

A VANISHING THEOREM FOR FINITELY SUPPORTED IDEALS IN REGULAR LOCAL RINGS

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To Mel Hochster, on the occasion of his 65th birthday.

ABSTRACT. Cohomological vanishing properties are proved for finitely supported ideals in an arbitrary d -dimensional regular local ring. (These imply refined Briançon-Skoda-type results, not otherwise known in mixed characteristic.) It is deduced that the adjoint \widetilde{I} of a finitely supported ideal I of order a has order $\sup(a + 1 - d, 0)$, and that taking adjoints of finitely supported ideals commutes with taking strict transforms at infinitely near points. Hence, \widetilde{I} is also finitely supported. Further: normal such I have reduction number $1 + \lfloor d(1 - 1/a) \rfloor$.

INTRODUCTION

In [L3, p. 747, (b)] there is a vanishing conjecture for an ideal I in a d -dimensional regular local ring (R, \mathfrak{m}) .¹ Suppose there is a map $f: X \rightarrow \text{Spec}(R)$ which factors as a finite sequence of blowups with smooth centers, and is such that $I\mathcal{O}_X$ is invertible. Let E be the closed fiber $f^{-1}\{\mathfrak{m}\}$. The conjecture is that

$$H_E^i(X, (I\mathcal{O}_X)^{-1}) = 0 \quad \text{for all } i \neq d.$$

This statement implies, with $\ell(I)$ the analytic spread of I , and $\widetilde{}$ denoting “adjoint ideal of” (a.k.a. multiplier ideal with exponent 1), that

$$\widetilde{I^{n+1}} = I\widetilde{I^n} \quad \text{for all } n \geq \ell(I) - 1,$$

which in turn implies a number of “Briançon-Skoda with coefficients” results, see [L3, pp. 745–746]. The conjectured statement holds true when $d = 2$; and it was proved by Cutkosky [C] for R essentially of finite type over a field of characteristic zero (in which case it is closely related to vanishing theorems which appear in the theory of multiplier ideals, see [Lz]). In these two situations, the assumed principalization f is known to exist for any $I \neq (0)$.

In this note we show that vanishing holds for those R -ideals which are *finitely supported*, i.e., for which there is a sequence of blowups as above, in which all the centers are closed points.

In addition, we deduce that the adjoint ideal of a finitely supported ideal I is itself finitely supported, with point basis obtained by subtracting $\min(d - 1, r_\beta)$ componentwise from the point basis (r_β) of I . (The terminology is explained in §3.)

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¹A stronger “CM” conjecture on that page was disproved by Hyry [Hy, p. 389, Ex. 3.6].

More consequences of vanishing are scattered throughout §§3–4. For example, for finitely supported I , Proposition 3.4 generalizes the above relation $\widetilde{I^{n+1}} = I\widetilde{I^n}$; and when, furthermore, I is the integral closure \overline{J} of a d -generated ideal J —whence $J\widetilde{I^{d-1}} = \widetilde{I^d}$ —Proposition 4.2 gives that $J\widetilde{I^{d-2}} = \widetilde{I^{d-1}} \cap J \neq \widetilde{I^{d-1}}$ (unless $I = R$), and that $J : I = \widetilde{J^{d-1}} + J = \widetilde{I^{d-1}} + J$. Moreover, for $1 \leq t \leq d$, $J\overline{J^{t-1}} = \overline{J^t}$ if and only if $t > d(1 - 1/\text{ord}_\alpha(J))$.

1. REFORMULATION OF VANISHING

Let K be a field. We denote by Greek letters $\alpha, \beta, \gamma, \dots$ regular local rings of dimension ≥ 2 , with fraction field K ; and we refer to such objects as “points.”

From now on α will be a d -dimensional point, with maximal ideal \mathfrak{m}_α , and $f: X \rightarrow \text{Spec}(\alpha)$ will be a proper birational map, with X regular (i.e., the local ring $\mathcal{O}_{X,x}$ is regular for every $x \in X$).

Let E_1, E_2, \dots, E_r be the $(d-1)$ -dimensional reduced irreducible components of the closed fiber $E := f^{-1}\{\mathfrak{m}_\alpha\}$. The local ring on X of the generic point of E_i is a discrete valuation ring R_i , whose corresponding valuation we denote by v_i . Since the regular ring α is universally catenary [GD4, (5.6.4)], the residue field of R_i has transcendence degree $d-1$ over $\alpha/\mathfrak{m}_\alpha$. There is then a unique point β_i infinitely near to α such that v_i is the order valuation ord_{β_i} associated with β_i , see [L1, §1, pp. 204, 208].²

We say that a point β' is *proximate to* another point β'' , and write $\beta' \succ \beta''$, when β' is infinitely near to β'' and the valuation ring of $\text{ord}_{\beta''}$ is the localization of β' at a height one prime ideal. For each i, j such that $\beta_i \succ \beta_j$, let \mathfrak{p}_{ij} be the height one prime ideal in β_i such that the localization $(\beta_i)_{\mathfrak{p}_{ij}}$ is the valuation ring R_j of v_j . Using induction on the length of the blowup sequence from β_j to β_i , one checks that $v_i(\mathfrak{p}_{ij}) = 1$.

Lemma 1.1. *Let I be a nonzero α -ideal. Then for each $i = 1, 2, \dots, r$, we have*

$$v_i(I) \geq \sum_{\{j \mid \beta_j \prec \beta_i\}} v_j(I).$$

(By convention, the sum of the empty family of integers is 0.)

Proof. After reindexing, we may assume that $\beta_1, \beta_2, \dots, \beta_s$ are all the β_j such that $\beta_j \prec \beta_i$; and then use that for some β_i -ideal I_i we have $I\beta_i = \mathfrak{p}_{i1}^{v_1(I)} \cdots \mathfrak{p}_{is}^{v_s(I)} I_i$. \square

Definition 1.2. A divisor $\sum_{i=1}^r n_i E_i$ is *full* if for each i , it holds that $n_i \geq 0$ and that, with preceding notation,

$$n_i \geq \sum_{\{j \mid \beta_j \prec \beta_i\}} n_j.$$

Examples 1.2.2. (a) For any nonzero α -ideal I , the divisor $\sum_{i=1}^r v_i(I) E_i$ is full.

(b) Any finite sum of full divisors is full.

(c) If $D = \sum_i n_i E_i$ is full, and $0 \leq c \in \mathbb{R}$, then $\lfloor cD \rfloor := \sum_i \lfloor cn_i \rfloor E_i$ is full.

(As usual, for any $\rho \in \mathbb{R}$, $\lfloor \rho \rfloor$ is the greatest integer $\leq \rho$.)

² The *first neighborhood* of α consists of all points of the form $\mathcal{O}_{Z,z}$ where $\varphi: Z \rightarrow \text{Spec}(\alpha)$ is the blowup of \mathfrak{m}_α and $z \in \varphi^{-1}\{\mathfrak{m}_\alpha\}$. A point β is *infinitely near to* α if there is a finite sequence of points beginning with α , ending with β , and such that each member other than α is in the first neighborhood of the preceding member.

Conjecture 1.3. *If $D = \sum_{i=1}^r n_i E_i$ is a full divisor then*

$$H_E^i(X, \mathcal{O}_X(D)) = 0 \text{ for all } i \neq d.$$

(This holds, obviously, when $i \leq 0$ or $i > d$.)

We assume henceforth that f is a composition

$$(1.3.1) \quad X = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 = \text{Spec}(\alpha)$$

where each $X_{i+1} \rightarrow X_i$ ($i < n$) is the blowup of a regular closed subscheme of X_i .

Example 1.3.2. For f as in (1.3.1), the conjecture holds when $D = 0$, in which case it is usually referred to as (an instance of) Grauert-Riemenschneider vanishing.

Indeed, for this to hold, [L2, p. 153, Lemma 4.2] shows it enough that *the natural derived-category map* $\tau: \alpha \rightarrow \mathbf{R}\Gamma(X, \mathcal{O}_X)$ *be an isomorphism;* and a straightforward induction, using the natural isomorphism $\mathbf{R}\Gamma(X, \mathcal{O}_X) \cong \mathbf{R}\Gamma(Z, \mathbf{R}h_* \mathcal{O}_X)$ associated to a suitable factorization of f as $X \xrightarrow{h} Z \xrightarrow{g} Y$, reduces proving that τ is an isomorphism to the case of a single blowup, where it follows from [GD3, (2.1.14) and (4.2.1)] (since the fibers of τ are single points or projective spaces), or from [L2, Theorems 4.1 and 5] (since regular local rings are pseudo-rational [LT, §4]).

Set $U := \text{Spec}(\alpha) - \{\mathfrak{m}_\alpha\}$, $V := f^{-1}U$. From 1.3.2 one gets a natural isomorphism $\mathcal{O}_U \xrightarrow{\sim} \mathbf{R}f_* \mathcal{O}_V$, whence $H^i(V, \mathcal{O}_V) \cong H^i(U, \mathcal{O}_U)$ for all i . But $H^0(U, \mathcal{O}_U) \cong \alpha$, and for $0 < i < d - 1$, $H^i(U, \mathcal{O}_U) \cong H_{\mathfrak{m}_\alpha}^{i+1}(\alpha) = 0$. Hence, for $D := \sum_{i=1}^r n_i E_i$ ($n_i \geq 0$) (so that $\mathcal{O}_X(D)|_V = \mathcal{O}_V$, and $H^0(\mathcal{O}_X(D)) = \alpha$), the natural exact sequences

$$H^{i-1}(X, \mathcal{O}_X(D)) \rightarrow H^{i-1}(V, \mathcal{O}_V) \rightarrow H_E^i(X, \mathcal{O}_X(D)) \xrightarrow{\psi^i} H^i(X, \mathcal{O}_X(D)) \rightarrow H^i(V, \mathcal{O}_V)$$

show that ψ^i is an isomorphism for $0 < i < d - 1$, and ψ^{d-1} is injective.

Furthermore, if $\mathfrak{m}_\alpha \mathcal{O}_X$ is invertible, and we take the harmless liberty of identifying the closed fiber E with the corresponding divisor, so that $\mathfrak{m}_\alpha \mathcal{O}_X = \mathcal{O}_X(-E)$, then applying \varinjlim_n to the exact row of the natural diagram

$$\begin{array}{ccccc} \text{Ext}^{d-1}(\mathcal{O}_{nE}, \mathcal{O}_X(D)) & \longrightarrow & \text{Ext}^{d-1}(\mathcal{O}_X, \mathcal{O}_X(D)) & \longrightarrow & \text{Ext}^{d-1}(\mathcal{O}_X(-nE), \mathcal{O}_X(D)) \\ & & \simeq \downarrow & & \downarrow \simeq \\ & & H^{d-1}(X, \mathcal{O}_X(D)) & & H^{d-1}(X, \mathcal{O}_X(D + nE)) \end{array}$$

we deduce a natural exact sequence

$$0 \longrightarrow H_E^{d-1}(X, \mathcal{O}_X(D)) \xrightarrow{\psi} H^{d-1}(X, \mathcal{O}_X(D)) \longrightarrow \varinjlim_n H^{d-1}(X, \mathcal{O}_X(D + nE))$$

where, one verifies, ψ is the above injective map ψ^{d-1} .

Thus for f as in (1.3.1) such that, further, $\mathfrak{m}_\alpha \mathcal{O}_X = \mathcal{O}_X(-E)$ is invertible, Conjecture 1.3 becomes:

Conjecture 1.4. *If $D = \sum_{i=1}^r n_i E_i$ is a full divisor then*

$$H^i(X, \mathcal{O}_X(D)) = 0 \text{ for } 0 < i < d - 1,$$

and for all $n > 0$ the natural map is an injection

$$H^{d-1}(X, \mathcal{O}_X(D)) \hookrightarrow H^{d-1}(X, \mathcal{O}_X(D + nE)).$$

2. A SPECIAL CASE

We prove Conjectures 1.3 and 1.4 in a special case.

Theorem 2.1. *With α as before, suppose the map $f: X \rightarrow \text{Spec}(\alpha)$ factors as*

$$X = X_r \rightarrow X_{r-1} \rightarrow \cdots \rightarrow X_0 = \text{Spec}(\alpha) \quad (r > 0),$$

where for $0 \leq i < r$ the map $X_{i+1} \rightarrow X_i$ is the blowup of a closed point of X_i . Then Conjecture 1.4—and thus Conjecture 1.3—holds true.

Proof. We proceed by induction on r . We often write $H^i(-)$ for $H^i(X, -)$.

Suppose $r = 1$, so that, with preceding notation, $E = E_1$ and $D = n_1 E$ ($n_1 \geq 0$). For any $q \geq 0$ there is a standard exact sequence, with $\mathcal{O}_E(mE) := \mathcal{O}_E \otimes \mathcal{O}_X(mE)$,

$$0 \rightarrow \mathcal{O}_X(qE) \rightarrow \mathcal{O}_X((q+1)E) \rightarrow \mathcal{O}_E((q+1)E) \rightarrow 0.$$

Here $E \cong \mathbb{P}^{d-1}$, the $(d-1)$ -dimensional projective space over the field $\alpha/\mathfrak{m}_\alpha$, and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-1)$; so $H^i(\mathcal{O}_E((q+1)E)) = 0$ for $i < d-1$. Thus for $0 < i < d-1$ there are natural isomorphisms

$$H^i(\mathcal{O}_X(qE)) \xrightarrow{\sim} H^i(\mathcal{O}_X((q+1)E));$$

and since, by Example 1.3.2, $H^i(\mathcal{O}_X) = 0$, it follows that $H^i(\mathcal{O}_X(n_1 E)) = 0$.

Moreover, for every q the natural map

$$H^{d-1}(\mathcal{O}_X(qE)) \rightarrow H^{d-1}(\mathcal{O}_X((q+1)E)) = H^{d-1}(\mathcal{O}_X(qE + E))$$

is injective, whence so is $H^{d-1}(\mathcal{O}_X(n_1 E)) \rightarrow H^{d-1}(\mathcal{O}_X(n_1 E + nE))$.

Next, when $r > 1$, let $g: Y \rightarrow \text{Spec}(\alpha)$ be the composition of $r-1$ closed-point blowups, and $h: X \rightarrow Y$ the blowup of a closed point $y \in Y$. Make the indexing such that E_1 is the closed fiber of h . With v_i as in §1, and $2 \leq i \leq r$, let E'_i be the center of v_i on Y . Arrange further that E'_2, \dots, E'_s are all of the E'_i which pass through y . Fullness of $D = \sum_{i=1}^r n_i E_i$ entails $n_1 \geq n_2 + \cdots + n_s$.

Let $D' := n_2 E'_2 + \cdots + n_r E'_r$; and let $h^{-1}D'$ be the divisor

$$h^{-1}D' := (n_2 + \cdots + n_s)E_1 + n_2 E_2 + \cdots + n_r E_r,$$

so that $\mathcal{O}_X(h^{-1}D') = h^* \mathcal{O}_Y(D')$. Fullness of D' follows from that of D , because for $i > 1$, β_i is not proximate to β_1 . So the inductive hypothesis gives that Conjecture 1.4 holds for D' . It follows that *it also holds for $h^{-1}D'$* : indeed, as $\mathbf{R}h_* \mathcal{O}_X = \mathcal{O}_Y$ (cf. 1.3.2), the standard projection isomorphism gives

$$\begin{aligned} \mathbf{R}\Gamma(X, \mathcal{O}_X(h^{-1}D')) &= \mathbf{R}\Gamma(Y, \mathbf{R}h_*(\mathcal{O}_X \otimes h^* \mathcal{O}_Y(D'))) \\ &\cong \mathbf{R}\Gamma(Y, \mathbf{R}h_*(\mathcal{O}_X) \otimes \mathcal{O}_Y(D')) = \mathbf{R}\Gamma(Y, \mathcal{O}_Y(D')), \end{aligned}$$

and similarly for the full divisor $D' + nE'$, where E' is the full divisor such that $\mathcal{O}_Y(-E') = \mathfrak{m}_\alpha \mathcal{O}_Y$ (see 1.2.2), so that $h^{-1}(D' + nE') = h^{-1}D' + nE$, whence

$$H^i(X, \mathcal{O}_X(h^{-1}D')) \cong H^i(Y, \mathcal{O}_Y(D')) = 0 \quad (0 < i < d-1),$$

and the natural map

$$H^{d-1}(X, \mathcal{O}_X(h^{-1}D')) \rightarrow H^{d-1}(X, \mathcal{O}_X(h^{-1}D' + nE))$$

is isomorphic to the natural injection

$$H^{d-1}(Y, \mathcal{O}_Y(D')) \hookrightarrow H^{d-1}(Y, \mathcal{O}_Y(D' + nE')).$$

It will therefore be enough to show the following:

Lemma 2.2. *If Conjecture 1.4 holds for a divisor $D_\nu := \nu E_1 + n_2 E_2 + \cdots + n_r E_r$ where $\nu \geq n_2 + \cdots + n_s$, then it holds for $D_{\nu+1}$.*

Proof. Denote the residue field of y by $\kappa(y)$, so that $E_1 \cong \mathbb{P}_{\kappa(y)}^{d-1}$. For any $n \geq 0$, there is the usual exact sequence

$$0 \rightarrow \mathcal{O}_X(D_\nu + nE) \rightarrow \mathcal{O}_X(D_{\nu+1} + nE) \rightarrow \mathcal{O}_{E_1} \otimes \mathcal{O}_X(D_{\nu+1} + nE) \rightarrow 0.$$

Moreover, with $N := n_2 + \cdots + n_s - \nu - 1$,

$$\mathcal{O}_{E_1} \otimes \mathcal{O}_X(D_{\nu+1} + nE) \cong \mathcal{O}_{E_1}(N).$$

To see this, just note, with $\lambda: E_1 \hookrightarrow X$ the inclusion, that $\lambda^* \mathcal{O}_X(E_1) \cong \mathcal{O}_{E_1}(-1)$, that if $2 \leq j \leq s$ then since E'_j (see above) is a regular subscheme of Y passing through y , therefore

$$\lambda^* \mathcal{O}_X(E_j) \cong \lambda^*(h^* \mathcal{O}_Y(E'_j) \otimes \mathcal{O}_X(-E_1)) \cong \lambda^*(\mathcal{O}_X(-E_1)) \cong \mathcal{O}_{E_1}(1),$$

that if $j > s$ then $\lambda^* \mathcal{O}_X(E_j) = \mathcal{O}_{E_1}$, and that since $\mathcal{O}_Y(E')|_U \cong \mathcal{O}_U$ for some open U containing y (E' as above), therefore $\lambda^* \mathcal{O}_X(E) \cong \lambda^* h^* \mathcal{O}_Y(E') \cong \mathcal{O}_{E_1}$.

Since $N < 0$, it follows in case $n = 0$ that

$$H^i(\mathcal{O}_X(D_\nu)) \cong H^i(\mathcal{O}_X(D_{\nu+1})) \quad (0 \leq i < d-1),$$

so that $H^i(\mathcal{O}_X(D_\nu)) = 0 \implies H^i(\mathcal{O}_X(D_{\nu+1})) = 0$. Furthermore, for any $n \geq 0$ there is a natural injection $H^{d-1}(\mathcal{O}_X(D_\nu + nE)) \hookrightarrow H^{d-1}(\mathcal{O}_X(D_{\nu+1} + nE))$. There is then a commutative diagram, with exact rows,

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{d-1}(\mathcal{O}_X(D_\nu)) & \longrightarrow & H^{d-1}(\mathcal{O}_X(D_{\nu+1})) & \longrightarrow & H^{d-1}(\mathcal{O}_{E_1}(N)) \\ & & \psi_\nu \downarrow & & \psi_{\nu+1} \downarrow & & \parallel \\ 0 & \longrightarrow & H^{d-1}(\mathcal{O}_X(D_\nu + nE)) & \longrightarrow & H^{d-1}(\mathcal{O}_X(D_{\nu+1} + nE)) & \longrightarrow & H^{d-1}(\mathcal{O}_{E_1}(N)) \end{array}$$

Hence if ψ_ν is injective then so is $\psi_{\nu+1}$. Lemma 2.2 results.

This completes the proof of Theorem 2.1 □

Remarks 2.3. With f as in Theorem 2.1, and E_i, β_i as before, set

$$E_i^* := \sum_{\{j | \beta_j \supset \beta_i\}} \text{ord}_{\beta_j}(\mathfrak{m}_{\beta_i}) E_j.$$

So if \mathbf{p} is the $r \times r$ proximity matrix with $p_{ii} = 1$, $p_{ji} = -1$ if $\beta_j \prec \beta_i$, and $p_{ji} = 0$ otherwise, then by [L4, p. 301, (4.6)] (whose proof is valid in any dimension),

$$(2.3.1) \quad (E_1^*, \dots, E_r^*)^t = \mathbf{p}^{-1}(E_1, \dots, E_r)^t$$

where “t” means “transpose”. Then, for any $n_1, \dots, n_r \in \mathbb{Z}$, premultiplying both sides of (2.3.1) by $(n_1, \dots, n_r) \mathbf{p}$ yields

$$\sum_{i=1}^r (n_i - \sum_{\{j | \beta_j \prec \beta_i\}} n_j) E_i^* = \sum_{i=1}^r n_i E_i.$$

Hence, the monoid of full divisors is freely generated by E_1^*, \dots, E_r^* .

For example, the relative canonical divisor $K_f := (d-1)(E_1^* + \cdots + E_r^*)$ is full. Note that by [LS, pp. 201–202], $\mathcal{J}_f := \mathcal{O}_X(-K_f)$ is the relative Jacobian ideal of f , and by [LS, p. 206, (2.3)], $\omega_f := \mathcal{J}_f^{-1} = \mathcal{O}_X(K_f)$ is a canonical dualizing sheaf for f . (In fact, since f is a local complete intersection map, $\omega_f \cong f^! \mathcal{O}_{\text{Spec}(\alpha)}$.)

Corollary 2.4. *Under the hypotheses of Theorem 2.1, the following hold for any full divisor D on X .*

- (i) $H^i(X, \mathcal{O}_X(K_f - D)) = 0$ for all $i \neq 0$.
- (ii) $H_E^i(X, \mathcal{O}_X(K_f - D)) = 0$ for all $i \neq 1, d$.
- (iii) $H_E^1(X, \mathcal{O}_X(K_f - D)) \cong \alpha/H^0(X, \mathcal{O}_X(K_f - D))$.
- (iv) $H_E^d(X, \mathcal{O}_X(K_f - D))$ is an injective hull of $\alpha/\mathfrak{m}_\alpha$.

Proof. For any invertible \mathcal{O}_X -module L , the α -module $H^i(X, L \otimes \omega_f)$ is Matlis-dual to $H_E^{d-i}(X, L^{-1})$ [L0, p. 188, Theorem]; and so (i) and (ii) result from Theorem 2.1 by duality (via Conjectures 1.3 and 1.4, respectively). Similarly, (iv) is dual to the obvious statement that $H^0(X, \mathcal{O}_X(D)) = \alpha$. Assertion (iii) results from the natural exact sequence

$$\begin{aligned} 0 = H_E^0(X, \mathcal{O}_X(K_f - D)) &\rightarrow H^0(X, \mathcal{O}_X(K_f - D)) \rightarrow \alpha = H^0(V, \mathcal{O}_V) \\ &\rightarrow H_E^1(X, \mathcal{O}_X(K_f - D)) \rightarrow H^1(X, \mathcal{O}_X(K_f - D)) \stackrel{(i)}{=} 0. \end{aligned}$$

□

3. FINITELY SUPPORTED IDEALS

Recall that an α -ideal I is *finitely supported* if there is a map $f: X \rightarrow \text{Spec}(\alpha)$ which factors as in Theorem 2.1 such that the \mathcal{O}_X -module $I\mathcal{O}_X$ is invertible. In this situation, $I\mathcal{O}_X = \mathcal{O}_X(-D)$, where, as in Example 1.2.2, D is a full divisor.³

Also, $H^0(X, I\mathcal{O}_X) = \bar{I}$, the integral closure of I ; and with $\omega_f = \mathcal{O}_X(K_f)$ as in Remark 2.3, $H^0(X, \mathcal{O}_X(K_f - D)) = H^0(X, I\omega_f)$ is the *adjoint ideal* \tilde{I} , see [L3, p. 742, (1.3.1)].

The vanishing conjecture and various consequences hold for finitely supported ideals (but see Remark 4.1).

Corollary 3.1. *If $f: X \rightarrow \text{Spec}(\alpha)$ is as in Theorem 2.1, and I is an α -ideal such that $\mathcal{I} := I\mathcal{O}_X$ is invertible, then the following all hold.*

- (i) $H_E^i(X, \mathcal{I}^{-1}) = H_E^i(X, \mathcal{I}^{-1}\omega_f) = 0$ for all $i \neq d$.
- (i)' $H_E^i(X, \mathcal{I}) = H_E^i(X, \mathcal{I}\omega_f) = 0$ for all $i \neq 1, d$.
- (ii) $H^i(X, \mathcal{I}\omega_f) = H^i(X, \mathcal{I}) = 0$ for all $i \neq 0$.
- (ii)' $H^i(X, \mathcal{I}^{-1}\omega_f) = H^i(X, \mathcal{I}^{-1}) = 0$ for all $i \neq d - 1, 0$.
- (iii) $H^{d-1}(X, \mathcal{I}^{-1}\omega_f)$ is Matlis-dual to $H_E^1(X, \mathcal{I}) \cong \alpha/\bar{I}$.
- (iv) $H^{d-1}(X, \mathcal{I}^{-1})$ is Matlis-dual to $H_E^1(X, \mathcal{I}\omega_f) \cong \alpha/\tilde{I}$.
- (v) $H^0(X, \mathcal{I}^{-1}\omega_f) = H^0(X, \mathcal{I}^{-1}) \cong \alpha$.
- (vi) $H_E^d(X, \mathcal{I}) = H_E^d(X, \mathcal{I}\omega_f)$ is an injective hull of $\alpha/\mathfrak{m}_\alpha$.

Proof. Since the divisors D and $K_f + D$ are both full, (i) and (ii)' follow from Theorem 2.1, via Conjectures 1.3 and 1.4 respectively; and, given the duality mentioned in the proof of Corollary 2.4, (iii) and (iv) both result from Corollary 2.4(iii). Statement (v) is obviously true. Finally, (ii), (i)' and (vi) result from their respective dual versions (i), (ii)', and (v). □

³For more on finitely supported ideals, see [AGL], [DC], [Ga], [Tn].

Remark 3.2. For the vanishing of $H_E^1(X, \mathcal{I}^{-1})$, and hence its dual $H^{d-1}(X, \mathcal{I}\omega_f)$, it suffices that f factor as in (1.3.1). Indeed, there is an exact, locally split, sequence

$$0 \rightarrow C \rightarrow \mathcal{O}_X^n \rightarrow \mathcal{I} \rightarrow 0,$$

whence, with $(-)^* := \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$, an exact sequence

$$(3.2.1) \quad 0 \rightarrow \mathcal{I}^{-1} = \mathcal{I}^* \rightarrow \mathcal{O}_X^n \rightarrow C^* \rightarrow 0,$$

giving another exact sequence

$$0 = H_E^0(X, C^*) \rightarrow H_E^1(X, \mathcal{I})^{-1} \rightarrow H_E^1(X, \mathcal{O}_X^n) = 0$$

where the first term vanishes because C^* is locally free, and the third by Example 1.3.2.

Tensoring (3.2.1) with ω_f (a dualizing sheaf, inverse to the relative Jacobian ideal) and noting that $H^{d-1}(X, \mathcal{O}_X)$ and hence its dual $H_E^1(X, \omega_f)$ vanish (Example 1.3.2), one shows similarly that $H_E^1(\mathcal{I}^{-1}\omega_f)$ and its dual $H^{d-1}(X, \mathcal{I})$ both vanish.

The *point basis* $\mathbf{B}(I)$ of a nonzero α -ideal I is the family of nonnegative integers $(\text{ord}_\beta(I^\beta))$ indexed by the set of all points β infinitely near to α , with I^β the transform of I in β (i.e., $I^\beta := t^{-1}I\beta$, where t is the gcd of the elements in $I\beta$).

Two nonzero α -ideals have the same point basis iff their integral closures are the same, see [L1, p. 209, Prop. (1.10)]. The proof in *loc. cit.* shows, moreover, that if I and J are α -ideals such that $\text{ord}_\beta(I^\beta) \leq \text{ord}_\beta(J^\beta)$ for all β , then $\bar{I} \supset \bar{J}$, where “ $\bar{}$ ” denotes integral closure.

The ideal I is finitely supported iff I has finitely many *base points*, i.e., β such that $\text{ord}_\beta(I^\beta) \neq 0$, see [L1, p. 213, (1.20), and p. 215, Remark]. Thus the product of two finitely supported ideals is still finitely supported.⁴

Here is the main result in this section (proved for $d = 2$ in [L3, p. 749, (3.1.2)]).

Theorem 3.3. *Let α be a d -dimensional regular local ring ($d \geq 2$) and I a finitely supported α -ideal, with point basis $\mathbf{B}(I) = (r_\beta)$. Then:*

- (1) $\text{ord}_\alpha(\tilde{I}) = \max(\text{ord}_\alpha(I) + 1 - d, 0)$; and
- (2) for any β infinitely near to α , $\tilde{I}^\beta = (\tilde{I})^\beta$.

Hence the adjoint ideal \tilde{I} is the unique integrally closed ideal with point basis $(\max(r_\beta + 1 - d, 0))$. In particular, \tilde{I} is finitely supported.

Remark 3.3.1. A propos of (2), $\overline{\tilde{I}^\beta} = \overline{\tilde{I}}^\beta \supset \bar{I}^\beta$ (where “ $\bar{}$ ” denotes integral closure), see [L1, p. 207, Prop. (1.5)(vi)]; but equality doesn’t always hold ([Tn, Example 1.2]).

Corollary 3.3.2. *For any finitely supported α -ideal I , $\tilde{I} = \alpha \iff \text{ord}_\alpha(I) < d$.*

We also have the following *weak subadditivity* consequence:

Corollary 3.3.3. *For finitely supported α -ideals I, J , it holds that*

$$\overline{\tilde{I}\tilde{J}} \supset \tilde{I}\tilde{J}.$$

⁴It can be seen, for any α -ideal I and any d -dimensional infinitely near β , that if I^β is \mathfrak{m}_β -primary then β is dominated by a Rees valuation of I . Hence I is finitely supported iff every base point of I is d -dimensional. More constructively, I is finitely supported iff $I\mathcal{O}_X$ is invertible where $X \rightarrow \text{Spec}(\alpha)$ is obtained by successively blowing up all the finitely many d -dimensional infinitely near β such that β is dominated by a Rees valuation of I —in which case, with the notation of Remark 2.3, $I\mathcal{O}_X = \mathcal{O}_X(-\sum_i \text{ord}_{\beta_i}(I^{\beta_i})E_i^*)$.

Proof. One checks that for any nonnegative integers r and s ,

$$\max(r + 1 - d, 0) + \max(s + 1 - d, 0) \leq \max(r + s + 1 - d, 0).$$

Since “transform” respects products, therefore $\text{ord}_\beta((\widetilde{IJ})^\beta) \leq \text{ord}_\beta((\widetilde{IJ})^\beta)$ for all β , whence the conclusion (see above). \square

In the opposite direction lie the next Corollary and also Proposition 3.4 below.

Corollary 3.3.4. *For finitely supported α -ideals I, J , it holds that*

$$\widetilde{IJ} \supset \overline{IJ},$$

with equality if and only if $\text{ord}_\beta(J^\beta) \geq d - 1$ at every base point β of I .

Proof. The inclusion results from the equality $\widetilde{IJ} : I = \widetilde{J}$ [L3, p. 741, (b) and (d)].

The point basis $\mathbf{B}(\overline{IJ}) =: (r_\beta)$ satisfies

$$r_\beta = \text{ord}_\beta(I^\beta) + \max(\text{ord}_\beta(J^\beta) - d + 1, 0)$$

[L1, p. 212, (1.15)], whence

$$r_\beta = \max(\text{ord}_\beta(IJ)^\beta - d + 1, 0) \iff \text{either } \text{ord}_\beta(I^\beta) = 0 \\ \text{or } [\text{ord}_\beta(I^\beta) > 0 \text{ and } \text{ord}_\beta(J^\beta) \geq d - 1].$$

\square

For the *proof of Theorem 3.3*, we begin by proving Corollary 3.3.2.

Any β with $I^\beta \neq \beta$ is d -dimensional [L1, p. 214, (1.22)], so since α is regular, the residue field of β is finite over that of α , see [GD4, (5.6.4)]. Hence, by [Hi, p. 217, Lemma 8]:

(*) *if $\text{ord}_\alpha(I) < d$ then $\text{ord}_\beta(I^\beta) < d$ for any infinitely near β .*

With this in mind, recall from Remark 2.3 that, with $f: X \rightarrow \text{Spec}(\alpha)$ as at the beginning of this §3, and $K_f =: \sum_i c_i E_i$ we have

$$\widetilde{I} = H^0(X, I\omega_f) = H^0\left(X, \mathcal{O}_X\left(\sum_i (c_i - \text{ord}_{\beta_i}(I))E_i\right)\right),$$

and

$$c_i = \sum_{\beta_j \subset \beta_i} \text{ord}_{\beta_j}(\mathfrak{m}_j^{d-1}) \geq \sum_{\beta_j \subset \beta_i} \text{ord}_{\beta_j}(I^{\beta_j}) \text{ord}_{\beta_j}(\mathfrak{m}_j) = \text{ord}_{\beta_i}(I),$$

the last equality by [L1, p. 209–210, Lemma (1.11)]. The implication “ \Leftarrow ” in 3.3.2 results.

Furthermore, if, say, E_1 corresponds to the valuation ring of ord_α , then

$$\text{ord}_\alpha(\widetilde{I}) \geq \text{ord}_\alpha(I) - c_1 = \text{ord}_\alpha(I) - (d - 1).$$

In particular, if $\widetilde{I} = \alpha$ then $\text{ord}_\alpha(I) \leq d - 1$, giving the implication “ \Rightarrow ” in 3.3.2.

Now Corollary 3.3.2 and (*) show that Theorem 3.3 holds if $\text{ord}_\alpha(I) < d$. For the rest, we need the following key fact.

Proposition 3.4. *Let I and J be finitely supported α -ideals such that for each β infinitely near to α , $\text{ord}_\beta(J^\beta) \geq (d - 1)\text{ord}_\beta(I^\beta)$. Then*

$$\widetilde{IJ} = I\widetilde{J}.$$

Proof. In the proof of Lemma 1.1, applied to the present situation, one has $I_i = I^{\beta_i}$; hence the condition “ $\text{ord}_\beta(J^\beta) \geq (d-1)\text{ord}_\beta(I^\beta)$ for each β ” translates to

$$(3.4.1) \quad \left(v_i(J) - \sum_{\{j \mid \beta_j \prec \beta_i\}} v_j(J) \right) \geq (d-1) \left(v_i(I) - \sum_{\{j \mid \beta_j \prec \beta_i\}} v_j(I) \right) \quad (1 \leq i \leq r),$$

which implies that if $f: X \rightarrow \text{Spec}(\alpha)$ is a composite of closed-point blowups such that $I\mathcal{O}_X$ and $J\mathcal{O}_X$ are both invertible (such an f exists because IJ is finitely supported) then, for $0 \leq k \leq d-1$, it holds that $I^k(J\mathcal{O}_X)^{-1} = \mathcal{O}_X(D_k)$ with D_k a full divisor on X .

Hence, by Corollary 2.4(i),

$$\mathrm{H}^{d-i}(X, (I\mathcal{O}_X)^{-k} J\omega_f) = \mathrm{H}^{d-i}(X, \mathcal{O}_X(K_f - D_k)) = 0 \quad (0 \leq i, k \leq d-1).$$

This being so, we see that the case $J = I^n$ ($n \geq d-1$) is treated in [L3, §2.3]; and the proof for arbitrary J is essentially the same.⁵ \square

Corollary 3.4.2. *If J is a finitely supported α -ideal with $\text{ord}_\alpha(J) \geq d-1$ then*

$$\widetilde{\mathfrak{m}_\alpha J} = \mathfrak{m}_\alpha \widetilde{J}.$$

Now we can prove Theorem 3.3 by induction on the least number of closed-point blowups needed to principalize I . Set $\text{ord}_\alpha(I) := a$. Since we have already disposed of the case $a < d$, it will clearly be enough to show that:

- (1) If $a \geq d-1$, then $\text{ord}_\alpha(\widetilde{I}) = a+1-d$; and
- (2) if $g: X_1 \rightarrow \text{Spec}(\alpha)$ is the blow up of $\mathfrak{m} := \mathfrak{m}_\alpha$, and β is the local ring of a closed point on X_1 , then

$$\widetilde{I}^\beta = (\widetilde{I})^\beta.$$

Let $h: X = X_r \rightarrow X_1$ be as in Theorem 2.1. For any \mathcal{O}_X -module L , the natural map is an isomorphism $\mathfrak{m}h_*(L) \xrightarrow{\sim} h_*(\mathfrak{m}L)$. (The assertion being local on X_1 , one can assume that $\mathfrak{m}\mathcal{O}_{X_1} \cong \mathcal{O}_{X_1} \dots$) Furthermore, from Remark 2.3 one deduces that $\omega_h = \omega_f(\mathfrak{m}\mathcal{O}_X)^{d-1}$. So with $\widetilde{\mathcal{I}}_1 := I(\mathfrak{m}\mathcal{O}_{X_1})^{-a}$, it holds that

$$\begin{aligned} (\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1} \widetilde{\mathcal{I}}_1 &:= (\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1} h_*(\widetilde{\mathcal{I}}_1 \omega_h) = (\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1} h_*(\widetilde{\mathcal{I}}_1 \omega_f (\mathfrak{m}\mathcal{O}_X)^{d-1}) \\ &= (\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1} h_*(I\omega_f (\mathfrak{m}\mathcal{O}_X)^{-a+d-1}) = h_*(I\omega_f). \end{aligned}$$

Using induction on $s > 0$, one deduces from Corollary 3.4.2 that

$$\mathfrak{m}^s \widetilde{I} = \widetilde{\mathfrak{m}^s I} := \mathrm{H}^0(X, \mathfrak{m}^s I\omega_f) = \mathrm{H}^0(X_1, h_*(\mathfrak{m}^s I\omega_f)) = \mathrm{H}^0(X_1, \mathfrak{m}^s h_*(I\omega_f)).$$

Since the invertible \mathcal{O}_{X_1} -module $\mathfrak{m}\mathcal{O}_{X_1}$ is g -ample, and $h_*(I\omega_f)$ is coherent, therefore $\mathfrak{m}^s h_*(I\omega_f)$ is generated by its global sections for all $s \gg 0$; that is, by the preceding,

$$\mathfrak{m}^s h_*(I\omega_f) = \mathfrak{m}^s \widetilde{I}\mathcal{O}_{X_1},$$

whence

$$(\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1} \widetilde{\mathcal{I}}_1 = h_*(I\omega_f) = \widetilde{I}\mathcal{O}_{X_1}.$$

Since $\widetilde{\mathcal{I}}_1 \not\subset \mathfrak{m}\mathcal{O}_{X_1}$, this implies (1) above; and then—as it is straightforward to check for any $z \in X_1$ that the stalk $(\widetilde{\mathcal{I}}_1)_z$ is just $(\widetilde{\mathcal{I}}_1)_z$ —localizing at β gives (2).

This completes the proof of Theorem 3.3. \square

⁵Here, a principalization of I is given to begin with, so the fact—of which a special case is used in *loc. cit.*—that the $\widetilde{}$ operation on α -ideals commutes with smooth base change follows from commutativity with H^0 and with formation of ω .

4. ADDITIONAL OBSERVATIONS

Let $J = (\xi_1, \dots, \xi_d)$ ($d := \dim \alpha$) be a finitely supported (hence \mathfrak{m}_α -primary) α -ideal. Proposition 4.2(ii) shows that (as was mentioned in the Introduction) $J\widetilde{J^{n-1}} = \widetilde{J^n}$ for all $n \geq d$, whereas for $1 \leq s < d$, $J\widetilde{J^{s-1}} \neq \widetilde{J^s}$; and Proposition 4.2(i) shows, via Corollary 3.3.2, that for $s > 0$, $J\overline{J^{s-1}} = \overline{J^s}$ if and only if $s > d(1 - 1/\text{ord}_\alpha(J))$. In particular, $J\overline{J^{d-1}} = \overline{J^d}$.

Remark 4.1. Bernd Ulrich informed me of an example of Huneke and Huckaba [HH, p. 88], in which α can be taken to be the localization at (x, y, z) of the polynomial ring $\mathbb{C}[x, y, z]$ (so that $d = 3$), $J = (x^4, y(y^3 + z^3), z(y^3 + z^3))$, and $J\overline{J^2} \neq \overline{J^3}$. This J cannot, then, be finitely supported. In fact it has a curve of base points in the blow up of \mathfrak{m}_α . Moreover, analysis of the proof of Proposition 4.2(i) shows that if $f: X \rightarrow \text{Spec}(\alpha)$ is a principalization of J by a sequence of smoothly centered blowups then $H^1(X, J\mathcal{O}_X) \supset \overline{J^3}/J\overline{J^2} \neq 0$. Thus, for instance, Corollary 3.1(ii) does not hold for principalizations of arbitrary \mathfrak{m}_α -primary ideals.

It is well-known that for any ideal $I \supset J$, the dual of α/I (i.e., $\text{Hom}_\alpha(\alpha/I, \mathcal{E})$, where \mathcal{E} is an injective hull of $\alpha/\mathfrak{m}_\alpha$) is (isomorphic to) $(J : I)/J$. Indeed, by local duality the dual of $\alpha/I = H_{\mathfrak{m}_\alpha}^0(\alpha/I)$ is $\text{Ext}^d(\alpha/I, \alpha)$, and, the sequence (ξ_1, \dots, ξ_d) being regular, there are standard isomorphisms

$$\text{Ext}^d(\alpha/I, \alpha) \cong \text{Hom}_\alpha(\alpha/I, \alpha/J) \cong (J : I)/J.$$

Proposition 4.2. *For the preceding J , setting $J^t := \alpha$ for all $t \leq 0$, we have, for all $s \in \mathbb{Z}$,*

- (i) $\overline{J^s}/J\overline{J^{s-1}}$ is dual to $\alpha/(\widetilde{J^{d-s}} + J)$; and
- (ii) $\widetilde{J^s}/J\widetilde{J^{s-1}}$ is dual to $\alpha/(\overline{J^{d-s}} + J)$.

Hence, since a finite-length module and its dual have the same annihilator,

- (iii) $J\overline{J^{s-1}} : \overline{J^s} = \widetilde{J^{d-s}} + J$; and
- (iv) $J\widetilde{J^{s-1}} : \widetilde{J^s} = \overline{J^{d-s}} + J$.

A proof is given below.

Corollary 4.3. *For any $s \in \mathbb{Z}$, the following conditions are equivalent.*

- (i) $J\overline{J^{s-1}} = \overline{J^s} \cap J$.
- (ii) $J\overline{J^{d-s-1}} : \overline{J^{d-s}} = J : \overline{J^{d-s}}$.
- (iii) $J\overline{J^{d-s-1}} = \overline{J^{d-s}} \cap J$.
- (iv) $J\widetilde{J^{s-1}} : \widetilde{J^s} = J : \widetilde{J^s}$.

Proof. Since, clearly, $J\overline{J^{s-1}} \subset \overline{J^s} \cap J$, therefore 4.2(ii) makes condition (i) hold if and only if $\overline{J^s}/(\overline{J^s} \cap J)$ is dual to $\alpha/(\overline{J^{d-s}} + J)$, i.e., isomorphic to $(J : \overline{J^{d-s}})/J$. All these modules have finite length, so the natural isomorphism

$$\overline{J^s}/(\overline{J^s} \cap J) \cong (\overline{J^s} + J)/J \stackrel{4.2(\text{iii})}{=} (J\overline{J^{d-s-1}} : \overline{J^{d-s}})/J$$

shows that (i) \iff (ii).

The proof of (iii) \iff (iv) is analogous. (Replace s by $d - s$, and interchange $\widetilde{}$ and $\overline{}$.)

The implications (i) \implies (iv) and (iii) \implies (ii) are obvious. \square

Corollary 4.4. *The following hold.*

- (i) $J\widetilde{J^{d-2}} = \widetilde{J^{d-1}} \cap J$.
- (ii) $J\overline{J^{d-2}} = \overline{J^{d-1}} \cap J$.
- (iii) $J\widetilde{J^{d-3}} = \widetilde{J^{d-2}} \cap J$.

Proof. For $s \geq d - 1$, condition 4.3(ii) obviously holds, whence so does 4.3(i).

Similarly, (ii) results from 4.3(iv) \implies 4.3(iii), with $s = 1$.

As pointed out by Bernd Ulrich, condition (iii) results similarly from the fact that 4.3(iii) holds for $s = d - 2$, a special case of the main result in [It]. \square

Proof of Proposition 4.2. Let $f: X \rightarrow \text{Spec}(\alpha)$ be a composition of closed-point blowups such that $L := J\mathcal{O}_X$ is invertible. Then for all $s \in \mathbb{Z}$, $H^0(X, L^s) = \overline{J^s}$ and $H^0(X, L^s\omega_f) = \widetilde{J^s}$.

Corollary 3.1(ii) and (ii)', give, for all $j \geq 0$,

$$\begin{aligned} H^i(X, L^j) &= 0 & (i \neq 0), \\ H^i(X, L^{-j}) &= 0 & (0 < i < d - 1). \end{aligned}$$

Arguing as in [LT, p. 112], one finds then that $\overline{J^s}/J\overline{J^{s-1}}$ is isomorphic to the kernel of the map

$$H^{d-1}(X, L^{s-d}) \xrightarrow{\xi_1 \oplus \dots \oplus \xi_d} \underbrace{H^{d-1}(X, L^{s+1-d}) \oplus \dots \oplus H^{d-1}(X, L^{s+1-d})}_{d \text{ times}}.$$

Hence $\overline{J^s}/J\overline{J^{s-1}}$ is dual to the cokernel of the dual map

$$\underbrace{H_E^1(X, L^{d-s-1}\omega_f) \times \dots \times H_E^1(X, L^{d-s-1}\omega_f)}_{d \text{ times}} \xrightarrow{(\xi_1, \dots, \xi_d)} H_E^1(X, L^{d-s}\omega_f).$$

Corollary 3.1(i) and (iv) give, for all $j \in \mathbb{Z}$, $H_E^1(X, L^j\omega_f) \cong \alpha/\widetilde{J^j}$. Accordingly, one verifies that $\overline{J^s}/J\overline{J^{s-1}}$ is dual to the cokernel of the map

$$\alpha^d \xrightarrow{(\xi_1, \dots, \xi_d)} \alpha/\widetilde{J^{d-s}},$$

proving (i). The proof of (ii) is analogous, except that one begins by tensoring the complex $K(F, \sigma)$ in [LT, p. 112] with $L^s\omega_f$ instead of with L^s . \square

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