{\mathsf{e}}}(F, V)\\ &\simeq \operatorname{Hom}_{B^{\mathsf{e}}}(S, V)\end{aligned}$$

The quasi-isomorphisms are induced by $X \otimes_K Y \simeq V$ and $F \simeq S$, respectively, because F is semiprojective over B^{e} and V is semiinjective over B^{e}. □

Step 9. The composed morphism of the chain of isomorphisms

$$\begin{aligned}\mathsf{R}\mathrm{Hom}_S(\mathsf{R}\mathrm{Hom}_S(M, D^{\sigma}), N) &\simeq \mathsf{R}\mathrm{Hom}_B(\mathsf{R}\mathrm{Hom}_P(M, L), N)\\ &\simeq \mathsf{R}\mathrm{Hom}_C(B, \mathsf{R}\mathrm{Hom}_P(\mathsf{R}\mathrm{Hom}_P(M, L), N))\\ &\simeq \mathsf{R}\mathrm{Hom}_C(B, \mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}) \otimes^{\mathsf{L}}_{P^{\mathsf{e}}} (M \otimes^{\mathsf{L}}_K N))\\ &\simeq \mathsf{R}\mathrm{Hom}_C(B, \mathsf{R}\mathrm{Hom}_{P^{\mathsf{e}}}(P, M \otimes^{\mathsf{L}}_K N))\\ &\simeq \mathsf{R}\mathrm{Hom}_{B^{\mathsf{e}}}(S, M \otimes^{\mathsf{L}}_K N)\\ &\simeq \mathsf{R}\mathrm{Hom}_{S \otimes^{\mathsf{L}}_K S}(S, M \otimes^{\mathsf{L}}_K N)\end{aligned}$$

provided by Steps 4 through 8 and Theorem 3.2, defines an isomorphism in $\mathsf{D}(S)$.

Proof. The diagram of DG algebras in Step 1 provides a morphism from B^{e} to every DG algebra appearing in the chain of canonical isomorphisms above. Thus, each isomorphism in the chain above defines a unique isomorphism in $\mathsf{D}(B^{\mathsf{e}})$. Its source and target are complexes of S-modules, on which B^{e} acts through the composed morphism of DG algebras $B^{\mathsf{e}} \to B \to S$. This map is equal to the composition $B^{\mathsf{e}} \to S^{\mathsf{e}} \to S$. Therefore, Lemma 2.3.3, applied first to the quasi-isomorphism $B^{\mathsf{e}} \to S^{\mathsf{e}}$, then to the homomorphisms $S \to S^{\mathsf{e}} \to S$ given by $s \mapsto s \otimes 1$ and $s \otimes s' \mapsto ss'$, shows that the complexes above are also isomorphic in $\mathsf{D}(S)$. □

Step 10. The morphism in Step 9 is natural with respect to M and N.

Proof. The morphism in question is represented by a composition of quasi-isomorphisms of DG modules over B^{e}, so it suffices to verify that each such quasi-isomorphism represents a natural morphism in $\mathsf{D}(B^{\mathsf{e}})$.

Three kinds of quasi-isomorphisms are used. The one chosen in Step 2 involves neither M nor N, and so works simultaneously for all complexes of S-modules; thus, no issues of naturality arises there. Some of the constituent quasi-isomorphisms themselves are natural isomorphisms, such as Hom-tensor adjunction or associativity of tensor products. Finally, there are quasi-isomorphisms of functors induced replacing some DG module with a semiprojective or a semiinjective resolution. The induced morphism of derived functors are natural, because morphisms of DG modules define unique up to homotopy morphisms of their resolutions; see 2.3.1. □

The isomorphism (4.1.1) and its properties have now been established.
Theorem 1.2(1) shows that formula (4.1.2) is equivalent to (4.1.1). □

The next result is an analog of Theorem 4.1 for the derived Hochschild functor from Remark 3.10; it can be proved along the same lines, so the argument is omitted.

Theorem 4.6. *If* $\mathrm{fd}_K S$ *is finite, then in* $\mathsf{D}(S)$ *there are isomorphisms*

$$S \otimes^{\mathsf{L}}_{S\otimes^{\mathsf{L}}_K S} \mathsf{R}\mathrm{Hom}_K(M,N) \simeq \mathsf{R}\mathrm{Hom}_S(M,D^\sigma)\otimes^{\mathsf{L}}_S N\,, \tag{4.6.1}$$

$$S \otimes^{\mathsf{L}}_{S\otimes^{\mathsf{L}}_K S} \mathsf{R}\mathrm{Hom}_K(\mathsf{R}\mathrm{Hom}_S(M,D^\sigma),N) \simeq M\otimes^{\mathsf{L}}_S N\,, \tag{4.6.2}$$

for all $M\in\mathsf{P}(\sigma)$ *and* $N\in\mathsf{D}(S)$*; this morphism is natural in* M *and* N*.* $\square$

Setting $M = S = N$ in (4.6.1) produces a remarkable expression for D^σ:

Corollary 4.7. *In* $\mathsf{D}(S)$ *there is an isomorphism*

$$D^\sigma \simeq S \otimes^{\mathsf{L}}_{S\otimes^{\mathsf{L}}_K S} \mathsf{R}\mathrm{Hom}_K(S,S)\,. \qquad \square$$

Remark 4.8. The right hand sides of (4.1.2) and (3.11.1) coincide, so one might wonder whether the induced isomorphism of the derived Hochschild functors on the right hand side might be induced by an isomorphism of their coefficients:

$$\mathsf{R}\mathrm{Hom}_S(M,D^\sigma)\otimes^{\mathsf{L}}_K N \simeq \mathsf{R}\mathrm{Hom}_K(M,N)\,.$$

To prove that no such isomorphism exists in general, it suffices to consider the case when K is a field, $S = K[x]$ a polynomial ring over K, and $M = S = N$. The factorization $K \to K[x] = K[x]$ gives $D^\sigma \simeq K[x]$, hence the left-hand side is isomorphic to $K[x]\otimes_K K[x]$. On the other hand, the S^{e}-module $\mathsf{R}\mathrm{Hom}_K(K[x],K[x])$ on the right hand side has an uncountable basis as a K-vector space.

5. Global duality

We now reconsider a portion of the preceding results from a global point of view. The facts needed from Grothendieck duality theory for schemes are summarized in this section, and the globalized results given in the next.

While it is not difficult to show that the complexes and functors we will deal with specialize over affine schemes to sheafifications of similar things that have appeared earlier, the corresponding statement for functorial maps between such objects is not so easy to establish, and we will not be settling this issue here. Indeed, giving concrete descriptions of abstractly characterized functorial maps is one of the major problems of duality theory.

Schemes are assumed throughout to be noetherian.

A scheme-map $f\colon X\to Y$ is *essentially of finite type* if every $y\in Y$ has an affine open neighborhood $V = \mathrm{Spec}(A)$ such that $f^{-1}V$ can be covered by finitely many affine open $U_i = \mathrm{Spec}(C_i)$ such that the corresponding ring homomorphisms $A\to C_i$ are essentially of finite type.

If, moreover, each C_i is a localization of A (that is, a ring of fractions) and $A\to C_i$ is the canonical map, then we say that f is *localizing.*

The property "essentially finite-type" behaves well with respect to composition and base change: if $f\colon X\to Y$ and $g\colon Y\to Z$ are scheme-maps, and if both f and g are essentially of finite type, then so is the composition gf; if gf and g are essentially of finite type then so is f; and if $Y'\to Y$ is any scheme-map then $X' := Y'\times_Y X$ is noetherian, and the projection $X'\to Y'$ is essentially of finite type.

Similar statements hold with "localizing" in place of "essentially finite-type."

If the scheme-map f is localizing and also injective (as a set-map) then we say that f is a *localizing immersion.*

A scheme-map is *essentially smooth*, resp. *essentially étale*, if it is essentially of finite type and formally smooth, resp. formally étale [14, §17.1].

For example, any localizing map is essentially étale: this assertion, being local (see [14, (17.1.6)]), results from [14, (17.1.2)] and [13, (19.10.3)(ii)].)

Remark 5.1. In several places we will refer to proofs in [17] which make use of the fact that the diagonal of a smooth map is a quasi-regular immersion. To ensure that those proofs apply here, note that the same property for essentially smooth maps is given by [14, 16.10.2, 16.9.4].

In [24, 4.1], extending a compactification theorem of Nagata, it is shown that *any essentially-finite-type separated map f of noetherian schemes factors as $f = \bar{f}u$ with $\bar{f}$ proper and u a localizing immersion.*

Example 5.2. (Local compactification.) A map $f\colon X = \operatorname{Spec} S \to \operatorname{Spec} K = Y$ coming from an essentially finite-type homomorphism of noetherian rings $K \to S$ factors as

$$X \xrightarrow{j} Z \overset{i}{\hookrightarrow} \bar{Z} \xrightarrow{\pi} Y,$$

where Z is the Spec of a finitely-generated K-algebra T of which S is a localization, j being the corresponding map, where i is an open immersion, and where π is a projective map, so that π is proper and ij is a localizing immersion.

In the rest of this section we review basic facts about Grothendieck duality, referring to [19] and [23] for details.

Henceforth all scheme-maps are assumed to be essentially of finite type, and separated.

For a scheme X, $\mathsf{D}(X)$ is the derived category of the category of $\mathcal{O}_X$-modules, $\mathsf{D_c}(X) \subset \mathsf{D}(X)$ (resp. $\mathsf{D_{qc}}(X) \subset \mathsf{D}(X)$) is the full subcategory whose objects are the $\mathcal{O}_X$-complexes with coherent (resp. quasi-coherent) homology modules, and $\mathsf{D}^+_\bullet$ (resp. $\mathsf{D}^-_\bullet$) is the full subcategory of $\mathsf{D}_\bullet$ whose objects are the complexes $E \in \mathsf{D}_\bullet$ with $H^n(E) := H_{-n}(E) = 0$ for all $n \ll 0$ (resp. $n \gg 0$).

5.3. To any scheme-map $f\colon X \to Y$ one associates the right-derived direct-image functor $\mathsf{R}f_*\colon \mathsf{D_{qc}}(X) \to \mathsf{D_{qc}}(Y)$ and its *left adjoint,* the left-derived inverse-image functor $\mathsf{L}f^*\colon \mathsf{D_{qc}}(Y) \to \mathsf{D_{qc}}(X)$ [19, 3.2.2, 3.9.1, 3.9.2].

These functors interact with the left-derived tensor product $\otimes^{\mathsf{L}}$ via a natural isomorphism

$$\mathsf{L}f^*(E \otimes^{\mathsf{L}}_Y F) \xrightarrow{\ \sim\ } \mathsf{L}f^*E \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \qquad \big(E, F \in \mathsf{D}(Y)\big), \tag{5.3.1}$$

see [19, 3.2.4]; via the functorial map

$$\mathsf{R}f_*G \otimes^{\mathsf{L}}_Y \mathsf{R}f_*H \to \mathsf{R}f_*(G \otimes^{\mathsf{L}}_X H) \qquad \big(G, H \in \mathsf{D}(X)\big) \tag{5.3.2}$$

adjoint to the natural composite map

$$\mathsf{L}f^*(\mathsf{R}f_*G \otimes^{\mathsf{L}}_Y \mathsf{R}f_*H) \underset{(5.3.1)}{\xrightarrow{\ \sim\ }} \mathsf{L}f^*\mathsf{R}f_*G \otimes^{\mathsf{L}}_X \mathsf{L}f^*\mathsf{R}f_*H \longrightarrow G \otimes^{\mathsf{L}}_X H;$$

and via the *projection isomorphism*

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y G \xrightarrow{\ \sim\ } \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) \qquad \big(F \in \mathsf{D_{qc}}(X),\ G \in \mathsf{D_{qc}}(Y)\big), \tag{5.3.3}$$

defined qua map to be the natural composition

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y G \longrightarrow \mathsf{R}f_*F \otimes^{\mathsf{L}}_Y \mathsf{R}f_*\mathsf{L}f^*G \underset{(5.3.2)}{\longrightarrow} \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G).$$

see [19, 3.9.4].

5.4. Interactions with the derived (sheaf-)homomorphism functor $\mathsf{R}\mathcal{H}om$ occur via natural bifunctorial maps

$$\mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(E,F) \to \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*E, \mathsf{L}f^*F) \qquad \big(E,F \in \mathsf{D}(Y)\big), \tag{5.4.1}$$

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(E,F) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*E, \mathsf{R}f_*F) \qquad \big(E,F \in \mathsf{D}(X)\big), \tag{5.4.2}$$

the former corresponding via (5.5.1) below to the composite map

$$\mathsf{L}f^*\mathsf{R}\mathcal{H}om_X(E,F)\otimes^{\mathsf{L}}_X \mathsf{L}f^*E \underset{(5.3.1)^{-1}}{\xrightarrow{\;\sim\;}} \mathsf{L}f^*\big(\mathsf{R}\mathcal{H}om_X(E,F)\otimes^{\mathsf{L}}_Y E\big) \xrightarrow{\mathsf{L}f^*\varepsilon} \mathsf{L}f^*F,$$

with ε corresponding via (5.5.1) to the identity map of $\mathsf{R}\mathcal{H}om_Y(E,F)$; and the latter corresponding to the composite map

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(E,F)\otimes^{\mathsf{L}}_Y \mathsf{R}f_*E \underset{(5.3.2)}{\longrightarrow} \mathsf{R}f_*\big(\mathsf{R}\mathcal{H}om_X(E,F)\otimes^{\mathsf{L}}_X E\big) \xrightarrow{\mathsf{R}f_*\varepsilon} \mathsf{R}f_*F.$$

The map (5.4.1) is an *isomorphism* if f is an open immersion, or if $E \in \mathsf{D}^-_{\mathsf{c}}(Y)$, $F \in \mathsf{D}^+_{\mathsf{qc}}(Y)$ and f has finite flat dimension [19, 4.6.7].

5.5. The fundamental adjunction relation between the derived tensor and derived homomorphism functors is expressed by the standard trifunctorial isomorphism

$$\mathsf{R}\mathcal{H}om_X\big(A\otimes^{\mathsf{L}}_X B, C\big) \xrightarrow{\;\sim\;} \mathsf{R}\mathcal{H}om_X\big(A, \mathsf{R}\mathcal{H}om_X(B,C)\big) \qquad \big(A,B,C \in \mathsf{D}(X)\big),$$

see e.g., [19, §2.6]. Application of the composite functor $\mathrm{H}^0\mathsf{R}\Gamma(X,-)$ to this isomorphism produces a canonical isomorphism

$$\mathrm{Hom}_{\mathsf{D}(X)}\big(A\otimes^{\mathsf{L}}_X B, C\big) \xrightarrow{\;\sim\;} \mathrm{Hom}_{\mathsf{D}(X)}\big(A, \mathsf{R}\mathcal{H}om_X(B,C)\big) \qquad \big(A,B,C \in \mathsf{D}(X)\big). \tag{5.5.1}$$

From among the many resulting maps, we will need the functorial one

$$\mathsf{R}\mathcal{H}om_X(M,E)\otimes^{\mathsf{L}}_X F \longrightarrow \mathsf{R}\mathcal{H}om_X(M, E\otimes^{\mathsf{L}}_X F) \qquad \big(M,E,F \in \mathsf{D}(X)\big), \tag{5.5.2}$$

corresponding via (5.5.1) to the natural composite map (with ε as above):

$$(\mathsf{R}\mathcal{H}om_X(M,E)\otimes^{\mathsf{L}}_X F)\otimes^{\mathsf{L}}_X M \xrightarrow{\;\sim\;} (\mathsf{R}\mathcal{H}om_X(M,E)\otimes^{\mathsf{L}}_X M)\otimes^{\mathsf{L}}_X F \xrightarrow{\varepsilon\otimes^{\mathsf{L}}_X 1} E\otimes^{\mathsf{L}}_X F.$$

The map (5.5.2) is an *isomorphism* if the complex M is *perfect* (see §6). Indeed, the question is local on X, so one can assume that M is a bounded complex of finite-rank free $\mathcal{O}_X$-modules. The assertion is then given by a simple induction—similar to the one in the second-last paragraph in the proof of [19, 4.6.7]—on the number of degrees in which M doesn't vanish.

Similarly, the map (5.5.2) is an isomorphism if F is perfect.

5.6. For any commutative square of scheme-maps

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & \Xi & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

one has the map $\theta_\Xi\colon \mathsf{L}u^*\mathsf{R}f_* \to \mathsf{R}g_*\mathsf{L}v^*$ adjoint to the natural composite map

$$\mathsf{R}f_* \longrightarrow \mathsf{R}f_*\mathsf{R}v_*\mathsf{L}v^* \xrightarrow{\;\sim\;} \mathsf{R}u_*\mathsf{R}g_*\mathsf{L}v^*.$$

When Ξ is a *fiber square* (which means that the map associated to Ξ is an isomorphism $X' \xrightarrow{\sim} X \times_Y Y'$), and u is *flat*, then θ_Ξ is an *isomorphism.* In fact, for any fiber square Ξ, θ_Ξ *is an isomorphism* $\iff$ Ξ *is tor-independent* [19, 3.10.3].

5.7. Duality theory focuses on the *twisted inverse-image pseudofunctor*

$$f^!\colon \mathsf{D}^+_{\mathsf{qc}}(Y) \to \mathsf{D}^+_{\mathsf{qc}}(X),$$

where "pseudofunctoriality" (also known as "2-functoriality") entails, in addition to functoriality, a family of functorial isomorphisms $c_{f,g}\colon (gf)^! \xrightarrow{\sim} f^!g^!$, one for each composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$, satisfying a natural "associativity" property vis-à-vis any composable triple, see, e.g., [19, 3.6.5].

This pseudofunctor is uniquely determined up to isomorphism by the following three properties:

(i) If f is essentially étale then $f^!$ is the usual restriction functor f^*.

(ii) If f is proper then $f^!$ is right-adjoint to $\mathsf{R}f_*$ (which takes $\mathsf{D}^+_{\mathsf{qc}}(X)$ into $\mathsf{D}^+_{\mathsf{qc}}(Y)$ [19, (3.9.2)]).

(iii) Suppose there is given a fiber square Ξ as above, with f (hence g) proper and u (hence v) essentially étale. Then the functorial *base-change map*

$$\beta_\Xi(F)\colon\ : v^*f^!F \to g^!u^*F \qquad \big(F \in \mathsf{D}^+_{\mathsf{qc}}(Y)\big), \tag{5.7.1}$$

defined to be adjoint to the natural composition

$$\mathsf{R}g_*v^*f^!F \underset{\theta_\Xi^{-1}}{\xrightarrow{\sim}} u^*\mathsf{R}f_*f^!F \longrightarrow u^*F,$$

is identical with the natural composite *isomorphism*

$$v^*f^!F = v^!f^!F \xrightarrow{\sim} (fv)^!F = (ug)^!F \xrightarrow{\sim} g^!u^!F = g^!u^*F.$$

For the existence of such a pseudofunctor, see [24, §5.2].

Remarks 5.8. (a) If f has finite flat dimension (in addition to being proper), then (5.7.1) is an *isomorphism* for *all* $F \in \mathsf{D}_{\mathsf{qc}}(Y)$—see [19, 4.7.4] and [20, 1.2].

(b) Theorem 5.3 in [24] (as elaborated in [23, 7.1.6]) states that, moreover, one can associate, in an essentially unique way, to *any* fiber square Ξ with u (hence v) flat, a functorial isomorphism β_Ξ, agreeing with (5.7.1) when f is proper, and with the natural isomorphism $v^*f^* \xrightarrow{\sim} g^*u^*$ when f is essentially étale.

(c) Let $f\colon X \to Y$ be essentially smooth, so that by [14, 16.10.2] the relative differential sheaf Ω_f is locally free over $\mathcal{O}_X$. On any connected component W of X, the rank of Ω_f is a constant, denoted $d(W)$. There is a *functorial isomorphism*

$$f^!F \xrightarrow{\sim} \Sigma^d {\textstyle\bigwedge}^d_{\mathcal{O}_X}(\Omega_f) \otimes_{\mathcal{O}_X} f^*F \qquad \big(F \in \mathsf{D}_{\mathsf{qc}}(Y)\big), \tag{5.8.1}$$

with $\Sigma^d {\textstyle\bigwedge}^d_{\mathcal{O}_X}(\Omega_f)$ the complex whose restriction to any W is $\Sigma^{d(W)} {\textstyle\bigwedge}^{d(W)}_{\mathcal{O}_W}(\Omega_f\big|_W)$.

To prove this, one may assume that X itself is connected, and set $d := d(X)$. Noting that the diagonal $\Delta\colon X \to X \times_Y X$ is defined locally by a regular sequence of length d [14, 16.9.4], so that $\Delta^!\mathcal{O}_{X\times_Y X} \otimes^{\mathsf{L}} \mathsf{L}\Delta^*G \cong \Delta^!G$ for all $G \in \mathsf{D}_{\mathsf{qc}}(X \times_Y X)$ [15, p. 180, 7.3], one can imitate the proof of [31, p. 397, Theorem 3], where, in view of (b) above, one can drop the properness condition and take $U = X$, and where finiteness of Krull dimension is superfluous.

In this connection, see also 5.10 below, and [11, §2.2].

5.9. The fact that $\beta_{\Xi}(F)$ in (5.7.1) is an isomorphism for all $F \in \mathsf{D}^+_{\mathsf{qc}}(Y)$ whenever u is an open immersion and f is proper, is shown in [19, §4.6, part V] to be equivalent to *sheafified duality,* which is that *for any proper* $f\colon X \to Y$, *and any* $E \in \mathsf{D}_{\mathsf{qc}}(X)$, $F \in \mathsf{D}^+_{\mathsf{qc}}(Y)$, *the natural composition, in which the first map comes from* 5.4.2,

$$\mathsf{R}f_*\mathcal{H}om_X(E,\, f^!F) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*E,\, \mathsf{R}f_*f^!F) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*E, F), \tag{5.9.1}$$

is an isomorphism.

Moreover, if the proper map f has finite flat dimension, then sheafified duality holds for *all* $F \in \mathsf{D}_{\mathsf{qc}}(Y)$, see [19, 4.7.4].

If f is a *finite* map, the isomorphism (5.9.1) with $E = \mathcal{O}_X$ determines the functor $f^!$ up to isomorphism. (See [11, §2.2].) In the affine case, for example, if $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ corresponds to a finite ring homomorphism $A \to B$, and $^\sim$ denotes sheafification, then for an A-complex M, the B-complex $f^!(M^\sim)$ can be defined by the equality

$$f^!(M^\sim) = \mathsf{R}\mathrm{Hom}_A(B, M)^\sim. \tag{5.9.2}$$

5.10. ($f^!$ and $\otimes^{\mathsf{L}}$). For any $f = \bar f u$ with $\bar f$ proper and u localizing, and E, $F \in \mathsf{D}^+_{\mathsf{qc}}(Y)$ such that $E \otimes^{\mathsf{L}}_Y F \in \mathsf{D}^+_{\mathsf{qc}}(Y)$ (e.g., E perfect, see §6), there is a canonical functorial map

$$f^!E \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \to f^!(E \otimes^{\mathsf{L}}_Y F) \tag{5.10.1}$$

equal, when u=1, to the map χ^f adjoint to the natural composite map

$$\mathsf{R}f_*(f^!E \otimes^{\mathsf{L}}_X \mathsf{L}f^*F) \xrightarrow{\ \sim\ } \mathsf{R}f_*f^!E \otimes^{\mathsf{L}}_Y F \longrightarrow E \otimes^{\mathsf{L}}_Y F,$$

(see (5.3.3)), and equal, in the general case, to the natural composition

$$\begin{aligned} f^!E \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \cong u^*\bar f^!E \otimes^{\mathsf{L}}_X u^*\mathsf{L}\bar f^*F &\cong u^*(\bar f^!E \otimes^{\mathsf{L}}_X \mathsf{L}\bar f^*F) \\ &\xrightarrow{u^*\chi^{\bar f}} u^*\bar f^!(E \otimes^{\mathsf{L}}_Y F) \cong f^!(E \otimes^{\mathsf{L}}_Y F). \end{aligned} \tag{5.10.2}$$

"Canonicity" means (5.10.2) depends only on f, not on the factorization $f = \bar f u$. This is shown by imitation of the proof of [19, 4.9.2.2], after one notes that for any composition $X \xrightarrow{i} X' \xrightarrow{v} Y'$ with i a closed immersion and v localizing, the induced map from X to its schematic image in Y' is localizing: the question being local, this just means that for a multiplicative system M in a ring B, and a B_M-ideal J with inverse image I in B, the natural map $(B/I)_M \to B_M/J$ is bijective. (See also [24, 5.8].)

5.10.3. By [24, Theorem 5.9], the map (5.10.1) is an *isomorphism* if f has finite flat dimension and $E = \mathcal{O}_Y$—hence more generally if E is *perfect,* cf. end of §5.5. In particular, for any $g\colon Y \to Z$ there is a natural isomorphism

$$(gf)^!\mathcal{O}_Z \cong f^!g^!\mathcal{O}_Z \xrightarrow{\ \sim\ } f^!\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}f^*g^!\mathcal{O}_Z.$$

In combination with 5.8(c), and (6.2.1) below, this appears to be a globalization of [6, Theorem 8.6]. But it is by no means clear (nor will we address the point further here) that for maps of affine schemes the present isomorphism agrees with the sheafification of the one in *loc. cit.*

5.11. ($f^!$ and $\mathsf{R}\mathcal{H}om$). Let $f\colon X \to Y$ be a scheme-map, $E \in \mathsf{D}^-_{\mathsf{c}}(Y)$, $F \in \mathsf{D}^+_{\mathsf{qc}}(Y)$. *There is a canonical isomorphism*

$$f^!\mathsf{R}\mathcal{H}om_Y(E, F) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*E,\, f^!F). \tag{5.11.1}$$

Indeed, by [15, p. 92, 3.3], $\mathsf{R}\mathcal{H}om_Y(E,F)\in\mathsf{D}^+_{\mathsf{qc}}(Y)$, so $f^!\mathsf{R}\mathcal{H}om_Y(E,F)\in\mathsf{D}^+_{\mathsf{qc}}(X)$; and furthermore, $f^!F\in\mathsf{D}^+_{\mathsf{qc}}(X)$ and, by [15, p. 99, 44], $\mathsf{L}f^*E\in\mathsf{D}^-_{\mathsf{c}}(X)$, so that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*E,f^!F)\in\mathsf{D}^+_{\mathsf{qc}}(X)$. (Those proofs in [15] which are "left to the reader" use [15, p. 73, 7.3].) So when f is *proper* (the only case we'll need), the map (5.11.1) and its inverse come out of the following composite functorial isomorphism, for any $G\in\mathsf{D}^+_{\mathsf{qc}}(X)$—in particular, $G=f^!\mathsf{R}\mathcal{H}om_Y(E,F)$ or $G=\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*E,f^!F)$:

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{D}(X)}\big(G,f^!\mathsf{R}\mathcal{H}om_Y(E,F)\big) &\xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathsf{D}(Y)}\big(\mathsf{R}f_*G,\mathsf{R}\mathcal{H}om_Y(E,F)\big) && \text{by 5.7(ii)}\\
&\xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathsf{D}(Y)}\big(\mathsf{R}f_*G\otimes^{\mathsf{L}}_Y E,F)\big) && \text{by (5.5.1)}\\
&\xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathsf{D}(Y)}\big(\mathsf{R}f_*(G\otimes^{\mathsf{L}}_X \mathsf{L}f^*E),F)\big) && \text{by (5.3.3)}\\
&\xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathsf{D}(X)}\big(G\otimes^{\mathsf{L}}_X \mathsf{L}f^*E,f^!F)\big) && \text{by 5.7(ii)}\\
&\xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathsf{D}(X)}\big(G,\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*E,f^!F)\big) && \text{by (5.5.1)}
\end{aligned}
$$

(For the general case, one compactifies, and shows canonicity. . .)

6. Reduction of derived Hochschild functors over schemes

Terminology and assumptions remain as in the first part of Section 5. Again, *all schemes are assumed to be noetherian, and all scheme-maps to be essentially of finite type, and separated.*

An $\mathcal{O}_X$-complex M is *perfect* if X can be covered by open sets U such that the restriction $M|_U$ is $\mathsf{D}(U)$-isomorphic to a bounded complex of finite-rank locally free $\mathcal{O}_U$-modules. For a scheme-map $f\colon X\to Y$, with f_0 the map f considered only as a map of topological spaces, and f_0^{-1} the left adjoint of the direct image functor f_{0*} from sheaves of abelian groups on X to sheaves of abelian groups on Y, there is a standard way of making $f_0^{-1}\mathcal{O}_Y$ into a sheaf of commutative rings on X, whose stalk at any point $x\in X$ is $\mathcal{O}_{Y,f(x)}$. An $\mathcal{O}_X$-complex M is *f-perfect* if $M\in\mathsf{D}_{\mathsf{c}}(X)$ and M is isomorphic in the derived category of $f_0^{-1}\mathcal{O}_Y$-modules to a bounded complex of flat $f_0^{-1}\mathcal{O}_Y$-modules. Perfection is equivalent to id^X-perfection, with id^X the identity map of X [16, p. 135, 5.8.1].

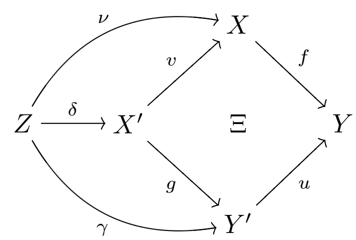
If f factors as $X\xrightarrow{i}Z\xrightarrow{g}Y$ with g essentially smooth and i a closed immersion, then M is f-perfect if and only if i_*M is (id^Z-)perfect: the proof of [17, pp. 252, 4.4] applies here (see Remark 5.1). Using [17, p. 242, 3.3], one sees that f-perfection is local on X: M is f-perfect if and only if every $x\in X$ has an open neighborhood U such that $M|_U$ is $f|_U$-perfect. Note that, f being a composite of essentially finite-type maps, and hence itself essentially of finite type, there is always such a U for which $f|U$ factors as (essentially smooth) $\circ$ (closed immersion).

Let $\mathsf{P}(f)$ be the full subcategory of $\mathsf{D}(X)$ whose objects are all the f-perfect complexes; and let $\mathsf{P}(X):=\mathsf{P}(\mathrm{id}^X)$ be the full subcategory of perfect $\mathcal{O}_X$-complexes.

If $f\colon X=\operatorname{Spec}S\to\operatorname{Spec}K=Y$ corresponds to a homomorphism of noetherian rings $\sigma\colon K\to S$, then $\mathsf{P}(f)$ is equivalent to the category $\mathsf{P}(\sigma)$ of §4: in view of the standard equivalence, given by sheafification, between coherent S-modules and coherent $\mathcal{O}_X$-modules, this follows from [17, p. 168, 2.2.2.1 and p. 242, 3.3].

The central result in this section is the following theorem.

Theorem 6.1. *Consider a commutative diagram of scheme-maps*



with δ proper, f of finite flat dimension, u flat, and Ξ a fiber square.

For $M \in \mathsf{P}(f)$, $E \in \mathsf{P}(Y)$ and $N \in \mathsf{D}^{+}_{\mathsf{qc}}(Y')$, the following assertions hold.

(i) $u^*E \otimes^{\mathsf{L}}_{Y'} N \in \mathsf{D}^{+}_{\mathsf{qc}}(Y')$.

(ii) $v^*\mathsf{R}\mathcal{H}om_X(M, f^!E) \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*N \in \mathsf{D}^{+}_{\mathsf{qc}}(X')$.

(iii) *There exist functorial isomorphisms*

$$\delta^!\big(v^*\mathsf{R}\mathcal{H}om_X(M, f^!E) \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*N\big) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Z\big(\mathsf{L}\nu^*M, \gamma^!(u^*E \otimes^{\mathsf{L}}_{Y'} N)\big).$$

(Note that v is flat, so that $v^* \cong \mathsf{L}v^*$; and similarly for u.)

Before presenting a proof, we derive global versions of some results established earlier for homomorphisms of rings.

Remark 6.2. If $\sigma\colon K \to S$ is a homomorphism of rings that is essentially of finite type and $g\colon V = \operatorname{Spec} S \to \operatorname{Spec} K = W$ is the corresponding scheme-map then, with $^\sim$ denoting sheafification—an equivalence of categories from $\mathsf{D}(S)$ to $\mathsf{D}_{\mathsf{qc}}(V)$, with quasi-inverse $\mathbf{R}\Gamma(V, -)$, see [9, 5.5]—there is an isomorphism in $\mathsf{D}(V)$:

(6.2.1) $$g^!\mathcal{O}_W \simeq (D^\sigma)^\sim.$$

To see this, factor σ as $K \to P := V^{-1}K[x_1, \dots, x_d] \twoheadrightarrow S$ (see (1.0.1)), so that, correspondingly, $g = g_1g_2$ with g_1 essentially smooth of relative dimension d and g_2 a closed immersion; then by (5.8.1), (5.9.2), and Theorem 1.1,

$$g^!\mathcal{O}_W \simeq g_2^!g_1^!\mathcal{O}_W \simeq \Sigma^d \operatorname{RHom}_P(S, \Omega^d_{P|K}) \simeq (D^\sigma)^\sim.$$

So the following assertion, for an arbitrary scheme-map $f\colon X \to Y$, globalizes Theorem 1.2(1)—and supports our calling any $\mathcal{O}_X$-complex isomorphic in $\mathsf{D}(X)$ to $f^!\mathcal{O}_Y$ a *relative dualizing complex for f*. Set

$$D_fM := \mathsf{R}\mathcal{H}om_X(M, f^!\mathcal{O}_Y) \qquad \big(M \in \mathsf{D}(X)\big).$$

Then *the contravariant functor D_f takes $\mathsf{P}(f)$ into itself, and for every $M \in \mathsf{P}(f)$ the canonical map is an isomorphism $M \xrightarrow{\sim} D_fD_fM$.*

Indeed, the proof of [17, p. 259, 4.9.2] (in whose first line (4.8) should be (4.9)) applies here, with "localizing immersion" in place of "open immersion," and with "essentially smooth" in place of "smooth," see Remark 5.1. (Actually, the assertion being local on both X and Y, for compactifiability of f one can use Example 5.2 rather than the compactification theorem [24, 4.1].)

For $E = \mathcal{O}_Y$ and D_fM in place of M, Theorem 6.1 and Remark 6.2 yield the next Corollary, which bears comparison—at least formally—with Verdier's "kernel theorem" [30, p. 44, Thm. 4.1]:

Corollary 6.3. *Under the assumptions of 6.1 there exists a natural isomorphism*

$$\delta^!(v^*M \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*N) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Z(\mathsf{L}\nu^*D_fM, \gamma^!N)\,. \qquad \square$$

Corollary 6.4. *Let $f\colon X \to Y$ be a flat scheme-map. Set $X' := X \times_Y X$, with canonical projections $X \xleftarrow{\pi_1} X' \xrightarrow{\pi_2} X$ and diagonal map $\delta\colon X \to X'$.*

There are natural isomorphisms, for $M \in \mathsf{P}(f)$, $E \in \mathsf{P}(Y)$ and $N \in \mathsf{D}^+_{\mathsf{qc}}(X)$:

$$\delta^!\big(\pi_1^*\mathsf{R}\mathcal{H}om_X(M, f^!E) \otimes^{\mathsf{L}}_{X'} \pi_2^*N\big) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X\big(M,\, f^*E \otimes^{\mathsf{L}}_X N\big).$$

Proof. The maps π_1 and π_2 are flat along with f. The assertion is just the special case of Theorem 6.1 corresponding to the data $Z := X$, $Y' := X$, $u := f$, $v := \pi_1$, and $g := \pi_2$—so that $\nu = \gamma = \mathrm{id}^X$. □

The first isomorphism in the next corollary is, for flat f, a globalization of Theorem 4.1 insofar as the objects involved are concerned. This is seen by using the description of $\delta^!$ given in 5.9 for the finite map δ, and the standard equivalence of $\mathsf{D}(S)$ and $\mathsf{D}_{\mathsf{qc}}\big(\operatorname{Spec} S\big)$ for a commutative ring S [9, 5.5]. We won't deal with the relation between the corresponding isomorphisms.

Corollary 6.5 (Global reduction formulae)**.** *With f and $\delta\colon X \to X'$ as in 6.4, there exist, for $M \in \mathsf{P}(f)$ and $N \in \mathsf{D}^+_{\mathsf{qc}}(X)$, natural isomorphisms*

$$\delta^!(\pi_1^*M \otimes^{\mathsf{L}}_{X'} \pi_2^*N) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(M, f^!\mathcal{O}_Y), N);$$

$$\delta^!\mathsf{R}\mathcal{H}om_{X'}(\pi_1^*M, \pi_2^*N) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(M \otimes^{\mathsf{L}}_X f^!\mathcal{O}_Y, N).$$

Proof. For the first isomorphism, apply 6.4 with $E = \mathcal{O}_Y$ and D_fM in place of M, and use the isomorphism $M \xrightarrow{\ \sim\ } D_fD_fM$ from Remark 6.2.

The second isomorphism is the composition

$$\begin{aligned}\delta^!\mathsf{R}\mathcal{H}om_{X'}(\pi_1^*M, \pi_2^*N) &\underset{a}{\xrightarrow{\ \sim\ }} \mathsf{R}\mathcal{H}om_X(\mathsf{L}\delta^*\pi_1^*M, \delta^!\pi_2^*N)\\ &\underset{b}{\xrightarrow{\ \sim\ }} \mathsf{R}\mathcal{H}om_X(M, \mathsf{R}\mathcal{H}om_X(f^!\mathcal{O}_Y, N))\\ &\underset{c}{\xrightarrow{\ \sim\ }} \mathsf{R}\mathcal{H}om_X(M \otimes^{\mathsf{L}}_X f^!\mathcal{O}_Y, N),\end{aligned}$$

where the isomorphism a comes from (5.11.1), b from the special case $M = \mathcal{O}_X$ of the first isomorphism in 6.5, and c from the first isomorphism in §5.5.

The following lemma contains the key ingredient for the proof of Theorem 6.1.

Lemma 6.6. *Let $g\colon X' \to Y'$ be a scheme-map of finite flat dimension. For all $M' \in \mathsf{P}(g)$, $E' \in \mathsf{P}(Y')$ and $F' \in \mathsf{D}^+_{\mathsf{qc}}(Y')$, the map from (5.5.2) is an isomorphism*

$$\psi\colon \mathsf{R}\mathcal{H}om_{X'}(M', g^!E') \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*F' \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_{X'}(M', g^!E' \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*F')\,. \tag{6.6.1}$$

Proof. Using the isomorphisms (5.3.1) and (for open immersions) (5.4.1), one checks that everything here commutes with restriction to open subsets on X', whence the question is local on both X' and Y' (see Remark 5.8(b).) Thus it may be assumed that both X' and Y' are affine and that g factors as $X' \xrightarrow{i} Z' \xrightarrow{h} Y'$ with i a closed immersion and h essentially smooth.

Since i_* preserves stalks of $\mathcal{O}_{X'}$-modules, therefore i_* is an exact functor, and furthermore, since D-maps are isomorphisms if they are so at the homology level, it will suffice to show that $i_*(\psi)$ $(= \mathsf{R}i_*(\psi))$ is an isomorphism in $\mathsf{D}(Z')$.

Before proceeding, note that $\mathsf{R}\mathcal{H}om_{X'}(M', i^!h^!E') \in \mathsf{D}^+_{\mathsf{qc}}(X')$. That's because $i_*M' \in \mathsf{D}^-_{\mathsf{c}}(Z')$, so the duality isomorphism (5.9.1) and [15, p. 92, 3.3] give

$$i_*\mathsf{R}\mathcal{H}om_{X'}(M', i^!h^!E') \cong \mathsf{R}\mathcal{H}om_{Z'}(i_*M', h^!E') \in \mathsf{D}^+_{\mathsf{qc}}(Z').$$

In fact, $\mathsf{R}\mathcal{H}om_{Z'}(i_*M', h^!E')$ is *perfect* because i_*M' and $h^!E'$ are both perfect (see (5.8.1)), [16, p. 130, 4.19.1] and [16, p. 148, 7.1]).

Recall from 5.10.3 that the map (5.5.2) is an isomorphism if the complex M is perfect; and that the map (5.10.1) is an isomorphism when f is flat and E is perfect.

Now, there is the sequence of natural isomorphisms:

$$
\begin{aligned}
&i_*\big(\mathsf{R}\mathcal{H}om_{X'}(M', g^!E')\otimes^{\mathsf{L}}_{X'}\mathsf{L}g^*F'\big)\\
&\qquad\xrightarrow{\ \sim\ } i_*\big(\mathsf{R}\mathcal{H}om_{X'}(M', i^!h^!E')\otimes^{\mathsf{L}}_{X'}\mathsf{L}i^*\mathsf{L}h^*F'\big)\\
&\qquad\xrightarrow{\ \sim\ } i_*\mathsf{R}\mathcal{H}om_{X'}(M', i^!h^!E')\otimes^{\mathsf{L}}_{Z'}\mathsf{L}h^*F' &&\text{by (5.3.3)}\\
&\qquad\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_{Z'}(i_*M', h^!E')\otimes^{\mathsf{L}}_{Z'}\mathsf{L}h^*F' &&\text{by (5.9.1)}\\
&\qquad\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_{Z'}\big(i_*M', h^!E'\otimes^{\mathsf{L}}_{Z'}\mathsf{L}h^*F'\big) &&\text{by (5.5.2)}\\
&\qquad\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_{Z'}\big(i_*M', h^!(E'\otimes^{\mathsf{L}}_{Y'}F')\big) &&\text{by (5.10.1)}\\
&\qquad\xrightarrow{\ \sim\ } i_*\mathsf{R}\mathcal{H}om_{X'}\big(M', i^!h^!(E'\otimes^{\mathsf{L}}_{Y'}F')\big) &&\text{by (5.9.1)}\\
&\qquad\xrightarrow{\ \sim\ } i_*\mathsf{R}\mathcal{H}om_{X'}\big(M', g^!(E'\otimes^{\mathsf{L}}_{Y'}F')\big)\\
&\qquad\xrightarrow{\ \sim\ } i_*\mathsf{R}\mathcal{H}om_{X'}(M', g^!E'\otimes^{\mathsf{L}}_{X'}\mathsf{L}g^*F') &&\text{by (5.10.1).}
\end{aligned}
$$

It can be shown that these isomorphisms compose to $i_*(\psi)$; but we avoid this somewhat lengthy verification and instead use a "way-out" argument. Fix M' and E'. Via the above sequence of isomorphisms, the source and target of $i_*(\psi)$, considered as functors in F', are isomorphic to the functor $\Upsilon\colon \mathsf{D}^+_{\mathsf{qc}}(Y')\to\mathsf{D}^+_{\mathsf{qc}}(Z')$ given by $\Upsilon(F')=\mathsf{R}\mathcal{H}om_{Z'}(i_*M', h^!E')\otimes^{\mathsf{L}}_{Z'}\mathsf{L}h^*F'$. Since $\mathsf{R}\mathcal{H}om_{Z'}(i_*M', h^!E')$ is perfect and h is flat, it follows that Υ is a *bounded functor* [19, (1.11.1)], whence the same is true of the source and target of $i_*\psi$.

Furthermore, one checks that ψ (and hence $i_*\psi$) is a morphism of Δ-functors (see [19, §1.5]). By [15, p. 69, (iii)], it suffices therefore to prove that *$i_*\psi$ is an isomorphism when F' is a quasi-coherent module.*

Since Y' is affine, any such F' is a homomorphic image of a free $\mathcal{O}_{Y'}$-module. Hence, by [15, p. 69, (iii)] (dualized), we may assume that F' itself is free.

Since Υ respects direct sums in that for any small family (F_α) in $\mathsf{D}(Z')$, the natural map is an isomorphism

$$\oplus_\alpha\Upsilon(F_\alpha)\xrightarrow{\ \sim\ }\Upsilon(\oplus_\alpha F_\alpha),$$

the same holds for the source and target of $i_*\psi$. There results a reduction to the trivial case when $F'=\mathcal{O}_{Y'}$.

This completes the proof of Lemma 6.6. □

Proof of Theorem 6.1. Assertion (i) holds because $u^*E\in\mathsf{P}(Y')$.

Since Ξ is a fiber square, the map v is flat along with u. For the same reason, the map g has finite flat dimension—so that $\mathsf{L}g^*N\in\mathsf{D}^+_{\mathsf{qc}}(X')$, see [19, §2.7.6], and the $\mathcal{O}_{X'}$-complex v^*M is g-perfect, see [17, p. 257, 4.7]. We then have natural isomorphisms

$$
\begin{aligned}
v^*\mathsf{R}\mathcal{H}om_X(M, f^!E)\otimes^{\mathsf{L}}_{X'}\mathsf{L}g^*N &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_{X'}(v^*M, v^*f^!E)\otimes^{\mathsf{L}}_{X'}\mathsf{L}g^*N\\
&\xrightarrow{\ \beta_\Xi\ } \mathsf{R}\mathcal{H}om_{X'}(v^*M, g^!u^*E)\otimes^{\mathsf{L}}_{X'}\mathsf{L}g^*N\\
&\xrightarrow{\ \psi\ } \mathsf{R}\mathcal{H}om_{X'}(v^*M, g^!u^*E\otimes^{\mathsf{L}}_{X'}\mathsf{L}g^*N)\\
&\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_{X'}\big(v^*M, g^!(u^*E\otimes^{\mathsf{L}}_{Y'}N)\big)
\end{aligned}
$$

described, respectively, in and around (5.4.1), (5.8)(b), (6.6.1), and 5.10.3.

Since $v^*M \in \mathsf{P}(g) \subset \mathsf{D}^-_{\mathsf{c}}(X')$ and $g^!(u^*E \otimes^{\mathsf{L}}_{Y'} N) \in \mathsf{D}^+_{\mathsf{qc}}(X')$, therefore

$$\mathsf{R}\mathcal{H}om_{X'}\big(v^*M,\, g^!(u^*E \otimes^{\mathsf{L}}_{Y'} N)\big) \in \mathsf{D}^+_{\mathsf{qc}}(X'),$$

cf. [15, p. 92, 3.3]. Assertion (ii) in 6.1 results.

The composition of the maps above induces the first isomorphism below:

$$\begin{aligned}\delta^!\big(v^*\mathsf{R}\mathcal{H}om_X(M, f^!E) \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*N\big) &\xrightarrow{\ \sim\ } \delta^!\mathsf{R}\mathcal{H}om_{X'}\big(v^*M, g^!(u^*E \otimes^{\mathsf{L}}_{Y'} N)\big)\\ &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Z\big(\mathsf{L}\delta^*v^*M, \delta^!g^!(u^*E \otimes^{\mathsf{L}}_{Y'} N)\big)\\ &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Z\big(\mathsf{L}\nu^*M, \gamma^!(u^*E \otimes^{\mathsf{L}}_{Y'} N)\big).\end{aligned}$$

The second isomorphism is from (5.11.1). The third isomorphism is canonical. □

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Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.
E-mail address: `avramov@math.unl.edu`

Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.
E-mail address: `iyengar@math.unl.edu`

Department of Mathematics, Purdue University, W. Lafayette, IN 47907, U.S.A.
E-mail address: `jlipman@purdue.edu`

Chennai Mathematical Institute, Siruseri 603103, INDIA
E-mail address: `snayak@cmi.ac.in`