GROTHENDIECK DUALITY THEORIES—ABSTRACT AND CONCRETE

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Abstract. Grothendieck Duality—the theory of the twisted inverse image pseudofunctor $(-)^!$ over a suitable category of scheme-maps—can be developed *concretely*, with emphasis on explicit constructions, or, in greater generality, *abstractly*, with emphasis on category-theoretic considerations. We aim to connect these approaches, a nontrivial matter involving some alluring relations, for instance among differential forms, residues and duality. In particular, it emerges that the culminating Ideal Theorem in Hartshorne's "Residues and Duality" holds for arbitrary essentially-finite-type maps of noetherian schemes and bounded-below complexes with quasi-coherent cohomology.

What appears here mostly concerns pseudo-coherent finite maps. The rest is being prepared.

Contents

1. Introduction

1.1. (Notation and terminology.) A *ringed space* is a topological space X furnished with a sheaf $\mathcal{O}_X$ of commutative rings. A map $(f,\theta)\colon X \to Y$ of ringed spaces consists of a continuous map $f\colon X \to Y$ and a homomorphism of sheaves of rings $\theta\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. (The appropriate θ, though understood to be present, is often left out of the notation.) Such spaces and maps form a category $\mathbf{S}$: the composition of $(g,\psi) : Y \to Z$ with (f,θ) is $(g\circ f,\, g_*\theta\circ\psi)$, and the identity id_X of X is the map for which $Y = X$ and both f and θ are identity maps. Schemes and their maps constitute a full subcategory.

A diagram depicting $\mathbf{S}$-maps is *natural* if each unlabeled arrow in it represents a map whose description, while omitted, is presumed evident. Arrows decorated with a "$\sim$" or "$\simeq$" represent isomorphisms.

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The abelian category of $\mathcal{O}_X$-modules on a ringed space X is denoted $\mathcal{A}(X)$; $\mathcal{A}_{\mathsf{qc}}(X) \subset \mathcal{A}(X)$ is the full subcategory spanned by the *quasi-coherent* $\mathcal{O}_X$-modules. The derived category of $\mathcal{A}(X)$ is denoted $\mathbf{D}(X)$; $\mathbf{D}_{\mathsf{qc}}(X) \subset \mathbf{D}(X)$ is the full subcategory spanned by the $\mathcal{O}_X$-complexes with quasi-coherent cohomology. $\mathbf{D}^+(X) \subset \mathbf{D}(X)$ is the full subcategory spanned by the *locally* cohomologically bounded-below complexes (those $C \in \mathbf{D}(X)$ for which there is an open cover $(X_\alpha)_{\alpha\in A}$ of X and for each α an integer n_α such that the restriction $(H^iC)|_{X_\alpha}$ vanishes for all $i < n_\alpha$); and $\mathbf{D}^+_{\mathsf{qc}}(X) := \mathbf{D}_{\mathsf{qc}}(X) \cap \mathbf{D}^+(X)$.

To reduce clutter, for any monoidal category $(\mathbf{C}, \otimes)$ and $A, B, C \in \mathbf{C}$ we will identify—harmlessly, via the natural isomorphism—$(A \otimes B) \otimes C$ with $A \otimes (B \otimes C)$, and denote either of these objects as $A \otimes B \otimes C$.

Assigning to each ringed-space map $f\colon X \to Y$ the derived direct image $\mathsf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ and the derived inverse image $\mathsf{L}f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X)$ leads to a pair of *adjoint monoidal pseudofunctors* on $\mathbf{S}$, see [L09, 3.6.7, 3.6.10].

The abbreviation "qcqs" denotes "quasi-compact and quasi-separated."[1] A basic fact of Grothendieck Duality theory is that *for any map $f\colon X \to Y$ of qcqs schemes, the restriction to $\mathbf{D}_{\mathsf{qc}}(X)$ of $\mathsf{R}f_*$ has a right adjoint,* i.e., there exist a functor $f^\times\colon \mathbf{D}(Y) \to \mathbf{D}_{\mathsf{qc}}(X)$ and a functorial map

$$\tau_f^\times\colon \mathsf{R}f_*f^\times G \to G \qquad (G \in \mathbf{D}(Y))$$

such that for any complex $F \in \mathbf{D}_{\mathsf{qc}}(X)$, the natural composite map

$$\mathrm{Hom}_{\mathbf{D}(X)}(F, f^\times G) \to \mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*F, \mathsf{R}f_*f^\times G) \xrightarrow{\text{via } \tau_f^\times} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*F, G)$$

is an isomorphism. (See, e.g., [L09, Corollary 4.1.2] and the notes following its proof.)

1.2. One of our goals, ultimately, is to prove the "Ideal Theorem," called in [H66, p. 6] the *primum mobile* of that book, and which—with restrictions we won't need (existence of dualizing complexes, coherence of cohomology of sheaf-complexes)—is one of its main results (*ibid.,* p. 383, Corollary 3.4).

Paraphrased, the Ideal Theorem asserts, *first of all,* the existence of a *duality pseudofunctor,* by which is meant a $\mathbf{D}^+_{\mathsf{qc}}$-valued pseudofunctor $(-)^!$ on the category $\mathcal{E}$ of finite-type separated maps of noetherian schemes, and for each proper $\mathcal{E}$-map f a functorial map $\tau_f\colon \mathsf{R}f_*f^! \to \mathrm{id}$, satisfying the following properties, of which (i), (ii) and (iv) jointly determine these data up to unique isomorphism:

(i) For any étale $\mathcal{E}$-map f, $f^!$ is the usual restriction functor f^*.

(ii) (Duality). For any proper $\mathcal{E}$-map f, τ_f makes $f^!$ right-adjoint to $\mathsf{R}f_*$ (i.e., in §1.1, one can take $(f^\times, \tau_f^\times) := (f^!, \tau_f)$).

[1] In the oft to be referred-to exposition [L09], this condition is called "concentrated." The frequent subsequent references to [L09] (of which these notes may be viewed as a continuation) are due much more to its approach and convenience than to any originality.

(iii) (Flat base change). For any fiber square in $\mathcal{E}$

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\big\downarrow & \clubsuit & \big\downarrow{\scriptstyle f} \\ Z' & \xrightarrow[u]{} & Z \end{array} \tag{1.2.1}$$

with f (hence g) proper and u (hence v) flat, and $F \in \mathbf{D}^+_{\mathsf{qc}}(Z)$, the map

$$\beta_\clubsuit(F)\colon\ : v^*f^!F \to g^!u^*F \tag{1.2.2}$$

adjoint (via (ii)) to the natural composition

$$\mathsf{R}g_*v^*f^!F \xrightarrow{\ \sim\ } u^*\mathsf{R}f_*f^!F \xrightarrow[u^*\tau_f]{} u^*F$$

is an *isomorphism.*

(iv) (For gluing (i) and (ii)). In (iii), if u (hence v) is an open immersion, then $\beta_\clubsuit(F)$ is equal to the natural composite isomorphism (which exists for *any* commutative square (1.2.1) with u and v étale)

$$v^*f^!F = v^!f^!F \xrightarrow{\ \sim\ } (fv)^!F = (ug)^!F \xrightarrow{\ \sim\ } g^!u^!F = g^!u^*F.$$

This much of the Ideal Theorem is contained in Theorems 4.8.1 and 4.8.3 of the notes [L09], and was extended to essentially-finite-type maps by Nayak in [Nk09, §5.2]. The methods of proof are largely category-theoretic, in line with the "abstract" development of Grothendieck Duality initiated by Verdier and Deligne (see Deligne's Appendix in [H66], and also [Nm96]).

The pseudofunctor $(-)^!$ extends from $\mathbf{D}^+_{\mathsf{qc}}$ to $\mathbf{D}_{\mathsf{qc}}$ if one restricts to proper maps or to $\mathcal{E}$-maps of finite tor-dimension [AJL11, §§5.7–5.9]—and even without such restrictions if one relaxes "pseudofunctor" to "oplax functor," i.e., one allows that for an $\mathcal{E}$-diagram $W \xrightarrow{g} X \xrightarrow{f} Z$ the associated map $(fg)^! \to g^!f^!$ need not be an isomorphism, see [Nm23]. (For maps of finite tor-dimension the agreement of the oplax $(-)^!$ with the preceding pseudofunctor results from [Nm23, Prop. 13.11].) In fact, Neeman's results apply to a broad class of noetherian stacks, including those of Deligne-Mumford.

Nayak has also established extensions to composites of pseudoproper maps and étale maps of formal schemes [Nk05, Theorem 7.1.6], and to composites of proper flat pseudo-coherent maps and étale maps of qcqs schemes [Nk05, Theorem 7.3.2]. In another direction, extensions are emerging in derived algebraic geometry, see, for example, [Ga13, §3], [LZ24], [Sch18].

Secondly—and this will be the focus of our attention—the Ideal Theorem gives *concrete realizations* of the pseudofunctor $(-)^!$ over the subcategories of $\mathcal{E}$ spanned, respectively, by its finite maps and its smooth maps, and concrete descriptions of abstractly specified pseudofunctorial maps associated to some combinations of these two types of map.

A concrete realization of $(-)^!$ is, informally stated, a *concretely describable* duality pseudofunctor $(-)^{\#}$, preferably, though not necessarily, canonical.[2]

A concrete realization of a functorial map built up from concrete elementary maps involving the identity functor id, the derived tensor product and derived direct image, by means of categorical operations like adjunction, composition and successive application of previously defined functors, is a concrete description of such a map. (This somewhat vague characterization will be clarified by a number of examples, starting in section 2.6.)

In particular, for proper $\mathcal{E}$-maps f one wants a *concrete right adjoint* $f^{\#}$ *of* $\mathsf{R}f_*$, and a *concrete counit map* $\mathsf{R}f_*f^{\#} \to \mathrm{id}$, varying pseudofunctorially. Two such pairs are necessarily canonically isomorphic.

The first part of this exposition explores constructs associated to certain *finite* maps $f\colon X \to Y$, for which, as in [H66, pp. 164–175]—though with weaker hypotheses, see §2.3—one defines "quasi-concretely" a functor

$$f^\flat\colon \mathbf{D}^+_{\mathsf{qc}}(Y) \to \mathbf{D}^+_{\mathsf{qc}}(X)$$

plus a functorial $f_*\mathcal{O}_X$-isomorphism

$$\bar{t}_f\colon \mathsf{R}f_*f^\flat \xrightarrow{\;\sim\;} \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, -),$$

such that with t_f the natural composite map

$$\mathsf{R}f_*f^\flat \underset{\bar{t}_f}{\xrightarrow{\;\sim\;}} \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, -) \longrightarrow \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, -) \xrightarrow{\;\sim\;} \mathrm{id},$$

$(f^\flat, t_f)$ is a right adjoint for $\mathsf{R}f_*$. Thus $(f^\flat, t_f)$ is a realization of $(f^!, \tau_f)$ (*pseudofunctorially,* see §2.5). The definition involves $\mathsf{R}\mathcal{H}om(f_*\mathcal{O}_X, -)$ and the left adjoint $\bar{f}^*$ of $\bar{f}_*$ where $\bar{f}\colon (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X) := \overline{Y}$ is the natural ringed-space map (see Example 2.1.1); these functors, being characterized by universal properties, are well-defined only up to canonical isomorphism, and thus limit the extent to which $(f^\flat, t_f)$ can be considered to be concrete or canonical.

Sometimes, simpler realizations exist. For example, restricting to finite maps $f\colon X \to Y$ that are locally finitely presentable and flat (that is, $f_*\mathcal{O}_X$ is locally free of finite rank over $\mathcal{O}_Y$), and to quasi-coherent $\mathcal{O}_Y$-complexes F, there is a right adjoint $(f^\flat, \bar{t}_f)$ for the $\mathbf{D}(\mathcal{A}_{\mathsf{qc}})$-valued pseudofunctor $(-)_*$ with $f^\flat F := \bar{f}^*\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \in \mathcal{A}_{\mathsf{qc}}(X)$, and $\bar{t}_f(F)$ the isomorphism

$$f_*f^\flat F = f_*\bar{f}^*\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \xrightarrow{\;\sim\;} \mathcal{H}om_Y(f_*\mathcal{O}_X, F)$$

arising from $\bar{f}^*\colon \mathcal{A}_{\mathsf{qc}}(\overline{Y}) \to \mathcal{A}_{\mathsf{qc}}(X)$ being quasi-inverse to $\bar{f}_*$. This $(f^\flat, \bar{t}_f)$ is as concrete or canonical as $\bar{f}^*$ is. If Y is separated and quasi-compact, the natural functor $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(Y)) \to \mathbf{D}_{\mathsf{qc}}(Y)$ is an equivalence, so every complex $G \in \mathbf{D}_{\mathsf{qc}}(Y)$ is isomorphic (functorially) to a quasi-coherent complex QG;

[2] *Concrete* and *canonical* are somewhat flexible concepts. In what sense, for instance, is the number 1 canonical? As Humpty Dumpty said, "a word means . . . just what I choose it to mean."

thus, over such schemes one gets a realization of $(f^!, \tau_f)$ that is concrete or canonical to the extent that the functors $\bar{f}^*$ and Q are.

Suppose, moreover, f is *étale*. The usual trace map $f_*\mathcal{O}_X \to \mathcal{O}_Y$ lifts to an $f_*\mathcal{O}_X$-isomorphism $f_*\mathcal{O}_X \xrightarrow{\sim} \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)$, giving, for $G \in \mathbf{D}(Y)$, the first of the functorial $f_*\mathcal{O}_X$-isomorphisms (the second being natural)

$$f_*\mathcal{O}_X \otimes_Y G \xrightarrow{\sim} \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) \otimes_Y G \xrightarrow{\sim} \mathcal{H}om_Y(f_*\mathcal{O}_X, G). \tag{1.2.3}$$

If $\phi\colon \overline{Y} \to Y$ is the ringed-space map corresponding to the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ (so that $f = \phi\bar{f}$), then the functor $f_*\mathcal{O}_X \otimes_Y (-)\colon \mathbf{D}(Y) \to \mathbf{D}(\overline{Y})$ is left-adjoint to ϕ_*, and so may be identified with ϕ^*. Then by applying $\bar{f}^*$ to (1.2.3), one gets a functorial $\mathcal{O}_X$-isomorphism

$$c_f^\flat\colon f^*G \xrightarrow{\sim} \bar{f}^*\mathcal{H}om_Y(f_*\mathcal{O}_X, G) = f^\flat G,$$

whence the concrete realization (f^*, tr_f) of $(f^!, \tau_f)$, where for $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$, $\mathrm{tr}_f(G)$ is the natural composite map

$$\mathsf{R}f_*f^*G \xrightarrow{\mathsf{R}f_*c_f^\flat} \mathsf{R}f_*f^\flat G \longrightarrow G,$$

which can be shown (using e.g., [L09, (3.7.1)] with $f := \bar{f}$ and $g := \phi$ plus [L09, (3.4.7)(ii)] with $A := \phi^*G$ and $f := \bar{f}$) to be the natural composite map

$$\mathsf{R}f_*f^*G \xrightarrow{\sim} f_*\mathcal{O}_X \otimes_Y G \xrightarrow{\text{trace}\otimes\text{id}} \mathcal{O}_Y \otimes_Y G \xrightarrow{\sim} G.$$

Again, this realization is canonical insofar as the left adjoint f^* of f_* is.

For an extension to "almost étale" f, see Proposition 2.9.13.

When f is a map of affine schemes, this all has a well-known commutative-algebra translation. Indeed, for any commutative ring R, sheafification and the derived global-section functor induce inverse *equivalences* between duality theory over the derived category of R-modules and that over $\mathbf{D}_{\mathsf{qc}}(\operatorname{Spec} R)$, allowing us to realize functors and functorial maps in the latter through concrete commutative-algebra constructions in the former.

For example, when S is a finite R-algebra, with corresponding scheme-map $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$, then one gets a concrete right adjoint for $\mathsf{R}f_*$ by sheafifying the fact that the restriction-of-scalars functor from $\mathbf{D}(S)$ to $\mathbf{D}(R)$ has as right adjoint the functor $\mathsf{R}\mathrm{Hom}_R(S, -)$ together with the natural functorial $\mathbf{D}(R)$-map

$$\mathsf{R}\mathrm{Hom}_R(S, M) \to \mathsf{R}\mathrm{Hom}_R(R, M) = M \qquad (M \in \mathbf{D}(R)).$$

Another example is the commutative-algebra map corresponding to the projection map ((iii) in §1.3 below), as described in Lemma 3.3.6.

There is much more along these lines in §3.

As another example, a detailed account of duality provides, for a *smooth* $\mathcal{E}$-map $f\colon X \to Y$ with fibers of pure dimension d and Ω_f^d the sheaf of relative d-forms, a *canonical functorial isomorphism*

$$f^\# F := \Omega_f^d[d] \otimes_X f^*F \xrightarrow[c_f(F)]{\sim} f^!F \qquad (F \in \mathbf{D}_{\mathsf{qc}}(Y)) \tag{1.2.4}$$

(where "$[d]$" denotes d-fold translation in $\mathbf{D}(X)$), having pseudofunctorial variance induced by canonical isomorphisms of the form

$$\Omega^e_f[e]\otimes_X f^*\Omega^d_g[d] \xrightarrow{\ \sim\ } \Omega^{d+e}_{gf}[d+e]$$

for smooth $\mathcal{E}$-maps $g\colon Y\to Z$ with fibers of pure dimension e; and further, when f is also *proper,* an explicit elucidation of the composite map

$$\mathrm{Tr}_f\colon \mathsf{R}f_*f^{\#} \underset{\mathsf{R}f_*c_f}{\xrightarrow{\ \sim\ }} \mathsf{R}f_*f^! \underset{\tau_f}{\longrightarrow} \mathrm{id}$$

via the theory of residues, as sketched in [V68, pp. 398–400] and developed in [HS93] or [LS92, Proposition 4.2.2].[3]

Assuming the first part of the Ideal Theorem, together with a canonical representation of $g^!$ when g is a regular immersion [H66, p. 180, Corollary 7.3] (reproduced, in essence, in Proposition 2.10.12 below), Verdier constructed such a c_f in [V68, Proof of Theorem 3]. (A much expanded treatment of this "fundamental class" is given in [LN17].) But proving pseudofunctoriality of c_f is far from straightforward. That, and much more about c_f and its relation to traces and residues, is addressed in [NkS19], in the context of formal schemes.

The finite étale situation is the overlap between the finite one and the smooth one. The "concrete" proof in [H66] of the Ideal Theorem depends on pseudofunctorially gluing $(-)^\flat$ and $\Omega^*_{(-)}[*]$ along that overlap. In contrast, the basic idea here will be to show (with weaker hypotheses) that the above pseudofunctor $(-)^!$, *abstractly constructed* via gluing of $(-)^*$ over étale maps and of a right adjoint of $\mathsf{R}(-)_*$ over proper maps, is isomorphic over finite (resp. smooth) maps to $(-)^\flat$ (resp. $\Omega^*_{(-)}[*]$).

1.3. The foregoing instantiates a more general theme, as follows.

For our relatively unsophisticated purposes, the *abstract yoga of duality* takes a dualizing structure on a category $\mathbf{C}$ to be an adjoint pair $(^*,{}_*)$ of monoidal pseudofunctors ([L09, §3.6] or [L09a, Lecture 3]), taking values in closed categories $\mathbf{D}_X$ $(X\in\mathbf{C})$ with product $\otimes_X$ and unit $\mathcal{O}_X$, *plus* for each $\mathbf{C}$-map $\psi\colon X\to Y$ a right adjoint $\psi^\times$ of the functor $\psi_*\colon \mathbf{D}_X\to\mathbf{D}_Y$, with specified counit map $\psi_*\psi^\times\to\mathrm{id}$, *plus* a class S of oriented commutative $\mathbf{C}$-squares[4]

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & \spadesuit & \big\downarrow{\scriptstyle f} \\ Z' & \xrightarrow[\ u\ ]{} & Z, \end{array}$$

with S closed under vertical and horizontal juxtaposition, such that for all $\psi\colon X\to Y$, the following maps are *isomorphisms*:

[3]In [H66, p. 383] this is required only for $X:=\mathbb{P}^d(Y)$ and $f\colon X\to Y$ the natural map, in which case Tr_f can be described explicitly via Čech complexes, see e.g., [HK90, §5].

[4]An oriented commutative square is a quadruple of maps (u,f,v,g) such that $ug=fv$, *plus* an ordering of the pair (u,f). A diagram such as the following one always represents a commutative square that is oriented by putting the bottom arrow u ahead of f.

(i) the map $\psi^*\mathcal{O}_Y \to \mathcal{O}_X$ adjoint to the natural map $\mathcal{O}_Y \to \psi_*\mathcal{O}_X$;

(ii) for all $F, G \in \mathbf{D}_Y$, the map $\psi^*(F \otimes_Y G) \to \psi^*F \otimes_X \psi^*G$ adjoint to the natural composite map

$$F \otimes_Y G \to \psi_*\psi^*F \otimes_Y \psi_*\psi^*G \to \psi_*(\psi^*F \otimes_X \psi^*G);$$

(iii) for all $E \in \mathbf{D}_X$ and $F \in \mathbf{D}_Y$, the map $\psi_*E \otimes_Y F \to \psi_*(E \otimes_X \psi^*F)$ ("projection map") adjoint to the natural composite map

$$\psi^*(\psi_*E \otimes_Y F) \to \psi^*\psi_*E \otimes_X \psi^*F \to E \otimes_X \psi^*F;$$

(iv) for each $\spadesuit \in \mathsf{S}$ and $G \in \mathbf{D}_X$, the map $u^*f_*G \to g_*v^*G$ adjoint to the natural composite map $g^*u^*f_*G \xrightarrow{\sim} v^*f^*f_*G \longrightarrow v^*G$;

(v) for each $\spadesuit \in \mathsf{S}$ and $F \in \mathbf{D}_Z$, the map $v^*f^\times F \to g^\times u^*F$, adjoint to the natural composite map $g_*v^*f^\times F \xrightarrow{\sim} u^*f_*f^\times F \longrightarrow u^*F$.

On these axiomatic foundations, one constructs, using only categorical operations (adjunction, composition, composite functors . . .), a superstructure of pseudofunctorial maps, and compatibilities among them expressed by commutative diagrams. (For more in this vein, see e.g., [L09, §3.5.4] or [L09a, Lectures 4 and 6]. Even more generally, see [Ho22] or [CD19, Part I].)

This abstract theory is modeled by a variety of specific situations. For example (somewhat oversimplified), $\mathbf{C}$ could be some category of commutative rings, $\mathbf{D}_X$ the category of X-modules (or its derived category) with the usual closed structure, and $(^*, _*)$ the usual (or derived) extension- and restriction-of-scalars pseudofunctors. (For elaboration, see §3.) Or, $\mathbf{C}$ could be some category of ringed spaces, $\mathbf{D}_X$ the category of quasi-coherent $\mathcal{O}_X$-modules (or $\mathbf{D}^+_{\mathsf{qc}}(X)$, see §1.2), and $(^*, _*)$ the usual (or derived) inverse- and direct-image pseudofunctors. Or, $\mathbf{C}$ could be the category of compactifiable maps of qcqs schemes, and $\mathbf{D}_X$ the derived category of torsion sheaves on X with the étale topology [De73]. Other categories that support duality theories are those whose objects are certain finite diagrams of noetherian schemes, with flat arrows [Ha09], or certain finite ringed spaces [SS20], or certain algebraic stacks [Nm23]. With a few extras, one can also consider categories of topological rings (local duality) or noetherian formal schemes, each $\mathbf{D}_X$ being a suitable ordinary or derived category [AJL99]. There are other examples, for instance those mentioned in §1.2. Undoubtedly, more will emerge in the future.

In specific situations, to enliven things and enhance applicability one needs *concrete interpretations* of the functors and maps in the preceding conditions (i)–(v), as well as of useful pseudofunctorial maps that can be categorically derived.

For instance, in the context of §1.2, concrete interpretations of $f^\times$ were indicated for smooth or finite $\mathcal{E}$-maps. As for noteworthy derived maps, in that situation there are concrete descriptions, at least via flat or injective resolutions, in various places in [H66]. However, the question of whether the maps so described are the same as the corresponding categorically-defined ones is not often addressed. (See the footnote in the proof of 2.1.9 below.) So one cannot say without further ado that, for example, the multitudinous diagrams in [L09] that are abstractly shown to commute remain commutative when their maps are interpreted as in [H66].

The overall goal here is to embed the concrete duality theory in [H66] *(amended in* [Co00]*) into the abstract one in* [L09], *by showing how important maps in* [H66] *can be described in category-theoretic terms.*[5]

Section 2 below is devoted to doing this for certain finite maps, section 3 to translation into commutative-algebra terms. Beyond (i)–(v), there being an endless number of maps that can be categorically deduced, only a few salient examples will be examined closely, such as the maps

$$\mathsf{L}f^*E\otimes^{\mathsf{L}}_X f^\times F \longrightarrow f^\times(E\otimes^{\mathsf{L}}_Y F) \qquad (E,\ F\in\mathbf{D}_{\mathsf{qc}}(Y)),$$
$$\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*E, f^\times F) \longrightarrow f^\times\mathsf{R}\mathcal{H}om_Y(E,F) \qquad (E,\ F\in\mathbf{D}_{\mathsf{qc}}(Y)),$$

associated with a finite map $f\colon X\to Y$. (See 2.7.7 and 2.8.2.)

It is intended that smooth $\mathcal{E}$-maps will be treated in a subsequent part of this exposition. For such maps, basic results—even for the context of formal schemes—can be found in [NkS19].

2. Pseudo-coherent finite maps

Recall that a scheme-map $f\colon X\to Y$ is *affine* (resp. *finite*) if for each affine open subscheme $U\subset Y$—or equivalently, for every member of some affine open cover of Y—the scheme $f^{-1}U$ is affine (resp. affine and such that the natural map makes $\Gamma(f^{-1}U,\mathcal{O}_X)$ into a finite $\Gamma(U,\mathcal{O}_Y)$-module), see [GrD61a, 1.3.2, 6.1.4]. Any affine f is *separated,* and when for each U the $\Gamma(U,\mathcal{O}_Y)$-module $\Gamma(f^{-1}U,\mathcal{O}_X)$ is locally finitely presentable, *proper.*

This section 2 is concerned with a concrete $\mathbf{D}^+_{\mathsf{qc}}$-valued pseudofunctor $(-)^\flat$ together with functorial maps $\mathsf{R}(-)_*(-)^\flat\to\mathrm{id}$, over the category $\mathbf{\Phi}$ of finite maps $f\colon X\to Y$ that are *pseudo-coherent*, meaning $f_*\mathcal{O}_X$ is locally resolvable by a complex of finite-rank locally free $\mathcal{O}_Y$-modules (see 2.3.7). (For example, finite maps of locally noetherian schemes, finite locally free maps, and regular immersions all are pseudo-coherent.) These data constitute a pseudofunctorial right adjoint for $\mathsf{R}(-)_*$. Restricting to qcqs schemes in $\mathbf{\Phi}$, one has then a *concrete realization of the pseudofunctorial pair* $((-)^\times,\tau^\times_-)$.

The locally noetherian case is treated in [H66, pp. 164–175], where it is indicated that the "usual reductions" cut things down to the elementary context of modules over commutative rings. (Cf. section 3 below.) The present approach is more general and technical, and also more explicit, than that classical one, but basically similar, as follows.

Fix a scheme Y. The direct-image functor gives an equivalence between (i): a category whose objects are pairs (X,F) with X a scheme affine over Y and F a quasi-coherent $\mathcal{O}_X$-module and, (ii): the opposite of a category

[5] For orientation, consider the analogous theme in the elementary duality theory of ordinary restriction- and extension-of-scalars pseudofunctors between modules over commutative rings.

whose objects are pairs (L,F) with L a quasi-coherent $\mathcal{O}_Y$-algebra and F a quasi-coherent L-module, see [GrD71, §9.2]. In greater generality, with F and F replaced by objects in $\mathbf{D}_{\mathsf{qc}}(X)$ and $\mathbf{D}_{\mathsf{qc}}(\mathsf{L})$ respectively, the derived direct-image functor induces an equivalence, see Proposition 2.1.6. There is an explicit $\mathbf{D}_{\mathsf{qc}}(-)$-valued duality pseudofunctor over quasi-coherent $\mathcal{O}_Y$-algebras, globalizing the well-known pseudofunctor over commutative rings. (For the latter, see 3.1.18–3.1.23). To transfer this pseudofunctor over to Y-schemes, via the equivalence, one needs to remain in a quasi-coherent context. This can be done, for instance, using right adjoints $\mathsf{R}\overline{Q}_L$ for the inclusions $\mathbf{D}_{\mathsf{qc}}(\mathsf{L}) \hookrightarrow \mathbf{D}(\mathsf{L})$; but discussion along these lines appears only briefly, in §2.11, because $\mathsf{R}\overline{Q}$ is awkward to explicate for non-affine schemes, and it doesn't commute with open immersions.

Rather, we'll just restrict to $\mathbf{\Phi}$, where $\mathsf{R}\overline{Q}$ is not needed because $\mathbf{\Phi}$-maps $f\colon X\to Y$ have the following key property (Lemma 2.3.8):

$$\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathbf{D}^+_{\mathsf{qc}}(Y)) \subset \mathbf{D}^+_{\mathsf{qc}}(Y).$$

Consequently, the equivalence $\mathsf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \xrightarrow{\approx} \mathbf{D}_{\mathsf{qc}}(f_*\mathcal{O}_X)$ in 2.1.6 reduces finding a concrete right adjoint $(f^\flat, t_f)$ for $\mathsf{R}f_*\colon \mathbf{D}^+_{\mathsf{qc}}(X) \to \mathbf{D}^+_{\mathsf{qc}}(Y)$ to finding one for the restriction-of-scalars functor $\phi_*\colon \mathbf{D}_{\mathsf{qc}}(f_*\mathcal{O}_X) \to \mathbf{D}_{\mathsf{qc}}(Y)$. But *derived adjoint associativity,* as enhanced in Proposition 2.2.1, implies that the functor $\phi_*\colon \mathbf{D}(f_*\mathcal{O}_X) \to \mathbf{D}(Y)$ has the right adjoint

$$\phi^\flat(-) := \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, -),$$

with counit the natural $\mathbf{D}(Y)$-map ("evaluation at 1")

$$\phi_*\phi^\flat G = \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G) \longrightarrow \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, G) = G \qquad (G\in\mathbf{D}(Y)),$$

giving the desired construction (see Proposition 2.3.5).

In fact Theorem 2.3.9 says more: for any $\mathbf{\Phi}$-map $f\colon X\to Y$, $F\in\mathbf{D}_{\mathsf{qc}}(X)$ and $G\in\mathbf{D}^+_{\mathsf{qc}}(Y)$, there is a *sheafified duality isomorphism*

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^\flat G) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, G),$$

which turns out to be a concrete realization of the standard abstract one, namely the natural composite

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^!G) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, \mathsf{R}f_*f^!G) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, G),$$

see Proposition 2.4.5.

Arguing as one does for $(-)^\times$, one gets, for pseudo-coherent finite maps, basic properties of $(-)^\flat$ such as *pseudofunctoriality* (Proposition 2.5.2) and *tor-independent base change* (Theorem 2.6.4).

In further illustration of §1.3, concrete interpretations are given, for suitable $f\colon X\to Y$ and $F,G\in\mathbf{D}_{\mathsf{qc}}(Y)$, of some basic abstractly defined maps: the pseudofunctoriality isomorphism (Proposition 3.1.23), and the base-change map of (1.2.2)—with $f^\flat$ in place of $f^!$ (Proposition 2.6.14);

and the maps

$$f^\flat F\otimes^{\mathsf L}_X \mathsf Lf^*G \to f^\flat(F\otimes^{\mathsf L}_Y G),$$
$$\mathsf R\mathcal Hom_X(\mathsf Lf^*F, f^\flat G)\to f^\flat\mathsf R\mathcal Hom_Y(F,G),$$

(2.7.7 and 2.8.2, respectively).

Section 2.9 deals with the role played in concrete duality for a perfect affine map $f\colon X\to Y$ by the functorial *trace map*

$$\mathrm{tr}_f(G)\colon \mathsf Rf_*\mathsf Lf^*G\longrightarrow G\qquad (G\in\mathbf D_{\mathsf{qc}}(Y))$$

from [Il71, p. 154, 8.1], and by its dual, the *fundamental class* map

$$C_f(G)\colon \mathsf Lf^*G\longrightarrow f^\flat G\qquad (G\in\mathbf D_{\mathsf{qc}}(Y)),$$

which is an isomorphism whenever f is étale.

Section 2.10 discusses duality for Koszul-regular immersions—a class of perfect closed immersions of schemes which includes all regular immersions. On this class, there is a well-known concrete realization of $(-)^\flat$, involving normal bundles (see Proposition 2.10.12). The surprisingly difficult task of showing that this realization is *pseudofunctorial* (see Theorem 2.10.22) gets reduced to the case of affine schemes, which is translated into commutative-algebra terms in section 3, and then disposed of at the end of that section, along the lines of the proof in [NkS19, Appendix C.6]. ([Co00, §§2.5–2.6] contains the only other proof I know of.)

2.1. Associate to an affine scheme-map $f\colon X\to Y$ the map of ringed spaces

$$(X,\mathcal O_X)\xrightarrow{\bar f:=(f,\,\mathrm{id})}(Y,f_*\mathcal O_X)=:\overline Y.$$

This $\bar f$ is *flat,* i.e., for all $x\in X$, the stalk $\mathcal O_{X,x}$ is flat over $\mathcal O_{\overline Y,\bar f(x)}$: to check this one can assume that $X=\operatorname{Spec}S$, $Y=\operatorname{Spec}R$, with f corresponding to a commutative-ring homomorphism $\varphi\colon R\to S$, then note that if p is a prime S-ideal, the stalk S_p of $\mathcal O_X$ at p is a localization of—thus flat over—the stalk $S\otimes_R R_{\varphi^{-1}p}$ of $f_*\mathcal O_X$ at $f(p)$. So the functor $\bar f^*\colon \mathcal A(\overline Y)\to\mathcal A(X)$ is exact.

The restriction of $\bar f_*$ to $\mathcal A_{\mathsf{qc}}(X)$ is an equivalence (necessarily exact) from $\mathcal A_{\mathsf{qc}}(X)$ to $\mathcal A_{\mathsf{qc}}(\overline Y)$ [GrD71, p. 361, (9.2.1)][6]. The left adjoint $\bar f^*$ of $\bar f_*$ takes $\mathcal A_{\mathsf{qc}}(\overline Y)$ to $\mathcal A_{\mathsf{qc}}(X)$ [GrD71, p. 108, (5.1.4)], and so provides a quasi-inverse equivalence.

These quasi-inverse equivalences, being exact, extend to quasi-inverse equivalences of derived categories

$$\mathbf D\big(\mathcal A_{\mathsf{qc}}(X)\big)\rightleftarrows\mathbf D\big(\mathcal A_{\mathsf{qc}}(\overline Y)\big).$$

Example 2.1.1. (Cf. [GrD71, top of p. 362]) When $f=\operatorname{Spec}(\varphi)$ with $\varphi\colon R\to S$ as above, here is what's happening, in concrete terms.

Denote by φ_* the functor "restriction-of-scalars via φ" from S-modules to R-modules. Let M be an S-module, and let $\widetilde{M}_S$ (resp. $\widetilde{M}_R:=(\varphi_*M)\widetilde{{}_R}$)

[6] whose proof states incorrectly, if harmlessly, that $\overline Y$ is a *locally* ringed space. This proof has other features that make it harder to read than the original [GrD61a, (1.4.1), (1.4.3)].

be the corresponding quasi-coherent $\mathcal{O}_X$-(resp. $f_*\mathcal{O}_X$-)module. Any quasi-coherent $f_*\mathcal{O}_X$-module $\mathcal{M}$ is naturally isomorphic to such an $\widetilde{M}_R$: take $M := \Gamma(Y, \mathcal{M})$ [GrD71, p. 207, (1.4.5)]; and [GrD71, p. 214, (1.7.7(ii))] gives a natural $f_*\mathcal{O}_X$-isomorphism $\widetilde{M}_R \cong \bar{f}_*\widetilde{M}_S$. Hence, since $\bar{f}^*$ is quasi-inverse to $\bar{f}_*$, there is a natural isomorphism $\bar{f}^*\widetilde{M}_R \cong \widetilde{M}_S$.

2.1.2. If $F_1 \to F_2 \to F \to F_3 \to F_4$ is an exact sequence of $f_*\mathcal{O}_X$-modules with F_1, F_2, F_3 and F_4 quasi-coherent, then F is quasi-coherent (as follows via [GrD71, p. 218, (2.2.4)] from the similar property of $\mathcal{O}_Y$-modules [GrD71, p. 217, (2.2.2)(iii)]). Hence $\mathbf{D}_{\mathsf{qc}}(\overline{Y}) \subset \mathbf{D}(\overline{Y})$ is a triangulated subcategory (identifiable with the derived category of the category of $f_*\mathcal{O}_X$-complexes with quasi-coherent homology, see [L09, (1.9.1)]).

Since $\bar{f}^*$ is exact and, as above, preserves quasi-coherence, therefore

$$\bar{f}^*\mathbf{D}_{\mathsf{qc}}(\overline{Y}) \subset \mathbf{D}_{\mathsf{qc}}(X) \quad \text{and} \quad \bar{f}^*\mathbf{D}^+_{\mathsf{qc}}(\overline{Y}) \subset \mathbf{D}^+_{\mathsf{qc}}(X). \tag{2.1.3}$$

Also,

$$\mathsf{R}\bar{f}_*(\mathbf{D}_{\mathsf{qc}}(X)) \subset \mathbf{D}_{\mathsf{qc}}(\overline{Y}) \quad \text{and} \quad \mathsf{R}\bar{f}_*(\mathbf{D}^+_{\mathsf{qc}}(X)) \subset \mathbf{D}^+_{\mathsf{qc}}(\overline{Y}). \tag{2.1.4}$$

For, with $\psi = \psi_f \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ the homomorphism associated to f and

$$\phi = \phi_f := (\mathrm{id}, \psi) \colon \overline{Y} \to Y \tag{2.1.5}$$

the resulting ringed-space map (so that $f = \phi\bar{f}$), there are, for all $i \in \mathbb{Z}$ and $E \in \mathbf{D}_{\mathsf{qc}}(X)$, natural isomorphisms

$$\phi_* H^i\mathsf{R}\bar{f}_*E \xrightarrow{\sim} H^i\phi_*\mathsf{R}\bar{f}_*E \xrightarrow{\sim} H^i\mathsf{R}(\phi\bar{f})_*E = H^i\mathsf{R}f_*E;$$

hence—since an $f_*\mathcal{O}_X$-module G is quasi-coherent if ϕ_*G is [GrD71, p. 218, (2.2.4)], and, clearly, vanishes if ϕ_*G does—it's enough to prove (2.1.4) with $\bar{f} \colon X \to \overline{Y}$ replaced by $f \colon X \to Y$, which is done in [L09, 3.9.2]. (Or, reduce to where Y and X are affine, say $X = \operatorname{Spec} R$, and apply [BN93, p. 225, 5.1].)

Proposition 2.1.6. *The functor $\bar{f}^* \colon \mathbf{D}(\overline{Y}) \to \mathbf{D}(X)$ induces an equivalence from $\mathbf{D}_{\mathsf{qc}}(\overline{Y})$ to $\mathbf{D}_{\mathsf{qc}}(X)$ (resp. $\mathbf{D}^+_{\mathsf{qc}}(\overline{Y})$ to $\mathbf{D}^+_{\mathsf{qc}}(X)$), with quasi-inverse induced by $\mathsf{R}\bar{f}_* \colon \mathbf{D}(X) \to \mathbf{D}(\overline{Y})$.*

Proof. The functor $\mathsf{R}\bar{f}_*$ is right adjoint to $\bar{f}^* \colon \mathbf{D}(\overline{Y}) \to \mathbf{D}(X)$ [L09, (3.2.1)]; so in view of (2.1.3) and (2.1.4), it suffices that the counit and unit maps

$$\epsilon_E \colon \bar{f}^*\mathsf{R}\bar{f}_*E \to E \ \ (E \in \mathbf{D}_{\mathsf{qc}}(X)) \quad \text{and} \quad \eta_F \colon F \to \mathsf{R}\bar{f}_*\bar{f}^*F \ \ (F \in \mathbf{D}_{\mathsf{qc}}(\overline{Y}))$$

both be isomorphisms (see [M98, p. 93, Theorem 1]).

To show that ϵ_E is an isomorphism one can assume that E is K-injective (:= q-injective [L09, §2.3]), so that $\mathsf{R}\bar{f}_*E \cong \bar{f}_*E$ and ϵ_E can be identified with the counit map associated to the adjunction between $\bar{f}^*$ and $\bar{f}_*$ (see [L09, (3.2.1.3)]), an isomorphism because these functors are, as above, quasi-inverse equivalences.

As for η_F, since $\epsilon_{\bar f^*F}$ is an isomorphism and the composite map

$$\bar f^*F \xrightarrow{\bar f^*\eta_F} \bar f^*\mathsf{R}\bar f_*\bar f^*F \xrightarrow{\epsilon_{\bar f^*F}} \bar f^*F$$

is the identity map, therefore $\bar f^*\eta_F$ is an isomorphism. Since $\bar f$ is flat one has, denoting cohomology functors by H^n, that for all $n \in \mathbb{Z}$, $\bar f^*H^n\eta_F$ is isomorphic to $H^n\bar f^*\eta_F$ and so is an isomorphism. Since $\bar f^*|_{\mathcal{A}_{\mathsf{qc}}(\overline{Y})}$ is an equivalence, therefore every $H^n\eta_F$ is an isomorphism, so η_F is an isomorphism. $\square$

For any ringed-space map $h\colon V \to W$ and E, $E' \in \mathbf{D}(V)$, one has the natural bifunctorial composite

$$\begin{aligned}\nu(E,E')\colon \mathsf{R}h_*\mathsf{R}\mathcal{H}om_V(E,E') &\longrightarrow \mathsf{R}h_*\mathsf{R}\mathcal{H}om_V(\mathsf{L}h^*\mathsf{R}h_*E,E') \\ &\xrightarrow{\;\sim\;} \mathsf{R}\mathcal{H}om_W(\mathsf{R}h_*E,\mathsf{R}h_*E')\end{aligned} \tag{2.1.7}$$

with the isomorphism as in [L09, 3.2.3(ii)]. Using [L09, 3.2.5(f)] and the last assertion in [L09, 3.2.3(i)], or otherwise, one verifies that application of the functor $H^0\mathsf{R}\Gamma(W,-)$ to this composite map produces the obvious map

$$\operatorname{Hom}_{\mathbf{D}(V)}(E,E') \longrightarrow \operatorname{Hom}_{\mathbf{D}(W)}(\mathsf{R}h_*E,\mathsf{R}h_*E').$$

If the natural map $\mathsf{L}h^*\mathsf{R}h_*E \to E$ is an isomorphism—for example if $h\colon V \to W$ is $\bar f\colon X \to \overline{Y}$ (as above) and $E \in \mathbf{D}_{\mathsf{qc}}(X)$—then so is the composite map (2.1.7). Hence:

Corollary 2.1.8 (Sheafified duality for $\bar f$)**.** *For $E \in \mathbf{D}_{\mathsf{qc}}(X)$, $F \in \mathbf{D}_{\mathsf{qc}}(\overline{Y})$, one has the composite bifunctorial isomorphism*

$$\mathsf{R}\bar f_*\mathsf{R}\mathcal{H}om_X(E,\bar f^*F) \underset{(2.1.7)}{\xrightarrow{\;\sim\;}} \mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar f_*E,\mathsf{R}\bar f_*\bar f^*F) \underset{2.1.6}{\xrightarrow{\;\sim\;}} \mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar f_*E,F),$$

to which application of $H^0\mathsf{R}\Gamma(Y,-)$ gives the adjunction isomorphism

$$\operatorname{Hom}_{\mathbf{D}_{\mathsf{qc}}(X)}(E,\bar f^*F) \xrightarrow{\;\sim\;} \operatorname{Hom}_{\mathbf{D}_{\mathsf{qc}}(\overline{Y})}(\mathsf{R}\bar f_*E,F).$$ $\square$

The equivalence 2.1.6 between $\mathbf{D}_{\mathsf{qc}}(\overline{Y})$ and $\mathbf{D}_{\mathsf{qc}}(X)$ is compatible with standard additional structures. For example, it respects the derived tensor product and sheafified hom functors, in the following sense:

Corollary 2.1.9. *For $E, F \in \mathbf{D}_{\mathsf{qc}}(X)$, the natural maps are isomorphisms*

$$\kappa\colon \mathsf{R}\bar f_*E \otimes^{\mathsf{L}}_{\overline{Y}} \mathsf{R}\bar f_*F \xrightarrow{\;\sim\;} \mathsf{R}\bar f_*(E \otimes^{\mathsf{L}}_X F),$$

$$\nu\colon \mathsf{R}\bar f_*\mathsf{R}\mathcal{H}om_X(E,F) \underset{2.1.7}{\xrightarrow{\;\sim\;}} \mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar f_*E,\mathsf{R}\bar f_*F);$$

and if $\mathsf{R}\mathcal{H}om_X(E,F) \in \mathbf{D}_{\mathsf{qc}}(X)$, then the natural composite

$$\bar f^*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar f_*E,\mathsf{R}\bar f_*F) \underset{\rho}{\longrightarrow} \mathsf{R}\mathcal{H}om_X(\bar f^*\mathsf{R}\bar f_*E,\bar f^*\mathsf{R}\bar f_*F) \underset{2.1.6}{\xrightarrow{\;\sim\;}} \mathsf{R}\mathcal{H}om_X(E,F)$$

is an isomorphism too.

Proof. By definition ([L09, 3.2.4(ii)]), κ is the unique map such that the following diagram, with ϵ as in the proof of 2.1.6, and the isomorphism on the left as in [L09, 3.2.4(i)], commutes:

$$\begin{array}{ccc} \bar f^*(\mathsf{R}\bar f_*E\otimes^{\mathsf{L}}_{\overline{Y}}\mathsf{R}\bar f_*F) & \xrightarrow{\bar f^*\kappa} & \bar f^*\mathsf{R}\bar f_*(E\otimes^{\mathsf{L}}_{X}F) \\ \simeq\downarrow & & \downarrow \epsilon_{E\otimes^{\mathsf{L}}_X F} \\ \bar f^*\mathsf{R}\bar f_*E\otimes^{\mathsf{L}}_{\overline{Y}}\bar f^*\mathsf{R}\bar f_*F & \xrightarrow[\epsilon_E\otimes^{\mathsf{L}}_X\epsilon_F]{} & E\otimes^{\mathsf{L}}_X F \end{array}$$

By 2.1.6, the counit maps ϵ_E, ϵ_F and $\epsilon_{E\otimes^{\mathsf{L}}_X F}$ are isomorphisms, and therefore so is $\bar f^*\kappa$, whence, by 2.1.6 again, so is κ.

That ν is an isomorphism was noted above (just before 2.1.8).

As for the last assertion, it holds by assumption that

$$\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar f_*E,\mathsf{R}\bar f_*F)\underset{\nu}{\cong}\bar f_*\mathsf{R}\mathcal{H}om_X(E,F)\in\mathbf{D}_{\mathsf{qc}}(\overline{Y})$$

and

$$\mathsf{R}\mathcal{H}om_X(\bar f^*\mathsf{R}\bar f_*E,\bar f^*\mathsf{R}\bar f_*F)\underset{2.1.6}{\cong}\mathsf{R}\mathcal{H}om_X(E,F)\in\mathbf{D}_{\mathsf{qc}}(X).$$

But it follows easily from 2.1.6 that for $A\in\mathbf{D}_{\mathsf{qc}}(\overline{Y})$ and $B\in\mathbf{D}_{\mathsf{qc}}(X)$, a $\mathbf{D}(X)$-map $\bar f^*A\to B$ is an isomorphism $\Leftrightarrow$ so is its adjoint $A\to\bar f_*B$. So the map ρ is an isomorphism, since by its definition [L09, (3.5.4.5)][7] it is adjoint to the natural composite $\mathbf{D}(\overline{Y})$-isomorphism

$$\begin{aligned}\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar f_*E,\mathsf{R}\bar f_*F) &\xrightarrow{\sim}\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar f_*E,\mathsf{R}\bar f_*\bar f^*(\mathsf{R}\bar f_*F))\\ &\xrightarrow{\sim}\mathsf{R}\bar f_*\mathsf{R}\mathcal{H}om_X(\bar f^*\mathsf{R}\bar f_*E,\bar f^*\mathsf{R}\bar f_*F).\end{aligned}$$

□

Corollary 2.1.10. *For $E\in\mathbf{D}_{\mathsf{qc}}(X)$ and $F\in\mathbf{D}_{\mathsf{qc}}(\overline{Y})$, the projection maps*

$$p_{1,\bar f}\colon \mathsf{R}\bar f_*E\otimes^{\mathsf{L}}_{\overline{Y}}F\longrightarrow\mathsf{R}\bar f_*(E\otimes^{\mathsf{L}}_X\bar f^*F)$$
$$p_{2,\bar f}\colon F\otimes^{\mathsf{L}}_{\overline{Y}}\mathsf{R}\bar f_*E\longrightarrow\mathsf{R}\bar f_*(\bar f^*F\otimes^{\mathsf{L}}_X E)$$

are isomorphisms.

Proof. As in [L09, 3.4.6.2], $p_{1,\bar f}$ is the natural composite isomorphism

$$\begin{aligned}\mathsf{R}\bar f_*E\otimes^{\mathsf{L}}_{\overline{Y}}F &\xrightarrow[2.1.6]{\sim}\mathsf{R}\bar f_*\bar f^*(\mathsf{R}\bar f_*E\otimes^{\mathsf{L}}_{\overline{Y}}F)\\ &\xrightarrow{\sim}\mathsf{R}\bar f_*(\bar f^*\mathsf{R}\bar f_*E\otimes^{\mathsf{L}}_X\bar f^*F)\xrightarrow[2.1.6]{\sim}\mathsf{R}\bar f_*(E\otimes^{\mathsf{L}}_X\bar f^*F);\end{aligned}$$

and similarly for $p_{2,\bar f}$. □

[7] Apropos of §1.3, it is left to the interested reader to show that for ringed spaces this ρ is the same as the explicit map in [H66, p. 109, 5.8]. For this, [L09, 3.5.6(g)], with $\alpha\colon [D,E]\otimes D\to E$ the natural map, might be useful.

2.2. Let Y be a ringed space, and $\psi\colon \mathcal{O}_Y \to \mathcal{S}$ an $\mathcal{O}_Y$-algebra. Let $\overline{Y}$ be the ringed space $(Y, \mathcal{S})$, and $\phi\colon \overline{Y} \to Y$ the ringed-space map (id_Y, ψ).

(This subsection is independent of the preceding one. Subsequently, only the case where $\psi\colon \mathcal{O}_Y \to \mathcal{S} := f_*\mathcal{O}_X$ is as before—just after (2.1.4)—will be needed.)

Note: $\mathcal{O}_{\overline{Y}} = \mathcal{S}$, $\mathcal{A}(\overline{Y})$ is the category of $\mathcal{S}$-modules, $\phi_*\colon \mathcal{A}(\overline{Y}) \to \mathcal{A}(Y)$ is just restriction of scalars, $\otimes_{\overline{Y}} = \otimes_{\mathcal{S}}$ and $\mathcal{H}om_{\overline{Y}} = \mathcal{H}om_{\mathcal{S}}$.

The functor

$$\mathcal{H}om_\psi\colon \mathcal{A}(\overline{Y})^{\mathsf{op}} \times \mathcal{A}(Y) \to \mathcal{A}(\overline{Y})$$

(where "$^{\mathsf{op}}$" denotes "opposite category") is given by

$$\mathcal{H}om_\psi(F, G) := \mathcal{H}om_Y(\phi_*F, G) \in \mathcal{A}(\overline{Y}) \qquad \big(F \in \mathcal{A}(\overline{Y}),\ G \in \mathcal{A}(Y)\big),$$

with scalar multiplication given by the following natural composite map, where $m_{\phi_*F}\colon \mathcal{S} \otimes_Y \phi_*F \to \phi_*F$ is scalar multiplication,

$$\begin{aligned}\mathcal{S} \otimes_Y \mathcal{H}om_Y(\phi_*F, G) &= \mathcal{H}om_Y(\phi_*F, G) \otimes_Y \mathcal{S}\\ &\xrightarrow{\text{via } m_{\phi_*F}} \mathcal{H}om_Y(\mathcal{S} \otimes_Y \phi_*F, G) \otimes_Y \mathcal{S}\\ &\xrightarrow{\ \sim\ } \mathcal{H}om_Y(\mathcal{S}, \mathcal{H}om_Y(\phi_*F, G)) \otimes_Y \mathcal{S} \longrightarrow \mathcal{H}om_Y(\phi_*F, G),\end{aligned}$$

and with the the obvious action on maps in $\mathcal{A}(\overline{Y})^{\mathsf{op}} \times \mathcal{A}(Y)$.

This scalar multiplication is adjoint to the natural composite

$$\mathcal{H}om_Y(\phi_*F, G) \otimes_Y \mathcal{S} \otimes_Y \phi_*F \xrightarrow{\text{via } m_{\phi_*F}} \mathcal{H}om_Y(\phi_*F, G) \otimes_Y \phi_*F \longrightarrow G,$$

that is, the border of the following natural diagram with $[-,-] := \mathcal{H}om_Y(-,-)$, $\otimes := \otimes_Y$ and $m := m_{\phi_*F}$, is commutative.

$$\begin{array}{ccccc}
[\phi_*F, G] \otimes \big(\mathcal{S} \otimes \phi_*F\big) & \xrightarrow{\text{via } m} & [\phi_*F, G] \otimes \phi_*F & \longrightarrow & G\\
\Big\downarrow{\scriptstyle \text{via } m} & ① & & \nearrow \quad ② & \Big\uparrow\\
[\mathcal{S} \otimes \phi_*F, G] \otimes \big(\mathcal{S} \otimes \phi_*F\big) & \xrightarrow{\ \sim\ } & \big([\mathcal{S}, [\phi_*F, G]] \otimes \mathcal{S}\big) \otimes \phi_*F & \longrightarrow & [\phi_*F, G] \otimes \phi_*F
\end{array}$$

Indeed, subdiagram ① commutes by [L09, 3.5.3(h)] (with $B = \mathcal{S} \otimes \phi_*F$, $A = \phi_*F$ and $C = G$), and ② by, e.g., the Kelly-Mac Lane coherence theorem [KM71, p. 107, Thm. 2.4] applied to ② after replacement of $\mathcal{S}$ and ϕ_*F, respectively, by arbitrary objects D and E in $\mathcal{A}(Y)$.

It follows that scalar multiplication factors naturally as

$$[\phi_*F, G] \otimes \mathcal{S} \longrightarrow [\phi_*F, G] \otimes [\phi_*F, \phi_*F] \longrightarrow [\phi_*F, G]$$

where the second map is "internal composition" [L09, 3.5.3(h)], that is, the map adjoint to the natural composite

$$[\phi_*F, G] \otimes ([\phi_*F, \phi_*F] \otimes \phi_*F) \longrightarrow [\phi_*F, G] \otimes \phi_*F \longrightarrow G.$$

Let $\mathsf{R}\mathcal{H}om_\psi\colon \mathbf{D}(\overline{Y})^{\mathsf{op}} \times \mathbf{D}(Y) \to \mathbf{D}(\overline{Y})$ be a right-derived functor of $\mathcal{H}om_\psi$, specified on objects by choosing for each $\mathcal{O}_Y$-complex G a K-injective resolution $G \to I_G$ and then setting, for any $\mathcal{S}$-complex F,

$$\mathsf{R}\mathcal{H}om_\psi(F, G) := \mathcal{H}om_\psi(F, I_G),$$

and specified on maps in the standard way.

The next proposition is a slightly upgraded version of *derived adjoint associativity*—which it becomes when ψ is an isomorphism. (This terminology is clarified in Remark 2.2.7.)

Proposition 2.2.1. *There is a unique trifunctorial* $\mathbf{D}(\overline{Y})$*-isomorphism*

$$\alpha(E,F,G)\colon \mathsf{R}\mathcal{H}om_{\psi}(E\otimes^{\mathsf{L}}_{\overline{Y}}F,G) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_{\overline{Y}}(E,\mathsf{R}\mathcal{H}om_{\psi}(F,G)) \qquad \big(E,F\in\mathbf{D}(\overline{Y}),\ G\in\mathbf{D}(Y)\big)$$

such that the following natural diagram, with $\mathcal{H}:=\mathcal{H}om$ *and* $\alpha_0(E,F,G)$ *the standard isomorphism of* $\mathcal{O}_{\overline{Y}}$*-complexes, commutes.*

$$\begin{array}{ccccc}
\mathcal{H}_{\psi}(E\otimes_{\overline{Y}}F,G) & \longrightarrow & \mathsf{R}\mathcal{H}_{\psi}(E\otimes_{\overline{Y}}F,G) & \longrightarrow & \mathsf{R}\mathcal{H}_{\psi}(E\otimes^{\mathsf{L}}_{\overline{Y}}F,G)\\
{\scriptstyle\alpha_0(E,F,G)}\downarrow\simeq & & & & \simeq\downarrow{\scriptstyle\alpha(E,F,G)}\\
\mathcal{H}_{\overline{Y}}\big(E,\mathcal{H}_{\psi}(F,G)\big) & \longrightarrow & \mathsf{R}\mathcal{H}_{\overline{Y}}\big(E,\mathcal{H}_{\psi}(F,G)\big) & \longrightarrow & \mathsf{R}\mathcal{H}_{\overline{Y}}\big(E,\mathsf{R}\mathcal{H}_{\psi}(F,G)\big)
\end{array}$$

In fact, $\alpha(E,F,G)=\alpha_0(P_E,F,I_G)$ where $P_E\to E$ (resp. $G\to I_G$) is a quasi-isomorphism with P_E a K-flat $\mathcal{O}_{\overline{Y}}$-complex (resp. I_G a K-injective $\mathcal{O}_Y$-complex). The *proof* is the same as that of [L09, Proposition (2.6.1)*], except that in the remarks preceding *loc. cit.* one sets

$\mathbf{L'_1}:=\mathbf{K}(\overline{Y})$ (the homotopy category of $\mathcal{O}_{\overline{Y}}$-complexes),

$\mathbf{L'_2}:=$ full subcategory of $\mathbf{K}(\overline{Y})$ spanned by all K-flat $\mathcal{O}_{\overline{Y}}$-complexes,

$\mathbf{L'_3}:=$ full subcategory of $\mathbf{K}(Y)$ spanned by all K-injective $\mathcal{O}_Y$-complexes,

adjusts $\mathbf{L''_i}$ and $\mathbf{D''_i}$ accordingly ($\mathbf{i}=1,2,3$), and then observes that for $(F,G)\in\mathbf{L'_2}\times\mathbf{L'_3}$, if the $\mathcal{O}_{\overline{Y}}$-complex E is exact then so is

$$\mathcal{H}_{\overline{Y}}\big(E,\mathcal{H}_{\psi}(F,G)\big)\cong\mathcal{H}_{\psi}(E\otimes^{\mathsf{L}}_{\overline{Y}}F,G),$$

that is, $\mathcal{H}_{\psi}(F,G)$ is K-injective. □

Upon replacing $\mathcal{S}$ by $\mathcal{O}_Y$ in the above definition of scalar multiplication for $\mathcal{H}om_{\psi}(F,G)$, one checks the equality of $\mathcal{O}_Y$-complexes

(2.2.2) $$\phi_*\mathcal{H}om_{\psi}(F,G)=\mathcal{H}om_Y(\phi_*F,G);$$

replacing G by I_G gives the natural isomorphism (in $\mathbf{D}(Y)$)

(2.2.3) $$\phi_*\mathsf{R}\mathcal{H}om_{\psi}(F,G)\cong\mathsf{R}\mathcal{H}om_Y(\phi_*F,G).$$

As is readily verified, the isomorphism (2.2.3) is naturally identifiable with the inverse of the isomorphism $\alpha_{\phi}(\mathcal{O}_{\overline{Y}},F,G)$ given by the next Corollary.

Corollary 2.2.4. *There is a unique trifunctorial* $\mathbf{D}(Y)$*-isomorphism*

$$\alpha_{\phi}(E,F,G)\colon \mathsf{R}\mathcal{H}om_Y\big(\phi_*(E\otimes^{\mathsf{L}}_{\overline{Y}}F),G\big)\xrightarrow{\sim}\phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}\big(E,\mathsf{R}\mathcal{H}om_{\psi}(F,G)\big) \qquad \big(E,F\in\mathbf{D}(\overline{Y}),\ G\in\mathbf{D}(Y)\big)$$

such that the following natural $\mathbf{D}(Y)$*-diagram, with* $\mathcal{H}:=\mathcal{H}om$*, commutes.*

$$\begin{array}{ccccc}
\mathcal{H}_Y\big(\phi_*(E\otimes_{\overline{Y}}F),G\big) & \longrightarrow & \mathsf{R}\mathcal{H}_Y\big(\phi_*(E\otimes_{\overline{Y}}F),G\big) & \longrightarrow & \mathsf{R}\mathcal{H}_Y\big(\phi_*(E\otimes^{\mathsf{L}}_{\overline{Y}}F),G\big)\\
\simeq\big\downarrow & & & & \simeq\big\downarrow \alpha_\phi(E,F,G)\\
\phi_*\mathcal{H}_{\overline{Y}}\big(E,\mathcal{H}_\psi(F,G)\big) & \longrightarrow & \phi_*\mathsf{R}\mathcal{H}_{\overline{Y}}\big(E,\mathcal{H}_\psi(F,G)\big) & \longrightarrow & \phi_*\mathsf{R}\mathcal{H}_{\overline{Y}}\big(E,\mathsf{R}\mathcal{H}_\psi(F,G)\big)
\end{array}$$

Proof. Applying ϕ_* to the diagram in 2.2.1, one sees that the diagram in 2.2.4 commutes when $\alpha_\phi(E,F,G)$ is the isomorphism $\phi_*\alpha(E,F,G)$.

Uniqueness need only be checked when E is K-flat and G is K-injective, in which case it holds because the maps in the top row of the latter diagram are isomorphisms. □

Corollary 2.2.5 (Sheafified duality for ϕ)**.** *Setting*

$$\phi^\flat(-):=\mathsf{R}\mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}},-)\colon \mathbf{D}(Y)\to\mathbf{D}(\overline{Y}),$$

one has, for $E\in\mathbf{D}(\overline{Y})$ *and* $G\in\mathbf{D}(Y)$*, the bifunctorial isomorphism*

$$\alpha_\phi(E,\mathcal{O}_{\overline{Y}},G)\colon \mathsf{R}\mathcal{H}om_Y(\phi_*E,G)\ \xrightarrow{\ \sim\ }\ \phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(E,\phi^\flat G).$$

Remark. The inverse of $\alpha_\phi(E,\mathcal{O}_{\overline{Y}},G)$ is described in Proposition 2.4.4.

Corollary 2.2.6 (Global duality for ϕ)**.** *For* $E,F\in\mathbf{D}(\overline{Y})$ *and* $G\in\mathbf{D}(Y)$ *one has, with* $\alpha_\phi:=\alpha_\phi(E,F,G)$*, the functorial isomorphism*

$$\mathrm{H}^0\mathsf{R}\Gamma\big(Y,\alpha_\phi\big)\colon \mathrm{Hom}_{\mathbf{D}(Y)}\big(\phi_*(E\otimes^{\mathsf{L}}_{\overline{Y}}F),G\big)\ \xrightarrow{\ \sim\ }\ \mathrm{Hom}_{\mathbf{D}(\overline{Y})}\big(E,\mathsf{R}\mathcal{H}om_\psi(F,G)\big).$$

In particular, one has the adjunction $\phi_*\dashv\phi^\flat$ *given by the functorial isomorphism, with* $\alpha'_\phi:=\alpha_\phi(E,\mathcal{O}_{\overline{Y}},G)$*,*

$$\mathrm{H}^0\mathsf{R}\Gamma\big(Y,\alpha'_\phi\big)\colon\ \mathrm{Hom}_{\mathbf{D}(Y)}\big(\phi_*E,G\big)\ \xrightarrow{\ \sim\ }\ \mathrm{Hom}_{\mathbf{D}(\overline{Y})}\big(E,\phi^\flat G\big).$$

Remark 2.2.7. For fixed E and F in $\mathbf{D}(\overline{Y})$, the functorial isomorphism $\alpha(E,F,G)$ in 2.2.1 is *right-conjugate* (see for instance [L09, 3.3.5, 3.3.7]), via natural adjunctions, to the standard associativity isomorphism

$$\phi_*\big(D\otimes^{\mathsf{L}}_{\overline{Y}}(E\otimes^{\mathsf{L}}_{\overline{Y}}F)\big)\ \xleftarrow{\ \sim\ }\ \phi_*\big((D\otimes^{\mathsf{L}}_{\overline{Y}}E)\otimes^{\mathsf{L}}_{\overline{Y}}F\big)\qquad (D\in\mathbf{D}(\overline{Y})).$$

2.2.8. As always, one can explicate an adjunction through the associated counit and unit maps. One does so for $\phi_*\dashv\phi^\flat$ by means of the well-known counit and unit maps for the standard adjunction $\phi_*\dashv\mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}},-)$ (of functors between $\mathcal{A}(\overline{Y})$ and $\mathcal{A}(Y)$), obtaining that the corresponding counit map at $G\in\mathbf{D}(Y)$ is the natural composite

$$\begin{aligned}\phi_*\phi^\flat G=\phi_*\mathsf{R}\mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}},G)&\underset{(2.2.3)}{\xrightarrow{\ \sim\ }}\mathsf{R}\mathcal{H}om_Y(\phi_*\mathcal{O}_{\overline{Y}},G)\\ &\underset{\text{via }\psi}{\longrightarrow}\mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y,G)\xrightarrow{\ \sim\ }G;\end{aligned}\tag{2.2.8.1}$$

and the corresponding unit map at $E \in \mathbf{D}(\overline{Y})$ is the natural composite
(2.2.8.2)

$$E \xrightarrow{\sim} \mathcal{H}om_{\overline{Y}}(\mathcal{O}_{\overline{Y}}, E) \xhookrightarrow{j} \mathcal{H}om_{\psi}(\mathcal{O}_{\overline{Y}}, \phi_*E) \to \mathsf{R}\mathcal{H}om_{\psi}(\mathcal{O}_{\overline{Y}}, \phi_*E) = \phi^\flat\phi_*E.$$

(The inclusion map j is easily seen to be $\mathcal{O}_{\overline{Y}}$-linear.)

2.2.9. The functor $\otimes_\psi \colon \mathcal{A}(\overline{Y}) \times \mathcal{A}(Y) \to \mathcal{A}(\overline{Y})$ is given by

$$F \otimes_\psi G := \phi_*F \otimes_Y G \in \mathcal{A}(\overline{Y}) \qquad \big(F \in \mathcal{A}(\overline{Y}),\, G \in \mathcal{A}(Y)\big),$$

where the scalar multiplication map is the natural composite

$$\mathcal{O}_{\overline{Y}} \otimes_Y (\phi_*F \otimes_Y G) \xrightarrow{\sim} (\mathcal{O}_{\overline{Y}} \otimes_Y \phi_*F) \otimes_Y G \longrightarrow \phi_*F \otimes_Y G,$$

and the action of $\otimes_\psi$ on maps in $\mathcal{A}(\overline{Y}) \times \mathcal{A}(Y)$ is the obvious one.

One has for $E \in \mathcal{A}(\overline{Y}),\ G \in \mathcal{A}(Y)$, the natural isomorphism

$$\mathrm{Hom}_{\overline{Y}}(\mathcal{O}_{\overline{Y}} \otimes_\psi G, E) \xrightarrow{\sim} \mathrm{Hom}_Y(G, \phi_*E).$$

Thus the functor $\mathcal{O}_{\overline{Y}} \otimes_\psi -$ is left-adjoint to ϕ_*, and so can be identified with ϕ^*, after which the counit $\mathcal{A}(\overline{Y})$-map $\phi^*\phi_* \to \mathrm{id}$ becomes the scalar multiplication

$$\mathcal{O}_{\overline{Y}} \otimes_\psi \phi_*F \longrightarrow F \qquad (F \in \mathcal{A}(\overline{Y})),$$

and the unit map $\mathrm{id} \to \phi_*\phi^*$ becomes the natural $\mathbf{D}(Y)$-map

$$G \xrightarrow{\sim} \mathcal{O}_Y \otimes_Y G \longrightarrow \phi_*\mathcal{O}_{\overline{Y}} \otimes_Y G = \phi_*(\mathcal{O}_{\overline{Y}} \otimes_\psi G) \qquad (G \in \mathcal{A}(Y)),$$

There is an obvious functorial isomorphism

(2.2.9.1) $$F \otimes_\psi G \xrightarrow{\sim} F \otimes_{\overline{Y}} (\mathcal{O}_{\overline{Y}} \otimes_\psi G) = F \otimes_{\overline{Y}} \phi^*G.$$

One has then the natural isomorphism

(2.2.9.2) $$\phi_*(F \otimes_{\overline{Y}} \phi^*G) = \phi_*(F \otimes_{\overline{Y}} (\mathcal{O}_{\overline{Y}} \otimes_\psi G)) \cong \phi_*(F \otimes_\psi G) = \phi_*F \otimes_Y G,$$

whose inverse is easily seen to be the *projection map*

$$p'_{1,\phi} \colon \phi_*F \otimes_Y G \longrightarrow \phi_*(F \otimes_{\overline{Y}} \phi^*G),$$

cf. [L09, 3.4.6], so that $p'_{1,\phi}$ *is an isomorphism.*

Similarly, one has the isomorphism

$$p'_{2,\phi} \colon G \otimes_Y \phi_*F \xrightarrow{\sim} \phi_*(\phi^*G \otimes_{\overline{Y}} F);$$

and, as in [L09, 3.4.6.1], the following natural diagram commutes:

$$\begin{array}{ccc} \phi_*F \otimes_Y G & \xrightarrow[p'_{1,\phi}]{\sim} & \phi_*(F \otimes_{\overline{Y}} \phi^*G) \\ \simeq\downarrow & & \downarrow\simeq \\ G \otimes_Y \phi_*F & \xrightarrow[p'_{2,\phi}]{\sim} & \phi_*(\phi^*G \otimes_{\overline{Y}} F) \end{array}$$

This all applies more generally, *mutatis mutandis,* to $\mathcal{O}_{\overline{Y}}$-complexes F and $\mathcal{O}_Y$-complexes G. In that case, upon replacing G by a quasi-isomorphic K-flat complex one sees that the *derived* projection map is an isomorphism

$$(2.2.9.3)\quad p_{1,\phi}\colon \phi_*F\otimes^{\mathsf L}_{Y} G \xrightarrow{\;\sim\;} \phi_*(F\otimes^{\mathsf L}_{\overline{Y}}\mathsf L\phi^*G) \qquad (F\in\mathbf D(\overline{Y}),\ G\in\mathbf D(Y)).$$

Similarly (and by [L09, 3.4.6.1], equivalently), one has the isomorphism

$$(2.2.9.4)\quad p_{2,\phi}\colon G\otimes^{\mathsf L}_{Y}\phi_*F \xrightarrow{\;\sim\;} \phi_*(\mathsf L\phi^*G\otimes^{\mathsf L}_{\overline{Y}}F) \qquad (F\in\mathbf D(\overline{Y}),\ G\in\mathbf D(Y)).$$

One checks that (2.2.9.4) is left-conjugate to the inverse of the duality isomorphism $\mathsf R\mathcal{H}om_Y(\phi_*F,E)\xrightarrow{\;\sim\;}\phi_*\mathsf R\mathcal{H}om_{\overline{Y}}(F,\phi^\flat E)$ given by Corollary 2.2.5.

2.3. Let $f\colon X\to Y$ be an affine scheme-map, $\bar f$ as in §2.1, and ϕ and ψ as in the lines following (2.1.4). So $f=\phi\bar f$; and one gets properties of f by combining the corresponding ones of $\bar f$ and ϕ.

For example, from (2.2.9.3), (2.2.9.4) and 2.1.10 one gets, using transitivity of projection maps [L09, 3.7.1], a simple proof of the well-known fact that *for all* $E\in\mathbf D_{\mathsf{qc}}(X)$ *and* $G\in\mathbf D_{\mathsf{qc}}(Y)$ *the projection maps are isomorphisms*

$$(2.3.1)\quad \mathsf Rf_*E\otimes^{\mathsf L}_{Y}G\xrightarrow{\;\sim\;}\mathsf Rf_*(E\otimes^{\mathsf L}_{X}\mathsf Lf^*G),\quad G\otimes^{\mathsf L}_{Y}\mathsf Rf_*E\xrightarrow{\;\sim\;}\mathsf Rf_*(\mathsf Lf^*G\otimes^{\mathsf L}_{X}E).$$

(For the general case of arbitrary qcqs maps, see e.g., [L09, 3.9.4].)

Here we will emphasize duality results, a basic one being Theorem 2.3.4 (sheafified affine duality).

First, with $\phi^\flat:=\mathsf R\mathcal{H}om_\psi(f_*\mathcal{O}_X,-)$ as in (2.2.5), set

$$(2.3.2)\qquad f^\flat:=\bar f^*\phi^\flat.$$

Lemma 2.3.3. *If* $G\in\mathbf D(Y)$ *is such that* $\mathsf R\mathcal{H}om_Y(f_*\mathcal{O}_X,G)\in\mathbf D_{\mathsf{qc}}(Y)$ *then* $\phi^\flat G\in\mathbf D_{\mathsf{qc}}(\overline{Y})$ *(whence* $f^\flat G\in\mathbf D_{\mathsf{qc}}(X)$*), and so one has the composite isomorphism*

$$\bar t_G\colon \mathsf Rf_*f^\flat G=\phi_*\mathsf R\bar f_*\bar f^*\phi^\flat G\xrightarrow[\text{2.1.6}]{\;\sim\;}\phi_*\phi^\flat G\xrightarrow[(2.2.3)]{\;\sim\;}\mathsf R\mathcal{H}om_Y(f_*\mathcal{O}_X,G).$$

Proof. The functor ϕ_* is exact, and $\phi_*\phi^\flat G\cong\mathsf R\mathcal{H}om_Y(f_*\mathcal{O}_X,G)\in\mathbf D_{\mathsf{qc}}(Y)$, so by [GrD71, p. 218, (2.2.4)], $\phi^\flat G\in\mathbf D_{\mathsf{qc}}(\overline{Y})$. ☐

Consequently:

Theorem 2.3.4 (Sheafified affine duality)**.** *Let* $G\in\mathbf D(Y)$ *be such that* $\mathsf R\mathcal{H}om_Y(f_*\mathcal{O}_X,G)\in\mathbf D_{\mathsf{qc}}(Y)$*. For all* $F\in\mathbf D_{\mathsf{qc}}(X)$ *one has the natural composite bifunctorial* duality isomorphism

$$\begin{aligned}\mathsf Rf_*\mathsf R\mathcal{H}om_X(F,f^\flat G)&\xrightarrow{\;\sim\;}\phi_*\mathsf R\bar f_*\mathsf R\mathcal{H}om_X(F,\bar f^*\phi^\flat G)\\&\xrightarrow[\text{2.1.8}]{\;\sim\;}\phi_*\mathsf R\mathcal{H}om_{\overline{Y}}(\mathsf R\bar f_*F,\phi^\flat G)\\&\xrightarrow[\text{2.2.5}]{\;\sim\;}\mathsf R\mathcal{H}om_Y(\phi_*\mathsf R\bar f_*F,G)\xrightarrow{\;\sim\;}\mathsf R\mathcal{H}om_Y(\mathsf Rf_*F,G).\quad\square\end{aligned}$$

Proposition 2.4.5 below gives another factorization of the duality isomorphism in 2.3.4. The next Proposition, in essence, globalizes that.

Proposition 2.3.5. *Let $G \in \mathbf{D}(Y)$ satisfy $\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G) \in \mathbf{D}_{\mathsf{qc}}(Y)$, and let t_G be the composite map*

$$\mathsf{R}f_*f^\flat G = \phi_*\mathsf{R}\bar{f}_*\bar{f}^*\phi^\flat G \underset{2.1.6}{\xrightarrow{\sim}} \phi_*\phi^\flat G \underset{(2.2.8.1)}{\longrightarrow} G$$

given by the counit maps associated to the adjunctions $\bar{f}^* \dashv \mathsf{R}f_*$ and $\phi_* \dashv \phi^\flat$ in 2.1.6 and 2.2.6 respectively. Then:

(i) $\big(f^\flat G, t_G\big)$ *represents the contravariant functor* $\mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*-,\ G)$ *from $\mathbf{D}_{\mathsf{qc}}(X)$ to the category of $\Gamma(Y, \mathcal{O}_Y)$-modules.*

(ii) *t_G is the natural composite map*

$$\mathsf{R}f_*f^\flat G \underset{2.3.3}{\xrightarrow{\sim}} \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G) \longrightarrow \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, G) \xrightarrow{\sim} G.$$

Proof. (i) The assertion is that for all $F \in \mathbf{D}_{\mathsf{qc}}(X)$, the map gotten by going clockwise around the following natural diagram from top left to bottom left is an isomorphism—which holds because, clearly, the diagram commutes.

$$\begin{CD}
\mathrm{Hom}_{\mathbf{D}(X)}(F, \bar{f}^*\phi^\flat G) @>>> \mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*F, \mathsf{R}f_*\bar{f}^*\phi^\flat G)\\
@V{2.1.6}V{\simeq}V @|\\
\mathrm{Hom}_{\mathbf{D}(\overline{Y})}(\mathsf{R}\bar{f}_*F, \mathsf{R}\bar{f}_*\bar{f}^*\phi^\flat G) @>>> \mathrm{Hom}_{\mathbf{D}(Y)}(\phi_*\mathsf{R}\bar{f}_*F, \phi_*\mathsf{R}\bar{f}_*\bar{f}^*\phi^\flat G)\\
@V{2.1.6}V{\simeq}V @V{\simeq}V{2.1.6}V\\
\mathrm{Hom}_{\mathbf{D}(\overline{Y})}(\mathsf{R}\bar{f}_*F, \phi^\flat G) @>>> \mathrm{Hom}_{\mathbf{D}(Y)}(\phi_*\mathsf{R}\bar{f}_*F, \phi_*\phi^\flat G)\\
@V{2.2.6}V{\simeq}V @VV{(2.2.8.1)}V\\
\mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*F, G) @= \mathrm{Hom}_{\mathbf{D}(Y)}(\phi_*\mathsf{R}\bar{f}_*F, G)
\end{CD}$$

(ii) Left to the reader. □

Corollary 2.3.6. *If Y is qcqs, then for G, t_G as in Proposition 2.3.5 and $\tau_G := \tau_f(G)\colon \mathsf{R}f_*f^\times G \to G$ the canonical map, there is a unique $\mathbf{D}(X)$-map*

$$\xi_f\colon f^\flat G \to f^\times G$$

such that $t_G = \tau_G \circ \mathsf{R}f_*\xi_f$; and this ξ_f is an isomorphism.

Proof. Both $(f^\flat G, t_G)$ and $(f^\times G, \tau_G)$ represent $\mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*-,\ G)$. □

2.3.7. We review some conditions under which $\mathsf{R}\mathcal{H}om^\bullet_Y(f_*\mathcal{O}_X,\ G) \in \mathbf{D}_{\mathsf{qc}}(Y)$ for *all* $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$ (resp. $\mathbf{D}_{\mathsf{qc}}(Y)$)—see Lemma 2.3.8 below.

For this we need the notion of *pseudo-coherence,* discussed in detail in the primary source [Il71], or, more accessibly, in [TT90, pp. 283*ff,* §2], or in [St24, tag 08E4]. (A brief summary appears in [L09, §4.3].) The simplest characterization is that an $\mathcal{O}_Y$-complex E is pseudo-coherent if over each affine open $U \subset Y$, the restriction $E|_U$ is $\mathbf{D}(U)$-isomorphic to a bounded-above complex F of finite-rank locally free $\mathcal{O}_X$-modules. (When E is an $\mathcal{O}_Y$-module, this means that $E|_U$ is resolvable by such an F.) It suffices for pseudo-coherence of E that this condition hold over each member of a covering of Y by affine open subschemes.

Such an F being K-flat, it holds for any scheme-map $h\colon Z \to Y$ that, with $h_U\colon h^{-1}U \to U$ the induced map, the natural map is an isomorphism $\mathsf{L}h_U^*F \xrightarrow{\sim} h_U^*F$, and hence that if E is pseudo-coherent then so is $\mathsf{L}h^*E$.

A finite scheme-map $f\colon X \to Y$ is pseudo-coherent if for any pseudo-coherent $\mathcal{O}_X$-complex E, $\mathsf{R}f_*E$ is a pseudo-coherent $\mathcal{O}_Y$-complex (see [L09, remark just before 4.7.3.4]). For this condition to hold, it suffices that $f_*\mathcal{O}_X$ be pseudo-coherent. (This assertion, being local, need only be shown when X and Y are affine, for which case see [LN07, 4.3.2, (i)⇔(iii)].) If Y is locally noetherian, then every finite f is pseudo-coherent.

If, in addition, $f_*\mathcal{O}_X$ has finite tor-dimension locally (and hence globally if Y is quasi-compact), i.e., $f_*\mathcal{O}_X$ is a perfect complex [Il71, p. 135, 5.8.1], then f is *quasi-perfect*, i.e., for any perfect $\mathcal{O}_X$-complex E, $\mathsf{R}f_*E$ is a perfect $\mathcal{O}_Y$-complex. (Use [LN07, Proposition 2.1] locally on Y.) It is equivalent that f be *perfect*, as defined in [Il71, p. 250, Définition 4.1]—see, for instance, [L09, Example 4.7.3(d)].

For example, perfection holds for any affine map f that is flat and locally finitely presentable, so that $f_*\mathcal{O}_X$ is a flat and locally finitely presentable—i.e., locally free of finite rank—$\mathcal{O}_Y$-module, see [GrD71, p. 357, (9.1.15)(i)].

Perfection also holds whenever f is a regular immersion, so that $f_*\mathcal{O}_X$ is locally quasi-isomorphic to a Koszul complex.

Clearly, a composition of two pseudo-coherent (resp. perfect) finite maps is again pseudo-coherent (resp. perfect).

Lemma 2.3.8. *If $F \in \mathbf{D}(Y)$ is pseudo-coherent (resp. perfect) then for all $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$ (resp. $\mathbf{D}_{\mathsf{qc}}(Y)$), $\mathsf{R}\mathcal{H}om_Y(F, G) \in \mathbf{D}^+_{\mathsf{qc}}(Y)$ (resp. $\mathbf{D}_{\mathsf{qc}}(Y)$).*

Proof. The assertion is essentially local, so one can assume Y affine and then proceed as in the proof of [L09, Lemma 4.3.5]. Alternatively, use [H66, p. 73–74, Proposition 7.3 (ii) and (iii)], or see [St24, tag 0A6H]. □

Corollary 2.3.9 (Sheafified finite duality)**.** *Let $f\colon X \to Y$ be a pseudo-coherent (resp. perfect) finite scheme-map, and $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$ (resp. $\mathbf{D}_{\mathsf{qc}}(Y)$). For all $F \in \mathbf{D}_{\mathsf{qc}}(X)$ one has the composite bifunctorial* duality isomorphism

$$\begin{aligned}\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^\flat G) &= \phi_*\mathsf{R}\bar{f}_*\mathsf{R}\mathcal{H}om_X(F, \bar{f}^*\phi^\flat G)\\ &\underset{(2.1.8)}{\xrightarrow{\sim}} \phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar{f}_*F, \phi^\flat G)\\ &\underset{(2.2.5)}{\xrightarrow{\sim}} \mathsf{R}\mathcal{H}om_Y(\phi_*\mathsf{R}\bar{f}_*F, G) = \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, G)\end{aligned}$$

Proof. This follows immediately from 2.3.4 and 2.3.8. □

Application of the functor $\mathrm{H}^0\mathsf{R}\Gamma(Y,-)$ yields an adjunction $\mathsf{R}f_* \dashv f^\flat$ composed of those given by 2.1.6 and 2.2.6:

Corollary 2.3.10. *For any f as in 2.3.9 the functor $f^\flat\colon \mathbf{D}^+_{\mathsf{qc}}(Y) \to \mathbf{D}^+_{\mathsf{qc}}(X)$ is right-adjoint to $\mathsf{R}f_*\colon \mathbf{D}^+_{\mathsf{qc}}(X) \to \mathbf{D}^+_{\mathsf{qc}}(Y)$, with unit map the natural functorial composite map*

$$u_F\colon F \underset{2.1.6}{\xrightarrow{\ \sim\ }} \bar f^*\mathsf{R}\bar f_*F \underset{(2.2.8.2)}{\longrightarrow} \bar f^*\phi^\flat\phi_*\mathsf{R}\bar f_*F = f^\flat\mathsf{R}f_*F,$$

and counit the functorial map t_G (see 2.3.5), which identifies naturally with the canonical map ("evaluation at 1")

$$\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G) \longrightarrow \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, G) = G.$$

When f is perfect, the superscript "$^+$" can be omitted. □

2.3.11. (Cf. [H66, p.172, 6.8].) Similarly, if f is *any* closed immersion then $f^\flat$ is right-adjoint to $f_* = \mathsf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$. That's because for such f, Lemma 2.1.6 and hence Proposition 2.3.4 hold for all $G \in \mathbf{D}(Y)$.

Example 2.3.12. *Let $G \in \mathbf{D}(Y)$ be such that $\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G) \in \mathbf{D}_{\mathsf{qc}}(Y)$, u the unit map in 2.3.10, and μ the natural composite*

$$f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X \otimes_Y f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X.$$

The following natural diagram commutes.

$$\begin{array}{ccc}
\mathsf{R}f_*f^\flat G & \xrightarrow{\ \mathsf{R}f_*u_{f^\flat G}\ } & \mathsf{R}f_*f^\flat\mathsf{R}f_*f^\flat G \\
 & & \Big\downarrow{\scriptstyle \simeq\ 2.3.3} \\
{\scriptstyle 2.3.3\ \simeq}\Big\downarrow & & \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G)) \\
 & & \Big\downarrow{\scriptstyle \simeq} \\
\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G) & \underset{\text{via}\,\mu}{\longrightarrow} & \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X, G)
\end{array}$$

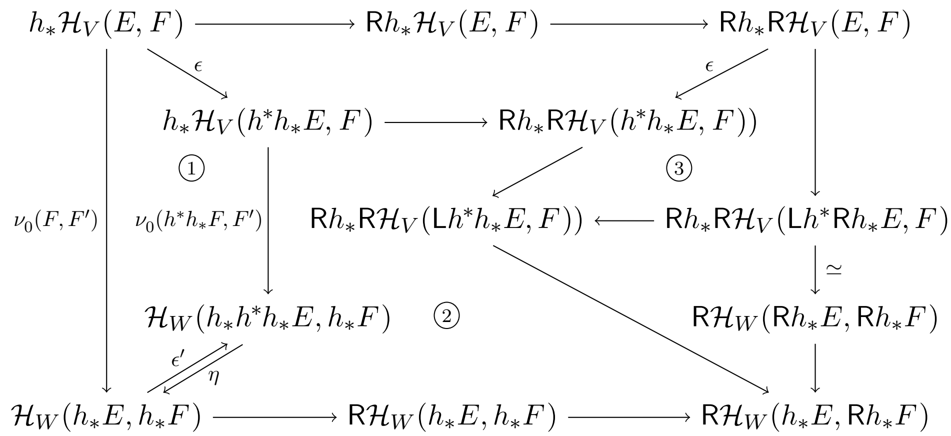
This assertion will not be used, so the (not entirely trivial) proof is omitted.

2.4. The following Proposition 2.4.4 (resp. 2.4.5) shows that the duality isomorphism in Corollary 2.2.5 (resp. Theorem 2.3.4) is concordant with the abstract duality map given by [L09, 4.2.1]. Proposition 2.4.6 characterizes the isomorphism $\xi_f\colon f^\flat G \xrightarrow{\ \sim\ } f^\times G$ in 2.3.6 by means of that abstract map.

Lemma 2.4.1. *Let $h\colon V \to W$ be a ringed-space map, and let E, F be $\mathcal{O}_V$-complexes. With $\nu_0 = \nu_0(E,F)$ the natural map (cf. [L09, (3.1.4)]), $\nu = \nu(E,F)$ as in (2.1.7), and $\mathcal{H} := \mathcal{H}om$, the following natural diagram commutes.*

$$
\begin{CD}
h_*\mathcal{H}_V(E,F) @>>> \mathsf{R}h_*\mathcal{H}_V(E,F) @>>> \mathsf{R}h_*\mathsf{R}\mathcal{H}_V(E,F)\\
@V{\nu_0}VV @. @VV{\nu}V\\
@. @. \mathsf{R}\mathcal{H}_W(\mathsf{R}h_*E,\mathsf{R}h_*F)\\
@VVV @. @VVV\\
\mathcal{H}_W(h_*E,h_*F) @>>> \mathsf{R}\mathcal{H}_W(h_*E,h_*F) @>>> \mathsf{R}\mathcal{H}_W(h_*E,\mathsf{R}h_*F)
\end{CD}
$$

Proof. The diagram expands naturally as follows, with maps labeled $\epsilon^\bullet$ (resp. η) induced by the counit map $h^*h_*E \to E$ (resp. the unit map $E \to h_*h^*E$):



Commutativity of subdiagram ① results from the obvious equalities

$$\nu_0(E,F) = \eta\epsilon'\nu_0(E,F) = \eta\nu_0(h^*h_*E,F)\epsilon.$$

Commutativity of ② results from [L09, 3.2.3(ii)] (modulo replacement in [L09, 3.1.5, 3.1.6] of (f,A,B) by $(h,h_*E,F)\dots$)

Commutativity of ③ results from that of [L09, 3.2.1.2].

Commutativity of the unlabeled subdiagrams is clear. $\square$

Lemma 2.4.2. *Let $h\colon V \to W$ be a ringed-space map, and let E, F be $\mathcal{O}_V$-complexes. The following natural diagram commutes.*

$$
\begin{CD}
\mathsf{R}h_*E \otimes^{\mathsf{L}}_W \mathsf{R}h_*F @>{\gamma}>{[\text{L09, 3.2.4(ii)}]}> \mathsf{R}h_*(E \otimes^{\mathsf{L}}_V F)\\
@AAA @VVV\\
h_*E \otimes^{\mathsf{L}}_W h_*F @. \mathsf{R}h_*(E \otimes_V F)\\
@VVV @AAA\\
h_*E \otimes_W h_*F @>>{\gamma_0}> h_*(E \otimes_V F)
\end{CD}
$$

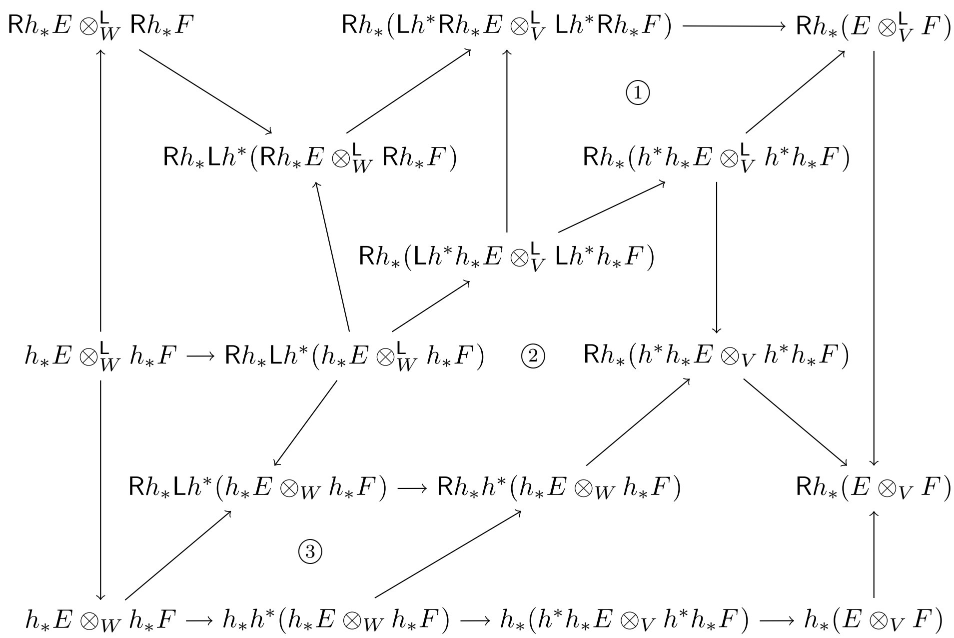
Proof. By definition, (or, if a definition other than that in [L09, 3.2.4] is preferred, one shows that) γ is adjoint to the natural composite map

$$\mathsf{L}h^*(\mathsf{R}h_*E \otimes^{\mathsf{L}}_W \mathsf{R}h_*F) \longrightarrow \mathsf{L}h^*\mathsf{R}h_*E \otimes^{\mathsf{L}}_V \mathsf{L}h^*\mathsf{R}h_*F \longrightarrow E \otimes^{\mathsf{L}}_V F.$$

By [L09, 3.1.9], and application of the "duality principle" of [L09, §3.4.5] to the argument at the beginning of that subsection[8] (or otherwise), γ_0 is adjoint to the natural composite map

$$h^*(h_*E \otimes_W h_*F) \longrightarrow h^*h_*E \otimes_V h^*h_*F \longrightarrow E \otimes_V F.$$

It suffices therefore to show commutativity of the natural diagram



Commutativity of the unlabeled subdiagrams is easily checked.

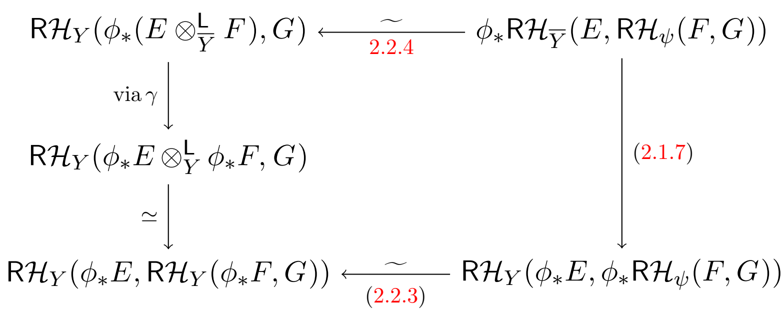
Commutativity of ① (respectively ③) is given by that of [L09, (3.2.1.2)], (respectively [L09, (3.2.1.3)]).

Commutativity of ② is given by [L09, (3.2.4(i)]. □

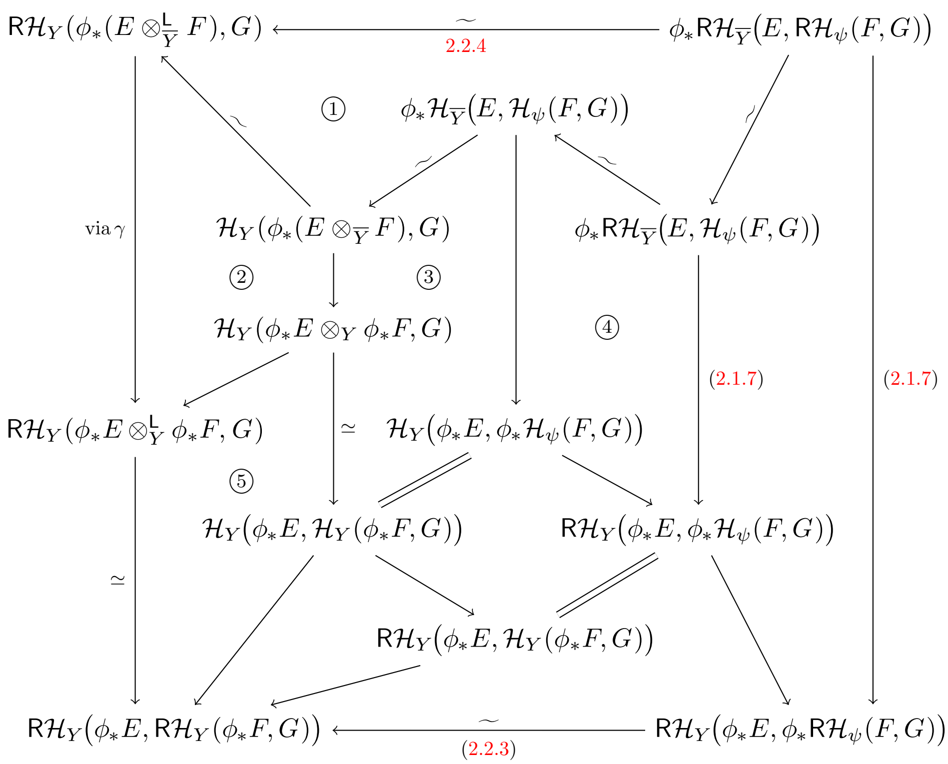
As in §2.2, let Y be any ringed space, $\psi\colon \mathcal{O}_Y \to \mathcal{S}$ an $\mathcal{O}_Y$-algebra, $\overline{Y}$ the ringed space $(Y, \mathcal{S})$, and $\phi\colon \overline{Y} \to Y$ the ringed-space map (id_Y, ψ). Then $\mathcal{O}_{\overline{Y}} = \mathcal{S}$, $\mathcal{A}(\overline{Y})$ is the category of $\mathcal{S}$-modules, $\phi_*\colon \mathcal{A}(\overline{Y}) \to \mathcal{A}(Y)$ is restriction of scalars, $\otimes_{\overline{Y}} = \otimes_{\mathcal{S}}$ and $\mathcal{H}om_{\overline{Y}} = \mathcal{H}om_{\mathcal{S}}$.

Lemma 2.4.3. *For $E, F \in \mathbf{D}(\overline{Y})$, $G \in \mathbf{D}(Y)$, the following natural diagram, with $\mathcal{H} := \mathcal{H}om$ and with $\gamma\colon \phi_*E \otimes^{\mathsf{L}}_Y \phi_*F \to \phi_*(E \otimes^{\mathsf{L}}_{\overline{Y}} F)$ as in* [L09, 3.2.4(ii))], *commutes.*

[8] In connection with [L09, §3.4.5], in the proof of [L09, 3.2.4(ii)], the erroneous phrase "the adjoint (3.5.4.1) of (3.4.2.1)" should be replaced by "(3.4.5.1)".



Proof. It may be assumed that F is a K-flat $\mathcal{S}$-complex and that G is a K-injective $\mathcal{O}_Y$-complex; and then (as in the proof of 2.2.1) $\mathcal{H}_\psi(\mathcal{S}, G)$ is a K-injective $\mathcal{S}$-complex. The diagram in question expands naturally as



It suffices therefore to show commutativity of all the subdiagrams.

Checking commutativity of the unlabeled subdiagrams is straightforward.

Commutativity of subdiagram ① is given by that of the diagram in 2.2.4.

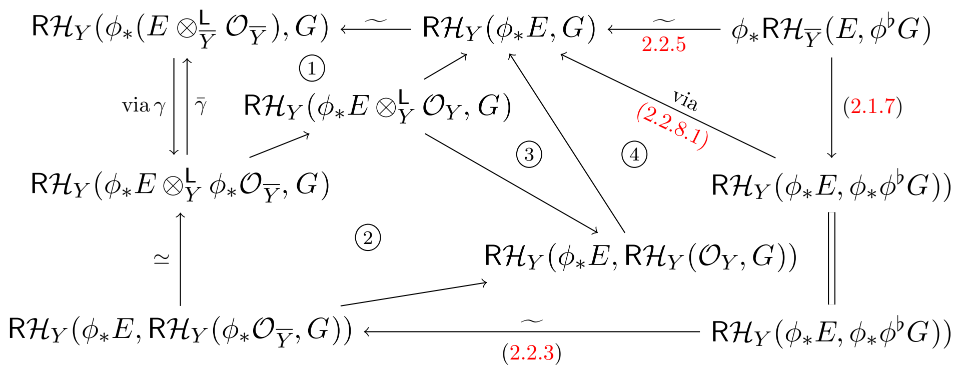
Commutativity of ② follows from Lemma 2.4.2; of ④ from Lemma 2.4.1; and of ⑤ from Corollary 2.2.4 (with $\mathcal{S} = \mathcal{O}_Y$ and $\psi = \mathrm{id}$).

Commutativity of ③ is easily verified: just do so after applying global sections over an arbitrary open $U \subset Y$. $\square$

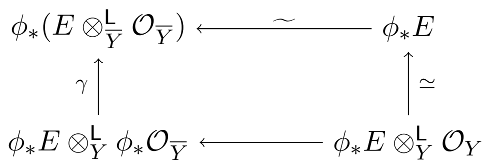
Proposition 2.4.4. *The isomorphism* $\alpha_\phi(E,\mathcal{O}_{\overline{Y}},G)^{-1}$ *in* 2.2.5 *factors as*

$$\phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(E,\phi^\flat G)\xrightarrow[(2.1.7)]{}\mathsf{R}\mathcal{H}om_Y(\phi_*E,\phi_*\phi^\flat G)\xrightarrow[(2.2.8.1)]{\text{via}}\mathsf{R}\mathcal{H}om_Y(\phi_*E,G).$$

Proof. In Lemma 2.4.3, set $F:=\mathcal{O}_{\overline{Y}}$ to get that the border of the following natural diagram commutes:



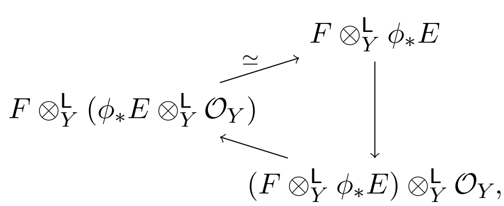
Here, the map $\bar\gamma$ is the left inverse of "via γ" induced by the right inverse of γ that is given by the natural commutative diagram



(cf. subdiagram ② in the proof of [L09, 3.4.7(iii)]). Thus subdiagram ① commutes; and careful diagram-chasing shows it enough to prove that subdiagrams ②, ③ and ④ commute.

Commutativity of ② and of ④ is obvious.

Finally, one finds, using the definition of the maps involved (see [L09, 3.5.3(e), 3.5.6(e)]), that the left-conjugate of ③ is the natural diagram



whose commutativity is easily shown when F is K-flat, so that it holds for all $F\in\mathbf{D}(Y)$, whence commutativity holds for ③. □

Now specialize to where Y is a scheme and, with $f\colon X\to Y$ an affine scheme-map, $\psi\colon\mathcal{O}_Y\to\mathcal{S}:=f_*\mathcal{O}_X$ is the associated map.

Proposition 2.4.5. *The duality isomorphism in Theorem 2.3.4 factors as*

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^\flat G) \xrightarrow{(2.1.7)} \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, \mathsf{R}f_*f^\flat G) \xrightarrow[2.3.5]{\text{via } t_G} \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, G).$$

Proof. As in Lemma 2.3.3, $\phi^\flat G \in \mathbf{D}_{\mathsf{qc}}(\overline{Y})$. The Proposition amounts to commutativity of the border of the diagram

$$\begin{array}{ccc}
\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, \bar{f}^*\phi^\flat G) & \xrightarrow{(2.1.7)} & \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, \mathsf{R}f_*\bar{f}^*\phi^\flat G) \\
{\scriptstyle 2.1.9}\downarrow\simeq & ① & \Vert \\
\phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar{f}_*F, \mathsf{R}\bar{f}_*\bar{f}^*\phi^\flat G) & \xrightarrow{(2.1.7)} & \mathsf{R}\mathcal{H}om_Y(\phi_*\mathsf{R}\bar{f}_*F, \phi_*\mathsf{R}\bar{f}_*\bar{f}^*\phi^\flat G) \\
{\scriptstyle 2.1.6}\downarrow\simeq & ② & \simeq\downarrow{\scriptstyle 2.1.6} \\
\phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{R}\bar{f}_*F, \phi^\flat G) & \xrightarrow{(2.1.7)} & \mathsf{R}\mathcal{H}om_Y(\phi_*\mathsf{R}\bar{f}_*F, \phi_*\phi^\flat G) \\
{\scriptstyle 2.2.5}\downarrow\simeq & ③ & \downarrow{\scriptstyle (2.2.8.1)} \\
\mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, G) & = & \mathsf{R}\mathcal{H}om_Y(\phi_*\mathsf{R}\bar{f}_*F, G)
\end{array}$$

Commutativity of ① results from that of the second diagram in [L09, 3.7.1.1]—where "(gf)" should be "(fg)"—with $(f, g, F') := (\phi, \bar{f}, f^\flat G)$; that of ② is clear; and that of ③ is given by 2.4.4 with $E := \mathsf{R}\bar{f}_*F$. $\square$

For a map $f\colon X \to Y$ of qcqs schemes, and $G \in \mathbf{D}(Y)$, the *abstract duality map* $\delta = \delta(f, G)$ is the natural composite (with $\tau_G := \tau_f(G)$)

$$\begin{aligned}
\mathsf{R}f_*f^\times G = \mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathcal{O}_X, f^\times G) &\xrightarrow{(2.1.7)} \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathcal{O}_X, \mathsf{R}f_*f^\times G) \\
&\xrightarrow{\text{via } \tau_G} \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathcal{O}_X, G).
\end{aligned}$$

If $\mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathcal{O}_X, G) \in \mathbf{D}_{\mathsf{qc}}(Y)$ *then* δ *is an isomorphism.* To see this via Yoneda's Lemma, one checks (cf. [V68, p. 404]) that for $E \in \mathbf{D}_{\mathsf{qc}}(Y)$, the map

$$\mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathsf{R}f_*f^\times G) \to \mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathcal{O}_X, G))$$

induced by δ is naturally isomorphic to the map

$$\mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*\mathsf{L}f^*E, G) \to \mathrm{Hom}_{\mathbf{D}(Y)}(E \otimes^{\mathsf{L}}_Y \mathsf{R}f_*\mathcal{O}_X, G)$$

induced by the projection *isomorphism*

$$E \otimes^{\mathsf{L}}_Y \mathsf{R}f_*\mathcal{O}_X \xrightarrow{\sim} \mathsf{R}f_*(\mathsf{L}f^*E \otimes^{\mathsf{L}}_X \mathcal{O}_X) = \mathsf{R}f_*\mathsf{L}f^*E.$$

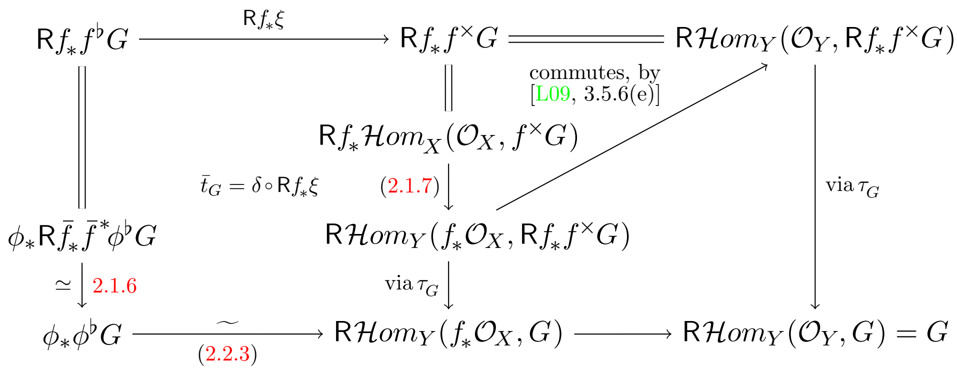
Proposition 2.4.6. *Let* $f\colon X \to Y$ *be an affine map of qcqs schemes, let* $G \in \mathbf{D}(Y)$ *be such that* $\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G) \in \mathbf{D}_{\mathsf{qc}}(Y)$, *and let* $\delta = \delta(f, G)$ *be the above duality isomorphism. The isomorphism* ξ_f *of Corollary 2.3.6 is the unique* $\mathbf{D}(X)$*-map* $\xi\colon f^\flat G \to f^\times G$ *such that the composite isomorphism*

$$\bar{t}_G\colon \mathsf{R}f_*f^\flat G = \phi_*\mathsf{R}\bar{f}_*\bar{f}^*\phi^\flat G \xrightarrow[2.1.6]{\sim} \phi_*\phi^\flat G \xrightarrow[(2.2.3)]{\sim} \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G).$$

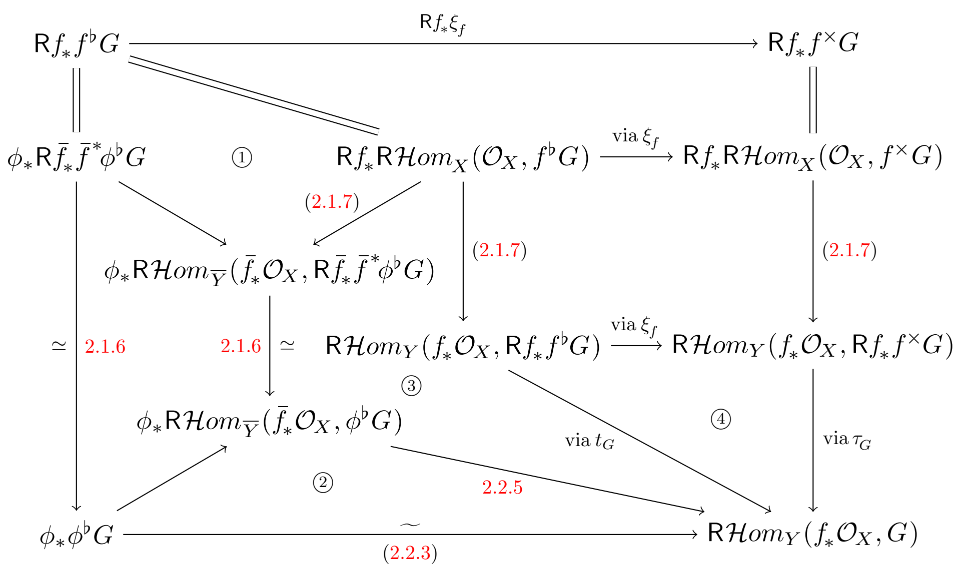
in 2.3.3 factors as

$$\mathsf{R}f_*f^\flat G \xrightarrow{\mathsf{R}f_*\xi} \mathsf{R}f_*f^\times G \xrightarrow{\delta} \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, G).$$

Proof. If $\bar t_G = \delta \circ \mathsf{R}f_*\xi$ then the following natural diagram commutes, i.e., $\tau_G \circ \mathsf{R}f_*\xi$ is the map t_G in 2.3.5, and so by 2.3.6, ξ is the isomorphism ξ_f.



It remains to be shown that $\bar t_G = \delta \circ \mathsf{R}f_*\xi_f$, that is, the outer border of the following natural diagram commutes.



Commutativity of the unlabeled subdiagrams is obvious. Commutativity of ① is a simple consequence of that of the first diagram in [L09, 3.5.6(e)]. That of ② is what is asserted immediately after (2.2.3), with $F := \bar f_*\mathcal{O}_X$. That of ③ is given by Proposition 2.4.5, and of ④ by Corollary 2.3.6. □

2.5. (Pseudofunctoriality) One verifies formally that over the category $\mathbf{F}$ of pseudo-coherent (resp. perfect) finite scheme-maps there is a unique $\mathbf{D}^+_{\mathsf{qc}}$(resp. $\mathbf{D}_{\mathsf{qc}}$)-valued contravariant pseudofunctor $(-)^\flat$ such that for any map $f \in \mathbf{F}$, Corollary 2.3.10 holds, and for any $W \xrightarrow{g} X \xrightarrow{f} Y$ in $\mathbf{F}$, the associated isomorphism $g^\flat f^\flat \xrightarrow{\ \sim\ } (fg)^\flat$ is the natural composite (see §2.2.8)

(2.5.1)
$$g^\flat f^\flat \to (fg)^\flat \mathsf{R}(fg)_* g^\flat f^\flat \xrightarrow{\ \sim\ } (fg)^\flat \mathsf{R}f_*\mathsf{R}g_* g^\flat f^\flat \to (fg)^\flat \mathsf{R}f_* f^\flat \to (fg)^\flat,$$

i.e., it is right-conjugate to the natural isomorphism $\mathsf{R}f_*\mathsf{R}g_* \xleftarrow{\sim} \mathsf{R}(fg)_*$, see [L09, 3.3.5] with (f_*, g_*, f^*, g^*) replaced by $(g^\flat f^\flat, (fg)^\flat, \mathsf{R}f_*\mathsf{R}g_*, \mathsf{R}(fg)_*)$, and cf. [L09, 3.6.8.1].

Restricting to qcqs schemes, one has the same statement with $(-)^\times$ in place of $(-)^\flat$. Consequently:

Proposition 2.5.2. *Over the category of pseudo-coherent (resp. perfect) finite maps of qcqs schemes, the map ξ_f in 2.3.6 is the f-component of an* isomorphism of pseudofunctors $(-)^\flat \xrightarrow{\sim} (-)^\times$. □

Remark 2.5.3. The **F**-maps f, g and fg entail maps of sheaves of rings $\psi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, $\xi\colon \mathcal{O}_X \to g_*\mathcal{O}_W$ and $\zeta := (f_*\xi)\circ\psi$, and ringed-space maps $\bar f\colon X \to (Y, f_*\mathcal{O}_X)$, $\bar g\colon W \to (X, g_*\mathcal{O}_W)$, $\overline{fg}\colon W \to (Y, f_*g_*\mathcal{O}_W)$ (see §2.1); and (2.5.1) is a functorial isomorphism

$$\bar g^*\mathsf{R}\mathcal{H}om_\xi(g_*\mathcal{O}_W, \bar f^*\mathsf{R}\mathcal{H}om_\psi(f_*\mathcal{O}_X, -)) \xrightarrow{\sim} \overline{fg}^*\mathsf{R}\mathcal{H}om_\zeta(f_*g_*\mathcal{O}_W, -).$$

The description of such an isomorphism in [H66, p. 167] needs noetherian hypotheses not assumed here. In any case, locally, where one deals, in essence, with a composition $R \to S \to T$ of ring homomorphisms, one realizes (2.5.1) as the sheafification of the natural $\mathbf{D}(T)$-isomorphism

$$\mathsf{R}\mathrm{Hom}_S(T, \mathsf{R}\mathrm{Hom}_R(S, \mathsf{G})) \xrightarrow{\sim} \mathsf{R}\mathrm{Hom}_R(T, \mathsf{G}) \qquad (\mathsf{G} \in \mathbf{D}(R)),$$

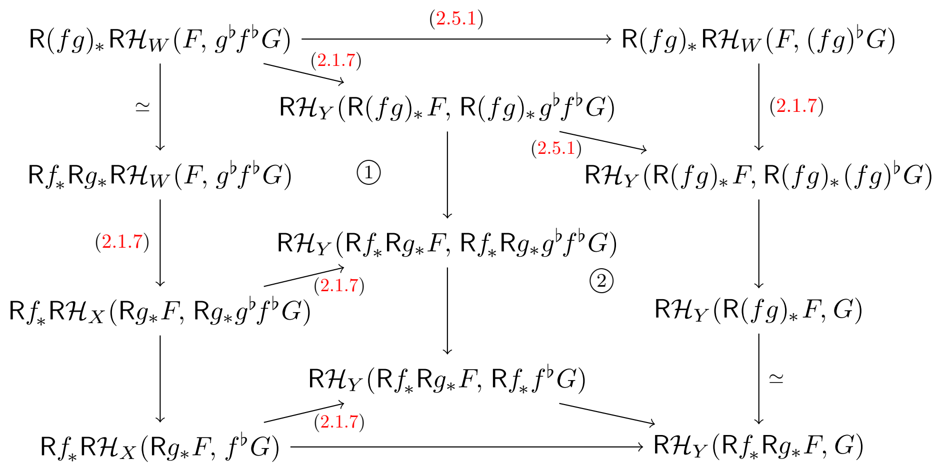
see the paragraph preceding Proposition 3.1.23.

The duality isomorphism in Corollary 2.3.9 has the following transitivity property, making it compatible with (2.5.1).

Proposition 2.5.4. *Let $W \xrightarrow{g} X \xrightarrow{f} Y$ in* **F** *be as above. For $F \in \mathbf{D}(W)$ and $G \in \mathbf{D}(Y)$, the following natural diagram commutes.*

$$\begin{array}{ccc}
\mathsf{R}(fg)_*\mathsf{R}\mathcal{H}om_W(F, g^\flat f^\flat G) & \xrightarrow{(2.5.1)} & \mathsf{R}(fg)_*\mathsf{R}\mathcal{H}om_W(F, (fg)^\flat G) \\
\simeq\big\downarrow & & \big\downarrow 2.3.9 \\
\mathsf{R}f_*\mathsf{R}g_*\mathsf{R}\mathcal{H}om_W(F, g^\flat f^\flat G) & & \mathsf{R}\mathcal{H}om_Y(\mathsf{R}(fg)_*F, G) \\
2.3.9\big\downarrow & & \big\downarrow\simeq \\
\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{R}g_*F, f^\flat G) & \xrightarrow[2.3.9]{} & \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathsf{R}g_*F, G)
\end{array}$$

Proof. Using Proposition 2.4.5, and with $\mathcal{H} := \mathcal{H}om$, expand the diagram naturally as follows:



Commutativity of ① is given by the second diagram in [L09, 3.7.1.1].

That of ② follows from (2.5.1) being (as stated above) right-conjugate to the natural isomorphism $\mathsf{R}f_*\mathsf{R}g_* \xrightarrow{\,\sim\,} \mathsf{R}(fg)_*$, see e.g., the second diagram in [L09, 3.3.7(a)], with (f_*, g_*, f^*, g^*) replaced by $(g^\flat f^\flat, (fg)^\flat, \mathsf{R}f_*\mathsf{R}g_*, \mathsf{R}(fg)_*)$.

Commutativity of the unlabeled subdiagrams is obvious. $\square$

2.6. Given Corollary 2.3.9, one can translate many standard results about the pseudofunctor $(-)^\times$ (see e.g., [L09, §4.7]) into corresponding results about the pseudofunctor $(-)^\flat$ of 2.5, and ask for concrete interpretations. In this section, an elaboration of [H66, p. 167, 6.3], tor-independent base change for $(-)^\flat$ is treated, both abstractly (Theorem 2.6.4) and concretely (Proposition 2.6.14). In the following two sections, which elaborate [H66, p. 174, 6.9], further illustration is provided by an explication of the interaction of $(-)^\flat$ with derived tensor (Proposition 2.7.7) and with derived hom (Proposition 2.8.2). Later, section 3 deals with the case of affine schemes, where the foregoing examples can be described in equivalent commutative-algebra terms.

Note, in perusing such examples with regard to a pseudo-coherent finite map $f = \phi\bar f\colon X \to Y$, that concretely representing a $\mathbf{D}_{\mathsf{qc}}(X)$-map $\xi\colon F \to G$ is more or less the same (via 2.1.6) as concretely representing the $\mathbf{D}_{\mathsf{qc}}(f_*\mathcal{O}_X)$-map $\mathsf{R}\bar f_*\xi$, and that the latter involves more than doing the same for the $\mathbf{D}_{\mathsf{qc}}(Y)$-map $\mathsf{R}f_*\xi$—because the natural map

$$\mathrm{Hom}_{\mathbf{D}_{\mathsf{qc}}(f_*\mathcal{O}_X)}(\mathsf{R}f_*F, \mathsf{R}f_*G) \to \mathrm{Hom}_{\mathbf{D}_{\mathsf{qc}}(Y)}(\mathsf{R}f_*F, \mathsf{R}f_*G)$$

isn't always injective.[9]

[9]Let k be a field and R a finite-dimensional local k-algebra with maximal ideal $m \neq 0$ such that the natural map $k \to R/m$ is an isomorphism. The natural composite

$$0 \neq \mathrm{Hom}_k(m/m^2, k) \xrightarrow{\,\sim\,} \mathrm{Hom}_R(m, k) \longrightarrow \mathrm{Ext}^1_R(k, k)$$

is an isomorphism, so that the natural map $\mathrm{Ext}^1_R(k,k) \to \mathrm{Ext}^1_k(k,k) = 0$ is not injective, i.e., the natural map $\mathrm{Hom}_{\mathbf{D}(R)}(k, k[1]) \to \mathrm{Hom}_{\mathbf{D}(k)}(k, k[1])$ is not injective.

2.6.1. To any oriented commutative square of scheme-maps

$$
\begin{array}{ccc}
X' & \xrightarrow{\;v\;} & X \\
{\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\;u\;]{} & Y
\end{array}
\tag{2.6.2}
$$

one associates the map

$$\theta_\sigma(E)\colon \mathsf{L}u^*\mathsf{R}f_*E \to \mathsf{R}g_*\mathsf{L}v^*E \qquad (E \in \mathbf{D}_{\mathsf{qc}}(X)) \tag{2.6.3}$$

adjoint to the natural composite map

$$\mathsf{R}f_*E \to \mathsf{R}f_*\mathsf{R}v_*\mathsf{L}v^*E \xrightarrow{\;\sim\;} \mathsf{R}u_*\mathsf{R}g_*\mathsf{L}v^*E.$$

For a concrete local description of θ_σ, see the end of §3.2 below.

As in [L09, §3.10], the square σ is called *independent* if for all $E \in \mathbf{D}_{\mathsf{qc}}(X)$, $\theta_\sigma(E)$ is an isomorphism.

If σ is independent, f and g finite and f pseudo-coherent (resp. perfect), then $g_*\mathcal{O}_{X'} \cong g_*\mathsf{L}v^*\mathcal{O}_X \cong \mathsf{L}u^*f_*\mathcal{O}_X$ is pseudo-coherent (resp. perfect), i.e., g is pseudo-coherent (resp. perfect).

If σ is a fiber square (i.e., the associated map $X' \to X \times_Y Y'$ is an isomorphism), then independence of σ is equivalent to *tor-independence,* i.e., for all $y' \in Y'$ and $x \in X$ such that $y := u(y') = f(x)$,

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y',y'}, \mathcal{O}_{X,x}) = 0 \quad \text{for all } i > 0.$$

For the proof, one reduces as in [L09, 3.10.3.2 and 3.10.3.3] to where Y, X, Y' and X' are all affine, in which case [L09, 3.10.3.1] applies.[10]

Hence, if σ is a fiber square then its being independent does not depend on its orientation; and such a σ is independent if either f or u is flat.

The following Theorem 2.6.4 is essentially contained in [L09, 4.4.1], in whose proof the assumption (at the beginning of Section 4.4) that all schemes are qcqs is needed only to ensure that $\mathsf{R}f_*$ has a right adjoint—which in the present circumstances is known to be so (Corollary 2.3.9) without the said assumption. The proof here, though related to that of *loc. cit.*, is more direct.

In Theorem 2.6.4 and Remark 2.6.5(a), these abbreviations are used:

$$(-)_* := \mathsf{R}(-)_*, \qquad (-)^* := \mathsf{L}(-)^*, \qquad \mathcal{H}om := \mathsf{R}\mathcal{H}om.$$

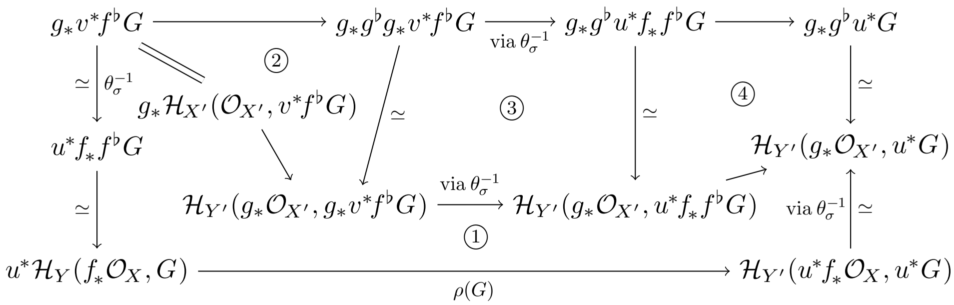
Theorem 2.6.4 (Tor-independent base change)**.** *Let σ be, as above, an independent fiber square, in which f—hence g—is finite and pseudo-coherent (resp. perfect) and u has finite tor-dimension (resp. u is arbitrary), and let $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$ (resp. $\mathbf{D}_{\mathsf{qc}}(Y)$). With $(-)^\flat$ and t_G as in Prop. 2.3.5, the adjoint of the composite map $g_*v^*f^\flat G \xrightarrow[\theta_\sigma^{-1}]{} u^*f_*f^\flat G \xrightarrow[u^*t_G]{} u^*G$ is an* isomorphism

$$\beta_\sigma(G)\colon v^*f^\flat G \xrightarrow{\;\sim\;} g^\flat u^*G.$$

---

[10]Misprint: references to (3.10) in *loc. cit.* should be to (3.10.2.3).

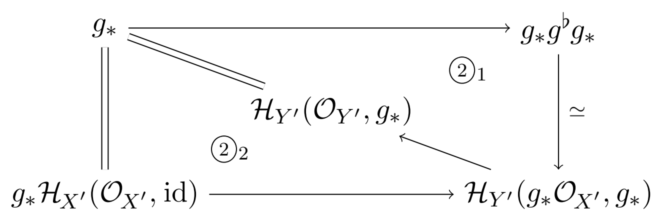
Proof. Since g is affine, [L09, 3.10.2.2] makes it enough to show that $g_*\beta_\sigma(G)$ is an isomorphism—i.e., that the top row of the following natural diagram, with $\mathcal{H} := \mathcal{H}om$, composes to an isomorphism.



The natural map $\rho(G)$ is an isomorphism: this assertion, being local, results, e.g., from [L09, 4.6.7] (in whose proof the untreated case where E is strictly perfect—i.e., a bounded complex of finite-rank locally free $\mathcal{O}_Y$-modules—and H arbitrary is easily disposed of by induction on the number of n such that $E^n \neq 0$.) Thus it suffices to show that the diagram commutes.

Using Proposition 2.4.6 and the second diagram in [L09, 3.5.6(e)], one verifies—with a bit of patience—that commutativity of subdiagram ① is given by [L09, Lemma 4.6.4] with $F := \mathcal{O}_X$ and $f^!$ replaced by $f^\flat$.

Commutativity of ② results from that of the natural functorial diagram



Here, by the description in 2.3.10 of the counit map t, commutativity of $②_1$ means just that the natural composite $g_* \to g_*g^\flat g_* \xrightarrow{u^*t} g_*$ is the identity; and that of $②_2$ is given by that of the first diagram in [L09, 3.5.6(e)].

Finally, commutativity of ③ and ④ is clear. $\square$

Remarks 2.6.5. (a) If the schemes in Theorem 2.6.4 are qcqs, the following diagram, with ξ as in 2.3.6 and $\beta_\sigma^\times(G)$ as in [L09, 4.4.3], commutes:

$$\begin{array}{ccc} v^*f^\flat G & \xrightarrow{\beta_\sigma(G)} & g^\flat u^*G \\ {\scriptstyle v^*\xi_f}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle \xi_g} \\ v^*f^\times G & \xrightarrow[\beta_\sigma^\times(G)]{} & g^\times u^*G \end{array}$$

Indeed, $\beta_\sigma^\times(G)$, resp. $\beta_\sigma(G)$, is by definition adjoint to

$$g_*v^*f^\times G \xrightarrow[\theta_\sigma^{-1}]{} u^*f_*f^\times G \xrightarrow[u^*\tau_G]{} u^*G, \quad \text{resp.} \quad g_*v^*f^\flat G \xrightarrow[\theta_\sigma^{-1}]{} u^*f_*f^\flat G \xrightarrow[u^*t_G]{} u^*G,$$

from which definitions one readily derives the assertion via 2.3.6.

(b) The special case of Theorem 2.6.4 where the map u is an open immersion is equivalent to the composite map in Proposition 2.4.5 being an isomorphism, cf. [L09, 4.3.6].

(c) A noteworthy result of Neeman [Nm23, Lemma 5.19] (using *ibid.,* Convention 5.5) implies that when u is flat and g is perfect, then $\beta_\sigma(G)$ is an isomorphism for all $G \in \mathbf{D}_{\mathsf{qc}}(X)$.

The rest of this section is devoted to realizing the map β_σ—or equivalently, the map $\bar g_*\beta_\sigma$—concretely. (As indicated before, this is rather more difficult than doing the same for $g_*\beta_\sigma$, which was shown in the proof of Theorem 2.6.4 to be naturally isomorphic to the map ρ, whose explicit description was indicated in the footnote in the proof of 2.1.9.) Locally, a more explicit such realization, in commutative-algebra terms, is given in Proposition 3.2.13.

Until further notice, the symbols $(-)_*$, $(-)^*$, $\otimes$ and $\mathcal{H}om$ will have their ordinary (non-derived) meaning.

2.6.6. Let Y be a ringed space, $\psi\colon \mathcal{O}_Y \to \mathcal{S}$ an $\mathcal{O}_Y$-algebra, and $\overline{Y}$ the ringed space $(Y, \mathcal{S})$. The category $\mathcal{A}(\overline{Y})$ of $\mathcal{S}$-modules is naturally isomorphic to the category having as objects the pairs $(\mathcal{N}, m_{\mathcal{N}})$ with $\mathcal{N}$ an $\mathcal{O}_Y$-module and $m_{\mathcal{N}}\colon \mathcal{S}\otimes_Y \mathcal{N} \to \mathcal{N}$ an $\mathcal{O}_Y$-homomorphism satisfying the usual conditions for scalar multiplication, and having the obvious morphisms.

For example, when F is an $\mathcal{S}$-module and G an $\mathcal{O}_Y$-module, the $\mathcal{S}$-module $\mathcal{H}om_\psi(F,G)$ is specified as such a pair in §2.2.

Let $u\colon Y' \to Y$ be a map of ringed spaces. Let $\psi'\colon \mathcal{O}_{Y'} \to \mathcal{S}'$ be the $\mathcal{O}_{Y'}$-algebra $u^*\psi$, $\mathcal{S} \to u_*u^*\mathcal{S} = u_*\mathcal{S}'$ the natural map, and

$$\bar u\colon \overline{Y}' := (Y', \mathcal{S}') \longrightarrow (Y, \mathcal{S}) =: \overline{Y}$$

the corresponding map of ringed spaces. By definition, essentially, the direct image $\bar u_*(\mathcal{M}, m_{\mathcal{M}})$ of an $\mathcal{S}'$-module is the $\mathcal{S}$-module $(u_*\mathcal{M}, m_{u_*\mathcal{M}})$ where $m_{u_*\mathcal{M}}$ is the natural composite

$$\mathcal{S}\otimes_Y u_*\mathcal{M} \longrightarrow u_*(u^*\mathcal{S}\otimes_{Y'}\mathcal{M}) \xrightarrow{u_*m_{\mathcal{M}}} u_*\mathcal{M}.$$

The direct image of a map of $\mathcal{S}'$-modules is, in these terms, specified in the obvious way.

One checks that the functor $\bar u_*$ has the left adjoint $\bar u^*$ given objectwise by $\bar u^*(\mathcal{N}, m_{\mathcal{N}}) = (u^*\mathcal{N}, m_{u^*\mathcal{N}})$ where $m_{u^*\mathcal{N}}$ is the natural composite

$$u^*\mathcal{S}\otimes_{Y'} u^*\mathcal{N} \xrightarrow{\ \sim\ } u^*(\mathcal{S}\otimes_Y \mathcal{N}) \xrightarrow{u^*m_{\mathcal{N}}} u^*\mathcal{N}, \tag{2.6.7}$$

and mapwise in the natural way.

Let $\phi\colon \overline{Y} = (Y, \mathcal{S}) \to (Y, \mathcal{O}_Y)$ be the ringed-space map (id_Y, ψ), and let $\phi'\colon \overline{Y}' = (Y', \mathcal{S}') \to (Y', \mathcal{O}_{Y'})$ be $(\mathrm{id}_{Y'}, \psi')$. With the preceding $\bar u^*$ it holds that $u^*\phi_* = \phi'_*\bar u^*$.

Lemma 2.6.8. *For any complexes $F \in \mathcal{A}(\overline{Y})$ and $G \in \mathcal{A}(Y)$ there is a unique $\mathcal{S}'$-map*

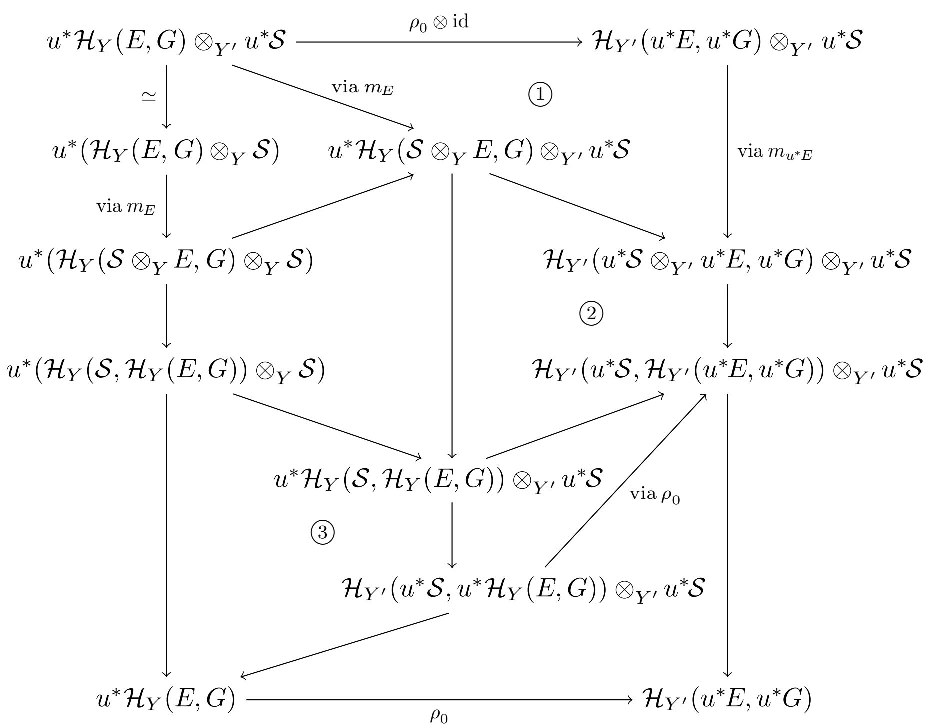
$$\bar\rho_0 = \bar\rho_0(F,G)\colon \bar u^*\mathcal{H}om_\psi(F,G) \to \mathcal{H}om_{\psi'}(\bar u^*F, u^*G)$$

such that $\phi'_*\bar\rho_0$ is the natural map

$$u^*\phi_*\mathcal{H}om_\psi(F,G) = u^*\mathcal{H}om_Y(\phi_*F,G) \xrightarrow{\rho_0} \mathcal{H}om_{Y'}(u^*\phi_*F, u^*G).$$

Proof. This follows easily from the standard explicit realization of ρ_0 and of the scalar multiplication map in §2.2.

More formally, the uniqueness of $\bar\rho_0$ is obvious, and its existence is given by commutativity of the border of the following natural diagram, in which $\mathcal{H} := \mathcal{H}om$ and $E := \phi_*F$, whose left (resp. right) column composes to scalar multiplication by $u^*\mathcal{S} = \mathcal{S}'$:



Commutativity of the unlabeled subdiagrams is clear.

That subdiagram ① commutes follows easily from the definition of m_{u^*E} (see (2.6.7), with $\mathcal{N} = E$).

Commutativity of ③ results from [L09, 3.5.6(a)] with (f, A, B) replaced by $(u, \mathcal{S}, \mathcal{H}_Y(E,G))$.

It suffices now to show commutativity of the following natural diagram, with $[-,-] := \mathcal{H}_Y(-,-)$ and $[-,-]' := \mathcal{H}_{Y'}(-,-)$, whose border is adjoint to ② without "$\otimes_{Y'} u^*\mathcal{S}$." (Note that $u^*[A,B] \to [u^*A, u^*B]'$ is adjoint to the natural composite map $[A,B] \longrightarrow [A, u_*u^*B] \xrightarrow{\ \sim\ } u_*[u^*A, u^*B]'$, see [L09, (3.5.4.5) *ff*].)

$$
\begin{array}{ccccc}
[S\otimes_Y E, G] & \longrightarrow & [S\otimes_Y E, u_*u^*G] & \longrightarrow & u_*[u^*(S\otimes_Y E), u^*G]' \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & ④ & [S,[E,u_*u^*G]] & ⑤ & u_*[u^*S\otimes_{Y'} u^*E, u^*G]' \\
 & \nearrow & \Big\downarrow & & \Big\downarrow \\
 & & [S, u_*[u^*E, u^*G]'] & \longrightarrow & u_*[u^*S,[u^*E,u^*G]']' \\
 & ⑥ & \Big\uparrow & ⑦ & \Big\uparrow \\
[S,[E,G]] & \longrightarrow & [S, u_*u^*[E,G]] & \longrightarrow & u_*[u^*S, u^*[E,G]]'
\end{array}
$$

Commutativity of subdiagrams ④ and ⑦ is clear. Commutativity of ⑥ results from the fact, noted above, that $u^*[E,G]\to[u^*E,u^*G]'$ is adjoint to $[E,G]\longrightarrow[E,u_*u^*G]\xrightarrow{\ \sim\ }u_*[u^*E,u^*G]'$. Commutativity of ⑤ is given by [L09, Exercise 3.5.6(c)], with $(f,E,F,G):=(u,S,E,u^*G)$. □

2.6.9. Let there be given an independent square of scheme-maps

$$
\begin{array}{ccc}
X' & \xrightarrow{\ v\ } & X \\
{\scriptstyle g}\Big\downarrow & \sigma & \Big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\ u\ ]{} & Y
\end{array}
$$

in which the maps f and g are affine. This σ decomposes as the border of the commutative diagram of ringed-space maps

$$
\begin{array}{ccc}
X' & \xrightarrow{\qquad\qquad v \qquad\qquad} & X \\
{\scriptstyle \bar g:=(g,\mathrm{id})}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar f:=(f,\mathrm{id})} \\
\overline{Y}{}':=(Y',\,g_*\mathcal{O}_{X'}) & \xrightarrow{\ \bar u:=(u,\bar\psi)\ } & \overline{Y}:=(Y,\,f_*\mathcal{O}_X) \\
{\scriptstyle \phi':=(\mathrm{id},\psi')}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi:=(\mathrm{id},\psi)} \\
Y' & \xrightarrow[\qquad\qquad u \qquad\qquad]{} & Y
\end{array}
$$

where $\psi\colon\mathcal{O}_Y\to f_*\mathcal{O}_X$ and $\psi'\colon\mathcal{O}_{Y'}\to g_*\mathcal{O}_{X'}$ are the maps associated with f and g respectively, and $\bar\psi$ is the natural composite map

$$
f_*\mathcal{O}_X\longrightarrow f_*v_*\mathcal{O}_{X'}\xrightarrow{\ \sim\ }u_*g_*\mathcal{O}_{X'}\,.
$$

The ringed-space maps $\bar f$ and $\bar g$ are flat (see §2.1), so that the functors $\bar f^*$ and $\bar g^*$ are exact, as are the functors ϕ_* and ϕ'_*.

There results, for any $E\in\mathbf{D}_{\mathsf{qc}}(X)$ a natural commutative diagram (see [L09, 3.7.2(ii)]):

$$
\begin{array}{ccccc}
\mathsf{L}u^*\mathsf{R}(\phi\bar f)_*E & & \xrightarrow[\theta_\sigma(E)]{\sim} & & \mathsf{R}(\phi'\bar g)_*\mathsf{L}v^*E \\
\simeq\big\downarrow & & & & \big\downarrow\simeq \\
\mathsf{L}u^*\phi_*\mathsf{R}\bar f_*E & \xrightarrow[\theta_1(\mathsf{R}\bar f_*E)]{} & \phi'_*\mathsf{L}\bar u^*\mathsf{R}\bar f_*E & \xrightarrow[\phi'_*\theta_2(E)]{} & \phi'_*\mathsf{R}\bar g_*\mathsf{L}v^*E
\end{array}
\tag{2.6.10}
$$

The definition [L09, 3.7.2(i)(c)] of θ_2 implies that the following natural diagram commutes:

$$
\begin{array}{ccc}
\bar g^*\mathsf{L}\bar u^*\mathsf{R}\bar f_*E & \xrightarrow{\bar g^*\theta_2(E)} & \bar g^*\mathsf{R}\bar g_*\mathsf{L}v^*E \\
\simeq\big\downarrow & & \simeq\big\downarrow 2.1.6 \\
\mathsf{L}v^*\bar f^*\mathsf{R}\bar f_*E & \xrightarrow[2.1.6]{\sim} & \mathsf{L}v^*E
\end{array}
$$

So $\bar g^*\theta_2(E)$—whence, by 2.1.6, $\theta_2(E)$—*is an isomorphism,* whence so is $\theta_1(\mathsf{R}\bar f_*E)$ (see (2.6.10)), i.e., by 2.1.6, *so is $\theta_1(H)$ for any $H\in\mathbf{D}_{\mathsf{qc}}(\overline{Y})$.*

For a ringed-space map $u\colon Y'\to Y$, an $\mathcal{O}_Y$-complex E is called u^*-*acyclic* if the canonical map is an isomorphism $\mathsf{L}u^*E\xrightarrow{\sim}u^*E$. For example, any K-flat E is u^*-acyclic.

For a scheme Y, a *strictly perfect* $\mathcal{O}_Y$-complex is a bounded complex of finite-rank locally free $\mathcal{O}_Y$-modules. Note that an $\mathcal{O}_Y$-complex is perfect if locally—even *globally* when Y is affine—it is the target of a quasi-isomorphism with source a strictly perfect one [Il71, p. 122, 4.8, p. 175, 2.2.10, p. 163, 2.0, and p. 96, 2.2].

Lemma 2.6.11. *In 2.6.9, let $F\in\mathbf{D}(\overline{Y})$, and assume* either *that ϕ_*F is pseudo-coherent, $G\in\mathbf{D}^+_{\mathsf{qc}}(Y)$ and u has finite tor-dimension* or *that ϕ_*F is perfect and $G\in\mathbf{D}_{\mathsf{qc}}(Y)$.*

(i) *There is a unique bifunctorial $\mathbf{D}(\overline{Y}')$-map*

$$\bar\rho=\bar\rho(F,G)\colon \mathsf{L}\bar u^*\mathsf{R}\mathcal{H}om_\psi(F,G)\to\mathsf{R}\mathcal{H}om_{\psi'}(\mathsf{L}\bar u^*F,\mathsf{L}u^*G)$$

such that if G is u^*-acyclic, the following natural diagram commutes:

$$
\begin{array}{ccccc}
\mathsf{L}\bar u^*\mathsf{R}\mathcal{H}om_\psi(F,G) & \xleftarrow[\bar a]{\sim} & \mathsf{L}\bar u^*\mathcal{H}om_\psi(F,G) & \xrightarrow[\bar b]{\sim} & \bar u^*\mathcal{H}om_\psi(F,G) \\
\Big\downarrow & & & & 2.6.8\big\downarrow\bar\rho_0(F,G) \\
\Big\downarrow & & & & \mathcal{H}om_{\psi'}(\bar u^*F,u^*G) \\
\bar\rho(F,G)\Big\downarrow & & & & \simeq\big\downarrow\bar c \\
\Big\downarrow & & & & \mathsf{R}\mathcal{H}om_{\psi'}(\bar u^*F,u^*G) \\
\Big\downarrow & & & & \big\downarrow\bar d \\
\mathsf{R}\mathcal{H}om_{\psi'}(\mathsf{L}\bar u^*F,\mathsf{L}u^*G) & & \xrightarrow[\bar e]{\sim} & & \mathsf{R}\mathcal{H}om_{\psi'}(\mathsf{L}\bar u^*F,u^*G);
\end{array}
$$

and this $\bar\rho(F,G)$ is an isomorphism.

(ii) *The following diagram commutes.*

$$
\begin{array}{ccc}
\phi'_*\mathsf{L}\bar u^*\mathsf{R}\mathcal{H}om_\psi(F,G) & \xrightarrow{\ \phi'_*\bar\rho\ } & \phi'_*\mathsf{R}\mathcal{H}om_{\psi'}(\mathsf{L}\bar u^*F,\mathsf{L}u^*G) \\
\simeq\big\downarrow \theta_1^{-1} & & (2.2.3)\big\downarrow\simeq \\
\mathsf{L}u^*\phi_*\mathsf{R}\mathcal{H}om_\psi(F,G) & & \mathsf{R}\mathcal{H}om_{Y'}(\phi'_*\mathsf{L}\bar u^*F,\mathsf{L}u^*G) \\
\simeq\big\downarrow (2.2.3) & & \text{via }\theta_1\big\downarrow\simeq \\
\mathsf{L}u^*\mathsf{R}\mathcal{H}om_Y(\phi_*F,G) & \xrightarrow[\ \rho\ ]{} & \mathsf{R}\mathcal{H}om_{Y'}(\mathsf{L}u^*\phi_*F,\mathsf{L}u^*G)
\end{array}
$$

Proof. (i) First, *the canonical map* $\mathcal{H}om_\psi(F,G)\to\mathsf{R}\mathcal{H}om_\psi(F,G)$ *is an isomorphism* (*whence so is* $\bar a$)—whether or not G is u^*-acyclic. For proving this, application of the functor ϕ_* justifies replacing "ψ" by "Y" and "F" by "ϕ_*F" (see (2.2.2) and (2.2.3)). Moreover, the question being local, one can assume ϕ_*F to be a complex of locally free $\mathcal{O}_Y$-modules. Then one can proceed as in the second- and third-last paragraphs of [L09, §4.6], with $(E,H):=(\phi_*F,G)$. (In line 3 of the third-last paragraph, "isomorphism" should be "quasi-isomorphism." Also, when ϕ_*F is strictly perfect and G is arbitrary, induct on the number of degrees in which F doesn't vanish.)

Likewise, $\bar c$ is an isomorphism.

Now every $\mathcal{O}_Y$-complex G is $\mathbf{D}(Y)$-isomorphic to a u^*-acyclic one, which can be assumed bounded-below if u has finite tor-dimension and $G\in\mathbf{D}^{+}(Y)$ [L09, 2.7.5,(vi) and (a)]. Thus to prove (i) one may assume that G is u^*-acyclic, so that $\bar e$ is an isomorphism, whence, via [L09, 2.6.5], the existence and uniqueness of a map $\bar\rho$ making the diagram commute.

A similar inductive argument shows that if G is u^*-acyclic, then for each integer n, the canonical map is an isomorphism

$$H^n\mathsf{L}u^*\mathcal{H}om_Y(\phi_*F,G)\ \xrightarrow{\ \sim\ }\ H^nu^*\mathcal{H}om_Y(\phi_*F,G),$$

i.e., $\mathcal{H}om_Y(\phi_*F,G)=\phi_*\mathcal{H}om_\psi(F,G)$ is u^*-acyclic. Also,

$$H:=\mathcal{H}om_\psi(F,G)\cong\mathsf{R}\mathcal{H}om_\psi(F,G)\in\mathbf{D}_{\mathsf{qc}}(\overline{Y}),$$

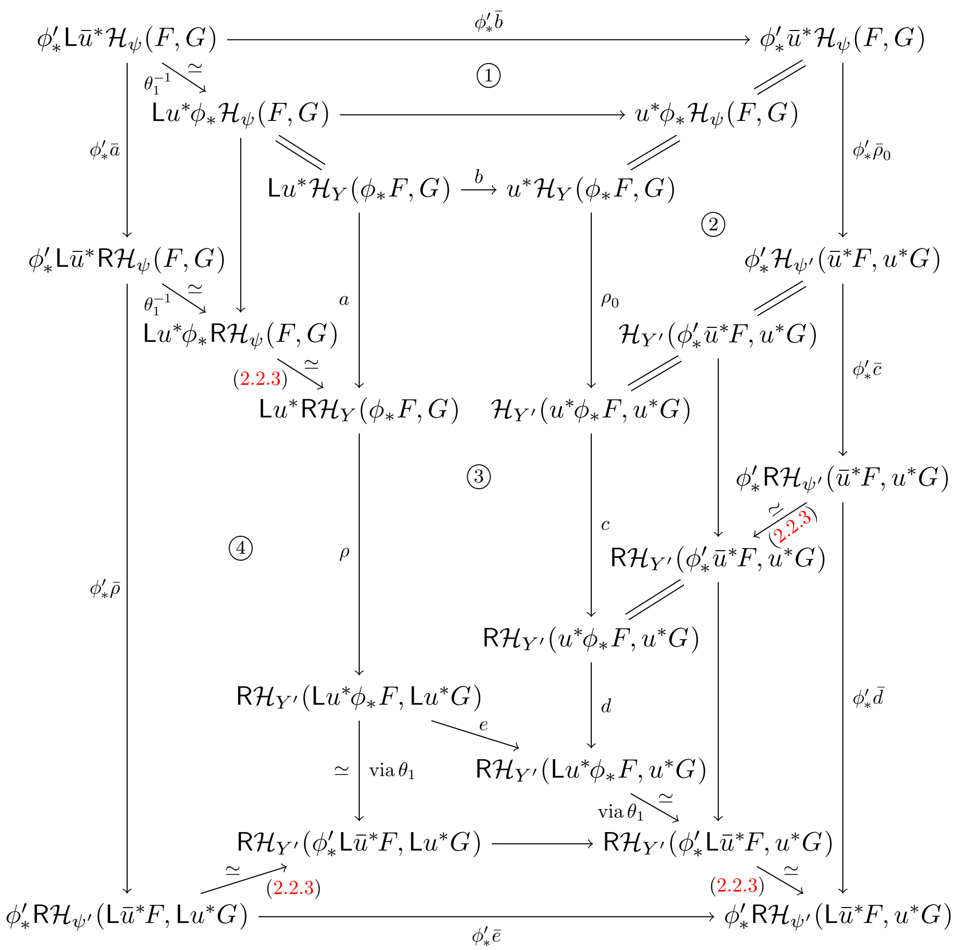
as follows via [GrD71, p. 218, (2.2.4)] from the exactness of ϕ_* and the fact that, by (2.2.3) and 2.3.8, $\phi_*\mathsf{R}\mathcal{H}om_\psi(F,G)\cong\mathsf{R}\mathcal{H}om_Y(\phi_*F,G)\in\mathbf{D}_{\mathsf{qc}}(Y)$. So as in the remarks right after (2.6.10), $\theta_1(H)$ is an isomorphism; and, as noted right after (2.6.7), $u^*\phi_*=\phi'_*\bar u^*$. Thus, from the natural diagram

$$
\begin{array}{ccc}
\mathsf{L}\bar u^*\phi_*H & \xrightarrow[\theta_1(H)]{\ \sim\ } & \phi'_*\mathsf{L}\bar u^*H \\
\simeq\big\downarrow & & \big\downarrow\phi'_*\bar b \\
\bar u^*\phi_*H & = & \phi'_*\bar u^*H,
\end{array}
$$

which commutes (see [L09, Lemma 3.10.1.1]), one gets that $\phi'_*\bar b$, and hence $\bar b$ itself, is an isomorphism.

Finally, another similar induction shows that $\phi'_*\bar\rho_0(F,G) = \rho_0(F,G)$ is an isomorphism, whence so is $\bar\rho_0(F,G)$. Therefore, if G is u^*-acyclic then $\bar\rho(F,G) \cong \bar\rho_0(F,G)$ is an isomorphism, whence so is $\bar\rho(F,G)$ even if G is not u^*-acyclic.

(ii) This says that in the following natural diagram, where $\mathcal{H} := \mathcal{H}om$, subdiagram ④ commutes. (The maps labeled θ_1^{-1} exist by the remarks after (2.6.10) because the isomorphic complexes $\mathcal{H}om_\psi(F,G)$ and $\mathsf{R}\mathcal{H}om_\psi(F,G)$ are in $\mathbf{D}_{\mathsf{qc}}(\overline{Y})$, as follows via [GrD71, p. 218, (2.2.4)] from the exactness of ϕ_* and the fact that, by 2.3.8, $\phi_*\mathsf{R}\mathcal{H}om_\psi(F,G) \cong \mathsf{R}\mathcal{H}om_Y(\phi_*F,G) \in \mathbf{D}_{\mathsf{qc}}(Y)$.)



As before, one may assume that G is u^*-acyclic, so that the maps e and $\bar e$ are isomorphisms.

Commutativity of subdiagram ① is given by [L09, Lemma 3.10.1.1] (with $(f,g,u,v) := (\phi,\phi',u,\bar u)$)—which holds over the category of arbitrary ringed spaces; and of ② by Lemma 2.6.8. Subdiagram ③ is just diagram (4.6.7.1) toward the end of [L09, §4.6], shown there to commute. By (i), the outer border commutes. Diagram-chasing shows then that ④ commutes. $\square$

Let $\bar\rho = \bar\rho(f_*\mathcal{O}_X, G)$ be as in 2.6.11, and $\theta_2\colon \mathsf{L}\bar u^*\mathsf{R}f_*\mathcal{O}_X \xrightarrow{\,\sim\,} \mathsf{R}\bar g_*\mathsf{L}v^*\mathcal{O}_X$ as in (2.6.10) (an isomorphism by the remarks following (2.6.10)). With notation as in (2.6.10), Proposition 2.6.14 below shows that applying $\bar g^*$ to the composite map

$$
\begin{aligned}
\bar\beta_\sigma(G)\colon \mathsf{L}\bar u^*\phi^\flat G = \mathsf{L}\bar u^*\mathsf{R}\mathcal{H}om_\psi(\bar f_*\mathcal{O}_X, G)& \\
\underset{\bar\rho}{\longrightarrow}\ \mathsf{R}\mathcal{H}om_{\psi'}(\mathsf{L}\bar u^*\bar f_*\mathcal{O}_X, \mathsf{L}u^*G)& \\
\underset{\text{via}\,\theta_2}{\xrightarrow{\,\sim\,}}\ \mathsf{R}\mathcal{H}om_{\psi'}(\bar g_*\mathcal{O}_{X'}, \mathsf{L}u^*G) = \phi'^\flat\mathsf{L}u^*G&
\end{aligned}
\tag{2.6.12}
$$

gives a realization of the base-change map $\beta_\sigma(G)$ of 2.6.4, a realization that is concrete, modulo taking resolutions, as far as indicated by 2.6.11(i) and the explicit local description of θ in [L09, Lemma 3.10.1.2].

A more explicit local realization of $\beta_\sigma(G)$, in commutative-algebra terms, results from Proposition 3.2.13 below.

Lemma 2.6.13. *In the situation of Theorem 2.6.4, $\beta_\sigma(G)$ is the unique $\mathbf{D}(X')$-map $\beta(G)\colon v^*f^\flat G \to g^\flat u^*G$ making the following diagram commute.*

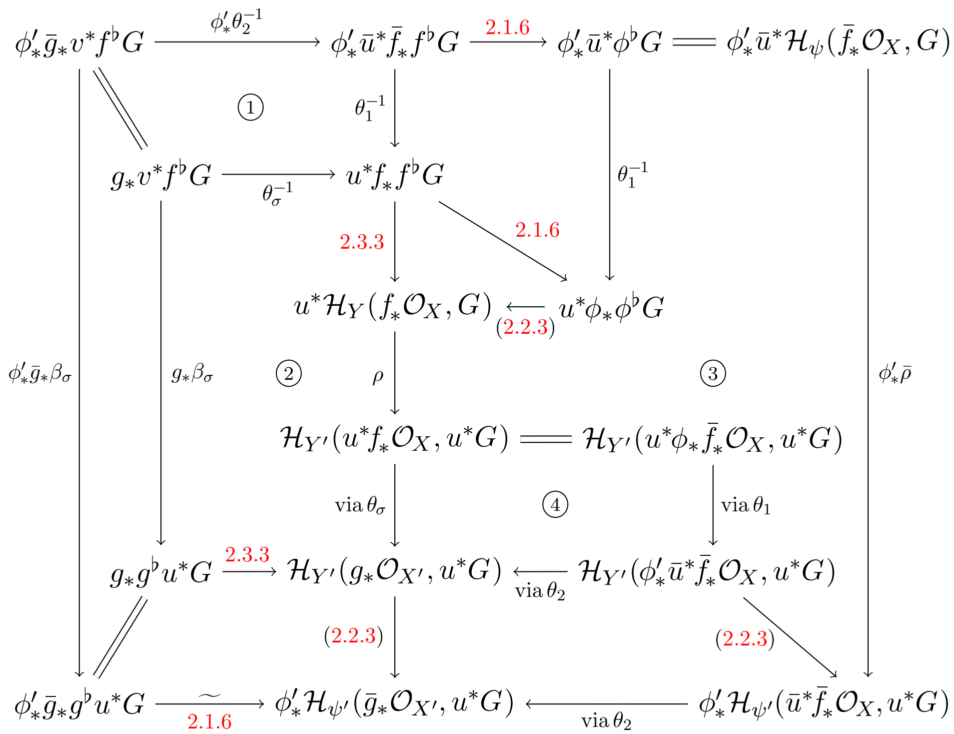
$$
\begin{array}{ccccc}
\mathsf{R}\bar g_*\mathsf{L}v^*f^\flat G & \xleftarrow[\theta_2(f^\flat G)]{\sim} & \mathsf{L}\bar u^*\mathsf{R}\bar f_*f^\flat G & \xrightarrow[2.1.6]{\sim} & \mathsf{L}\bar u^*\phi^\flat G \\
{\scriptstyle \mathsf{R}\bar g_*\beta(G)}\big\downarrow & & & & \big\downarrow{\scriptstyle (2.6.12)\ \bar\beta_\sigma(G)} \\
\mathsf{R}\bar g_*g^\flat\mathsf{L}u^*G & & \xrightarrow[2.1.6]{\sim} & & \phi'^\flat\mathsf{L}u^*G
\end{array}
$$

Proof. Uniqueness results from Proposition 2.1.6.

Applying $\mathrm{H}^0\mathsf{R}\Gamma(W,-)$ to the composite isomorphism in Corollary 2.4.4, and using the sentence right after (2.1.7), one gets that for $F \in \mathbf{D}(Y')$ the natural map

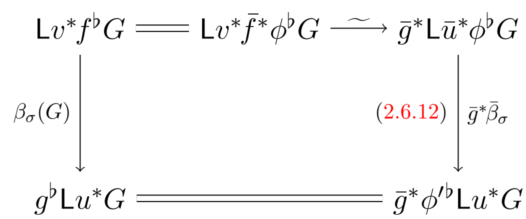
$$
\mathrm{Hom}_{\mathbf{D}(\overline{Y}')}(E, \phi'^\flat F) \longrightarrow \mathrm{Hom}_{\mathbf{D}(Y')}(\phi'_*E, \phi'_*\phi'^\flat F)
$$

has a left inverse, and so is injective. Hence, for Lemma 2.6.13 to hold it suffices that the border of the following diagram, in which $\mathcal{H} := \mathsf{R}\mathcal{H}om$, $\bar\rho := \bar\rho(f_*\mathcal{O}_X, G)$, $(-)_* := \mathsf{R}(-)_*$ and $(-)^* := \mathsf{L}(-)^*$, commute:

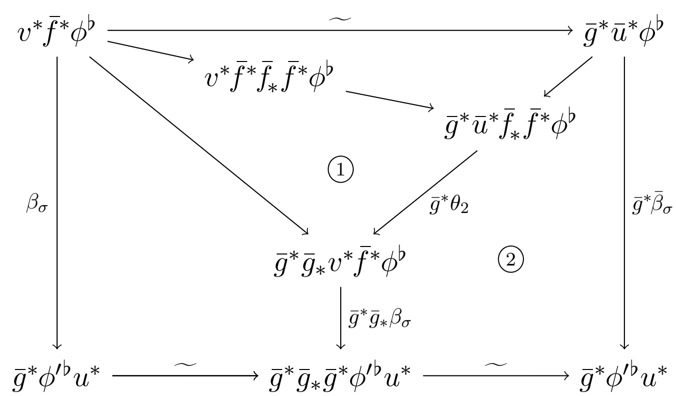


The unlabeled subdiagrams are easily seen to commute. Commutativity of subdiagrams ① and ④ is given by that of the diagram (2.6.10), with $E := f^\flat G$ or $\mathcal{O}_X$; commutativity of ② was shown in the proof of 2.6.4; and commutativity of subdiagram ③ is 2.6.11(ii). The conclusion follows. □

Proposition 2.6.14. *The following natural diagram commutes.*



Proof. The diagram, without "G," expands naturally to the following one, where $(-)^* := \mathsf{L}(-)^*$ and $(-)_* := \mathsf{R}(-)_*$:



Commutativity of subdiagram ① results from 2.1.6 and adjointness of θ_2 to the natural composite $\bar g^*\bar u^*\bar f_* \xrightarrow{\ \sim\ } v^*\bar f^*\bar f_* \to v^*$ (see [L09, 3.7.2(i)]); and that of ② from Lemma 2.6.13. That of the other two subdiagrams is clear. $\square$

2.7. For a pseudo-coherent finite map $f\colon X \to Y$, and $F, G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$, with G perfect (so that $f^\flat F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G \in \mathbf{D}^+_{\mathsf{qc}}(X)$ and $F \otimes^{\mathsf{L}}_Y G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$), Proposition 2.7.7 provides a concrete representation of the map

$$\chi = \chi(f, F, G)\colon f^\flat F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G \longrightarrow f^\flat(F \otimes^{\mathsf{L}}_Y G)$$

that is defined abstractly to be adjoint under 2.3.10 to the natural composite

$$\mathsf{R}f_*(f^\flat F \otimes^{\mathsf{L}}_Y \mathsf{L}f^*G) \xrightarrow[p]{\ \sim\ } \mathsf{R}f_*f^\flat F \otimes^{\mathsf{L}}_Y G \longrightarrow F \otimes^{\mathsf{L}}_Y G,$$

where p is the first (projection) isomorphism in (2.3.1) with $E := f^\flat F$.

The *pseudofunctoriality* of χ is explicated in Proposition 2.7.8.

Let $(Y, \mathcal{O}_Y)$ be a ringed space, and

(2.7.1) $$\gamma_0(E, F, G)\colon \mathcal{H}om_Y(E, F) \otimes_Y G \longrightarrow \mathcal{H}om_Y(E, F \otimes_Y G)$$

the natural map of $\mathcal{O}_Y$-complexes. The next result is, *mutatis mutandis,* an instance of [L09, Corollary 2.6.5].

Lemma 2.7.2. *There exists a unique trifunctorial map*

$$\gamma(E,F,G)\colon \mathsf{R}\mathcal{H}om_Y(E,F)\otimes^{\mathsf{L}}_Y G \to \mathsf{R}\mathcal{H}om_Y(E,F\otimes^{\mathsf{L}}_Y G) \quad (E,F,G \in \mathbf{D}(Y))$$

such that if F is K-injective and G is K-flat then the following natural diagram commutes.

$$\begin{array}{ccccc}
\mathcal{H}om_Y(E,F)\otimes_Y G & \xrightarrow{\ \sim\ } & \mathcal{H}om_Y(E,F)\otimes^{\mathsf{L}}_Y G & \xrightarrow{\ \sim\ } & \mathsf{R}\mathcal{H}om_Y(E,F)\otimes^{\mathsf{L}}_Y G \\
{\scriptstyle\gamma_0}\big\downarrow & & & & \big\downarrow{\scriptstyle\gamma} \\
\mathcal{H}om_Y(E,F\otimes_Y G) & \longrightarrow & \mathsf{R}\mathcal{H}om_Y(E,F\otimes_Y G) & \xrightarrow{\ \sim\ } & \mathsf{R}\mathcal{H}om_Y(E,F\otimes^{\mathsf{L}}_Y G)
\end{array}$$

Corollary 2.7.3. *The map* $\gamma(\mathcal{O}_Y, F, G)$ *factors naturally as*

$$\mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, F)\otimes^{\mathsf{L}}_Y G \xrightarrow{\ \sim\ } F\otimes^{\mathsf{L}}_Y G \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, F\otimes^{\mathsf{L}}_Y G).$$

Proof. Replace F (respectively G) by a quasi-isomorphic K-injective (respectively K-flat) complex, and then use the commutative diagram in 2.7.2 to reduce to the corresponding easily-verified statement for γ_0. □

Remark 2.7.4. As a relatively easy example of “concrete vs. abstract,” it is readily shown that γ_0 is adjoint to the natural composite

$$(\mathcal{H}om_Y(E,F)\otimes_Y G)\otimes_Y E \xrightarrow{\ \sim\ } (\mathcal{H}om_Y(E,F)\otimes_Y E)\otimes_Y G \longrightarrow F\otimes_Y G,$$

and hence that γ is adjoint to the natural composite

$$(\mathsf{R}\mathcal{H}om_Y(E,F)\otimes^{\mathsf{L}}_Y G)\otimes^{\mathsf{L}}_Y E \xrightarrow{\ \sim\ } (\mathsf{R}\mathcal{H}om_Y(E,F)\otimes^{\mathsf{L}}_Y E)\otimes^{\mathsf{L}}_Y G \longrightarrow F\otimes^{\mathsf{L}}_Y G.$$

Now let $\psi\colon \mathcal{O}_Y\to\mathcal{S}$, $\phi\colon(\overline{Y},\mathcal{O}_{\overline{Y}}) = (Y,\mathcal{S})\to(Y,\mathcal{O}_Y)$ be as in §2.2, and $\phi^\flat(-) := \mathsf{R}\mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}}, -)$ as in 2.2.5.

Define the map of $\mathcal{O}_{\overline{Y}}$-modules

$$\bar\gamma_0(F,G)\colon \mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}}, F)\otimes_{\overline{Y}}\phi^*G \longrightarrow \mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}}, F\otimes_Y G) \tag{2.7.5}$$

to be the natural composite $\mathcal{O}_{\overline{Y}}$-linear map

$$\begin{aligned}\phi_*(\mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}}, F)\otimes_{\overline{Y}}\phi^*G) &\underset{2.2.9.2}{\xrightarrow{\ \sim\ }} \phi_*\mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}}, F)\otimes_Y G\\ &=\!= \mathcal{H}om_Y(\phi_*\mathcal{O}_{\overline{Y}}, F)\otimes_Y G\\ &\underset{\gamma_0}{\longrightarrow} \mathcal{H}om_Y(\phi_*\mathcal{O}_{\overline{Y}}, F\otimes_Y G)\\ &=\!= \phi_*\mathcal{H}om_\psi(\mathcal{O}_{\overline{Y}}, F\otimes_Y G).\end{aligned}$$

Lemma 2.7.6. *In the situation of* §2.2:

(i) *The map adjoint under* 2.2.6 *to the composite map*

$$\phi_*(\phi^\flat F\otimes^{\mathsf{L}}_{\overline{Y}}\mathsf{L}\phi^*G) \underset{(2.2.9.3)}{\xrightarrow{\ \sim\ }} \phi_*\phi^\flat F\otimes^{\mathsf{L}}_Y G \underset{(2.2.8.1)}{\longrightarrow} F\otimes^{\mathsf{L}}_Y G \qquad (F,G\in\mathbf{D}(Y))$$

is the unique $\mathbf{D}(\overline{Y})$*-map*

$$\bar\chi\colon \phi^\flat F\otimes^{\mathsf{L}}_{\overline{Y}}\mathsf{L}\phi^*G \longrightarrow \phi^\flat(F\otimes^{\mathsf{L}}_Y G)$$

such that the following diagram commutes.

$$\begin{array}{ccc}
\phi_*(\phi^\flat F\otimes^{\mathsf{L}}_{\overline{Y}}\mathsf{L}\phi^*G) & \xrightarrow{\ \phi_*\bar\chi\ } & \phi_*\phi^\flat(F\otimes^{\mathsf{L}}_Y G)\\
{\scriptstyle (2.2.9.3)}\big\downarrow\simeq & & \\
\phi_*\phi^\flat F\otimes^{\mathsf{L}}_Y G & \textcircled{1} & \simeq\big\downarrow{\scriptstyle (2.2.3)}\\
{\scriptstyle (2.2.3)}\big\downarrow\simeq & & \\
\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F)\otimes^{\mathsf{L}}_Y G & \underset{\gamma}{\xrightarrow{\ 2.7.2\ }} & \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F\otimes^{\mathsf{L}}_Y G)
\end{array}$$

(ii) (Explicit $\bar\chi$) *For K-injective F and K-flat G, $\bar\chi$ factors naturally as*

$$\begin{aligned}
&\phi^\flat F \otimes^{\mathsf L}_{\overline{Y}} \mathsf L\phi^*G = \mathsf R\mathcal{H}om_\psi(\mathcal O_{\overline{Y}}, F) \otimes^{\mathsf L}_{\overline{Y}} \mathsf L\phi^*G\\
&\quad\xrightarrow{\ \sim\ } \mathcal{H}om_\psi(\mathcal O_{\overline{Y}}, F) \otimes_{\overline{Y}} \phi^*G\\
&\quad\xrightarrow[(2.7.5)]{\bar\gamma_0} \mathcal{H}om_\psi(\mathcal O_{\overline{Y}}, F\otimes_Y G) \longrightarrow \mathsf R\mathcal{H}om_\psi(\mathcal O_{\overline{Y}}, F\otimes_Y G) = \phi^\flat(F\otimes^{\mathsf L}_Y G).
\end{aligned}$$

Proof. First, the uniqueness in (i). If ① commutes then, as made clear by 2.7.3, so does the following diagram, where going clockwise from upper right to lower left gives the counit map from (2.2.8.1).

$$\begin{array}{ccccc}
\phi_*(\phi^\flat F \otimes^{\mathsf L}_{\overline{Y}} \mathsf L\phi^*G) & & \xrightarrow{\quad\phi_*\bar\chi\quad} & & \phi_*\phi^\flat(F\otimes^{\mathsf L}_Y G)\\
{\scriptstyle(2.2.9.3)}\big\downarrow\simeq & & ① & & \simeq\big\downarrow{\scriptstyle(2.2.3)}\\
\phi_*\phi^\flat F\otimes^{\mathsf L}_Y G & \xrightarrow[(2.2.3)]{\sim} & \mathsf R\mathcal{H}om_Y(\phi_*\mathcal O_{\overline{Y}}, F)\otimes^{\mathsf L}_Y G & \xrightarrow[\gamma]{} & \mathsf R\mathcal{H}om_Y(\phi_*\mathcal O_{\overline{Y}}, F\otimes^{\mathsf L}_Y G)\\
{\scriptstyle(2.2.8.1)}\big\downarrow & & \big\downarrow{\scriptstyle\text{via }\psi} & & \big\downarrow{\scriptstyle\text{via }\psi}\\
F\otimes^{\mathsf L}_Y G & \xleftarrow[\text{natural}]{\sim} & \mathsf R\mathcal{H}om_Y(\mathcal O_Y, F)\otimes^{\mathsf L}_Y G & \xleftarrow[\gamma^{-1}]{\sim} & \mathsf R\mathcal{H}om_Y(\mathcal O_Y, F\otimes^{\mathsf L}_Y G)
\end{array}$$

Thus any $\mathbf D(\overline{Y})$-map $\bar\chi$ such that ① commutes must be the one adjoint under 2.2.6 to the natural composite

$$\phi_*(\phi^\flat F \otimes^{\mathsf L}_{\overline{Y}} \mathsf L\phi^*G) \xrightarrow[(2.2.9.3)]{\sim} \phi_*\phi^\flat F\otimes^{\mathsf L}_Y G \xrightarrow[(2.2.8.1)]{} F\otimes^{\mathsf L}_Y G.$$

This being so, (i) and (ii) can be proved by showing, after replacing F (resp. G) by a quasi-isomorphic K-injective (resp. K-flat) $\mathcal O_Y$-complex, that for $\bar\chi$ as in (ii), ① commutes; and for this just note that each subdiagram of the following natural diagram, where $\mathcal H := \mathcal Hom$, commutes—more or less by definition of the functorial maps involved, whence so does the border.

$$\begin{array}{ccccc}
\phi_*(\phi^\flat F \otimes^{\mathsf L}_{\overline{Y}} \mathsf L\phi^*G) & = & \phi_*(\mathsf R\mathcal H_\psi(\mathcal O_{\overline{Y}}, F)\otimes^{\mathsf L}_{\overline{Y}} \mathsf L\phi^*G) & \xrightarrow{\sim} & \phi_*(\mathcal H_\psi(\mathcal O_{\overline{Y}}, F)\otimes_{\overline{Y}} \phi^*G)\\
{\scriptstyle(2.2.9.3)}\big\downarrow\simeq & & & \swarrow{\scriptstyle\simeq\ (2.2.9.2)} & \big\downarrow\\
\phi_*\phi^\flat F\otimes^{\mathsf L}_Y G & \xrightarrow{\sim} & \phi_*\mathcal H_\psi(\mathcal O_{\overline{Y}}, F)\otimes_Y G & & \big\downarrow\\
{\scriptstyle(2.2.3)}\big\downarrow\simeq & & \big\| & & \big\downarrow{\scriptstyle\phi_*\bar\gamma_0}\\
\mathsf R\mathcal H_Y(\phi_*\mathcal O_{\overline{Y}}, F)\otimes^{\mathsf L}_Y G & \xleftarrow{\sim} & \mathcal H_Y(\phi_*\mathcal O_{\overline{Y}}, F)\otimes_Y G & & \big\downarrow\\
{\scriptstyle\gamma}\big\downarrow & \text{(see 2.7.2)} & \big\downarrow{\scriptstyle\gamma_0} & & \big\downarrow\\
\mathsf R\mathcal H_Y(\phi_*\mathcal O_{\overline{Y}}, F\otimes^{\mathsf L}_Y G) & \longleftarrow & \mathcal H_Y(\phi_*\mathcal O_{\overline{Y}}, F\otimes_Y G) & = & \phi_*\mathcal H_\psi(\mathcal O_{\overline{Y}}, F\otimes_Y G)\\
{\scriptstyle(2.2.3)}\big\uparrow\simeq & & & & \big\downarrow\\
\phi_*\phi^\flat(F\otimes^{\mathsf L}_Y G) & & \xleftarrow[(2.2.3)]{\sim} & & \phi_*\mathsf R\mathcal H_\psi(\mathcal O_{\overline{Y}}, F\otimes_Y G)
\end{array}$$

□

The following proposition addresses, both concretely and abstractly, the relation between $(-)^\flat$ and $\otimes^{\mathsf L}$.

Proposition 2.7.7. *Let $f\colon X\to Y$ be a pseudo-coherent finite scheme-map, and $\psi\colon \mathcal O_Y\to f_*\mathcal O_X$ the associated homomorphism, so that f factors as*

$$X:=(X,\mathcal O_X)\xrightarrow{\bar f:=(f,\mathrm{id})}\overline{Y}:=(Y,f_*\mathcal O_X)\xrightarrow{\phi:=(\mathrm{id},\psi)}(Y,\mathcal O_Y)=:Y$$

(see §2.3). For $F,G\in\mathbf D(Y)$ let $\chi=\chi(f,F,G)$ be the natural composite map

$$f^\flat F\otimes^{\mathsf L}_X\mathsf Lf^*G\ \xrightarrow{\ \sim\ }\ \bar f^*(\phi^\flat F\otimes^{\mathsf L}_{\overline Y}\mathsf L\phi^*G)\ \underset{2.7.6}{\xrightarrow{\bar f^*\bar\chi}}\ \bar f^*\phi^\flat(F\otimes^{\mathsf L}_Y G)=f^\flat(F\otimes^{\mathsf L}_Y G).$$

(i) *If $F\in\mathbf D^+_{\mathsf{qc}}(Y)$ and $G\in\mathbf D_{\mathsf{qc}}(Y)$ are such that $F\otimes^{\mathsf L}_Y G\in\mathbf D^+_{\mathsf{qc}}(Y)$, then $\chi(f,F,G)$ is the unique $\mathbf D(X)$-map $\chi'\colon f^\flat F\otimes^{\mathsf L}_X\mathsf Lf^*G\to f^\flat(F\otimes^{\mathsf L}_Y G)$ such that the following diagram commutes:*

$$\begin{array}{ccc}
\mathsf Rf_*(f^\flat F\otimes^{\mathsf L}_X\mathsf Lf^*G) & \xrightarrow{\qquad\mathsf Rf_*\chi'\qquad} & \mathsf Rf_*f^\flat(F\otimes^{\mathsf L}_Y G)\\
{\scriptstyle p}\big\downarrow{\scriptstyle (2.3.1)} & & \\
\mathsf Rf_*f^\flat F\otimes^{\mathsf L}_Y G & \text{①} & {\scriptstyle 2.3.3}\big\downarrow\simeq\\
\simeq\big\downarrow{\scriptstyle 2.3.3} & & \\
\mathsf R\mathcal Hom_Y(f_*\mathcal O_X,F)\otimes^{\mathsf L}_Y G & \xrightarrow[\gamma]{\qquad} & \mathsf R\mathcal Hom_Y(f_*\mathcal O_X,F\otimes^{\mathsf L}_Y G);
\end{array}$$

and χ corresponds under 2.3.5 to the composite map

$$\mathsf Rf_*(f^\flat F\otimes^{\mathsf L}_X\mathsf Lf^*G)\ \underset{p}{\xrightarrow{\ \sim\ }}\ \mathsf Rf_*f^\flat F\otimes^{\mathsf L}_Y G\ \underset{t_F\otimes^{\mathsf L}_Y\mathrm{id}}{\xrightarrow{\qquad}}\ F\otimes^{\mathsf L}_Y G.$$

(ii) *If f is perfect then* (i) *holds for all $F,G\in\mathbf D_{\mathsf{qc}}(Y)$, and both $\bar\chi$ and χ are isomorphisms.*

Proof. For diagram ① to commute when $\chi'=\chi$, it clearly suffices that the following natural diagram commute.

$$\begin{array}{ccccc}
\phi_*\mathsf R\bar f_*(\bar f^*\phi^\flat F\otimes^{\mathsf L}_X\bar f^*\mathsf L\phi^*G) & \xleftarrow{\ \sim\ } & \mathsf Rf_*(f^\flat F\otimes^{\mathsf L}_X\mathsf Lf^*G) & \xrightarrow{\mathsf Rf_*\chi} & \mathsf Rf_*f^\flat(F\otimes^{\mathsf L}_Y G)\\
 & \searrow{\scriptstyle\sim} & \simeq\big\downarrow & & \big\|\\
\simeq\big\downarrow{\scriptstyle 2.1.10}\quad\text{②} & & \phi_*\mathsf R\bar f_*\bar f^*(\phi^\flat F\otimes^{\mathsf L}_{\overline Y}\mathsf L\phi^*G) & \xrightarrow{\phi_*\mathsf R\bar f_*\bar f^*\bar\chi} & \phi_*\mathsf R\bar f_*\bar f^*\phi^\flat(F\otimes^{\mathsf L}_Y G)\\
 & & \simeq\big\downarrow{\scriptstyle 2.1.6} & & {\scriptstyle 2.1.6}\big\downarrow\simeq\\
\phi_*(\mathsf R\bar f_*\bar f^*\phi^\flat F\otimes^{\mathsf L}_{\overline Y}\mathsf L\phi^*G) & \underset{2.1.6}{\xrightarrow{\ \sim\ }} & \phi_*(\phi^\flat F\otimes^{\mathsf L}_{\overline Y}\mathsf L\phi^*G) & \underset{\phi_*\bar\chi}{\xrightarrow{\qquad}} & \phi_*\phi^\flat(F\otimes^{\mathsf L}_Y G)\\
\simeq\big\downarrow{\scriptstyle (2.2.9.3)} & & \simeq\big\downarrow{\scriptstyle (2.2.9.3)} & & \\
\phi_*\mathsf R\bar f_*\bar f^*\phi^\flat F\otimes^{\mathsf L}_Y G & \underset{2.1.6}{\xrightarrow{\ \sim\ }} & \phi_*\phi^\flat F\otimes^{\mathsf L}_Y G & \text{③} & {\scriptstyle (2.2.3)}\big\downarrow\simeq\\
\big\| & & \simeq\big\downarrow{\scriptstyle (2.2.3)} & & \\
\mathsf Rf_*f^\flat F\otimes^{\mathsf L}_Y G & \underset{2.3.3}{\xrightarrow{\ \sim\ }} & \mathsf R\mathcal Hom_Y(f_*\mathcal O_X,F)\otimes^{\mathsf L}_Y G & \underset{\gamma}{\xrightarrow{\qquad}} & \mathsf R\mathcal Hom_Y(f_*\mathcal O_X,F\otimes^{\mathsf L}_Y G)
\end{array}$$

But subdiagram ② commutes, by [L09, 3.4.7(i)], subdiagram ③ commutes, by 2.7.6, and all the unlabeled subdiagrams obviously commute as well.

As for the rest of (i), unwinding the definitions of the maps involved, one verifies that the unlabeled subdiagrams of the next diagram commute; and via 2.3.5(ii) and 2.7.3, that going around clockwise from upper right to lower left gives the map $t_{F\otimes^{\mathsf L}_Y G}$.

$$
\begin{array}{ccccc}
\mathsf{R}f_*(f^\flat F\otimes^{\mathsf L}_X \mathsf{L}f^*G) & & \xrightarrow{\quad \mathsf{R}f_*\chi' \quad} & & \mathsf{R}f_*f^\flat(F\otimes^{\mathsf L}_Y G)\\
{\scriptstyle p}\big\downarrow{\scriptstyle \simeq} & & \textcircled{1} & & {\scriptstyle 2.3.3}\big\downarrow{\scriptstyle \simeq}\\
\mathsf{R}f_*f^\flat F\otimes^{\mathsf L}_Y G & \underset{2.3.3}{\xrightarrow{\ \sim\ }} & \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F)\otimes^{\mathsf L}_Y G & \underset{\gamma}{\longrightarrow} & \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F\otimes^{\mathsf L}_Y G)\\
{\scriptstyle t_F\otimes^{\mathsf L}_Y \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \text{via}\,\psi} & & \big\downarrow{\scriptstyle \text{via}\,\psi}\\
F\otimes^{\mathsf L}_Y G & \underset{\text{natural}}{\xleftarrow{\ \sim\ }} & \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, F)\otimes^{\mathsf L}_Y G & \underset{\gamma^{-1}}{\xleftarrow{\ \sim\ }} & \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, F\otimes^{\mathsf L}_Y G)
\end{array}
$$

Thus, and in view of 2.3.8 with $F := f_*\mathcal{O}_X$, any χ' such that ① commutes must correspond under 2.3.5 to the composite map

$$\mathsf{R}f_*(f^\flat F\otimes^{\mathsf L}_X \mathsf{L}f^*G) \underset{p}{\xrightarrow{\ \sim\ }} \mathsf{R}f_*f^\flat F\otimes^{\mathsf L}_Y G \xrightarrow[t_F\otimes^{\mathsf L}_Y \mathrm{id}]{} F\otimes^{\mathsf L}_Y G.$$

(ii) If f is perfect, the proof of (i) is valid for all F and G in $\mathbf{D}_{\mathsf{qc}}(Y)$.

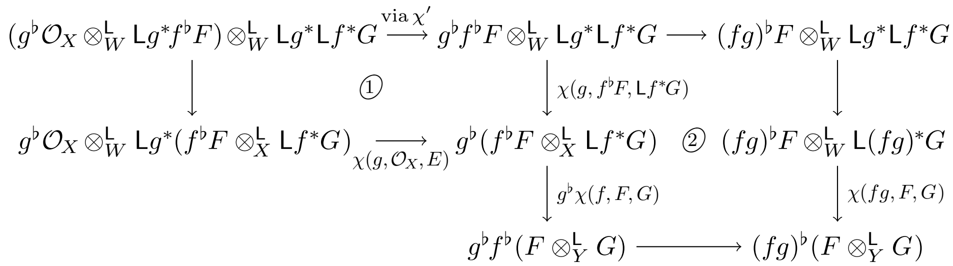
For χ and $\bar\chi$ to be isomorphisms, it suffices that $\mathsf{R}f_*\chi$ be an isomorphism: for if $\mathsf{R}f_*\chi = \phi_*\mathsf{R}\bar f_*\chi$ induces homology isomorphisms, then so does $\mathsf{R}\bar f_*\chi$, i.e., $\mathsf{R}\bar f_*\chi \cong \mathsf{R}\bar f_*\bar f^*\bar\chi$ is an isomorphism, whence by Proposition 2.1.6, so are χ and $\bar\chi$.

Since ② commutes when $\chi' = \chi$, therefore $\mathsf{R}f_*\chi$ is an isomorphism if γ is an isomorphism—which γ *is* when f is perfect. (This is well-known: the question being local, one can replace $f_*\mathcal{O}_X$ by an isomorphic bounded complex E of finite-rank free $\mathcal{O}_Y$-modules, then by induction on the number of nonvanishing components of E, using the triangle [H66, p. 70, (1)], reduce to the trivial case where E itself is a finite-rank free $\mathcal{O}_Y$-module.) $\square$

The following variant of [L09, 4.7.3.4, (a) and (d)] contains the *pseudo-functoriality of* χ (cf. the part of §5.7 in [AJL11] that follows (5.7.3)).

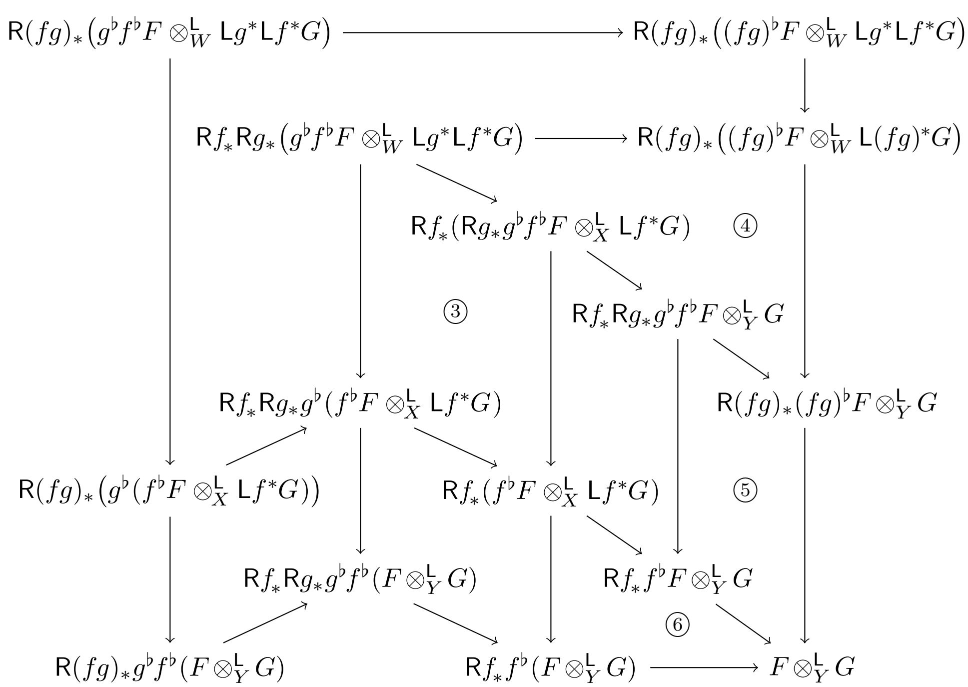
The proof of 2.7.8 that appears here is abstract; a concrete treatment, via 2.7.6(ii), is left to the curious reader.

Proposition 2.7.8 (Transitivity of χ)**.** *Let f, F, G and χ be as in 2.7.7. Let $g\colon W \to X$ be a pseudo-coherent finite scheme-map, assumed perfect if f is perfect. The following natural diagram, with $\chi' := \chi(g, \mathcal{O}_X, f^\flat F)$ and $E := f^\flat F\otimes^{\mathsf L}_X \mathsf{L}f^*G$, commutes.*



Remark. It should be noted that under the assumptions of 2.7.7(i), one has $E \cong f^\flat(F \otimes^{\mathsf{L}}_Y G) \in \mathbf{D}^+_{\mathsf{qc}}(X)$; so in any case, $\chi(g, \mathcal{O}_X, E)$ is well-defined.

Proof. The commutativity of subdiagram ② is equivalent to that of its adjoint, which, in view of 2.7.7, is the border of the natural diagram



In that diagram, the commutativity of ④ and ⑥ is given by 2.7.7, that of ③ follows from [L09, Proposition 3.7.1], and, with notation as in the first paragraph of §2.5, ⑤ is the second (commutative) diagram in [L09, 3.3.7(a)]. Commutativity of the unlabeled subdiagrams is easily checked.

Similarly, the commutativity of ① results from that of all the subdiagrams of the following natural diagram.

$$
\begin{array}{ccc}
\mathsf{R}g_*\big((g^\flat\mathcal{O}_X\otimes^{\mathsf{L}}_W \mathsf{L}g^*f^\flat F)\otimes^{\mathsf{L}}_W \mathsf{L}g^*\mathsf{L}f^*G\big) & \xrightarrow{\text{via }\chi'} & \mathsf{R}g_*\big(g^\flat f^\flat F\otimes^{\mathsf{L}}_W \mathsf{L}g^*\mathsf{L}f^*G\big)\\
 & ⑧ & \\
\mathsf{R}g_*(g^\flat\mathcal{O}_X\otimes^{\mathsf{L}}_W \mathsf{L}g^*f^\flat F)\otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \xrightarrow{\text{via }\chi'} & \mathsf{R}g_*g^\flat f^\flat F\otimes^{\mathsf{L}}_W \mathsf{L}f^*G\\
⑦ & ⑨ & \\
\mathsf{R}g_*g^\flat\mathcal{O}_X\otimes^{\mathsf{L}}_X f^\flat F\otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \longrightarrow & \mathcal{O}_X\otimes^{\mathsf{L}}_X f^\flat F\otimes^{\mathsf{L}}_X \mathsf{L}f^*G\\
\mathsf{R}g_*\big(g^\flat\mathcal{O}_X\otimes^{\mathsf{L}}_W \mathsf{L}g^*(f^\flat F\otimes^{\mathsf{L}}_X \mathsf{L}f^*G)\big) & & f^\flat F\otimes^{\mathsf{L}}_X \mathsf{L}f^*G
\end{array}
$$

The commutativity of subdiagram ⑦ is given, *mutatis mutandis,* by [L09, 3.4.7(iv)] (with f replaced by g, $A := \mathsf{L}f^*G$, $B := f^\flat F$ and $C := g^\flat\mathcal{O}_X$). The commutativity of ⑧ is clear, and that of ⑨ results from 2.7.7. So the border, and hence ①, commutes. $\square$

2.8. Let $f\colon X\to Y$ be a pseudo-coherent finite scheme-map (see §2.3.7), let $F\in\mathbf{D}(Y)$ be pseudo-coherent, and let $G\in\mathbf{D}^+_{\mathsf{qc}}(Y)$. Let $f=\phi\bar f$ be as in §2.1, $\phi^\flat$ as in 2.2.5, and $f^\flat := \bar f^*\phi^\flat$ as in (2.3.2).

By 2.3.8, $\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,G)\in\mathbf{D}^+_{\mathsf{qc}}(Y)$, so $\phi^\flat G\in\mathbf{D}^+_{\mathsf{qc}}(\overline{Y})$. (See the proof of 2.3.3). Similarly, (2.2.9.4) gives

$$\phi_*\mathsf{L}\phi^*F\cong F\otimes^{\mathsf{L}}_Y\phi_*\mathcal{O}_{\overline{Y}} = F\otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X\in\mathbf{D}_{\mathsf{qc}}(Y),$$

so $\mathsf{L}\phi^*F\in\mathbf{D}_{\mathsf{qc}}(\overline{Y})$.

Hence, by 2.1.6, there are natural isomorphisms

$$\mathsf{L}\phi^*F\cong \mathsf{R}\bar f_*\bar f^*\mathsf{L}\phi^*F\cong\mathsf{R}\bar f_*\mathsf{L}f^*F,\qquad \phi^\flat G\cong\mathsf{R}\bar f_*\bar f^*\phi^\flat G=\mathsf{R}\bar f_*f^\flat G.$$

Moreover, $\mathsf{L}f^*F\in\mathbf{D}_{\mathsf{qc}}(X)$ is pseudo-coherent and $f^\flat G\in\mathbf{D}^+_{\mathsf{qc}}(X)$, so 2.3.8 gives $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F,f^\flat G)\in\mathbf{D}^+_{\mathsf{qc}}(X)$, and 2.1.9 gives an isomorphism

$$\bar f^*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{L}\phi^*F,\phi^\flat G)\xrightarrow{\ \sim\ }\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F,f^\flat G). \tag{2.8.1}$$

Proposition 2.8.2. *Under the preceding conditions*:

(i) *The map adjoint under* 2.2.6 *to the natural composite*

$$\phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{L}\phi^*F,\phi^\flat G)\xrightarrow{\ \sim\ }\mathsf{R}\mathcal{H}om_Y(F,\phi_*\phi^\flat G)\xrightarrow[(2.2.8.1)]{}\mathsf{R}\mathcal{H}om_Y(F,G)$$

is the unique $\mathbf{D}(\overline{Y})$*-map*

$$\bar\zeta\colon\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{L}\phi^*F,\phi^\flat G)\to\phi^\flat\mathsf{R}\mathcal{H}om_Y(F,G)$$

making the next, natural, diagram commute—whence $\bar\zeta$ *is an isomorphism*:

$$
\begin{array}{ccc}
\phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{L}\phi^*F,\phi^\flat G) & \xrightarrow{\ \phi_*\bar\zeta\ } & \phi_*\phi^\flat\mathsf{R}\mathcal{H}om_Y(F,G)\\
\simeq\big\downarrow & \textcircled{1} & \\
\mathsf{R}\mathcal{H}om_Y(F,\phi_*\phi^\flat G) & & (2.2.3)\big\downarrow\simeq\\
\simeq\big\downarrow(2.2.3) & \mathsf{R}\mathcal{H}om_Y(F\otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X,G) & \\
\mathsf{R}\mathcal{H}om_Y(F,\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,G)) & & \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathsf{R}\mathcal{H}om_Y(F,G))
\end{array}
$$

$$\mathsf{R}\mathcal{H}om_Y(F,\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,G)) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(F\otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X,G) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathsf{R}\mathcal{H}om_Y(F,G))$$

(ii) (Explicit $\bar\zeta$) *Suppose that G (hence $\mathcal{H}om_\psi(f_*\mathcal{O}_X,G)$) is K-injective and that F is K-flat. Then $\bar\zeta$ is the natural composite map*

$$
\begin{aligned}
\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{L}\phi^*F,\phi^\flat G) &\xrightarrow{\ \sim\ } \mathcal{H}om_{\overline{Y}}(\phi^*F,\mathcal{H}om_\psi(f_*\mathcal{O}_X,G))\\
&\underset{\bar\zeta_0}{\xrightarrow{\ \sim\ }} \mathcal{H}om_\psi(f_*\mathcal{O}_X,\mathcal{H}om_Y(F,G)) \xrightarrow{\ \sim\ } \phi^\flat\mathsf{R}\mathcal{H}om_Y(F,G),
\end{aligned}
$$

with $\bar\zeta_0$ the natural composite map—$f_*\mathcal{O}_X$-linear via the multiplication action of $f_*\mathcal{O}_X$ on itself,

$$
\begin{aligned}
\phi_*\mathcal{H}om_{\overline{Y}}(\phi^*F,\mathcal{H}om_\psi(f_*\mathcal{O}_X,G)) &\xrightarrow{\ \sim\ } \mathcal{H}om_Y(F,\phi_*\mathcal{H}om_\psi(f_*\mathcal{O}_X,G))\\
&=\mathcal{H}om_Y(F,\mathcal{H}om_Y(f_*\mathcal{O}_X,G))\\
&\xrightarrow{\ \sim\ } \mathcal{H}om_Y(F\otimes_Y f_*\mathcal{O}_X,G)\\
&\xrightarrow{\ \sim\ } \mathcal{H}om_Y(f_*\mathcal{O}_X,\mathcal{H}om_Y(F,G))\\
&=\phi_*\mathcal{H}om_\psi(f_*\mathcal{O}_X,\mathcal{H}om_Y(F,G)).
\end{aligned}
$$

(iii) *The composite isomorphism*

$$
\begin{aligned}
\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F,f^\flat G) &\underset{(2.8.1)}{\xrightarrow{\ \sim\ }} \bar f^*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{L}\phi^*F,\phi^\flat G)\\
&\underset{\bar f^*\bar\zeta}{\xrightarrow{\ \sim\ }} \bar f^*\phi^\flat\mathsf{R}\mathcal{H}om_Y(F,G)=f^\flat\mathsf{R}\mathcal{H}om_Y(F,G)
\end{aligned}
$$

is the unique $\mathbf{D}(X)$-map ζ making the following natural diagram commute:

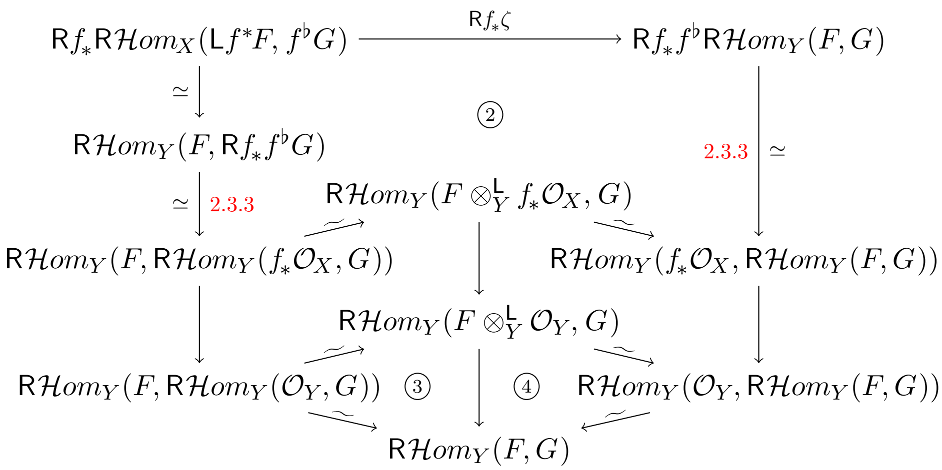
$$
\begin{array}{ccc}
\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F,f^\flat G) & \xrightarrow{\ \mathsf{R}f_*\zeta\ } & \mathsf{R}f_*f^\flat\mathsf{R}\mathcal{H}om_Y(F,G)\\
\simeq\big\downarrow & \textcircled{2} & \\
\mathsf{R}\mathcal{H}om_Y(F,\mathsf{R}f_*f^\flat G) & & 2.3.3\big\downarrow\simeq\\
\simeq\big\downarrow 2.3.3 & \mathsf{R}\mathcal{H}om_Y(F\otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X,G) & \\
\mathsf{R}\mathcal{H}om_Y(F,\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,G)) & & \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathsf{R}\mathcal{H}om_Y(F,G));
\end{array}
$$

$$\mathsf{R}\mathcal{H}om_Y(F,\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,G)) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(F\otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X,G) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathsf{R}\mathcal{H}om_Y(F,G))$$

and this ζ corresponds under 2.3.5 to the natural composite map

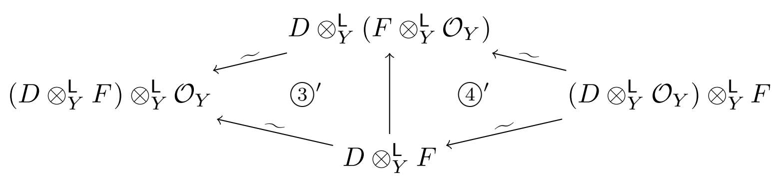
$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F,f^\flat G) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(F,\mathsf{R}f_*f^\flat G) \longrightarrow \mathsf{R}\mathcal{H}om_Y(F,G).$$

Proof. First, uniqueness in (iii) and (i). Consider the natural diagram



After replacing G by a quasi-isomorphic K-injective complex, one can drop all the Rs in ③ and ④ and check that the resulting subdiagrams—hence the original ones—are commutative.

More generally, in any closed category, using the definitions of the maps involved (see [L09, 3.5.6(e) and 3.5.3(e)]) one checks that subdiagrams ③ and ④ are right-conjugate to the (clearly) commutative natural diagrams



It follows, in view of the definition of the counit t for the adjunction $\mathsf{R}f_* \dashv f^\flat$ in 2.3.10, and of 2.3.8, that any $\mathbf{D}(X)$-map ζ such that ② commutes must be the one corresponding under 2.3.5 to the natural composite

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(F, \mathsf{R}f_*f^\flat G) \longrightarrow \mathsf{R}\mathcal{H}om_Y(F,G),$$

whence the uniqueness in (iii).[11]

Similarly, using (2.2.8.1) one shows that any $\bar\zeta$ making ① commute is adjoint under 2.2.6 to the natural composite

$$\phi_*\mathsf{R}\mathcal{H}om_{\overline{Y}}(\mathsf{L}\phi^*F, \phi^\flat G) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(F, \phi_*\phi^\flat G) \longrightarrow \mathsf{R}\mathcal{H}om_Y(F,G),$$

whence the uniqueness in (i).

---

[11] In fact, ζ is right-conjugate to the projection isomorphism

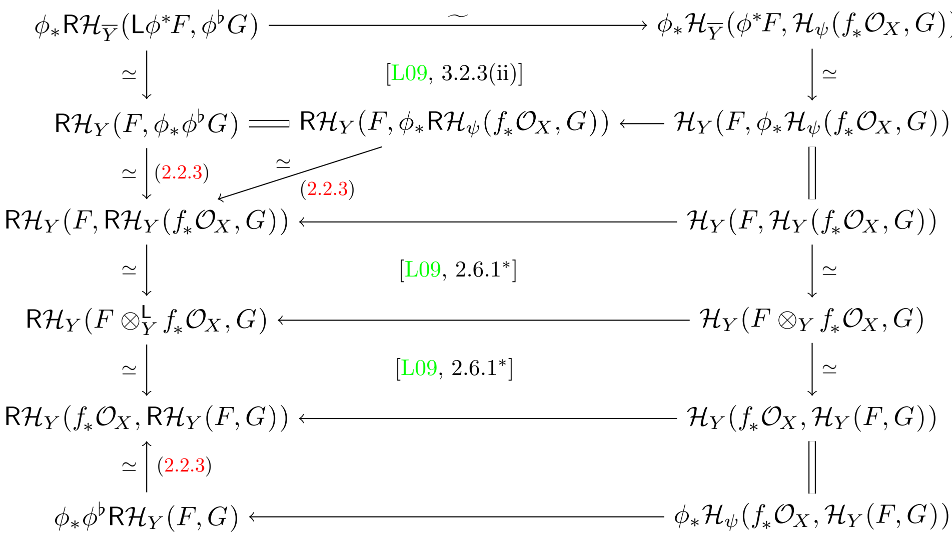
$$\mathsf{R}f_*E\otimes^{\mathsf{L}}_X F \xrightarrow{\ \sim\ } \mathsf{R}f_*(E\otimes^{\mathsf{L}}_X \mathsf{L}f^*F),$$

cf. [L09, Exercise 4.2.3(f)]. Moreover, ζ^{-1} is adjoint to the natural composite map

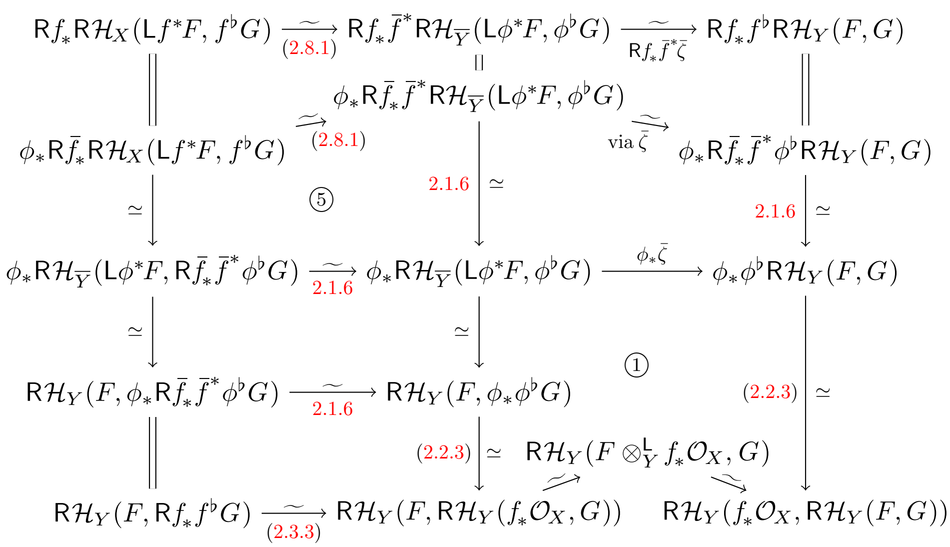
$$f^\flat\mathsf{R}\mathcal{H}om_Y(F,G)\otimes^{\mathsf{L}}_X \mathsf{L}f^*F \xrightarrow[\substack{2.7.7\\ \text{(iii)}}]{\chi} f^\flat(\mathsf{R}\mathcal{H}om_Y(F,G)\otimes^{\mathsf{L}}_Y F) \longrightarrow f^\flat G,$$

cf. [L09, Exercise 4.9.3(b)] (in whose third last line "$\bar f^!$" should be "$f^!$").

This being so, (i) and (ii) can be proved thus: assuming (as one may) that G (resp. F) is a K-injective (resp. K-flat) $\mathcal{O}_Y$-complex, show that with $\bar\chi$ as in (ii), diagram ① commutes; and for this, note that each subdiagram of the following natural diagram, where $\mathcal{H} := \mathcal{H}om$, commutes (more or less by definition of the functorial maps involved), whence so does the border.



As for (iii), to see that the following natural diagram expands ②, apply the first diagram in [L09, 3.7.1.1] to the leftmost column. Since by (i), subdiagram ① commutes, therefore for (iii) to hold—i.e., for the border to commute—it suffices that all the other subdiagrams commute.



Commutativity of ⑤ results, in view of [L09, Example 3.5.2(d)], from the definition of the map ρ given in the proof of Corollary 2.1.9.

Commutativity of the remaining subdiagrams is obvious. □

Proposition 2.8.3 (Transitivity of ζ)**.** *Let $\zeta = \zeta(f,F,G)$ be as in 2.8.2(iii). Let $g\colon W \to X$ be a pseudo-coherent finite scheme-map. Then, modulo natural isomorphisms,*

$$\zeta(gf,F,G) = g^\flat\zeta(f,F,G)\circ\zeta(g,\mathsf{L}f^*F,f^\flat G).$$

Proof. Left to the reader. □

2.9. This section deals with the role played in concrete duality for a perfect affine map $f\colon X\to Y$ by the functorial *trace map*

$$\mathrm{tr}_f(G)\colon \mathsf{R}f_*\mathsf{L}f^*G \to G \qquad (G\in\mathbf{D}_{\mathsf{qc}}(Y))$$

from [Il71, p. 154, 8.1]. Dual to this map is the *fundamental class* map

$$C_f(G)\colon \mathsf{L}f^*G \longrightarrow f^\flat G \qquad (G\in\mathbf{D}_{\mathsf{qc}}(Y)),$$

an isomorphism if f is étale.

The trace map is "transitive" with respect to a composition of perfect affine maps, and so the map C_f is *pseudofunctorial,* see Proposition 2.9.7.

For "almost étale" $f\colon X\to Y$, there results a canonical pair involving a "complementary sheaf" and a trace map, which pair represents the functor $\mathrm{Hom}_Y(f_*-,G)$ from $\mathcal{A}_{\mathsf{qc}}(X)$ to abelian groups, or, if $f_*\mathcal{O}_X$ is locally free, the functor $\mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*-,\,G)$ from $\mathbf{D}_{\mathsf{qc}}(X)$ to abelian groups, see Proposition 2.9.13.

2.9.1. Let $f\colon X\to Y$, $\bar f\colon X\to \overline{Y}$, $\psi\colon \mathcal{O}_Y\to f_*\mathcal{O}_X$, $\phi\colon\overline{Y}\to Y$ and $f^\flat\colon \mathbf{D}(Y)\to\mathbf{D}(X)$ be as in §2.3. Let F be a quasi-coherent $\mathcal{O}_X$-module and G an $\mathcal{O}_Y$-module such that the $\mathcal{O}_Y$-module $\mathcal{H}om_Y(f_*\mathcal{O}_X,G)$—equivalently, the $\mathcal{O}_{\overline{Y}}$-module $\mathcal{H}om_\psi(f_*\mathcal{O}_X,G)$—is quasi-coherent.

One has the isomorphism, *affine duality for quasi-coherent sheaves,*

$$\begin{aligned} f_*\mathcal{H}om_{\mathcal{O}_X}(F,\bar f^*\mathcal{H}om_\psi(f_*\mathcal{O}_X,G)) &\xrightarrow{\ \sim\ } \phi_*\mathcal{H}om_{f_*\mathcal{O}_X}(\bar f_*F,\mathcal{H}om_\psi(f_*\mathcal{O}_X,G))\\ &\xrightarrow{\ \sim\ } \mathcal{H}om_{\mathcal{O}_Y}(f_*F,G). \end{aligned}$$

that over open $U\subset Y$ is the standard isomorphism, with $S_U := \Gamma(f^{-1}U,\mathcal{O}_X)$, $F_U := \Gamma(f^{-1}U,F)$, $R_U:=\Gamma(U,\mathcal{O}_Y)$ and $G_U:=\Gamma(U,G)$,

$$\mathrm{Hom}_{S_U}(F_U,\mathrm{Hom}_{R_U}(S_U,G_U)) \xrightarrow{\ \sim\ } \mathrm{Hom}_{R_U}(F_U,G_U).$$

If $\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,G)\in\mathbf{D}_{\mathsf{qc}}(Y)$, then this duality isomorphism arises from application of H^0 to the isomorphism in Theorem 2.3.4.

It follows that the pair consisting of $\bar f^*\mathcal{H}om_\psi(f_*\mathcal{O}_X,G)$ $(=H^0f^\flat G)$ and the natural composite

$$\begin{aligned} t'_G\colon f_*\bar f^*\mathcal{H}om_\psi(f_*\mathcal{O}_X,G) &\xrightarrow{\ \sim\ } \phi_*\mathcal{H}om_\psi(f_*\mathcal{O}_X,G)\\ &=\mathcal{H}om_Y(f_*\mathcal{O}_X,G)\longrightarrow \mathcal{H}om_Y(\mathcal{O}_Y,G)=G \end{aligned}$$

represents the functor $\mathrm{Hom}_Y(f_*-,G)$ from $\mathcal{A}_{\mathsf{qc}}(X)$ to abelian groups.

If f is *flat and locally finitely presentable,* that is, the $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ is locally free of finite rank, then the natural map is an isomorphism

$$\bar f^*\mathcal{H}om_\psi(f_*\mathcal{O}_X,G) = H^0f^\flat G \xrightarrow{\ \sim\ } f^\flat G,$$

and Proposition 2.3.5 implies that the pair above also represents the functor $\mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*-,\,G)$ from $\mathbf{D}_{\mathsf{qc}}(X)$ to abelian groups.

In this case, moreover, the natural map is an $f_*\mathcal{O}_X$-isomorphism

$$\mathcal{H}om_\psi(f_*\mathcal{O}_X,\mathcal{O}_Y)\otimes_{\overline{Y}}\phi^*G \xrightarrow{\ \sim\ } \mathcal{H}om_\psi(f_*\mathcal{O}_X,G),$$

whence the representing pair is naturally isomorphic to the pair

$$(H^0f^\flat\mathcal{O}_Y\otimes_X f^*G,\ t'_{\mathcal{O}_Y}\otimes_Y \mathrm{id}_G).$$

If f is *finite and étale,* so that the $\mathcal{O}_{\overline{Y}}$-module $\mathcal{H}om_\psi(f_*\mathcal{O}_X,\mathcal{O}_Y)$ is free of rank one, generated by the usual *trace map* $\mathrm{tr}_f\colon f_*\mathcal{O}_X\to\mathcal{O}_Y$, then

$$H^0f^\flat\mathcal{O}_Y\otimes_X f^*G\cong \bar f^*\mathcal{O}_{\overline{Y}}\otimes_X f^*G\cong\mathcal{O}_X\otimes_X f^*G\cong f^*G,$$

and the representing pair is naturally isomorphic to the pair consisting of f^*G and the natural composite map

$$\mathsf{R}f_*f^*G \xrightarrow{\ \sim\ } f_*\mathcal{O}_X\otimes_Y G\xrightarrow{\mathrm{tr}_f\otimes\mathrm{id}_G}\mathcal{O}_Y\otimes_Y G\xrightarrow{\ \sim\ } G.$$

2.9.2. For dealing with more general f, recall from 2.7.4 the trifunctorial map, over any ringed space $(Y,\mathcal{O}_Y)$,

$$\gamma_Y\colon \mathsf{R}\mathcal{H}om_Y(L,M)\otimes^{\mathsf{L}}_Y N\to\mathsf{R}\mathcal{H}om_Y(L,M\otimes^{\mathsf{L}}_Y N)\qquad (L,M,N\in\mathbf{D}(Y)),$$

that is adjoint to the natural composition

$$\mathsf{R}\mathcal{H}om_Y(L,M)\otimes^{\mathsf{L}}_Y N\otimes^{\mathsf{L}}_Y L\xrightarrow{\ \sim\ }\mathsf{R}\mathcal{H}om_Y(L,M)\otimes^{\mathsf{L}}_Y L\otimes^{\mathsf{L}}_Y N\longrightarrow M\otimes^{\mathsf{L}}_Y N.$$

Elaborating 2.7.2, one finds that γ can be obtained from a K-injective resolution $M\to\overline{M}$ and a K-flat resolution $\overline{N}\to N$ as the sheafification

$$\mathcal{H}om_Y(L,\overline{M})\otimes_Y\overline{N}\to\mathcal{H}om_Y(L,\overline{M}\otimes_Y\overline{N})$$

of the map of presheaves of complexes that sends, for each open $U\subset Y$, any

$$(\alpha\colon L^i|_U\to\overline{M}^j|_U)\otimes_{\mathcal{O}_U}(\mu_k\in\overline{N}^k(U))\qquad(i,j,k\in\mathbb{Z})$$

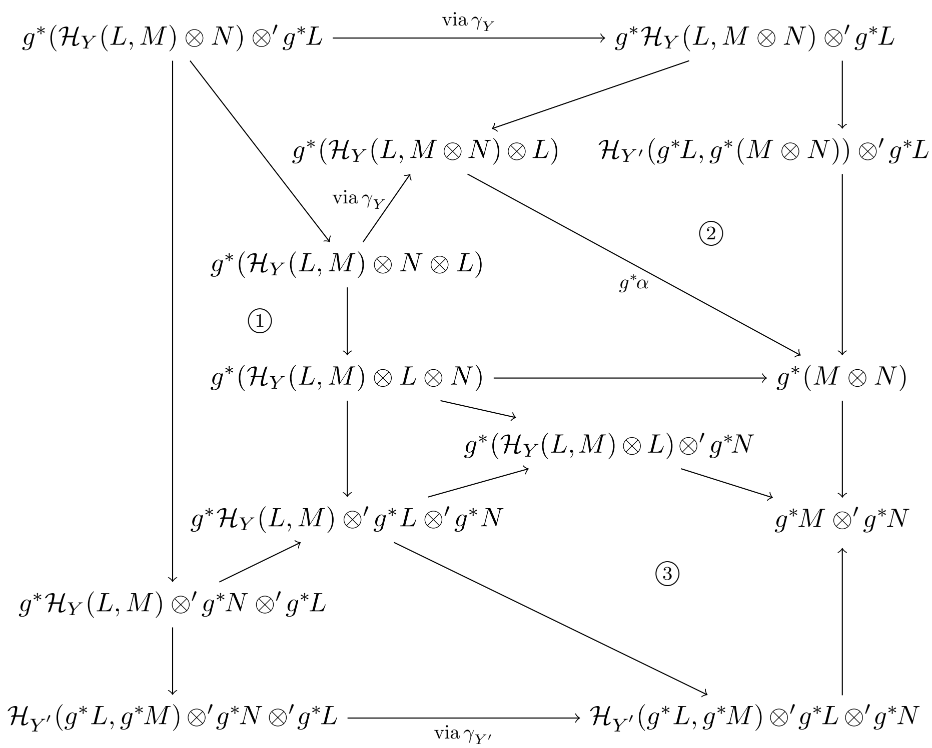
to the map taking $\lambda_i\in L^i(U)$ to $(-1)^{ik}\alpha(\lambda_i)\otimes_{\mathcal{O}_U}\mu_k$.

Lemma 2.9.3. (i) *For any ringed-space map $g\colon Y'\to Y$, the following natural diagram commutes.*

$$\begin{array}{ccc}
\mathsf{L}g^*(\mathsf{R}\mathcal{H}om_Y(L,M)\otimes^{\mathsf{L}}_Y N) & \xrightarrow{\mathsf{L}g^*\gamma_Y} & \mathsf{L}g^*\mathsf{R}\mathcal{H}om_Y(L,M\otimes^{\mathsf{L}}_Y N)\\
\simeq\downarrow & & \downarrow\\
\mathsf{L}g^*\mathsf{R}\mathcal{H}om_Y(L,M)\otimes^{\mathsf{L}}_{Y'}\mathsf{L}g^*N & & \mathsf{R}\mathcal{H}om_{Y'}(\mathsf{L}g^*L,\mathsf{L}g^*(M\otimes^{\mathsf{L}}_Y N))\\
\downarrow & & \downarrow\simeq\\
\mathsf{R}\mathcal{H}om_{Y'}(\mathsf{L}g^*L,\mathsf{L}g^*M)\otimes^{\mathsf{L}}_{Y'}\mathsf{L}g^*N & \xrightarrow[\gamma_{Y'}]{} & \mathsf{R}\mathcal{H}om_{Y'}(\mathsf{L}g^*L,\mathsf{L}g^*M\otimes^{\mathsf{L}}_{Y'}\mathsf{L}g^*N)
\end{array}$$

(ii) *If the complex L is perfect, then all the maps in this diagram are isomorphisms.*

Proof. (i) It suffices to show the commutativity of the adjoint diagram, i.e., of the border of the following natural diagram, in which g^* stands for $\mathsf{L}g^*$, $\mathcal{H}$ for $\mathsf{R}\mathcal{H}om$, $\otimes$ for $\otimes^{\mathsf{L}}_{Y}$ and $\otimes'$ for $\otimes^{\mathsf{L}}_{Y'}$, and α is adjoint to the identity map of $\mathcal{H}(L, M\otimes N)$:



Here, the commutativity of the unlabeled subdiagrams is easily verified.

The commutativity of ① results from that of the diagram that is *dual* (see [L09, 3.4.5]) to the second one following [L09, (3.4.2.1)].

That of ② results from [L09, 3.5.6(g)], with $C := \mathcal{H}_Y(L, M\otimes N)$, $D := L$, and $E := M\otimes N$ (details left to the reader).

That of ③ is given by [L09, 3.5.6(a)].

Thus (i) is proven.

(ii) A *strictly perfect* $\mathcal{O}_Y$-complex is a bounded complex of direct summands of finite-rank free $\mathcal{O}_Y$-modules. An $\mathcal{O}_Y$-complex is perfect if locally it is the target of a quasi-isomorphism with source a strictly perfect one [Il71, p. 122, 4.8, p. 163, 2.0, and p. 96, 2.2]. So to prove (ii), one can reduce, by localizing, to where L is strictly perfect, and then by a simple induction on the number n of degrees in which L doesn't vanish, to where $n = 1$, in which case the assertion is readily verified. □

2.9.4. Let $(Y, \mathcal{O}_Y)$ be a ringed space and L a *perfect* $\mathcal{O}_Y$-complex, so that, as in 2.9.3(ii), the map $\gamma = \gamma_Y$ is an isomorphism.

The trace map $\mathbf{tr}_L = \mathbf{tr}_{L/\mathcal{O}_Y}$ is defined to be the natural $\mathbf{D}(Y)$-composite

$$\mathsf{R}\mathcal{H}om_Y(L,L) \underset{\gamma^{-1}}{\xrightarrow{\sim}} \mathsf{R}\mathcal{H}om_Y(L,\mathcal{O}_Y) \otimes^{\mathsf{L}}_Y L \longrightarrow \mathcal{O}_Y,$$

see [Il71, p. 154, 8.1].[12] There results, for any perfect $\mathcal{O}_Y$-algebra A, the natural composite $\mathbf{D}(Y)$-map

$$\mathrm{tr}^{\mathrm{alg}}_A \colon A \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_A(A,A) \xrightarrow{\mu_A} \mathsf{R}\mathcal{H}om_Y(A,A) \xrightarrow{\mathbf{tr}_A} \mathcal{O}_Y.$$

From 2.9.3(ii) and commutativity of subdiagram ② (with $N := \mathcal{O}_Y$) in the proof of 2.9.3(i), one gets a natural identification $g^*\mathbf{tr}_{L/\mathcal{O}_Y} \cong \mathbf{tr}_{g^*L/\mathcal{O}_{Y'}}$. Together with the natural identification $g^*\mu_A \cong \mu_{g^*A}$, this gives a natural identification $g^*\mathrm{tr}^{\mathrm{alg}}_A \cong \mathrm{tr}^{\mathrm{alg}}_{g^*A}$.

Recall that for any $\mathcal{O}_Y$-complex G, the natural $\mathbf{D}(Y)$-map is an isomorphism $\mathcal{H}om(L,G) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om(L,G)$. (Proof: localizing on Y and induction on the number of degrees in which L doesn't vanish reduces the assertion to the trivial case $L = \mathcal{O}_Y$.) Consequently, and by the last part of §2.9.2, $\mathbf{tr}_L$ can be identified naturally with (the $\mathbf{D}(Y)$-image of) the natural composite $\mathcal{A}(Y)$-map

$$\mathcal{H}om_Y(L,L) \xrightarrow{\sim} \mathcal{H}om_Y(L,\mathcal{O}_Y) \otimes_Y L \longrightarrow \mathcal{O}_Y.$$

In particular, if L is a finite-rank locally free $\mathcal{O}_Y$-module then $\mathbf{tr}_L$ can be identified naturally with (the $\mathbf{D}(Y)$-image of) the the usual trace map $\mathrm{tr}_L \colon \mathcal{H}om(L,L) \to \mathcal{O}_Y$. Moreover, it follows from the explicit description of the map γ_Y in 2.9.2 that there is a natural identification

$$\mathbf{tr}_{L[j]} = (-1)^j \mathrm{tr}_L \qquad (j \in \mathbb{Z}).$$

More generally, if L is strictly perfect, so that the degree-i component L^i is a finite-rank locally free $\mathcal{O}_Y$-module for all i, and vanishes for all but finitely many i, then there is a natural identification

$$\mathbf{tr}_L = \sum_i (-1)^i \mathrm{tr}_{L^i}, \tag{2.9.4.1}$$

as is easily shown by induction on the number of i such that $L^i \neq 0$.

2.9.5. If $f\colon X \to Y$ is a *perfect* affine scheme-map, so that $\mathsf{R}f_*\mathcal{O}_X$ is a perfect $\mathcal{O}_Y$-complex , then one has the $\mathbf{D}(Y)$-map $\mathrm{tr}_f := \mathrm{tr}^{\mathrm{alg}}_{f_*\mathcal{O}_X} \colon f_*\mathcal{O}_X \to \mathcal{O}_Y$, whence for any $G \in \mathbf{D}_{\mathsf{qc}}(Y)$, the natural functorial composite

$$\mathrm{tr}_f(G)\colon \mathsf{R}f_*\mathsf{L}f^*G \underset{(2.3.1)}{\xrightarrow{\sim}} f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y G \xrightarrow[\mathrm{tr}_f \otimes^{\mathsf{L}} \mathrm{id}]{} \mathcal{O}_Y \otimes^{\mathsf{L}}_Y G \xrightarrow{\sim} G,$$

whence, by Corollary 2.3.10, a natural functorial $\mathbf{D}_{\mathsf{qc}}(X)$-map

$$C_f(G)\colon \mathsf{L}f^*G \longrightarrow f^\flat G. \tag{2.9.6}$$

[12] A similar definition holds in any closed monoidal category for an object L such that, with $[-,-]$:=internal hom, the natural map is an isomorphism $[L,\mathbf{1}] \otimes L \xrightarrow{\sim} [L,L]$. For even greater generality, see e.g., [PS14].

For example, if f is *flat and finitely presentable,* so that $f_*\mathcal{O}_X$ is locally free, then $\mathrm{tr}_f(\mathcal{O}_Y)\colon f_*\mathcal{O}_X \to \mathcal{O}_Y$ identifies naturally with the usual trace map; and if, also, Y is noetherian, then $C_f(\mathcal{O}_Y)\colon \mathcal{O}_X \to f^\flat\mathcal{O}_Y$ becomes the (naive) *fundamental class* ([AJL14, (0.2.3) with $n = 0$, plus Example 2.6]).

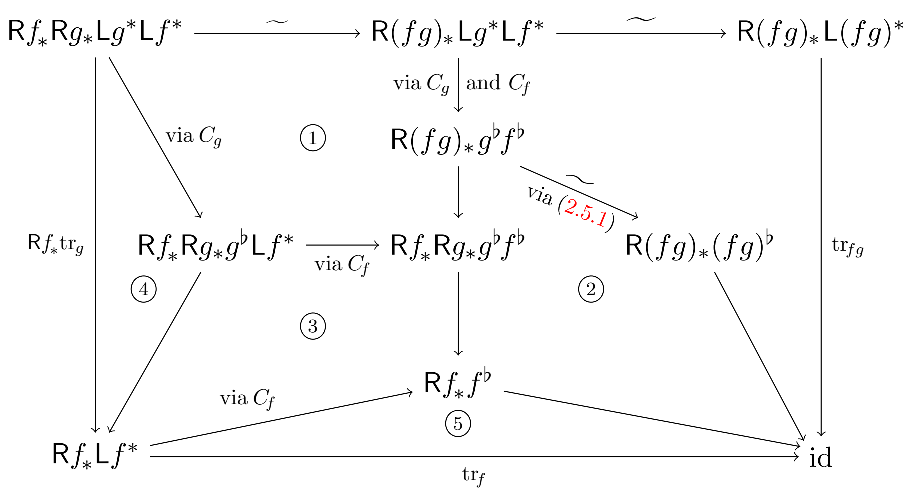
For finite *étale* f, C_f is the *isomorphism* at the end of §2.9.1.

The map $C_f\colon \mathsf{L}f^* \to f^\flat$ is *pseudofunctorial,* in the following sense.

Proposition 2.9.7. *Let $W \xrightarrow{g} X \xrightarrow{f} Y$ (hence fg) be perfect affine maps. The following diagram of functors from $\mathbf{D}_{\mathsf{qc}}(Y)$ to $\mathbf{D}_{\mathsf{qc}}(W)$ commutes.*

$$\begin{array}{ccc}
\mathsf{L}g^*\mathsf{L}f^* & \xrightarrow[\text{natural}]{\ \sim\ } & \mathsf{L}(fg)^* \\
{\scriptstyle \text{via } C_g \text{ and } C_f}\downarrow & & \downarrow{\scriptstyle C_{fg}} \\
g^\flat f^\flat & \xrightarrow[(2.5.1)]{\ \sim\ } & (fg)^\flat
\end{array}$$

Proof. It suffices to prove commutativity of the dual diagram, which is the unlabeled subdiagram in the following natural diagram.

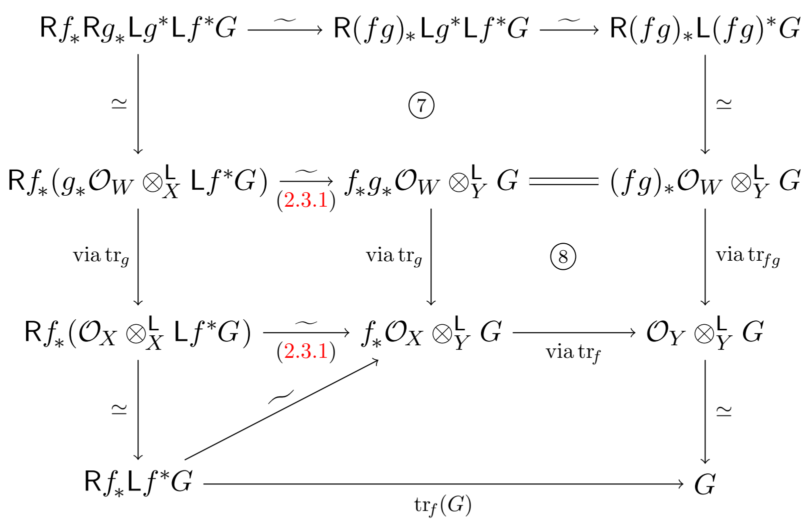


The commutativity of subdiagrams ① and ③ is clear, subdiagrams ④ and ⑤ commute by definition, and the commutativity of ② holds because the map (2.5.1) is, by definition, dual to the composite map

$$\mathsf{R}(fg)_*(fg)^\flat \to \mathsf{R}f_*\mathsf{R}g_*g^\flat f^\flat \to \mathsf{R}f_*f^\flat \to \mathrm{id}\,.$$

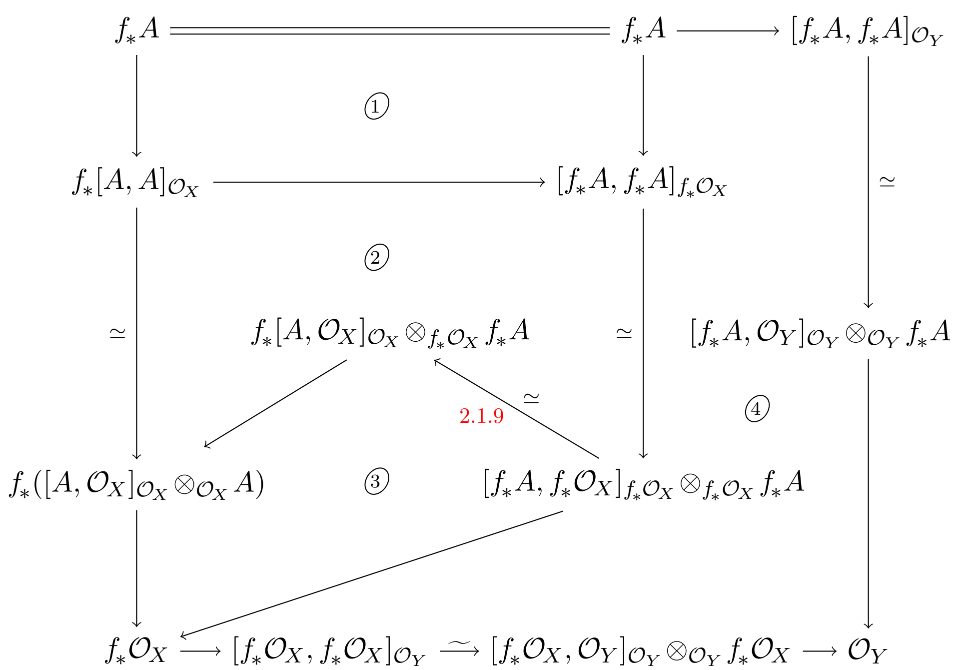
Hence diagram chasing shows it sufficient that the border commute (i.e., that *transitivity of the trace map* hold).

The border in question, applied to an arbitrary $G \in \mathbf{D}_{\mathsf{qc}}(Y)$, expands to the border of the following natural diagram.



Here, commutativity of the unlabeled subdiagrams is clear, and that of ⑦ results readily from [L09, 3.7.1] with $F := \mathcal{O}_X$. So to prove the commutativity of the border, it suffices to prove that of ⑧ with all occurrences of "$\otimes^{\mathsf{L}}_Y G$" ignored (i.e., to prove transitivity of the trace when $G = \mathcal{O}_Y$), which one can do by taking $A := g_*\mathcal{O}_W$ in the next lemma, where, for an $\mathcal{O}_X$- or $\mathcal{O}_Y$-algebra $\mathcal{S}$, $[-,-]_{\mathcal{S}} := \mathsf{R}\mathcal{H}om_{\mathcal{S}}(-,-)$ and $\otimes_{\mathcal{S}} := \otimes^{\mathsf{L}}_{\mathcal{S}}$.

Lemma 2.9.8. *If A is a perfect $\mathcal{O}_X$-complex then f_*A is perfect over both $f_*\mathcal{O}_X$ and $\mathcal{O}_Y$, and the following natural diagram commutes.*



Proof. To see that f_*A is perfect over $f_*\mathcal{O}_X$ and over $\mathcal{O}_Y$, one can localize on Y, and so can assume that X and Y are affine schemes, in which case the assertions have a standard translation into the analogous, and simple, ones in commutative algebra.

As for the diagram, we first show subdiagrams ①, ② and ③ commute, so that "trace is preserved under the equivalence in 2.1.6."

It is left to the reader to verify that the natural map

$$\alpha\colon f_*\mathsf{R}\mathcal{H}om_X(A,B)\to \mathsf{R}\mathcal{H}om_Y(f_*A,f_*B)\qquad (A,B\in \mathbf{D}(X))$$

associated to an *arbitrary ringed-space map* $f\colon (X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ via, e.g., [L09, (3.1.4) and 2.6.5], is characterized abstractly (cf. [L09, (3.5.4.1)] as being adjoint to the natural composite map

$$f_*\mathsf{R}\mathcal{H}om_X(A,B)\otimes^{\mathsf{L}}_Y f_*A\to f_*(\mathsf{R}\mathcal{H}om_Y(A,B)\otimes^{\mathsf{L}}_X A)\to f_*B.$$

The commutativity of ③ is the special case where $B:=\mathcal{O}_X$ and $\mathcal{O}_Y:=f_*\mathcal{O}_X$.

(Likewise, when other abstract properties of functorial maps, as in [L09], are used in the rest of this proof, it should be checked that the natural concrete interpretations of those maps do have those properties—so that the concrete result is an instance of an abstract one.)

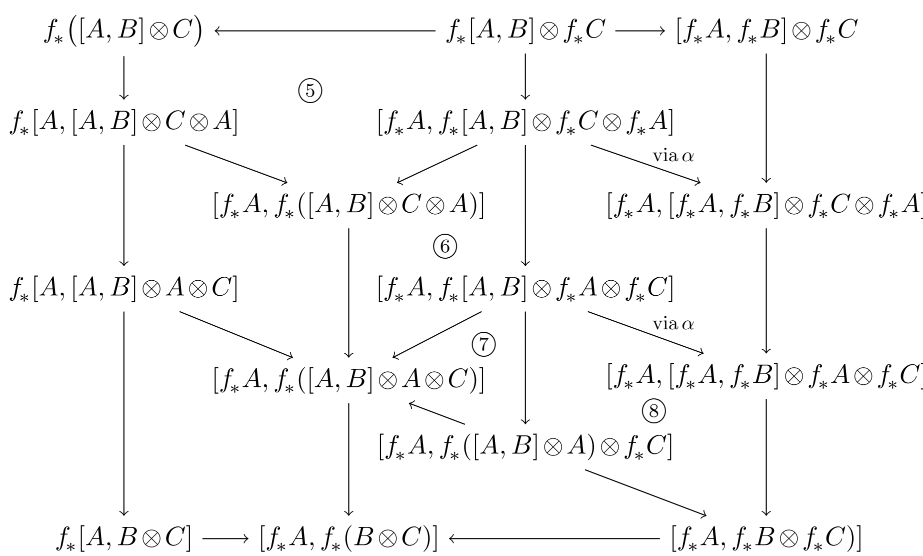
The commutativity of ① is that of the second diagram in [L09, 3.7.1.1] (where f_* and g_* should be switched), derived from the natural composite map denoted here by $(X,A)\to(X,\mathcal{O}_X)\to(Y,f_*\mathcal{O}_X)$, with $F=F':=A$.

The commutativity of ② is essentially the case $B:=\mathcal{O}_X$, $C:=A$ of the next lemma, applied to the functor $\mathsf{R}\bar f_*\colon \mathbf{D}(\mathcal{O}_X)\to\mathbf{D}(f_*\mathcal{O}_X)$ associated to the natural ringed-space map $\bar f\colon (X,\mathcal{O}_X)\to(Y,f_*\mathcal{O}_X)$.

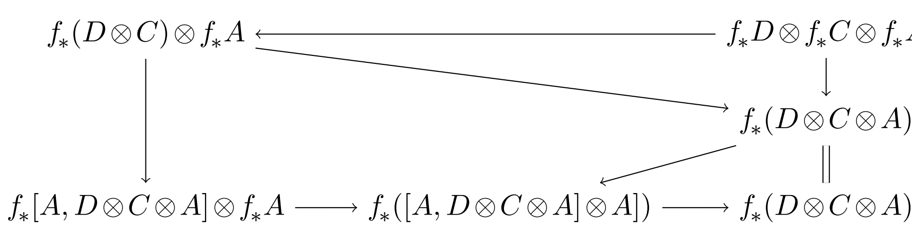
Lemma 2.9.9. *Let $f_*\colon \mathbf{X}\to\mathbf{Y}$ be a symmetric monoidal functor between monoidal closed categories. With notation as in* [L09, §3.5], *for any A, B and C in $\mathbf{X}$, the following natural diagram commutes.*

$$\begin{array}{ccccc} f_*\big([A,B]\otimes C\big) & \longleftarrow & f_*[A,B]\otimes f_*C & \longrightarrow & [f_*A,f_*B]\otimes f_*C\\ \downarrow & & & & \downarrow\\ f_*[A,B\otimes C] & \longrightarrow & [f_*A,f_*(B\otimes C)] & \longleftarrow & [f_*A,f_*B\otimes f_*C] \end{array}$$

Proof. [13] Expand the diagram, naturally, as follows:



Commutativity of the unlabeled subdiagrams is simple to verify. That of ⑥ and ⑦ follows at once from the definition of symmetric monoidal functor, see e.g., [L09, 3.4.2]. That of ⑧ results from the above description of α. That of ⑤ is equivalent to that of the adjoint diagram, that is, the border of the following natural diagram, where $D := [A,B]$; and this border does commute, since all the subdiagrams clearly do.



This completes the proof of Lemma 2.9.9. □

To complete the proof of Lemma 2.9.8, whence of Proposition 2.9.7, one needs subdiagram ④ to commute—which it does, by the next Lemma (with $\mathcal{S} := f_*\mathcal{O}_X$ and $\mathcal{T} := f_*A$).

Lemma 2.9.10. *Let $(Y, \mathcal{O}_Y)$ be a ringed space, let $\mathcal{S}$ be a perfect $\mathcal{O}_Y$-algebra, and let E be a perfect $\mathcal{S}$-module. Then E is a perfect $\mathcal{O}_Y$-module, and $\mathbf{tr}_{E/\mathcal{O}_Y}$ factors naturally as*

(2.9.10.1) $$\mathcal{H}om_{\mathcal{O}_Y}(E,E) \longrightarrow \mathcal{H}om_{\mathcal{S}}(E,E) \xrightarrow{\mathbf{tr}_{E/\mathcal{S}}} \mathcal{S} \xrightarrow{\mathrm{tr}^{\mathrm{alg}}_{\mathcal{S}/\mathcal{O}_Y}} \mathcal{O}_Y.$$

In particular, if $\mathcal{T}$ is a perfect $\mathcal{O}_Y$-algebra, then

$$\mathrm{tr}^{\mathrm{alg}}_{\mathcal{T}/\mathcal{O}_Y} = \mathrm{tr}^{\mathrm{alg}}_{\mathcal{S}/\mathcal{O}_Y} \circ \mathrm{tr}^{\mathrm{alg}}_{\mathcal{T}/\mathcal{S}} \colon \mathcal{T} \to \mathcal{O}_Y.$$

[13] Conceivably, the assertion is contained in [Lw72, Theorem 4.18].

Proof. That E is perfect over $\mathcal{O}_Y$ can be shown as in the beginning of the proof of 2.9.8.

As for (2.9.10.1), compatibility of **tr** with base change (see section 2.9.4) allows one to assume that E is $\mathbf{D}(\mathcal{S})$-isomorphic to a bounded complex of direct summands of finite-rank free $\mathcal{S}$-modules (see proof of 2.9.3(ii)). It suffices then, in view of (2.9.4.1), to show that for any direct summand F of a finite-rank free $\mathcal{S}$-module, the following natural diagram commutes, where by previous considerations, the maps involved can be identified with their nonderived precursors:

$$\begin{array}{ccc} [F,F]_{\mathcal{S}} & \longrightarrow & [F,F]_{\mathcal{O}_Y} \\ \simeq\downarrow & & \downarrow\simeq \\ [F,\mathcal{S}]_{\mathcal{S}}\otimes_{\mathcal{S}} F & & [F,\mathcal{O}_Y]_{\mathcal{O}_Y}\otimes_{\mathcal{O}_Y} F \\ \downarrow & & \downarrow \\ \mathcal{S} \longrightarrow [\mathcal{S},\mathcal{S}]_{\mathcal{O}_Y} & \xrightarrow{\ \sim\ } & [\mathcal{S},\mathcal{O}_Y]_{\mathcal{O}_Y}\otimes_{\mathcal{O}_Y}\mathcal{S} \longrightarrow \mathcal{O}_Y \end{array} \tag{2.9.10.2}$$

If $F = F_1 \oplus_{\mathcal{S}} F_2$, then (one verifies) this diagram is the direct sum of the four natural diagrams obtained by substituting F_i for the first occurrence of F at each node and F_j for the second occurrence, with $(i,j) = (1,1)$ or $(1,2)$ or $(2,1)$ or $(2,2)$, the resulting arrow

$$[F_i,\mathcal{T}]_{\mathcal{T}} \otimes_{\mathcal{T}} F_j \longrightarrow \mathcal{T} \qquad (\mathcal{T} := \mathcal{S} \text{ or } \mathcal{O}_Y)$$

representing the natural composite map

$$[F_i,\mathcal{T}]_{\mathcal{T}} \otimes_{\mathcal{T}} F_j \longrightarrow [F,\mathcal{T}]_{\mathcal{T}} \otimes_{\mathcal{T}} F_j \longrightarrow [F_j,\mathcal{T}]_{\mathcal{T}} \otimes_{\mathcal{T}} F_j \longrightarrow \mathcal{T},$$

which vanishes when $i \neq j$. Hence, (2.9.10.2) commutes for F if and only if it commutes for both F_1 and F_2.

It follows that the question of commutativity of (2.9.10.2) reduces to the trivial case where $F = \mathcal{S}$. □ □ □

Our underlying theme, concrete realizations of abstract constructions, spurs continuing on with interpretations, via traces for perfect affine maps, of maps involving $(-)^\flat$ via maps involving $(-)^*$. Proposition 2.9.7 is just a first example. But this could be an endless process.

Let some further examples, provided by the following assertions, suffice. As before, justification of these assertions requires deploying the formalism of adjoint functors between closed categories (see e.g., [L09, §§3.5.5–3.5.6]), and/or its scheme-theoretic realization (see [L09, §3.6.10]).

Exercise 2.9.11. Let $f\colon X \to Y$ be a perfect affine map, $F, G \in \mathbf{D}_{\mathsf{qc}}(Y)$, and C_f as in (2.9.6) (an isomorphism if f is étale).

(i) Let σ and β_σ be as in Theorem 2.6.4. Then the affine map g is perfect, and the following natural diagram commutes.

$$\begin{array}{ccc} \mathsf{L}v^*\mathsf{L}f^*G & \xrightarrow{\ \sim\ } & \mathsf{L}g^*\mathsf{L}u^*G \\ {\scriptstyle \mathsf{L}v^*C_f}\downarrow & & \downarrow{\scriptstyle C_g} \\ \mathsf{L}v^*f^\flat G & \underset{\beta_\sigma}{\xrightarrow{\ \sim\ }} & g^\flat\mathsf{L}u^*G \end{array}$$

(ii) Let $\chi\colon f^\flat F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G \to f^\flat(F \otimes^{\mathsf{L}}_Y G)$ be the isomorphism in 2.7.7(ii). The following natural diagram commutes.

$$\begin{array}{ccc} \mathsf{L}f^*F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \xrightarrow{\ \sim\ } & \mathsf{L}f^*(F \otimes^{\mathsf{L}}_Y G) \\ {\scriptstyle \text{via } C_f}\downarrow & & \downarrow{\scriptstyle C_f} \\ f^\flat F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \underset{\chi}{\xrightarrow{\ \sim\ }} & f^\flat(F \otimes^{\mathsf{L}}_Y G) \end{array}$$

(iii) Assume $\mathsf{R}\mathcal{H}om_Y(F,G) \in \mathbf{D}_{\mathsf{qc}}(Y)$ and $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) \in \mathbf{D}_{\mathsf{qc}}(Y)$ (for example, F perfect, or F pseudo-coherent and $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$).

Let $\zeta\colon \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) \to f^\flat\mathsf{R}\mathcal{H}om_Y(F,G)$ correspond via 2.3.5(i) to the natural composite map

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(F, \mathsf{R}f_*f^\flat G) \xrightarrow{t_G} \mathsf{R}\mathcal{H}om_Y(F,G).$$

The following natural diagram commutes.

$$\begin{array}{ccc} \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, \mathsf{L}f^*G) & \longleftarrow & \mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(F,G) \\ {\scriptstyle \text{via } C_f}\downarrow & & \downarrow{\scriptstyle C_f} \\ \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) & \underset{\zeta}{\longrightarrow} & f^\flat\mathsf{R}\mathcal{H}om_Y(F,G) \end{array}$$

2.9.12. For additional illustration, generalizing the last paragraph in §2.9.1, Proposition 2.9.13 below gives, for any "almost étale" $f\colon X \to Y$, a concrete representation of the representing pair $(H^0f^\flat\mathcal{O}_Y, t'_{\mathcal{O}_Y})$ in §2.9.1.

Consider a fiber square of scheme-maps, with qcqs f:

$$\begin{array}{ccc} V = U \times_Y X & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \underset{u}{\longrightarrow} & Y \end{array} \tag{2.9.12.1}$$

Assume that u is flat, or that f is affine. Then the natural map of functors from $\mathcal{A}_{\mathsf{qc}}(X)$ to $\mathcal{A}_{\mathsf{qc}}(U)$ is an isomorphism $u^*f_* \xrightarrow{\ \sim\ } g_*v^*$ [GrD71, 9.3.3], and one has the natural composite map

$$\begin{aligned}\varrho_0 = \varrho_0(f,u)\colon\ \mathrm{Hom}_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) &\longrightarrow \mathrm{Hom}_U(u^*f_*\mathcal{O}_X, u^*\mathcal{O}_Y)\\ &\xrightarrow{\ \sim\ } \mathrm{Hom}_U(g_*\mathcal{O}_V, \mathcal{O}_U),\end{aligned}$$

which sends $t\colon f_*\mathcal{O}_X \to \mathcal{O}_Y$ to the natural composite

$$g_*\mathcal{O}_V \xrightarrow{\ \sim\ } u^*f_*\mathcal{O}_X \xrightarrow{u^*t} u^*\mathcal{O}_Y \xrightarrow{\ \sim\ } \mathcal{O}_U.$$

That $s = \varrho_0 t$ is equivalent to the commutativity of the natural diagram

$$\begin{array}{ccc} u^*f_*\mathcal{O}_X & \xrightarrow{\ \sim\ } & g_*\mathcal{O}_V \\ {\scriptstyle u^*t}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ u^*\mathcal{O}_Y & \xrightarrow{\ \sim\ } & \mathcal{O}_U \end{array}$$

or of its adjoint

$$\begin{array}{ccccc} f_*\mathcal{O}_X & \longrightarrow & u_*u^*f_*\mathcal{O}_X & \xrightarrow{\ \sim\ } & u_*g_*\mathcal{O}_V \\ {\scriptstyle t}\big\downarrow & & & & \big\downarrow{\scriptstyle u_*s} \\ \mathcal{O}_Y & \longrightarrow & u_*u^*\mathcal{O}_Y & \xrightarrow{\ \sim\ } & u_*\mathcal{O}_U \end{array}$$

Suppose in addition that

(i) *u is schematically surjective and qcqs,* in other words, the natural map $\mathcal{O}_Y \to u_*\mathcal{O}_U$ is injective—whence the map ϱ_0 is injective and $v_*\mathcal{O}_V$ is quasi-coherent; and that

(ii) *g is finite and étale*—so that there is a unique *$g_*\mathcal{O}_V$-isomorphism*

$$c_g\colon g_*\mathcal{O}_V \xrightarrow{\ \sim\ } \mathcal{H}om_U(g_*\mathcal{O}_V, \mathcal{O}_U)$$

that sends $1 \in \Gamma(U, g_*\mathcal{O}_V)$ to the usual trace map $\mathrm{tr}_g\colon g_*\mathcal{O}_V \to \mathcal{O}_U$.

Denote base change to any open subscheme $W \subset Y$ by "subscript W." With $\tilde\varrho_0$ the sheafification of the map of presheaves associating to any W the map $\varrho_0(f_W, u_W)$, one has then the natural composite injective map

$$\varrho'\colon \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) \underset{\tilde\varrho_0}{\hookrightarrow} u_*\mathcal{H}om_U(g_*\mathcal{O}_V, \mathcal{O}_U) \underset{c_g^{-1}}{\xrightarrow{\ \sim\ }} u_*g_*\mathcal{O}_V = f_*v_*\mathcal{O}_V.$$

This map is in a natural way $f_*\mathcal{O}_X$-linear, so can also be represented, with notation as in §2.9.1, as

$$\varrho'\colon \mathcal{H}om_\psi(f_*\mathcal{O}_X, \mathcal{O}_Y) \hookrightarrow \bar f_*v_*\mathcal{O}_V.$$

Setting

$$\mathcal{C}_{\phi,u} := \varrho'\mathcal{H}om_\psi(f_*\mathcal{O}_X, \mathcal{O}_Y),$$

one gets a natural $\mathcal{O}_X$-isomorphism

$$H^0f^\flat\mathcal{O}_Y = \bar f^*\mathcal{H}om_\psi(f_*\mathcal{O}_X, \mathcal{O}_Y) \xrightarrow{\ \sim\ } \bar f^*\mathcal{C}_{\phi,u} =: \mathcal{C}_{f,u} \subset v_*\mathcal{O}_V.$$

For an explicit description of the $\mathcal{O}_X$-module $\mathcal{C}_{f,u}$, note that by the above description of the image of ϱ_0, $\mathcal{C}_{\phi,u}$ is the sheafification of the presheaf $\mathcal{C}^0_{\phi,u}$ for which, with “subscript W” as above, $\mathcal{C}^0_{\phi,u}(W)$ is the set of $r \in \Gamma(V_W, \mathcal{O}_V)$ such that the natural composite

$$f_{W*}\mathcal{O}_{X_W} \longrightarrow u_{W*}u_W^* f_{W*}\mathcal{O}_{X_W} \xrightarrow{\ \sim\ } u_{W*}g_{W*}\mathcal{O}_{V_W} \xrightarrow{u_{W*}\left(r\cdot \mathrm{tr}_{g_W}\right)} u_{W*}\mathcal{O}_{U_W}$$

factors (necessarily uniquely) as $f_{W*}\mathcal{O}_{X_W} \to \mathcal{O}_W \hookrightarrow u_{W*}\mathcal{O}_{U_W}$.

The next proposition results.

Proposition 2.9.13. *Let $f\colon X \to Y$ be an affine scheme-map such that the $\mathcal{O}_Y$-module $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)$ is quasi-coherent, and let (2.9.12.1) be a fiber square in which u is schematically surjective (i.e., the associated map $\mathcal{O}_Y \to u_*\mathcal{O}_U$ is injective) and qcqs, and the map g is finite and étale. The representing pair $(H^0 f^\flat \mathcal{O}_Y, t'_{\mathcal{O}_Y})$ in §2.9.1 is naturally isomorphic to the pair whose components are the “complementary sheaf” $\mathcal{C}_{f,u} \subset v_*\mathcal{O}_V$ (see above) and the restriction of $u_*\mathrm{tr}_g$ to $f_*\mathcal{C}_{f,u} \subset f_*v_*\mathcal{O}_V = u_*g_*\mathcal{O}_V$.*

Examples 2.9.14. (a) In 2.9.13, if u is the identity map then $\mathcal{C}_{\phi,u} = f_*\mathcal{O}_X$, giving the last paragraph in section 2.9.1.

(b) Let R be an integral domain with fraction field K, and $R \to S$ a ring-homomorphism with S finitely presentable as an R-module and $L := S \otimes_R K$ a separable K-algebra, with trace map $\mathrm{tr}_{L/K}\colon L \to K$. Let (2.9.12.1) be the scheme-diagram corresponding to the natural diagram

$$\begin{array}{ccc} L & \xleftarrow{\ j\ } & S \\ \uparrow & & \uparrow \\ K & \xleftarrow[\ i\ ]{} & R \end{array}$$

Then the complementary sheaf $\mathcal{C}_{f,u}$ is the sheafification of the S-module $\{x \in L \mid \mathrm{tr}_{L/K}(x(jS)) \subset R\}$.

2.10. This section treats duality for a class of *perfect closed immersions* of schemes, including all regular immersions. Described, on this class, is a concrete realization of the pseudofunctor $(-)^\flat$ (see Proposition 2.10.12 and Theorem 2.10.22), as well as its interaction with $\otimes^{\mathsf{L}}$ and with $\mathsf{R}\mathcal{H}om$ (see Proposition 2.10.25). One prior version of such material can be found in [Co00, §§2.5–2.6].

Throughout, $f\colon X \to Y$ will be a closed immersion of schemes, and $\mathcal{I}$ the kernel of the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. The functor $f_*\colon \mathcal{A}_{\mathsf{qc}}(X) \to \mathcal{A}_{\mathsf{qc}}(Y)$ is “extension by 0,” and its natural left adjoint f^* associates to $G \in \mathcal{A}_{\mathsf{qc}}(Y)$ the restriction to X of $G/\mathcal{I}G$. So f^*f_* is the identity functor of $\mathcal{A}_{\mathsf{qc}}(X)$; and for $G \in \mathcal{A}_{\mathsf{qc}}(Y)$, the unit map $G \to f_*f^*G$ identifies naturally with the canonical surjection $G \twoheadrightarrow G/\mathcal{I}G$.

2.10.1. Define the $\mathcal{O}_Y$-isomorphism

$$\nabla_{\!f}\colon \mathcal{I}/\mathcal{I}^2 \xrightarrow{\;\sim\;} \mathcal{T}or_1^{\mathcal{O}_Y}(f_*\mathcal{O}_X, f_*\mathcal{O}_X) = f_*H^{-1}\mathsf{L}f^*f_*\mathcal{O}_X, \tag{2.10.2}$$

to be $-\partial^{-1}$ where ∂ is the usual connecting isomorphism

$$\mathcal{T}or_1^{\mathcal{O}_Y}(f_*\mathcal{O}_X, f_*\mathcal{O}_X) \xrightarrow{\;\sim\;} \mathcal{T}or_0^{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{I}) = \mathcal{I}/\mathcal{I}^2$$

associated to the natural exact sequence $0 \to \mathcal{I} \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to 0$.

Any $\mathcal{O}_Y$-surjection $\vartheta\colon P \twoheadrightarrow \mathcal{I}$ with P *flat* expands to a flat resolution $P_\bullet : \cdots \to P' \to P \to \mathcal{O}_Y$ of $f_*\mathcal{O}_X \cong \mathcal{O}_Y/\mathcal{I}$, entailing natural isomorphisms

$$\mathcal{T}or_1^{\mathcal{O}_Y}(f_*\mathcal{O}_X, f_*\mathcal{O}_X) \xrightarrow{\;\sim\;} H_1(P_\bullet/\mathcal{I}P_\bullet) \xrightarrow{\;\sim\;} \mathcal{I}/\mathcal{I}^2$$

whose composition one finds, by dissecting definitions, to be $-\partial = \nabla_{\!f}^{-1}$.

For any flat $u\colon Y' \to Y$, the projection $f'\colon X' := X \times_Y Y' \to Y'$ being a closed immersion, one gets (via $P_\bullet$, for example) a natural identification

$$\nabla_{\!f'} = u^*\nabla_{\!f}. \tag{2.10.3}$$

The natural composite map

$$\begin{aligned} \mathsf{L}f^*f_*\mathcal{O}_X \otimes^{\mathsf{L}}_X \mathsf{L}f^*f_*\mathcal{O}_X &\xrightarrow{\;\sim\;} \mathsf{L}f^*(f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X) \\ &\longrightarrow \mathsf{L}f^*f_*(\mathcal{O}_X \otimes^{\mathsf{L}}_X \mathcal{O}_X) \xrightarrow{\;\sim\;} \mathsf{L}f^*f_*\mathcal{O}_X. \end{aligned} \tag{2.10.4}$$

makes the graded group $\oplus_{n\ge 0} H^{-n}\mathsf{L}f^*f_*\mathcal{O}_X$ into a strict (= alternating) graded $\mathcal{O}_X$-algebra. (Localize, and see, e.g., [B07, p. 201, Exercise 9(c)]).[14]) Thus with $\bigwedge$ denoting "exterior algebra," the isomorphism

$$f^*\nabla_{\!f}\colon f^*\big(\mathcal{I}/\mathcal{I}^2\big) \xrightarrow{\;\sim\;} f^*f_*H^{-1}\mathsf{L}f^*f_*\mathcal{O}_X = H^{-1}\mathsf{L}f^*f_*\mathcal{O}_X$$

extends uniquely to a homomorphism of graded $\mathcal{O}_X$-algebras

$$\textstyle\bigwedge_X f^*\big(\mathcal{I}/\mathcal{I}^2\big) := \oplus_{n\ge 0} \bigwedge_X^n f^*\big(\mathcal{I}/\mathcal{I}^2\big) \longrightarrow \oplus_{n\ge 0} H^{-n}\mathsf{L}f^*f_*\mathcal{O}_X. \tag{2.10.5}$$

2.10.6. Suppose the $\mathcal{O}_Y$-ideal $\mathcal{I}$ is generated by a sequence of global sections $\mathbf{t} := (t_1, t_2, \dots, t_d)$ that is *Koszul-regular*, i.e., with K_i the $\mathcal{O}_Y$-complex that is $\mathcal{O}_Y \xrightarrow{t_i} \mathcal{O}_Y$ in degrees -1 and 0 and that vanishes elsewhere, the Koszul complex $K(\mathbf{t}) := \otimes_{i=1}^d K_i$ is a finite free resolution of $f_*\mathcal{O}_X = H^0K(\mathbf{t})$. This holds if the germ of $\mathbf{t}$ at any point y in the image of f is regular, see [St24, tag 063K]; and conversely if Y is locally noetherian [St24, tag 063L] or, for arbitrary Y, modulo "smooth localization" [St24, tag 0629].

As a graded group, $K(\mathbf{t})$ identifies naturally with $\bigwedge_Y \mathcal{O}_Y^d$; and one checks that exterior-algebra multiplication

$$\mu_{\mathbf{t}}\colon K(\mathbf{t}) \otimes_{\mathcal{O}_Y} K(\mathbf{t}) \to K(\mathbf{t}) \tag{2.10.7}$$

is a map of *flat $\mathcal{O}_Y$-complexes* that is $\mathbf{D}(Y)$-isomorphic (via the natural map $K(\mathbf{t}) \to f_*\mathcal{O}_X$) to the natural composite map

$$f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X \longrightarrow f_*(\mathcal{O}_X \otimes^{\mathsf{L}}_X \mathcal{O}_X) \xrightarrow{\;\sim\;} f_*\mathcal{O}_X.$$

[14] We'll need this only for "Koszul-regular" f (see §§2.10.6, 2.10.9), in which case one can use—locally—the exterior-algebra structure on a Koszul complex that resolves $f_*\mathcal{O}_X$.

Using, e.g., [L09, (3.2.4.1)], one deduces a $\mathbf{D}(X)$-isomorphism from $f^*\mu_{\mathbf{t}}$ to the composite map (2.10.4). Hence the natural isomorphism

$$f^*K(\mathbf{t}) = \oplus_{n=0}^d \big(H^{-n}f^*K(\mathbf{t})\big)[n] \xrightarrow{\sim} \oplus_{n=0}^d \big(H^{-n}\mathsf{L}f^*f_*\mathcal{O}_X\big)[n]$$

is an isomorphism of graded $\mathcal{O}_X$-algebras. (Details left to the reader.)

Now let $P_\bullet := K(\mathbf{t}) \to f_*\mathcal{O}_X$ be the natural map, and let the map ϑ in 2.10.1 be the induced map $P := P_0 = \mathcal{O}_Y^d \twoheadrightarrow \mathcal{I}$ (which sends the i-th canonical generator of $\mathcal{O}_Y^d$ to t_i $(1 \le i \le d)$). From $H^{-1}K(\mathbf{t}) = 0$ it follows that $f^*\vartheta\colon \mathcal{O}_X^d \to f^*\mathcal{I} = f^*(\mathcal{I}/\mathcal{I}^2)$ is an isomorphism, the resulting natural composite $\mathbf{D}(X)$-isomorphism

(2.10.8)

$$\oplus_{n=0}^d \wedge_X^n f^*\big(\mathcal{I}/\mathcal{I}^2\big)[n] \xrightarrow{\sim} \oplus_{n=0}^d \wedge_X^n\big(\mathcal{O}_X^d\big)[n] \xrightarrow{\sim} f^*K(\mathbf{t}) \xrightarrow{\sim} \mathsf{L}f^*f_*\mathcal{O}_X$$

being sent by H^{-1} to the isomorphism $f^*\triangledown_f$. Furthermore, the preceding paragraph implies that application of the functor $\oplus_{n=0}^d H^{-n}$ to (2.10.8) gives an *isomorphism of graded $\mathcal{O}_X$-algebras,* which, being determined by what it does in degree 1, *must be the map* (2.10.5) (which does not depend on the choice of the generating sequence $\mathbf{t}$).

2.10.9. Let $g\colon V \to W$ be a map of ringed spaces. For each $d \in \mathbb{Z}$ and $F, G \in \mathbf{D}(W)$, one has the natural map

$$\operatorname{Ext}_W^d(F,G) = \operatorname{Hom}_{\mathbf{D}(W)}(F, G[d]) \longrightarrow \operatorname{Hom}_V(H^{-d}\mathsf{L}g^*F, H^0\mathsf{L}g^*G).$$

Base-changing to arbitrary open subsets of W, one gets a map of presheaves, whose sheafification is a bifunctorial $\mathbf{D}(W)$-map

$$\Psi(g,F,G,d)\colon \mathcal{E}xt_W^d(F,G) \longrightarrow g_*\mathcal{H}om_V(H^{-d}\mathsf{L}g^*F, H^0\mathsf{L}g^*G).$$

In particular, for g the closed immersion $f\colon X \to Y$, $F := f_*\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$, and G an $\mathcal{O}_Y$-module, one has the $\mathbf{D}(Y)$-map

$$\Psi(f, f_*\mathcal{O}_X, G, d)\colon \mathcal{E}xt_Y^d(f_*\mathcal{O}_X, G) \longrightarrow f_*\mathcal{H}om_X(H^{-d}\mathsf{L}f^*f_*\mathcal{O}_X, f^*G),$$

from which one gets, via (2.10.5), the natural composite map

$$\begin{aligned}\Phi(f,G,d)\colon \mathcal{E}xt_Y^d(f_*\mathcal{O}_X, G) &\longrightarrow f_*\mathcal{H}om_X(f^*\wedge_Y^d\big(\mathcal{I}/\mathcal{I}^2\big), f^*G)\\ &\xrightarrow{\sim} \mathcal{H}om_Y(\wedge_Y^d\big(\mathcal{I}/\mathcal{I}^2\big), G/\mathcal{I}G),\end{aligned}$$

sheafifying the natural composite map

$$\begin{aligned}\operatorname{Hom}_{\mathbf{D}(Y)}(f_*\mathcal{O}_X, G[d]) &\longrightarrow \operatorname{Hom}_Y(H^{-d}\mathsf{L}f^*f_*\mathcal{O}_X, H^0\mathsf{L}f^*G)\\ &\longrightarrow \operatorname{Hom}_Y(\wedge_Y^d\big(\mathcal{I}/\mathcal{I}^2\big), G/\mathcal{I}G).\end{aligned} \tag{2.10.10}$$

We'll say that f is a *Koszul-regular closed immersion of codimension d* if X is covered by open subsets of Y over each of which the ideal $\mathcal{I}$ is generated by a length d, Koszul-regular, sequence of sections. Such an f is perfect.

Lemma 2.10.11 (cf. [H66, p. 179, 7.2]). *If $f\colon X\to Y$ is a Koszul-regular closed immersion of codimension d, $\mathcal{I}$ is the kernel of the associated map $\mathcal{O}_Y\to f_*\mathcal{O}_X$, and G is an $\mathcal{O}_Y$-module, then $\Phi(f,G,d)$ is an isomorphism*

$$\mathcal{E}xt^d_Y(f_*\mathcal{O}_X,G)\xrightarrow{\ \sim\ }\mathcal{H}om_Y(\wedge^d_Y(\mathcal{I}/\mathcal{I}^2),G/\mathcal{I}G).$$

Proof. As (2.10.5) is an isomorphism (see last paragraph in section 2.10.6), the assertion means that $\Psi:=\Psi(f,f_*\mathcal{O}_X,G,d)$ *is an isomorphism.* The question being local (see (2.10.3)), one may assume that $\mathcal{I}$ is generated by a Koszul-regular sequence $\mathbf{t}=(t_1,t_2,\dots,t_d)$ of sections. Then, by [Il71, p. 121, Cor. 4.6], the natural map, with $\mathbf{K}(Y)$ the homotopy category of $\mathcal{O}_Y$-complexes, $\mathrm{Hom}_{\mathbf{K}(Y)}(K(\mathbf{t}),G[d])\xrightarrow[\alpha]{}\mathrm{Hom}_{\mathbf{D}(Y)}(K(\mathbf{t}),G[d])$ sheafifies to an isomorphism; and since the composition of the natural sequence of maps

$$\begin{aligned}\mathrm{Hom}_{\mathbf{K}(Y)}(K(\mathbf{t}),G[d])&\xrightarrow[\alpha]{}\mathrm{Hom}_{\mathbf{D}(Y)}(K(\mathbf{t}),G[d])\\&\xrightarrow[\beta]{}\mathrm{Hom}_{\mathcal{O}_X}(H^{-d}\mathsf{L}f^*K(\mathbf{t}),H^0\mathsf{L}f^*G)\\&\xrightarrow{\ \sim\ }\Gamma(Y,G)/\mathbf{t}\Gamma(Y,G),\end{aligned}$$

is just the natural isomorphism, therefore Ψ, the sheafification of β, is indeed an isomorphism. □

For a Koszul-regular immersion $f\colon X\to Y$, the next proposition gives another representation of $f^\flat$ and the counit map $f_*f^\flat\to\mathrm{id}$.

As a preliminary, note that there exists locally a Koszul-regular sequence $\mathbf{t}$ that generates the kernel $\mathcal{I}$ of the natural map $\mathcal{O}_Y\to f_*\mathcal{O}_X$; and one has natural isomorphisms

$$f_*f^\flat\mathcal{O}_Y\xrightarrow[2.3.3]{\ \sim\ }\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathcal{O}_Y)\xrightarrow{\ \sim\ }\mathcal{H}om_Y(K(\mathbf{t}),\mathcal{O}_Y)\xrightarrow{\ \sim\ }K(\mathbf{t})[-d],$$

the last being inverse to the adjoint of the natural composite map

$$K(\mathbf{t})[-d]\otimes_Y K(\mathbf{t})\xrightarrow{\ \sim\ }(K(\mathbf{t})\otimes_Y K(\mathbf{t}))[-d]\xrightarrow[(2.10.7)]{\mu_{\mathbf{t}}[-d]}K(\mathbf{t})[-d]\longrightarrow\mathcal{O}_Y.$$

Therefore, $f_*f^\flat\mathcal{O}_Y$—and hence $f^\flat\mathcal{O}_Y$—has vanishing cohomology in every degree other than d. So there are natural global $\mathbf{D}(X)$-isomorphisms

$$(H^df^\flat\mathcal{O}_Y)[-d]\xrightarrow{\ \sim\ }f^\flat\mathcal{O}_Y$$

and

$$\begin{array}{ccc}(H^df_*f^\flat\mathcal{O}_Y)[-d]&\xrightarrow{\quad\sim\quad}&f_*f^\flat\mathcal{O}_Y\\ \simeq\uparrow&&\downarrow\simeq\\ \big(H^d\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathcal{O}_Y)\big)[-d]&&\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X,\mathcal{O}_Y)\end{array}$$

Proposition 2.10.12. *Let $f\colon X\to Y$ be a Koszul-regular closed immersion of codimension d, let $\mathcal{I}$ be the kernel of the associated map $\mathcal{O}_Y\to f_*\mathcal{O}_X$, and let ω_f be the locally free $\mathcal{O}_X$-complex*

$$\omega_f:=\mathcal{H}om_X(\wedge^d_Xf^*(\mathcal{I}/\mathcal{I}^2),\mathcal{O}_X)[-d].$$

Let t' be the natural composite map

$$
\begin{aligned}
f_*\omega_f &\xrightarrow{\ \sim\ } \mathcal{H}om_Y\big(\wedge^d_Y(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}\big)[-d] \\
&\underset{2.10.11}{\xrightarrow{\ \sim\ }} \mathcal{E}xt^d_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)[-d] \\
&\xrightarrow{\ \sim\ } \big(H^d\mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)\big)[-d] \\
&\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) \\
&\longrightarrow \mathsf{R}\mathcal{H}om_Y(\mathcal{O}_Y, \mathcal{O}_Y) \xrightarrow{\ \sim\ } \mathcal{O}_Y.
\end{aligned}
$$

The functorial map dual to the natural composite $\mathbf{D}_{\mathsf{qc}}(Y)$*-map*

$$
f_*(\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) \underset{2.1.10}{\xrightarrow{\ \sim\ }} f_*\omega_f \otimes^{\mathsf{L}}_Y G \underset{\text{via } t'}{\longrightarrow} \mathcal{O}_Y \otimes^{\mathsf{L}}_Y G \xrightarrow{\ \sim\ } G
$$

is a $\mathbf{D}_{\mathsf{qc}}(X)$*-isomorphism*

$$
c^\flat_f(G)\colon \omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G \xrightarrow{\ \sim\ } f^\flat G \qquad (G \in \mathbf{D}_{\mathsf{qc}}(Y).
$$

Proof. Let $c^\flat\colon \omega_f \to f^\flat\mathcal{O}_Y$ be the natural composite $\mathcal{O}_X$-isomorphism

$$
\begin{aligned}
\omega_f &=\!= \mathcal{H}om_X\big(\wedge^d_X f^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X\big)[-d] \\
&\xrightarrow{\ \sim\ } f^*\mathcal{H}om_Y\big(\wedge^d_Y(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}\big)[-d] \\
&\underset{2.10.11}{\xrightarrow{\ \sim\ }} f^*\mathcal{E}xt^d_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)[-d] \qquad (2.10.12.1)\\
&\xrightarrow{\ \sim\ } f^*(H^d f_*f^\flat\mathcal{O}_Y)[-d] \\
&\xrightarrow{\ \sim\ } f^*(f_*H^d f^\flat\mathcal{O}_Y)[-d] \\
&\xrightarrow{\ \sim\ } (H^d f^\flat\mathcal{O}_Y)[-d] \xrightarrow{\ \sim\ } f^\flat\mathcal{O}_Y.
\end{aligned}
$$

Then with $\chi(f, \mathcal{O}_Y, G)\colon f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}f^*G \xrightarrow{\ \sim\ } f^\flat G$ the isomorphism from Proposition 2.7.7(ii), one has

$$
c^\flat_f(G) := \chi(f, \mathcal{O}_Y, G) \circ (c^\flat \otimes^{\mathsf{L}}_X \mathrm{id}_{\mathsf{L}f^*G}),
$$

that is, the border of the following natural diagram commutes:

$$
\begin{array}{ccccc}
f_*(\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) & \xrightarrow{\text{via } c^\flat} & f_*(f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) & \xrightarrow{f_*\chi} & f_*f^\flat G \\
\simeq \big\downarrow (2.3.1) & ① & (2.3.1)\big\downarrow\simeq & ② & \big\downarrow \\
 & & f_*f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_Y G & & G \\
 & \nearrow^{\text{via } c^\flat} & ③ & \searrow & \big\uparrow\simeq \\
f_*\omega_f \otimes^{\mathsf{L}}_Y G & & \xrightarrow[\text{via } t']{\qquad\qquad} & & \mathcal{O}_Y \otimes^{\mathsf{L}}_Y G
\end{array}
$$

Indeed, the commutativity of subdiagram ① is clear, that of ② is given by the last assertion in 2.7.7(i), and that of ③ (signifying that $c^\flat$ is dual to t') is readily verified.

Thus $c^\flat_f(G)$ is an isomorphism, as asserted. □

The next lemma provides an alternative, local, description of the isomorphism in 2.10.11, hence of (2.10.12.1) and $c_f^\flat(\mathcal{O}_Y)$.

Lemma 2.10.13. *In Proposition 2.10.12, suppose that $\mathcal{I}$ is generated by a Koszul-regular sequence $\mathbf{t} = (t_1, \dots, t_d)$ in $\Gamma(Y, \mathcal{I})$, and let $\vartheta\colon \mathcal{O}_Y^d \to \mathcal{I}$ be the $\mathcal{O}_Y$-homomorphism taking the i-th canonical generator of $\mathcal{O}_Y^d$ to t_i $(1 \le i \le d)$ (so that, since $H^{-1}K(\mathbf{t}) = 0$, $f^*\vartheta$ is an isomorphism).*

The following natural diagram of $\mathcal{O}_X$-isomorphisms commutes.

$$
\begin{array}{ccc}
\mathcal{H}om_X(\wedge_X^d f^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X) & \xrightarrow{\text{via } f^*\vartheta} & \mathcal{H}om_X(\wedge_X^d(\mathcal{O}_X^d), \mathcal{O}_X) \\
\uparrow & & \uparrow \\
f^*\mathcal{H}om_Y(\wedge_Y^d(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}) & & H^d\mathcal{H}om_X(f^*K_Y(\mathbf{t}), \mathcal{O}_X) \\
{\scriptstyle 2.10.11}\uparrow & & \uparrow \\
f^*\mathcal{E}xt_Y^d(f_*\mathcal{O}_X, \mathcal{O}_Y) & & H^d f^*\mathcal{H}om_Y(K_Y(\mathbf{t}), \mathcal{O}_Y) \\
\downarrow & & \uparrow \\
f^*H^d f_* f^\flat\mathcal{O}_Y & \longrightarrow & f^*H^d\mathcal{H}om_Y(K_Y(\mathbf{t}), \mathcal{O}_Y)
\end{array}
$$

Proof. Noting that $H^d\mathcal{H}om_Y(K_Y(\mathbf{t}), \mathcal{O}_Y)$ is annihilated by $\mathcal{I}$, and that H^d commutes with f_*, one checks that the adjoint diagram is isomorphic to the sheafification of the natural diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_X(\wedge_X^d f^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X) & \xrightarrow{\text{via } f^*\vartheta} & \mathrm{Hom}_X(\wedge_X^d(\mathcal{O}_X^d), \mathcal{O}_X) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_Y(\wedge_Y^d(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_Y/\mathcal{I}) & & H^d\,\mathrm{Hom}_X(f^*K_Y(\mathbf{t}), \mathcal{O}_X) \\
{\scriptstyle (2.10.10)}\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathbf{D}(Y)}(f_*\mathcal{O}_X, \mathcal{O}_Y[d]) & & H^d\,\mathrm{Hom}_Y(K_Y(\mathbf{t}), \mathcal{O}_Y) \\
\downarrow & & \| \\
H^d f_* f^\flat\mathcal{O}_Y & \longrightarrow & H^d\,\mathrm{Hom}_Y(K_Y(\mathbf{t}), \mathcal{O}_Y)
\end{array}
$$

So it suffices to see that this last diagram commutes—which one can do by pushing an arbitrary $\mathbf{D}(Y)$-map $f_*\mathcal{O}_X \to \mathcal{O}_Y[d]$ clockwise and counterclockwise around the diagram to the upper right corner. □

2.10.14. Next, the setup for Theorem 2.10.22—pseudofunctoriality of $c^\flat_{-}$.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be Koszul-regular closed immersions of codimensions d and e respectively, let $\mathcal{J}$ be the kernel of the natural map $\mathcal{O}_Z \to g_*\mathcal{O}_Y$, and let $\mathcal{L}$ be the kernel of the natural map $\mathcal{O}_Z \to g_* f_*\mathcal{O}_X$—so that $\mathcal{J} \subset \mathcal{L}$

and $\mathcal{I} := g^*(\mathcal{L}/\mathcal{J})$ is the kernel of the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then the map gf is a Koszul-regular closed immersion of codimension $d+e$, see [St24, tag 067Q].

The inclusion $\mathcal{J} \subset \mathcal{L}$ induces an exact sequence

$$0 \to f^*g^*\big(\mathcal{J}/\mathcal{J}^2\big) \xrightarrow{i} f^*g^*\big(\mathcal{L}/\mathcal{L}^2\big) \xrightarrow{p} f^*(\mathcal{I}/\mathcal{I}^2) \to 0$$

of locally free $\mathcal{O}_X$-modules of ranks e, $d+e$, and d, respectively, see [St24, tag 063N], whence a *locally split* exact sequence, with $E^\vee := \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ for any $\mathcal{O}_X$-module E:

$$0 \to \big(f^*(\mathcal{I}/\mathcal{I}^2)\big)^\vee \xrightarrow{p^\vee} \big(f^*g^*(\mathcal{L}/\mathcal{L}^2)\big)^\vee \xrightarrow{i^\vee} \big(f^*g^*(\mathcal{J}/\mathcal{J}^2)\big)^\vee \to 0.$$

Locally, there exists a right inverse q of p, giving rise to the left inverse j of i such that $ij = \mathrm{id} - qp$, whence the isomorphism

$$\big(f^*(\mathcal{I}/\mathcal{I}^2)\big)^\vee \oplus \big(f^*g^*(\mathcal{J}/\mathcal{J}^2)\big)^\vee \underset{(p^\vee,\, j^\vee)}{\xrightarrow{\;\sim\;}} \big(f^*g^*(\mathcal{L}/\mathcal{L}^2)\big)^\vee,$$

whence the standard isomorphism of presheaves (cf. [B70, Chap. III, §7.7]), hence of sheaves:

(2.10.15)
$$\textstyle\bigwedge_X^d\big(\big(f^*(\mathcal{I}/\mathcal{I}^2)\big)^\vee\big) \otimes_X \bigwedge_X^e\big(\big(f^*g^*(\mathcal{J}/\mathcal{J}^2)\big)^\vee\big) \xrightarrow{\;\sim\;} \bigwedge_X^{d+e}\big(\big(f^*g^*(\mathcal{L}/\mathcal{L}^2)\big)^\vee\big),$$

readily seen to be *independent of the choice of* q, so that these local maps glue together into a natural global isomorphism of invertible $\mathcal{O}_X$-modules.

2.10.16. In more explicit algebraic terms, the maps g and f correspond locally to a pair of surjective ring homomorphisms $R \overset{\varphi}{\twoheadrightarrow} S \overset{\xi}{\twoheadrightarrow} T$, with kernels J and I generated by Koszul-regular sequences $(r_1, \dots, r_e)$ and $(\bar{s}_1, \dots, \bar{s}_d)$ respectively. Let $s_i \in R$ be such that $\bar{s}_i = \xi(s_i)$ $(1 \le i \le d)$. The R-sequence $(r_1, \dots, r_e, s_1, \dots, s_d)$ is Koszul-regular [St24, tag 0669], and it generates the kernel L of $\xi\varphi$. The inclusion $J \subset L$ induces an exact sequence

(2.10.17)
$$0 \to T \otimes_S (J/J^2) \xrightarrow{\mathfrak{i}} L/L^2 \xrightarrow{\mathfrak{p}} I/I^2 \to 0$$

of free T-modules of respective ranks e, $d+e$ and d (see just before (2.10.8)). Let $\mathfrak{j}\colon L/L^2 \to T \otimes_S (J/J^2)$ be the left inverse of $\mathfrak{i}$ such that

$$\mathfrak{j}(r_m + L^2) = 1 \otimes (r_m + J^2) \qquad (1 \le m \le e),$$
$$\mathfrak{j}(s_n + L^2) = 0 \qquad (1 \le n \le d).$$

Over $\mathrm{Spec}(R)$, one checks, (2.10.15) is the sheafification of the isomorphism

$$\lambda\colon \textstyle\bigwedge_T^d \mathrm{Hom}_T(I/I^2, T) \otimes_T \bigwedge_T^e \mathrm{Hom}_T(T \otimes_S J/J^2, T) \xrightarrow{\;\sim\;} \bigwedge_T^{d+e} \mathrm{Hom}_T(L/L^2, T),$$

such that

$$\lambda\big((\alpha_1 \wedge \dots \wedge \alpha_d) \otimes (\beta_1 \wedge \dots \wedge \beta_e)\big) = (\alpha_1\mathfrak{p}) \wedge \dots \wedge (\alpha_d\,\mathfrak{p}) \wedge (\beta_1\mathfrak{j}) \wedge \dots \wedge (\beta_e\mathfrak{j}).$$

2.10.18. Back in the global situation, if an $\mathcal{O}_X$-module E is locally free of rank d, there is an isomorphism $\wedge_X^d(E^\vee) \xrightarrow{\sim} (\wedge_X^d E)^\vee$, induced by the map $(E^\vee)^d \times E^d \to \mathcal{O}_X$ that takes local sections $\big((\alpha_1,\dots,\alpha_d),(e_1,\dots,e_d)\big)$ to the determinant $\det(\alpha_i e_j)$. Also, for a finite-rank locally free $\mathcal{O}_Y$-module F, and $F^\vee := \mathcal{H}om_{\mathcal{O}_Y}(F,\mathcal{O}_Y)$, there is a natural isomorphism $f^*(F^\vee) \xrightarrow{\sim} (f^*F)^\vee$.

Using such isomorphisms, one gets from (2.10.15)—considered as a global isomorphism—a natural isomorphism of invertible $\mathcal{O}_X$-modules

$$(2.10.19)\quad \big(\wedge_X^d f^*(\mathcal{I}/\mathcal{I}^2)\big)^\vee \otimes_X f^*\big(\wedge_Y^e g^*(\mathcal{J}/\mathcal{J}^2)\big)^\vee \xrightarrow{\sim} \big(\wedge_X^{d+e}(gf)^*(\mathcal{L}/\mathcal{L}^2)\big)^\vee,$$

whence, for the $\mathcal{O}_X$-complexes

$$\omega_f := \big(\wedge_X^d f^*(\mathcal{I}/\mathcal{I}^2)\big)^\vee[-d], \qquad \omega_{gf} := \big(\wedge_X^{d+e}(gf)^*(\mathcal{L}/\mathcal{L}^2)\big)^\vee[-d-e]$$

and the $\mathcal{O}_Y$-complex

$$\omega_g := \big(\wedge_Y^e g^*(\mathcal{J}/\mathcal{J}^2)\big)^\vee[-e],$$

a natural isomorphism

$$(2.10.20)\qquad \omega_f \otimes_X f^*\omega_g \xrightarrow{\sim} \omega_{gf},$$

equal in degree $d+e$ to $(-1)^{de}$ times the isomorphism (2.10.19) (use the map θ_{ij} from [L09, (1.5.4)], with $i=-d$, $j=-e$).

In the local situation 2.10.16, routine manipulations show that (2.10.20) identifies naturally with the sheafification of the isomorphism of T-complexes

$$(2.10.21)\quad h\colon \mathrm{Hom}_T\big(\wedge_T^d(I/I^2),T\big)[-d] \otimes_T \mathrm{Hom}_S\big(\wedge_S^e(J/J^2),T\big)[-e] \\ \xrightarrow{\sim} \mathrm{Hom}_T\big(\wedge_T^{d+e}(L/L^2),T\big)[-d-e]$$

such that in degree $d+e$, with $r_i^L := (r_i + L^2) \in L/L^2$, etc.,

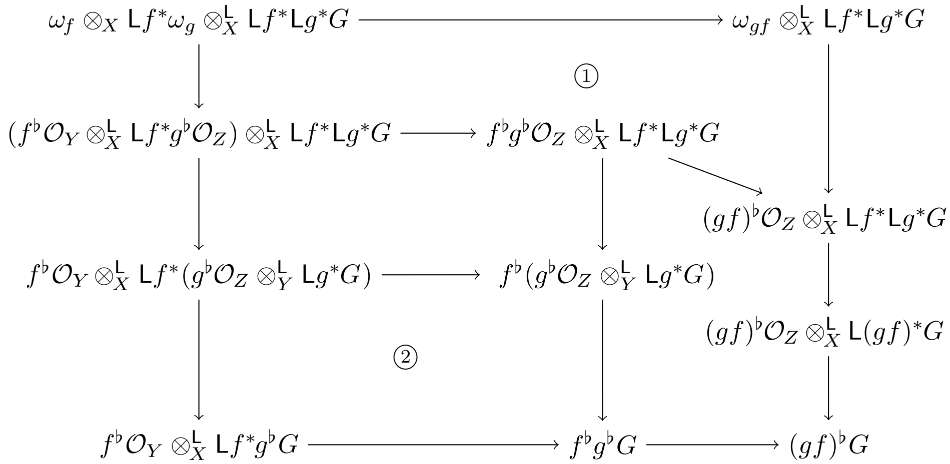
$$h(\alpha\otimes\beta)(r_1^L\wedge\cdots\wedge r_e^L\wedge s_1^L\wedge\cdots\wedge s_d^L) = \alpha(\bar s_1^I\wedge\cdots\wedge\bar s_d^I)\beta(r_1^J\wedge\cdots\wedge r_e^J).$$

The *pseudofunctoriality of* $c_-^\flat$ is given by the next theorem—essentially [Co00, p. 55, Theorem 2.5.1], whose proof in *ibid.*, section 2.6 is long and technical. The proof to be presented here is, *mutatis mutandis*, the more direct one given in [NkS19, Appendix C.6].

Theorem 2.10.22. *For* $G \in \mathbf{D}_{\mathsf{qc}}(Z)$, *the following natural* $\mathbf{D}(X)$*-diagram commutes*:

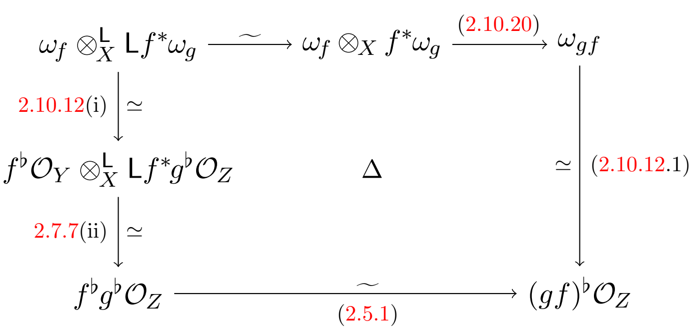
$$\begin{array}{ccc}
\omega_f \otimes_X \mathsf{L}f^*\omega_g \otimes^{\mathsf{L}}_X \mathsf{L}f^*\mathsf{L}g^*G & \xrightarrow{\text{via (2.10.20)}} & \omega_{gf} \otimes^{\mathsf{L}}_X \mathsf{L}f^*\mathsf{L}g^*G \\
{\scriptstyle 2.10.12(\mathrm{i})}\downarrow\simeq & & \simeq\downarrow \\
f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}f^*g^\flat G & & \omega_{gf} \otimes^{\mathsf{L}}_X \mathsf{L}(gf)^*G \\
{\scriptstyle 2.7.7(\mathrm{ii})}\downarrow\simeq & & \simeq\downarrow{\scriptstyle 2.10.12(\mathrm{i})} \\
f^\flat g^\flat G & \underset{(2.5.1)}{\xrightarrow{\sim}} & (gf)^\flat G
\end{array}$$

Proof of 2.10.22. It suffices to prove commutativity of the natural diagram



The commutativity of ① results from the following Lemma 2.10.23. That of ② is clear. That of the other two subdiagrams is contained, *mutatis mutandis,* in Proposition 2.7.8. The conclusion results. □

Lemma 2.10.23. *The following natural* $\mathbf{D}(X)$*-diagram commutes*:

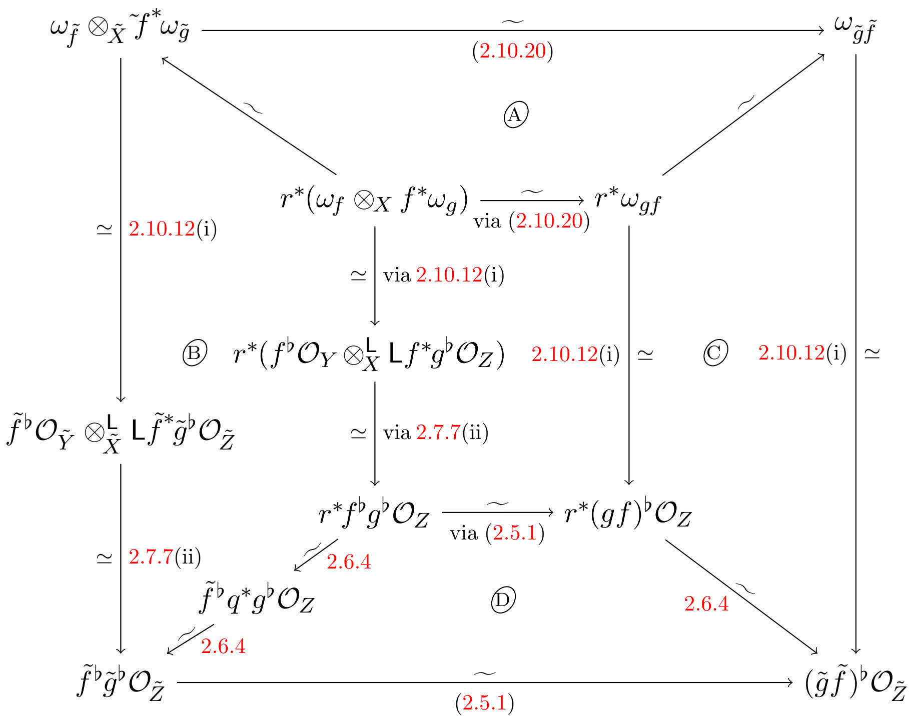


The strategy for proving 2.10.23 is to reduce to the local situation 2.10.16, which is disposed of in §3.5 below by means of arguments appearing in [NkS19, Appendix C.6.]. The reduction is given by the following lemma, with $\tilde{Z}$ the disjoint union of the members of an affine open covering of Z, over each of which both $\mathcal{I}$ and $\mathcal{J}$ are generated by Koszul-regular sequences of sections, and with p the natural map. (Note that the vertices in Δ all have homology that vanishes in degrees other than $d+e$, so that Δ is essentially a diagram of quasi-coherent $\mathcal{O}_X$-modules, whence for any faithfully flat map $r\colon \tilde{X}\to X$, Δ commutes if $r^*\Delta$ does.)

Lemma 2.10.24. *The situation being as in 2.10.14, let $p\colon \tilde Z \to Z$ be a flat scheme-map, and*

$$
\begin{array}{ccccc}
\tilde X & \xrightarrow{\ \tilde f\ } & \tilde Y & \xrightarrow{\ \tilde g\ } & \tilde Z\\
{\scriptstyle r}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p}\\
X & \xrightarrow[\ f\ ]{} & Y & \xrightarrow[\ g\ ]{} & Z
\end{array}
$$

a composite fiber square. Then $\tilde f$ and $\tilde g$ are Koszul-regular immersions, and the inner rectangle in the following natural diagram is isomorphic to the outer one:



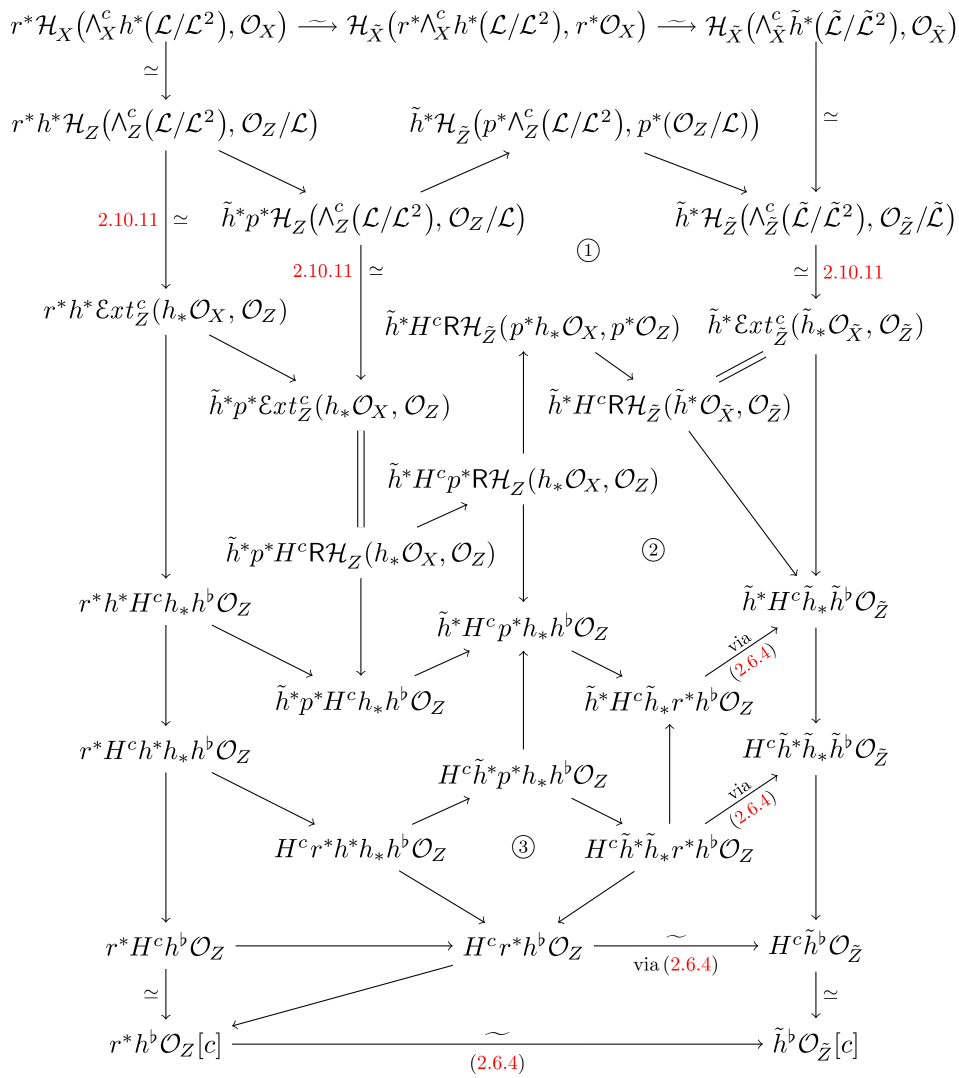
Proof. That $\tilde f$ and $\tilde g$ are Koszul-regular immersions (of respective codimensions d and e) is easy to verify.

It needs then to be shown that subdiagrams Ⓐ, Ⓑ, Ⓒ and Ⓓ commute. (In other words, the maps in 2.10.23 are compatible with flat base change.)

Set $h := gf$, $\tilde h := \tilde g \tilde f$. These are Koszul-regular immersions of codimension $c := d + e$, the kernel of the natural map $\mathcal O_Z \to h_*\mathcal O_X$ (resp. $\mathcal O_{\tilde Z} \to \tilde h_* \mathcal O_{\tilde X}$) being $\mathcal L$ (resp. $\tilde{\mathcal L} := \mathcal L \mathcal O_{\tilde Z}$).

Using the definitions of the maps in Ⓒ, and the easily-checked fact that, h_* being exact, the two natural composite maps $h^*H^c h_* \xrightarrow{\ \sim\ } h^*h_*H^c \to H^c$ and $h^*H^c h_* \to H^c h^* h_* \to H^c$ are equal, one sees that commutativity of Ⓒ

is equivalent to that of the border of the natural diagram, with $\mathcal{H} := \mathcal{H}om$,



Whether subdiagram ① commutes is a local question. Hence one can assume that $\mathcal{L}$ is generated by a Koszul-regular sequence $\mathbf{t} = (t_1, \dots, t_c)$, and replace $\mathsf{R}\mathcal{H}_Z(h_*\mathcal{O}_X, \mathcal{O}_Z)$ by $\mathcal{H}_Z(K_Z(\mathbf{t}), \mathcal{O}_Z)$ (resp. $\mathsf{R}\mathcal{H}_{\tilde{Z}}(\tilde{h}_*\mathcal{O}_X, \mathcal{O}_{\tilde{Z}})$ by $\mathcal{H}_Z(K_Z(p^*\mathbf{t}), \mathcal{O}_Z)$). Then Lemma 2.10.13 gives the commutativity of ①.

Subdiagram ② commutes, since without "$\tilde{h}^*H^c$" it is the diagram, with $(f, g, u, v, G) := (p, r, h, \tilde{h}, \mathcal{O}_{\tilde{Z}})$, shown in the proof of 2.6.4 to commute.

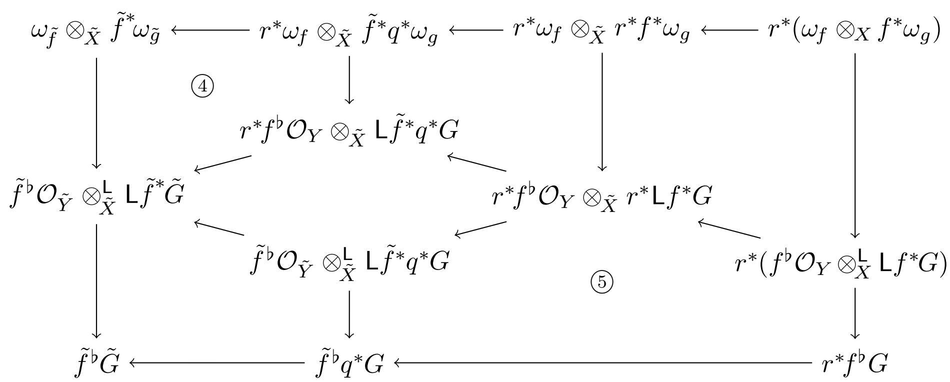
Commutativity of ③ holds because the natural map $p^*h_* \to \tilde{h}_*r^*$ is, by definition, adjoint to the natural composite map $\tilde{h}^*p^*h_* \xrightarrow{\sim} r^*h^*h_* \to r^*$.

Checking commutativity of the unlabeled subdiagrams is straightforward.

Diagram chasing shows now that the border commutes, whence so does subdiagram Ⓒ.

Subdiagram Ⓐ involves only sheaves, so its commutativity is readily checked, locally, via (2.10.21).

As for Ⓑ, expand it naturally as follows, where $G := g^\flat\mathcal{O}_Z$—so that one has the isomorphism $q^*G \underset{2.6.4}{\xrightarrow{\,\sim\,}} \tilde g^\flat\mathcal{O}_{\tilde Z} =: \tilde G$.



The commutativity of the unlabeled diagrams is easily checked. That of ④ is shown by arguments like those used above to show commutativity of Ⓒ. For that of ⑤, it suffices to show the commutativity of the adjoint diagram, as per the hint in [L09, 4.7.3.4(c)], *mutatis mutandis.*

Thus Ⓑ commutes.

The commutativity of Ⓓ can be shown in the same way as that of the last diagram in [L09, 4.6.8], thereby completing the proof of Lemma 2.10.24. ☐

For a Koszul-regular closed immersion $f\colon X \to Y$, the interaction of the isomorphism

$$c_f^\flat(G)\colon \omega_f \otimes^{\mathsf L}_X \mathsf L f^*G \xrightarrow{\,\sim\,} f^\flat G \qquad (G \in \mathbf D_{\mathsf{qc}}(X))$$

in 2.10.12(i) with independent base change, $\otimes^{\mathsf L}$ and $\mathsf R\mathcal{H}om$ is described in the following proposition.

Proposition 2.10.25. *Let $f\colon X \to Y$ be a Koszul-regular closed immersion of codimension d, and $F, G \in \mathbf D_{\mathsf{qc}}(Y)$.*

(i) *Let*

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X \\ {\scriptstyle g}\big\downarrow & \sigma & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;u\;]{} & Y \end{array}$$

be an independent square of scheme-maps (see §2.6.1) in which g is affine.

Then σ is a fiber square, and g is a Koszul-regular closed immersion of codimension d.

Moreover, with $\beta_\sigma(G)\colon \mathsf{L}v^*f^\flat G \longrightarrow g^\flat\mathsf{L}u^*G$ *adjoint to the natural composite map* $\mathsf{R}g_*\mathsf{L}v^*f^\flat G \xrightarrow{\sim} \mathsf{L}u^*\mathsf{R}f_*f^\flat G \longrightarrow \mathsf{L}u^*G$, *the following natural* $\mathbf{D}(X')$*-diagram commutes*:

$$\begin{array}{ccccc}
\mathsf{L}v^*(\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) & \xrightarrow{\sim} & \mathsf{L}v^*\omega_f \otimes^{\mathsf{L}}_{X'} \mathsf{L}v^*\mathsf{L}f^*G & \xrightarrow{\sim} & \omega_g \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*\mathsf{L}u^*G \\
{\scriptstyle \mathsf{L}v^*c^\flat_f(G)}\downarrow\simeq & & & & \simeq\downarrow{\scriptstyle c^\flat_g(\mathsf{L}u^*G)} \\
\mathsf{L}v^*f^\flat G & & \xrightarrow[\beta_\sigma(G)]{\sim} & & g^\flat\mathsf{L}u^*G
\end{array}$$

(ii) *The following natural* $\mathbf{D}(X)$*-diagram commutes*:

$$\begin{array}{ccc}
\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \xrightarrow{\sim} & \omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*(F \otimes^{\mathsf{L}}_Y G) \\
{\scriptstyle c^\flat_f(F)\otimes^{\mathsf{L}}_X \mathrm{id}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle c^\flat_f(F\otimes^{\mathsf{L}}_Y G)} \\
f^\flat F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \xrightarrow[\chi(f,F,G)]{2.7.7\text{(ii)}} & f^\flat(F \otimes^{\mathsf{L}}_Y G)
\end{array}$$

(iii) *If* F *is pseudo-coherent and* $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$, *then the following natural* $\mathbf{D}(X)$*-diagram, where the map* ξ *is adjoint to the natural map*

$$\omega_f \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om_Y(\mathsf{L}f^*F, \mathsf{L}f^*G) \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \longrightarrow \omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G,$$

commutes:

$$\begin{array}{ccccc}
\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(F,G) & \searrow & & \overset{\xi}{\nearrow} & \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, \omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) \\
{\scriptstyle c^\flat_f(-)}\downarrow\simeq & & \omega_f \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om_Y(\mathsf{L}f^*F, \mathsf{L}f^*G) & & \simeq\downarrow{\scriptstyle \text{via } c^\flat_f(\mathsf{L}f^*G)} \\
f^\flat\mathsf{R}\mathcal{H}om_Y(F,G) & & \xrightarrow[\zeta^{-1}]{2.8.2\text{(iii)}} & & \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G)
\end{array}$$

Proof. (i) Working locally on Y, one can assume that the kernel of the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is $\mathbf{t}\mathcal{O}_Y$, where $\mathbf{t}$ is a Koszul-regular sequence of length d. Then $f_*\mathcal{O}_X$ is resolved by the Koszul complex $K_Y(\mathbf{t})$, and, with $\mathbf{t}'$ the sequence $\mathbf{t}\mathcal{O}_{Y'}$, one has in $\mathbf{D}(Y')$ the natural isomorphisms

$$g_*\mathcal{O}_{X'} \cong \mathsf{L}g_*\mathsf{L}v^*\mathcal{O}_X \cong \mathsf{L}u^*\mathsf{R}f_*\mathcal{O}_X \cong u^*K_Y(\mathbf{t}) \cong K_{Y'}(\mathbf{t}').$$

Therefore, $\mathbf{t}'$ is a Koszul-regular sequence of length d, whence the projection $\tilde g\colon \tilde X = Y' \times_Y X \to Y'$ is a Koszul-regular closed immersion of codimension d; and the natural map $\tilde g_*\mathcal{O}_{\tilde X} \to g_*\mathcal{O}_{X'}$ is an isomorphism, so that, g being affine, the natural map $X' \to \tilde X$ is an isomorphism. The first two assertions, which need only be verified locally, result.

As for the diagram in question, the definition of $c_f^\flat(G)$ gives the following natural expansion:

$$\begin{array}{ccccc}
\mathsf{L}v^*(\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) & \longrightarrow & \mathsf{L}v^*\omega_f \otimes^{\mathsf{L}}_{X'} \mathsf{L}v^*\mathsf{L}f^*G & \longrightarrow & \omega_g \otimes^{\mathsf{L}}_{X'} \mathsf{L}g^*\mathsf{L}u^*G \\
\big\downarrow & \textcircled{1} & \big\downarrow & \textcircled{2} & \big\downarrow \\
\mathsf{L}v^*(f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}f^*G) & \longrightarrow & \mathsf{L}v^*f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}v^*\mathsf{L}f^*G & \longrightarrow & g^\flat\mathcal{O}_{Y'} \otimes^{\mathsf{L}}_X \mathsf{L}g^*\mathsf{L}u^*G \\
\big\downarrow & & \textcircled{3} & & \big\downarrow \\
\mathsf{L}v^*f^\flat G & & \longrightarrow & & g^\flat\mathsf{L}u^*G
\end{array}$$

The commutativity of ① is clear. For that of ②, cf. that of Ⓒ in 2.10.24. For that of ③, cf. [L09, 4.7.3.4(c)] (with $E := \mathcal{O}_Y$).

(ii) The diagram expands naturally as

$$\begin{array}{ccc}
\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \xrightarrow{\ \sim\ } & \omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*(F \otimes^{\mathsf{L}}_Y G) \\
{\scriptstyle \text{via } c_f^\flat(\mathcal{O}_Y)}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \text{via } c_f^\flat(\mathcal{O}_Y)} \\
f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \xrightarrow{\ \sim\ } & f^\flat\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{L}f^*(F \otimes^{\mathsf{L}}_Y G) \\
{\scriptstyle \text{via } c_f^\flat(F)}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle c_f^\flat(F \otimes^{\mathsf{L}}_X G)} \\
f^\flat F \otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \underset{\chi(f,F,G)}{\xrightarrow{\ 2.7.7\text{(ii)}\ }} & f^\flat(F \otimes^{\mathsf{L}}_Y G)
\end{array}$$

Commutativity of the top half is clear; and that of the bottom half is left for the reader to verify (cf. [L09, 4.7.3.4(a)] with $E := \mathcal{O}_Y$).

(iii) It suffices to prove the commutativity of the adjoint diagram, i.e., of the border of the following natural diagram (where $\mathcal{H} := \mathcal{H}om$):

$$\begin{array}{ccccc}
 & \nearrow & \omega_f \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}_X(\mathsf{L}f^*F, \mathsf{L}f^*G) \otimes^{\mathsf{L}}_X \mathsf{L}f^*F & \searrow & \\
\omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*\mathsf{R}\mathcal{H}_Y(F,G) \otimes^{\mathsf{L}}_X \mathsf{L}f^*F & & \textcircled{4} & & \omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*G \\
\big\downarrow & \searrow & \omega_f \otimes^{\mathsf{L}}_X \mathsf{L}f^*(\mathsf{R}\mathcal{H}_Y(F,G) \otimes^{\mathsf{L}}_Y F) & \nearrow & \big\downarrow \\
\big\downarrow & \textcircled{5} & \big\downarrow & \textcircled{6} & \big\downarrow \\
\big\downarrow & & f^\flat(\mathsf{R}\mathcal{H}_Y(F,G) \otimes^{\mathsf{L}}_Y F) & & \big\downarrow \\
\big\downarrow & \nearrow & \textcircled{7} & \searrow & \big\downarrow \\
f^\flat\mathsf{R}\mathcal{H}_Y(F,G) \otimes^{\mathsf{L}}_X \mathsf{L}f^*F & \longrightarrow & \mathsf{R}\mathcal{H}_X(\mathsf{L}f^*F, f^\flat G) \otimes^{\mathsf{L}}_X \mathsf{L}f^*F & \longrightarrow & f^\flat G
\end{array}$$

The commutativity of subdiagram ④ is given by [L09, 3.5.6(g)], with $\alpha\colon [D,E] \otimes D \to E$ the natural map. That of ⑤ is given by (ii), with $(\mathsf{R}\mathcal{H}_Y(F,G), F)$ in place of (F,G). That of ⑥ is clear.

For ⑦, it suffices to prove the commutativity of its adjoint, and so of all the subdiagrams of the following natural one:

$$\begin{tikzcd}
\mathsf{R}f_*(f^\flat\mathsf{R}\mathcal{H}_Y(F,G)\otimes^{\mathsf{L}}_X \mathsf{L}f^*F) \arrow[rrr] \arrow[dr, "(2.3.1)^{-1}"] \arrow[dddd] & & & \mathsf{R}f_*f^\flat(\mathsf{R}\mathcal{H}_Y(F,G)\otimes^{\mathsf{L}}_Y F) \arrow[dl] \arrow[dddd] \\
 & \mathsf{R}f_*f^\flat\mathsf{R}\mathcal{H}_Y(F,G)\otimes^{\mathsf{L}}_Y F \arrow[r] \arrow[dd] \arrow[r, phantom, "⑦_1", shift left=6] & \mathsf{R}\mathcal{H}_Y(F,G)\otimes^{\mathsf{L}}_Y F \arrow[dr] & \\
 & ⑦_2 & & G \\
 & \mathsf{R}f_*\mathsf{R}\mathcal{H}_X(\mathsf{L}f^*F, f^\flat G)\otimes^{\mathsf{L}}_Y F \arrow[r] & \mathsf{R}\mathcal{H}_Y(F,\mathsf{R}f_*f^\flat G)\otimes^{\mathsf{L}}_Y F \arrow[uu] \arrow[dr] \arrow[d, phantom, "⑦_3"] & \\
\mathsf{R}f_*(\mathsf{R}\mathcal{H}_X(\mathsf{L}f^*F, f^\flat G)\otimes^{\mathsf{L}}_X \mathsf{L}f^*F) \arrow[ur, "(2.3.1)^{-1}"] \arrow[rrr] & & & \mathsf{R}f_*f^\flat G \arrow[uu]
\end{tikzcd}$$

The commutativity of the unlabeled subdiagrams is easily checked. That of ⑦$_1$ results from 2.7.7(ii), and that of ⑦$_2$ (without "$\otimes^{\mathsf{L}}_Y F$") from 2.8.2(iii).

The commutativity of ⑦$_3$ results from that of its adjoint diagram, namely, with the abbreviations f_* for $\mathsf{R}f_*$, f^* for $\mathsf{L}f^*$, $[-,-]_Z$ for $\mathsf{R}\mathcal{H}_Z(-,-)$, and $\otimes_Z$ for $\otimes^{\mathsf{L}}_Z$ ($Z = X$ or Y), from that of the natural diagram

$$\begin{array}{ccc}
f^*(f_*[f^*F, f^\flat G]_X \otimes F) & \longrightarrow & f^*([F, f_*f^\flat G]_Y \otimes F) \\
{\scriptstyle (2.3.1)}\big\downarrow & & \big\downarrow \\
[f^*F, f^\flat G]_X \otimes f^*F & \longrightarrow & f^\flat G,
\end{array}$$

which commutativity follows, e.g., from [L09, 3.5.5, 3.5.6(d) and 3.4.6.2]. □

2.11. Corollary 2.3.9 extends to *all* affine maps of qcqs schemes if we replace $\mathbf{D}_{\mathsf{qc}}$ by $\mathbf{D}(\mathcal{A}_{\mathsf{qc}})$, see Corollary 2.11.2. Corollary 2.11.3 records that this replacement is unnecessary for schemes X such that the natural functor $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X)) \to \mathbf{D}_{\mathsf{qc}}(X)$ is an equivalence of categories—for instance finite-dimensional noetherian schemes [Il71, p. 191, 3.7] or quasi-compact separated schemes [BN93, p. 230, 5.5] (but not arbitrary qcqs schemes, see [Il71, p. 195, 0.3]).

Some details follow; the rest are left to the reader.

For any qcqs scheme X, the inclusion functor $j_X\colon \mathcal{A}_{\mathsf{qc}}(X) \to \mathcal{A}(X)$ has a right adjoint Q_X, the *quasi-coherator* [Il71, p. 187, 3.2]. For example, if X is affine one can take Q_X to be the sheafification of the global section functor.

Lemma 2.11.1. *Let $f\colon X \to Y$ be an affine map of qcqs schemes, let $\phi\colon \overline{Y} \to Y$ be as in (2.1.5) and let $Q := Q_Y$ be the quasi-coherator.*

(i) *The inclusion $\bar{\jmath}\colon \mathcal{A}_{\mathsf{qc}}(\overline{Y}) \hookrightarrow \mathcal{A}(\overline{Y})$ has a right adjoint $\overline{Q}$ such that $\phi_*\overline{Q} = Q\phi_*$.*

(ii) *The derived functor $\mathsf{R}\overline{Q}$ is right-adjoint to $\mathsf{R}\bar{\jmath}\colon \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(\overline{Y})) \to \mathbf{D}(\overline{Y})$.*

Proof. (i) For $N \in \mathcal{A}(Y)$, let $\epsilon_N\colon QN = j_Y QN \to N$ be the counit map for the adjunction $j_Y \dashv Q$. Set $\overline{\mathcal{O}} := \phi_*\mathcal{O}_{\overline{Y}} = f_*\mathcal{O}_X$, and $\otimes := \otimes_{\mathcal{O}_Y}$ (nonderived).

For any $M \in \mathcal{A}(\overline{Y})$, scalar multiplication is an $\mathcal{A}(Y)$-map

$$\mu\colon \overline{\mathcal{O}} \otimes \phi_* M \to \phi_* M.$$

Since Q is right-adjoint to j_Y, there is a unique $\mathcal{A}_{\mathsf{qc}}(Y)$-map λ making the following diagram commute:

$$\begin{array}{ccc} \overline{\mathcal{O}} \otimes Q\phi_* M & \xrightarrow{\ \lambda\ } & Q\phi_* M \\ {\scriptstyle \mathrm{id}\otimes\epsilon}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon} \\ \overline{\mathcal{O}} \otimes \phi_* M & \xrightarrow[\ \mu\ ]{} & \phi_* M \end{array}$$

One checks that λ makes $Q\phi_*M$ into a quasi-coherent $\overline{\mathcal{O}}$-module $\overline{Q}M$; that this construction is functorial; and that the resulting functor $\overline{Q}$ is as asserted in (i).

(ii) Since by (i), $\overline{Q}$ has an exact left adjoint, therefore if G is a K-injective complex in $\mathcal{A}(\overline{Y})$ then $\overline{Q}G$ is K-injective in $\mathcal{A}_{\mathsf{qc}}(\overline{Y})$; and for any complex F in $\mathcal{A}_{\mathsf{qc}}(\overline{Y})$, the functorial isomorphism of complexes of abelian groups

$$\mathrm{Hom}_{\mathcal{A}(\overline{Y})}(\bar{\jmath}F, G) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{A}_{\mathsf{qc}}(\overline{Y})}(F, \overline{Q}G)$$

that results from (i) gives a functorial derived-category isomorphism

$$\mathsf{R}\mathrm{Hom}_{\mathbf{D}(\overline{Y})}(\mathsf{R}\bar{\jmath}F, G) \xrightarrow{\ \sim\ } \mathsf{R}\mathrm{Hom}_{\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(\overline{Y}))}(F, \mathsf{R}\overline{Q}G),$$

to which application of the homology functor H^0 gives a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\overline{Y})}(\mathsf{R}\bar{\jmath}F, G) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(\overline{Y}))}(F, \mathsf{R}\overline{Q}G)$$

that extends via K-injective resolution $G' \to G$ to arbitrary $G' \in \mathbf{D}(\overline{Y})$. □

Corollary 2.11.2. *With f as in 2.11.1 and $\phi^\flat$ as in 2.2.5, the functor $\bar{f}^*\mathsf{R}\overline{Q}\phi^\flat$ is right-adjoint to $\mathsf{R}f_*\colon \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X)) \to \mathbf{D}(Y)$.*

Proof. The functor $\mathsf{R}f_*$ factors as

$$\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X)) \xrightarrow{\mathsf{R}\bar{f}_*} \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(\overline{Y})) \xrightarrow{\mathsf{R}\bar{\jmath}} \mathbf{D}(\overline{Y}) \xrightarrow{\phi_*} \mathbf{D}(Y).$$

So the assertion follows from the paragraph just before 2.1.1, Lemma 2.11.1, and Corollary 2.2.6. □

Corollary 2.11.3. *In 2.11.2, if the natural functor $\boldsymbol{j}_X\colon \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(X)) \to \mathbf{D}_{\mathsf{qc}}(X)$ is an equivalence then $\boldsymbol{j}_X\bar{f}^*\mathsf{R}\overline{Q}\phi^\flat$ is right-adjoint to $\mathsf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}(Y)$.*

3. From commutative algebra to affine schemes

By Theorem 2.6.4 with u an open immersion, or by direct verification, the foregoing constructions involving $f^\flat$ for finite pseudo-coherent $f\colon X \to Y$ are compatible with open immersions on Y, and so can be locally elucidated by making them more explicit when Y and X are affine schemes. We do this via an "equivalence," given by *sheafification,* from the (concretely realized) duality theory for derived categories of modules over commutative rings to the duality theory for $\mathbf{D}_{\mathsf{qc}}$-categories over affine schemes—essentially a special case of the equivalence mentioned near the beginning of §2. A quasi-inverse for this equivalence is provided by the derived global section functor. Details appear in section 3.1.

The underlying idea is, given a ring-homomorphism $\varphi\colon R\to S$ and a construction involving the corresponding scheme-map $f\colon \operatorname{Spec} S\to\operatorname{Spec} R$, to describe a concrete commutative-algebra construction involving φ, whose sheafification is naturally isomorphic to the given one, and so constitutes a concrete realization. (Abusing terminology, "concrete" signifies "concrete modulo choosing K-injective or K-projective resolutions of complexes.")

For instance, sheafifying the right adjoint $\varphi^{\#}(-):=\mathsf{R}\mathrm{Hom}_{\varphi}(S,-)$ of the restriction-of-scalars functor $\varphi_*\colon \mathbf{D}(S)\to\mathbf{D}(R)$ gives a right adjoint for $\mathsf{R}f_*$ (see Corollary 3.1.13 and Proposition 3.1.18).

More such realizations are presented, for the base-change isomorphism β_σ of Theorem 2.6.4 (see Proposition 3.2.13), and for the interaction of $\varphi^{\#}$ with $\otimes^{\mathsf{L}}$ and $\mathsf{R}\mathrm{Hom}$ (see Propositions 3.3.2 and 3.4.1).

The algebraic version of pseudofunctoriality for Koszul-regular immersions (to which the scheme-theoretic one has been reduced, see remarks immediately after Lemma 2.10.23) is proved in section 3.5. This proof, essentially the one in [NkS19, Appendix C.6], is more complicated than anything that came before.

3.1. For a commutative ring T, let $\mathcal{A}(T)$ be the category of T-modules and $\mathbf{D}(T)$ the corresponding derived category.

Set $Z:=\operatorname{Spec} T$. For the usual adjunction $\mathfrak{s}_T\dashv\Gamma_Z$ with $\mathfrak{s}_T:\mathcal{A}(T)\to\mathcal{A}(Z)$ the sheafification functor and $\Gamma_Z:=\Gamma(Z,-)\colon\mathcal{A}(Z)\to\mathcal{A}(T)$ the global-section functor [GrD71, 1.7.4], the unit map $\mathrm{id}\xrightarrow{\sim}\Gamma_Z\mathfrak{s}_T$ and counit map $\mathfrak{s}_T\Gamma_Z\to\mathrm{id}$ are the natural ones.

Since Γ_Z has an exact left adjoint, it preserves K-injectivity, and there results an adjunction $\mathfrak{s}_T\dashv\mathsf{R}\Gamma_Z$ of derived functors between $\mathbf{D}(T)$ and $\mathbf{D}(Z)$. Here the unit map is the natural functorial composite isomorphism

$$\mathsf{G}\xrightarrow{\sim}\Gamma_Z\mathfrak{s}_T\mathsf{G}\xrightarrow{\sim}\mathsf{R}\Gamma_Z\mathfrak{s}_T\mathsf{G}\qquad \mathsf{G}\in\mathbf{D}(T)\tag{3.1.1}$$

($\Gamma_Z\mathfrak{s}_T\to\mathsf{R}\Gamma_Z\mathfrak{s}_T$ is an isomorphism by [L09, 2.7.5, (ii)⇒(a)], dualized, with $d=0$, and [L09, 2.2.6])); and for $G\in\mathbf{D}(Z)$), if $G\to J$ is a quasi-isomorphism of $\mathcal{O}_Z$-complexes with J K-injective, then the counit map is the natural composite $\mathbf{D}(Z)$-map

$$\mathfrak{s}_T\mathsf{R}\Gamma_Z G\xrightarrow{\sim}\mathfrak{s}_T\Gamma_Z J\longrightarrow J\xrightarrow{\sim}G.$$

The functor $\mathfrak{s}_T$ factors naturally as $\mathcal{A}(T)\xrightarrow{\bar{\mathfrak{s}}_T}\mathcal{A}_{\mathsf{qc}}(Z)\hookrightarrow\mathcal{A}(Z)$, and $\bar{\mathfrak{s}}_T$ is an equivalence—whence so is its derived functor $\mathbf{D}(T)\to\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(Z))$, which will also be denoted $\bar{\mathfrak{s}}_T$.

The derived $\mathfrak{s}_T$, considered as a functor from $\mathbf{D}(T)$ to $\mathbf{D}_{\mathsf{qc}}(Z)\subset\mathbf{D}(Z)$, factors naturally as

$$\mathbf{D}(T)\underset{\bar{\mathfrak{s}}_T}{\xrightarrow{\approx}}\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(Z))\underset{j_Z}{\longrightarrow}\mathbf{D}_{\mathsf{qc}}(Z).\tag{3.1.2}$$

Since $\boldsymbol{j}_Z$ is an equivalence [BN93, p. 225, 5.1], therefore so is this $\mathfrak{s}_T$, for which a quasi-inverse (i.e., right adjoint) is the restriction to $\mathbf{D}_{\mathsf{qc}}(Z)$ of $\mathsf{R}\Gamma_Z$.

3.1.3. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Denote by $\varphi_*\colon \mathcal{A}(S) \to \mathcal{A}(R)$ the exact restriction-of-scalars functor, and also (abusing notation) its derived functor $\mathbf{D}(S) \to \mathbf{D}(R)$.

Let
$$\operatorname{Spec} S =: X \xrightarrow{\;f\;} Y := \operatorname{Spec} R$$
be the scheme-map corresponding to φ. For any S-complex E the natural $\mathbf{D}(Y)$-maps are isomorphisms

$$\mathfrak{s}_R\,\varphi_*\mathsf{E} \underset{\upsilon_\varphi(\mathsf{E})}{\xrightarrow{\;\sim\;}} f_*\mathfrak{s}_S\,\mathsf{E} \underset{q_f(\mathsf{E})}{\xrightarrow{\;\sim\;}} \mathsf{R}f_*\mathfrak{s}_S\,\mathsf{E}. \tag{3.1.4}$$

The first isomorphism is elementary: it is same as the natural isomorphism $\widetilde{\mathsf{E}}_R \cong \bar{f}_*\widetilde{\mathsf{E}}_S$ from Example 2.1.1, with scalars restricted from $f_*\mathcal{O}_X$ to $\mathcal{O}_Y$. In more detail (see [GrD71, 1.7.7(ii)]), $\upsilon_\varphi\colon \mathfrak{s}_R\,\varphi_* \xrightarrow{\;\sim\;} f_*\mathfrak{s}_S$ is the functorial map such that $\Gamma_Y\upsilon_\varphi$ is the natural composite functorial R-isomorphism

$$\Gamma_Y\mathfrak{s}_R\,\varphi_* \xrightarrow{\;\sim\;} \varphi_* \xrightarrow{\;\sim\;} \varphi_*\Gamma_X\mathfrak{s}_S = \Gamma_Y f_*\mathfrak{s}_S, \tag{3.1.5}$$

whence υ_φ is the natural composite isomorphism

$$\mathfrak{s}_R\,\varphi_* \xrightarrow{\;\sim\;} \mathfrak{s}_R\,\varphi_*\Gamma_X\mathfrak{s}_S = \mathfrak{s}_R\Gamma_Y f_*\mathfrak{s}_S \xrightarrow{\;\sim\;} f_*\mathfrak{s}_S. \tag{3.1.5$'$}$$

And, the map $q_f(\mathsf{E}^0)$ is an isomorphism for any S-module E^0 [GrD61b, 1.3.2], that is, such an E^0 is f_*-acyclic [L09, 2.2.6]; so by the dualized version of [L09, 2.7.5, (ii)⇒(a)], $q_f(\mathsf{E})$ is an isomorphism for any S-complex E.

The isomorphisms in (3.1.4) are pseudofunctorial, in that for any homomorphism $\xi\colon S \to T$ of commutative rings, with corresponding scheme-map $\operatorname{Spec} T =: V \xrightarrow{\;g\;} X := \operatorname{Spec} S$, the following natural diagram commutes:

$$\begin{array}{ccccc}
\mathfrak{s}_R\,\varphi_*\xi_* & = & = & = & \mathfrak{s}_R(\xi\varphi)_* \\
\downarrow{\scriptstyle \upsilon_\varphi} & & ① & & \downarrow{\scriptstyle \upsilon_{\xi\varphi}} \\
f_*\mathfrak{s}_S\,\xi_* & \xrightarrow{f_*\upsilon_\xi} & f_*g_*\mathfrak{s}_T & \xrightarrow{\;\sim\;} & (fg)_*\mathfrak{s}_T \\
\downarrow{\scriptstyle q_f} & & \downarrow & & \\
\mathsf{R}f_*\mathfrak{s}_S\,\xi_* & ② & & ③ & \downarrow{\scriptstyle q_{fg}} \\
\downarrow{\scriptstyle \mathsf{R}f_*\upsilon_\xi} & & & & \\
\mathsf{R}f_*g_*\mathfrak{s}_T & \underset{\mathsf{R}f_*q_g}{\longrightarrow} & \mathsf{R}f_*\mathsf{R}g_*\mathfrak{s}_T & \xrightarrow{\;\sim\;} & \mathsf{R}(fg)_*\mathfrak{s}_T
\end{array} \tag{3.1.6}$$

Commutativity of ① (i.e., pseudofunctoriality of υ) is readily checked via application of the equivalence Γ_Y and use of (3.1.5), or otherwise; that of ② is clear; and that of ③ is rudimentary, see. e.g., [L09, (3.6.4.1)].

3.1.7. (The reader is advised to proceed directly to section 3.1.10, returning here only as needed).

The functor $\mathsf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ has a monoidal structure (see [L09, Definition 3.4.2]), given by the natural composite map

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \xrightarrow{\;\sim\;} \mathsf{R}f_*\mathcal{O}_X$$

and by the natural bifunctorial map (see e.g., [L09, 3.2.4(ii)])

$$\mathsf{R}f_*E \otimes^{\mathsf{L}}_Y \mathsf{R}f_*F \longrightarrow \mathsf{R}f_*(E \otimes^{\mathsf{L}}_X F) \qquad (E, F \in \mathbf{D}(X)).$$

The functor $\varphi_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ has an analogous monoidal structure.

The exact functor $\mathfrak{s}_S$ also has a monoidal structure. For its description, recall that for every S-complex G, there is a quasi-isomorphism $\pi_{\mathsf{G}}\colon \bar{\mathsf{G}} \to \mathsf{G}$ such that $\bar{\mathsf{G}}$ is a direct limit of bounded-above flat S-complexes, and so is K-flat (see [L09, Prop. 2.5.5] and its proof). Then $\mathfrak{s}_S\bar{\mathsf{G}}$ is a direct limit of bounded-above flat $\mathcal{O}_X$-complexes, and so is a K-flat $\mathcal{O}_X$-complex; and $\mathfrak{s}_S\pi_{\mathsf{G}}$ is a quasi-isomorphism. Therefore, one can declare the said monoidal structure on $\mathfrak{s}_S$ to be given by the natural isomorphism $\mathcal{O}_X \xrightarrow{\;\sim\;} \mathfrak{s}_S S$ and by the unique bifunctorial $\mathbf{D}(X)$-isomorphism (see [L09, 2.6.5(ii)], dualized)

$$\mathfrak{s}_S\mathsf{E} \otimes^{\mathsf{L}}_X \mathfrak{s}_S\mathsf{F} \xrightarrow{\;\sim\;} \mathfrak{s}_S(\mathsf{E} \otimes^{\mathsf{L}}_S \mathsf{F}) \qquad (\mathsf{E}, \mathsf{F} \in \mathbf{D}(S)) \tag{3.1.8}$$

that makes the following otherwise natural diagram commute for all E, F:

$$\begin{array}{ccc}
\mathfrak{s}_S\mathsf{E} \otimes^{\mathsf{L}}_X \mathfrak{s}_S\mathsf{F} & \xrightarrow{\quad\sim\quad} & \mathfrak{s}_S(\mathsf{E} \otimes^{\mathsf{L}}_S \mathsf{F}) \\
\big\downarrow & & \big\downarrow \\
\mathfrak{s}_S\mathsf{E} \otimes_X \mathfrak{s}_S\mathsf{F} & \xrightarrow{\quad\sim\quad} & \mathfrak{s}_S(\mathsf{E} \otimes_S \mathsf{F})
\end{array}$$

Similar remarks apply to the functor $\mathfrak{s}_R$.

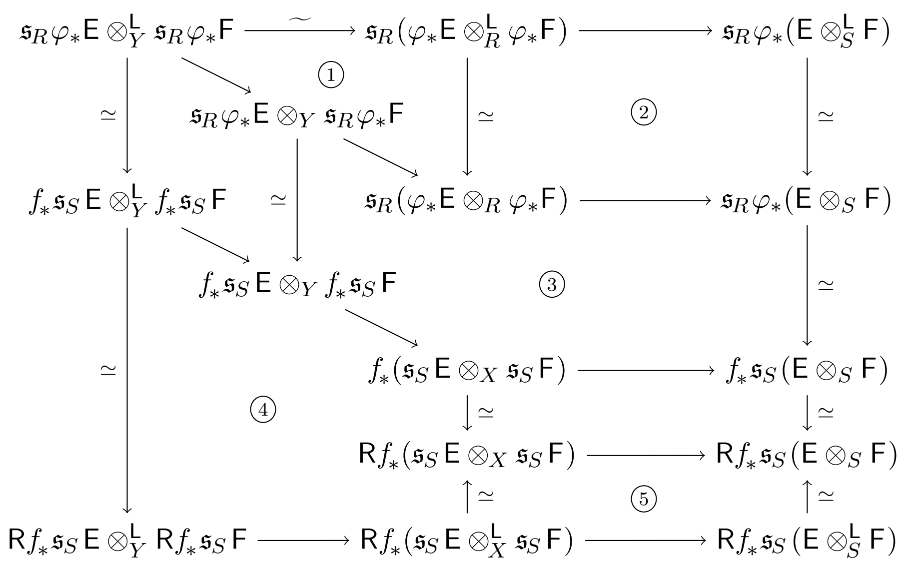
The isomorphism $\upsilon_\varphi\colon \mathfrak{s}_R\varphi_* \xrightarrow{\;\sim\;} \mathsf{R}f_*\mathfrak{s}_S$ from (3.1.5)′ is compatible with the monoidal structures on the functors involved:

Lemma 3.1.9. *For* E, $\mathsf{F} \in \mathbf{D}(S)$, *the following natural diagram commutes*:

$$\begin{array}{ccccc}
\mathfrak{s}_R\varphi_*\mathsf{E} \otimes^{\mathsf{L}}_Y \mathfrak{s}_R\varphi_*\mathsf{F} & \xrightarrow{\;\sim\;} & \mathfrak{s}_R(\varphi_*\mathsf{E} \otimes^{\mathsf{L}}_R \varphi_*\mathsf{F}) & \longrightarrow & \mathfrak{s}_R\varphi_*(\mathsf{E} \otimes^{\mathsf{L}}_S \mathsf{F}) \\
\simeq\big\downarrow{\scriptstyle \upsilon_\varphi \otimes^{\mathsf{L}}_Y \upsilon_\varphi} & & & & {\scriptstyle \upsilon_\varphi}\big\downarrow\simeq \\
\mathsf{R}f_*\mathfrak{s}_S\mathsf{E} \otimes^{\mathsf{L}}_Y \mathsf{R}f_*\mathfrak{s}_S\mathsf{F} & \longrightarrow & \mathsf{R}f_*(\mathfrak{s}_S\mathsf{E} \otimes^{\mathsf{L}}_X \mathfrak{s}_S\mathsf{F}) & \xrightarrow{\;\sim\;} & \mathsf{R}f_*\mathfrak{s}_S(\mathsf{E} \otimes^{\mathsf{L}}_S \mathsf{F})
\end{array}$$

Proof. In view of the above quasi-isomorphisms $\pi_{\mathsf{E}}\colon \bar{\mathsf{E}} \to \mathsf{E}$ and $\pi_{\mathsf{F}}\colon \bar{\mathsf{F}} \to \mathsf{F}$, one can assume that all the complexes E, F, $\mathfrak{s}_S\mathsf{E}$ and $\mathfrak{s}_S\mathsf{F}$ are K-flat.

It's enough then to prove commutativity of the natural diagram

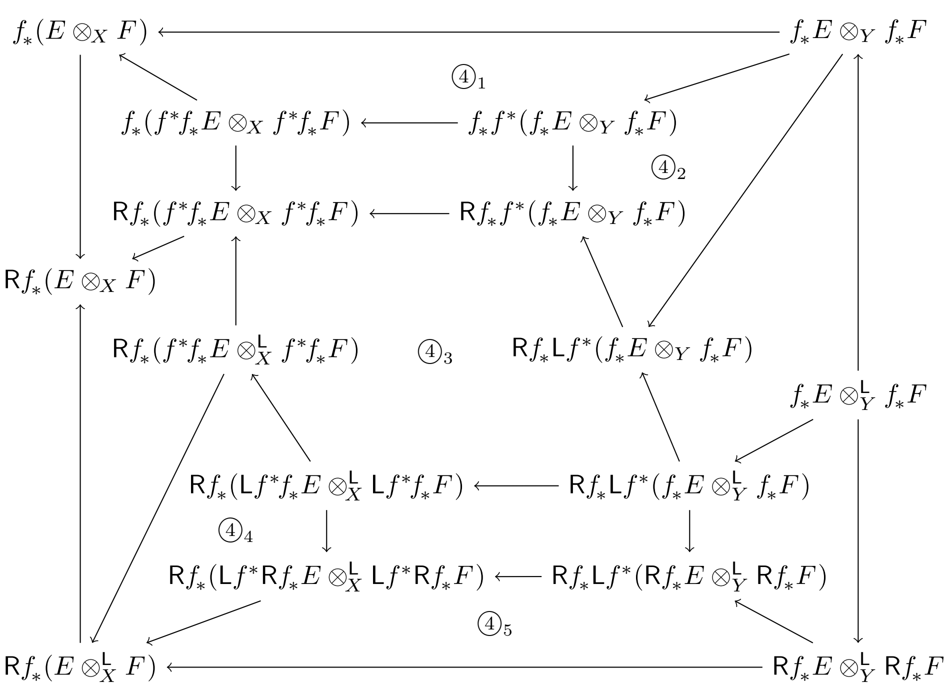


The commutativity of the unlabeled subdiagrams is clear.

Subdiagrams ① and ⑤ commute by the description of the map (3.1.8).

The commutativity of ③ can be easily be checked after application of the equivalence Γ_X.

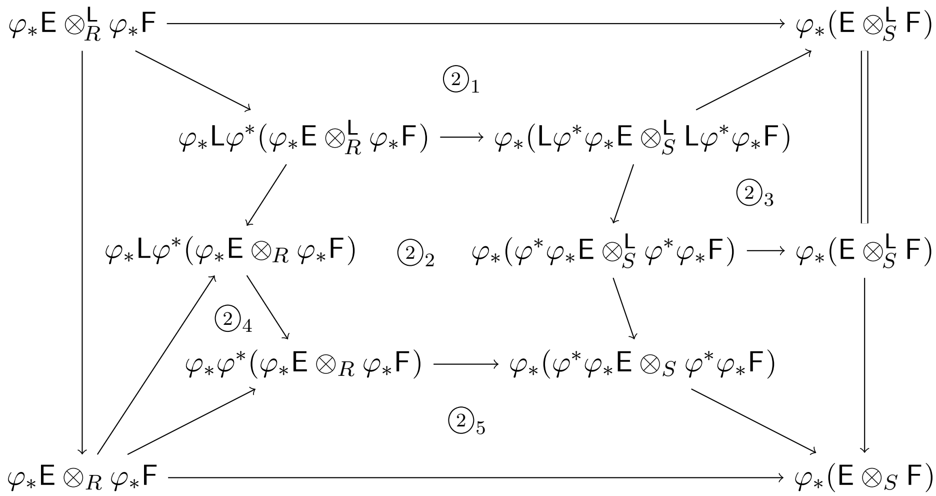
Subdiagram ④ expands naturally as follows, with $E := \mathfrak{s}_S\,\mathsf{E}$ and $F := \mathfrak{s}_S\,\mathsf{F}$:



The commutativity of the unlabeled subdiagrams is clear.

The commutativity of subdiagram ④$_1$ is given, e.g., by [L09, (3.4.5.2)] (taking into account *ibid.,* 3.1.9 and 3.4.4(a)). That of ④$_5$ holds by definition of its bottom arrow [L09, 3.2.4(ii)]; of ④$_2$ by that of [L09, (3.2.1.3)]; of ④$_4$ by that of [L09, (3.2.1.2)]; and of ④$_3$ by that of [L09, (3.2.4.1)].

Finally, subdiagram ② without "$\mathfrak{s}_R$" expands naturally as follows:



The commutativity of the unlabeled subdiagrams is clear. Subdiagram ②$_1$ commutes by definition, cf. [L09, 3.2.4(ii)]. Checking the commutativity of its nonderived version ②$_5$ is left to the reader. The commutativity of ②$_3$ (respectively ②$_4$) is given by that of [L09, (3.2.1.2)] (respectively [L09, (3.2.1.3)]. The commutativity of ②$_2$ is given by the commutative-algebra counterpart (proved similarly) of [L09, (3.2.4.1)]. Thus ② commutes.

This concludes the proof of Lemma 3.1.9. □

3.1.10. With $^{\mathsf{op}}$ denoting "opposite category," the functor

$$\mathrm{Hom}_\varphi\colon \mathcal{A}(S)^{\mathsf{op}}\times\mathcal{A}(R)\to\mathcal{A}(S)$$

is given by

$$\mathrm{Hom}_\varphi(\mathsf{E},\mathsf{G}) := \mathrm{Hom}_R(\varphi_*\mathsf{E},\mathsf{G}) \qquad \big(\mathsf{E}\in\mathcal{A}(S),\ \mathsf{G}\in\mathcal{A}(R)\big),$$

$\mathrm{Hom}_R(\varphi_*\mathsf{E},\mathsf{G})$ being an S-module in the usual way (cf. §2.2).

The proof of the next proposition and its corollaries, being similar to that of Proposition 2.2.1 and its corollaries, is left to the reader.

Proposition 3.1.11. *There is a unique trifunctorial* $\mathbf{D}(S)$*-isomorphism*

$$\bar\alpha(\mathsf{E},\mathsf{F},\mathsf{G})\colon \mathsf{R}\mathrm{Hom}_\varphi(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{F},\mathsf{G}) \xrightarrow{\ \sim\ } \mathsf{R}\mathrm{Hom}_S(\mathsf{E},\mathsf{R}\mathrm{Hom}_\varphi(\mathsf{F},\mathsf{G})) \qquad \big(\mathsf{E},\mathsf{F}\in\mathbf{D}(S),\ \mathsf{G}\in\mathbf{D}(R)\big)$$

such that the following natural diagram, with $\mathrm{H}:=\mathrm{Hom}$ *and* $\bar\alpha_0(\mathsf{E},\mathsf{F},\mathsf{G})$ *the standard isomorphism of* S*-complexes, commutes.*

$$\begin{array}{ccccc} \mathrm{H}_\varphi(\mathsf{E}\otimes_S\mathsf{F},\mathsf{G}) & \longrightarrow & \mathsf{R}\mathrm{H}_\varphi(\mathsf{E}\otimes_S\mathsf{F},\mathsf{G}) & \longrightarrow & \mathsf{R}\mathrm{H}_\varphi(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{F},\mathsf{G}) \\ {\scriptstyle\bar\alpha_0(\mathsf{E},\mathsf{F},\mathsf{G})}\downarrow\simeq & & & & \simeq\downarrow{\scriptstyle\bar\alpha(\mathsf{E},\mathsf{F},\mathsf{G})} \\ \mathrm{H}_S\big(\mathsf{E},\mathrm{H}_\varphi(\mathsf{F},\mathsf{G})\big) & \longrightarrow & \mathsf{R}\mathrm{H}_S\big(\mathsf{E},\mathrm{H}_\varphi(\mathsf{F},\mathsf{G})\big) & \longrightarrow & \mathsf{R}\mathrm{H}_S\big(\mathsf{E},\mathsf{R}\mathrm{H}_\varphi(\mathsf{F},\mathsf{G})\big) \end{array}$$

Corollary 3.1.12. *There is a unique trifunctorial* $\mathbf{D}(R)$*-isomorphism*

$$\bar\alpha_\varphi(\mathsf{E},\mathsf{F},\mathsf{G})\colon \mathsf{R}\mathrm{Hom}_R\big(\varphi_*(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{F}),\mathsf{G}\big) \xrightarrow{\ \sim\ } \varphi_*\mathsf{R}\mathrm{Hom}_S(\mathsf{E},\mathsf{R}\mathrm{Hom}_\varphi(\mathsf{F},\mathsf{G})\big) \qquad \big(\mathsf{E},\mathsf{F}\in\mathbf{D}(S),\ \mathsf{G}\in\mathbf{D}(R)\big)$$

such that the following natural $\mathbf{D}(R)$*-diagram, with* $\mathrm{H}:=\mathrm{Hom}$*, commutes.*

$$\begin{array}{ccccc} \mathrm{H}_R\big(\varphi_*(\mathsf{E}\otimes_S\mathsf{F}),\mathsf{G}\big) & \longrightarrow & \mathsf{R}\mathrm{H}_R\big(\varphi_*(\mathsf{E}\otimes_S\mathsf{F}),\mathsf{G}\big) & \longrightarrow & \mathsf{R}\mathrm{H}_R\big(\varphi_*(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{F}),\mathsf{G}\big) \\ {\scriptstyle\varphi_*\bar\alpha_0(\mathsf{E},\mathsf{F},\mathsf{G})}\downarrow\simeq & & & & \simeq\downarrow{\scriptstyle\bar\alpha_\varphi(\mathsf{E},\mathsf{F},\mathsf{G})} \\ \varphi_*\mathrm{H}_S\big(\mathsf{E},\mathrm{H}_\varphi(\mathsf{F},\mathsf{G})\big) & \longrightarrow & \varphi_*\mathsf{R}\mathrm{H}_S\big(\mathsf{E},\mathrm{H}_\varphi(\mathsf{F},\mathsf{G})\big) & \longrightarrow & \varphi_*\mathsf{R}\mathrm{H}_S\big(\mathsf{E},\mathsf{R}\mathrm{H}_\varphi(\mathsf{F},\mathsf{G})\big) \end{array}$$

This entails the functorial R*-isomorphism*

$$\mathrm{H}^0\bar\alpha_\varphi(\mathsf{E},\mathsf{F},\mathsf{G})\colon \mathrm{Hom}_{\mathbf{D}(R)}\big(\varphi_*(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{F}),\mathsf{G}\big) \xrightarrow{\ \sim\ } \varphi_*\mathrm{Hom}_{\mathbf{D}(S)}\big(\mathsf{E},\mathsf{R}\mathrm{Hom}_\varphi(\mathsf{F},\mathsf{G})\big).$$

Let

$$\varphi^{\#}\colon \mathbf{D}(R)\to\mathbf{D}(S)$$

be the functor $\mathsf{R}\mathrm{Hom}_\varphi(S,-)$.

Corollary 3.1.13. *For* $\mathsf{E}\in\mathbf{D}(S)$ *and* $\mathsf{G}\in\mathbf{D}(R)$*, one has the bifunctorial* $\mathbf{D}(R)$*-isomorphism*

$$\bar\alpha_\varphi(\mathsf{E},S,\mathsf{G})\colon \mathsf{R}\mathrm{Hom}_R(\varphi_*\mathsf{E},\mathsf{G}) \xrightarrow{\ \sim\ } \varphi_*\mathsf{R}\mathrm{Hom}_S(\mathsf{E},\varphi^{\#}\mathsf{G}),$$

In particular, there is an adjunction

$$\varphi_*\dashv\varphi^{\#}, \tag{3.1.14}$$

given by the functorial R*-isomorphism*

$$\mathrm{H}^0\bar\alpha_\varphi(\mathsf{E},S,\mathsf{G})\colon\ \mathrm{Hom}_{\mathbf{D}(R)}\big(\varphi_*\mathsf{E},\mathsf{G}\big) \xrightarrow{\ \sim\ } \varphi_*\mathrm{Hom}_{\mathbf{D}(S)}\big(\mathsf{E},\varphi^{\#}\mathsf{G}\big).$$

Here, the counit at $\mathsf{G}\in\mathbf{D}(R)$ is the natural composite map

$$\mathsf{t}_{\varphi,\mathsf{G}}\colon \varphi_*\varphi^{\#}\mathsf{G} \xrightarrow{\ \sim\ } \mathsf{R}\mathrm{Hom}_R(\varphi_*S,\mathsf{G}) \longrightarrow \mathsf{R}\mathrm{Hom}_R(R,\mathsf{G}) \xrightarrow{\ \sim\ } \mathsf{G}; \tag{3.1.15}$$

and the unit at $\mathsf{E}\in\mathbf{D}(S)$ is the natural composite map

$$\mathsf{u}_{\varphi,\mathsf{E}}\colon \mathsf{E} \xrightarrow{\ \sim\ } \mathsf{R}\mathrm{Hom}_S(S,\mathsf{E}) \longrightarrow \mathsf{R}\mathrm{Hom}_\varphi(S,\varphi_*\mathsf{E}) = \varphi^{\#}\varphi_*\mathsf{E},$$

i.e., the natural composite map

$$\mathsf{E} \xrightarrow{\eta'_{\mathsf{E}}} \mathrm{Hom}_{\varphi}(S, \varphi_*\mathsf{E}) \longrightarrow \mathsf{R}\mathrm{Hom}_{\varphi}(S, \varphi_*\mathsf{E}) = \varphi^{\#}\varphi_*\mathsf{E}$$

where for $e \in \mathsf{E}$, $\eta'_{\mathsf{E}}e$ is the R-homomorphism taking $s \in S$ to se. (Cf. 2.2.8.)

Arguing as in the proof of Proposition 2.4.4, one gets:

Proposition 3.1.16. *The inverse of the above isomorphism* $\bar\alpha_{\varphi}(\mathsf{E}, S, \mathsf{G})$ *factors naturally as*

$$\varphi_*\mathsf{R}\mathrm{Hom}_S(\mathsf{E}, \varphi^{\#}\mathsf{G}) \longrightarrow \mathsf{R}\mathrm{Hom}_R(\varphi_*\mathsf{E}, \varphi_*\varphi^{\#}\mathsf{G}) \longrightarrow \mathsf{R}\mathrm{Hom}_R(\varphi_*\mathsf{E}, \mathsf{G}).$$

The equivalences $\mathfrak{s}$ and $\mathsf{R}\Gamma$ (section 3.1) transform $\varphi^{\#}$ into a right adjoint of $\mathsf{R}f_*$, as follows.

One has, for $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and $G \in \mathbf{D}(Y)$, the natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}_{\mathsf{qc}}(X)}(E, \mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y G) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S)}(\mathsf{R}\Gamma_X E, \varphi^{\#}\mathsf{R}\Gamma_Y G)\\
&\underset{(3.1.14)}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathsf{R}\Gamma_X E, \mathsf{R}\Gamma_Y G)\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathfrak{s}_R\varphi_*\mathsf{R}\Gamma_X E, G)\\
&\underset{(3.1.4)}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*\mathfrak{s}_S\mathsf{R}\Gamma_X E, G)\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_* E, G).
\end{aligned} \tag{3.1.17}$$

Hence:

Proposition 3.1.18. (Cf. 2.11.2.) *The above-defined functor*

$$\mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y \colon \mathbf{D}(Y) \to \mathbf{D}_{\mathsf{qc}}(X)$$

is right-adjoint to $\mathsf{R}f_*$, *with unit at* $E \in \mathbf{D}_{\mathsf{qc}}(X)$ *the natural composite map*

$$\begin{aligned}
E \xrightarrow{\sim} \mathfrak{s}_S\mathsf{R}\Gamma_X E &\longrightarrow \mathfrak{s}_S\varphi^{\#}\varphi_*\mathsf{R}\Gamma_X E\\
&\xrightarrow{\sim} \mathfrak{s}_S\varphi^{\#}\mathsf{R}(\varphi_*\Gamma_X)E\\
&= \mathfrak{s}_S\varphi^{\#}\mathsf{R}(\Gamma_Y f_*)E \xrightarrow{\sim} \mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y\mathsf{R}f_*E
\end{aligned}$$

and counit at $G \in \mathbf{D}(Y)$ *the natural composite map*

$$G \longleftarrow \mathfrak{s}_R\mathsf{R}\Gamma_Y G \longleftarrow \mathfrak{s}_R\varphi_*\varphi^{\#}\mathsf{R}\Gamma_Y G \underset{(3.1.4)}{\xleftarrow{\sim}} \mathsf{R}f_*\mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y G.$$

Proof. The first assertion results at once from (3.1.17). Verifying that the unit and counit are as stated is straightforward (if slightly tedious). □

3.1.19. *Till this section 3 ends, set* $f^{\flat}G := \mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y G$ $(G \in \mathbf{D}(Y))$.

Proposition 3.1.18 shows that this is consistent with the previous meaning of $f^{\flat}G$ when G is as in Proposition 2.3.5; but now $f^{\flat}$ will be right-adjoint to the *entire* functor $\mathsf{R}f_* \colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}(Y)$ for *any* map f of affine schemes (cf. Corollary 2.11.2). The unit map $E \to f^{\flat}\mathsf{R}f_*E$ and the counit map $\mathsf{R}f_*f^{\flat}G \to G$ are then as in Proposition 3.1.18.

3.1.20. As for pseudofunctoriality, given ring homomorphisms $R \xrightarrow{\varphi} S \xrightarrow{\xi} T$, with corresponding scheme-maps $W \xrightarrow{g} X \xrightarrow{f} Y$, define (abstractly) the functorial isomorphism

$$\pi_{\xi,\varphi}\colon \xi^{\#}\varphi^{\#} \xrightarrow{\sim} (\xi\varphi)^{\#} \tag{3.1.21}$$

to be, analogously to (2.5.1), naturally right-conjugate to the identity map $(\xi\varphi)_* = \varphi_*\xi_*$. (For a concrete realization, see Proposition 3.1.23.) One checks that such π make $(-)^{\#}$ into a *contravariant pseudofunctor*.

It follows that $(-)^{\flat}$ is made into a *pseudofunctorial* right adjoint of $\mathsf{R}(-)_*$ by composite natural isomorphisms of the form

$$\pi'_{\xi,\varphi}\colon g^{\flat}f^{\flat} = \mathfrak{s}_T\xi^{\#}\mathsf{R}\Gamma_X\mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y \xrightarrow{\sim} \mathfrak{s}_T\xi^{\#}\varphi^{\#}\mathsf{R}\Gamma_Y \underset{\text{via }\pi_{\xi,\varphi}}{\xrightarrow{\sim}} \mathfrak{s}_T(\xi\varphi)^{\#}\mathsf{R}\Gamma_Y = (fg)^{\flat}.$$

One has then a natural commutative diagram of isomorphisms

$$\begin{array}{ccccc} \mathfrak{s}_T\xi^{\#}\mathsf{R}\Gamma_X\mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y & \xrightarrow{\sim} & \mathfrak{s}_T\xi^{\#}\varphi^{\#}\mathsf{R}\Gamma_Y & \underset{\mathfrak{s}_T(\pi_{\xi,\varphi})}{\xrightarrow{\sim}} & \mathfrak{s}_T(\xi\varphi)^{\#}\mathsf{R}\Gamma_Y \\ \simeq\downarrow & & & & \downarrow\simeq \\ g^{\flat}f^{\flat} & & \underset{(2.5.1)}{\xrightarrow{\sim}} & & (fg)^{\flat} \end{array} \tag{3.1.22}$$

Thus the next proposition, which contains a concrete realization of $\pi_{\xi,\varphi}$, entails, for pseudo-coherent maps of affine schemes, a concrete realization of the pseudofunctoriality isomorphism $g^{\flat}f^{\flat} \xrightarrow{\sim} (fg)^{\flat}$ in (2.5.1).

Proposition 3.1.23. *The* $\mathbf{D}(T)$*-isomorphism*

$$\pi_{\xi,\varphi}(\mathsf{G})\colon \mathsf{R}\mathrm{Hom}_{\xi}(T, \mathsf{R}\mathrm{Hom}_{\varphi}(S,\mathsf{G})) \xrightarrow{\sim} \mathsf{R}\mathrm{Hom}_{\xi\varphi}(T,\mathsf{G}) \qquad (\mathsf{G}\in\mathbf{D}(R))$$

is the unique functorial map α *such that the following natural diagram commutes for all* R*-complexes* G*.*

$$\begin{array}{ccc} \mathrm{Hom}_{\xi}(T,\mathrm{Hom}_{\varphi}(S,\mathsf{G})) & \longrightarrow & \mathrm{Hom}_{\xi\varphi}(T,\mathsf{G}) \\ \downarrow & & \downarrow \\ \mathsf{R}\mathrm{Hom}_{\xi}(T,\mathsf{R}\mathrm{Hom}_{\varphi}(S,\mathsf{G})) & \underset{\alpha(\mathsf{G})}{\longrightarrow} & \mathsf{R}\mathrm{Hom}_{\xi\varphi}(T,\mathsf{G}) \end{array}$$

Proof. It suffices to verify commutativity of the second diagram in [L09, 3.3.7(a)], with β the identity map of $(\xi\varphi)_* = \varphi_*\xi_*$.

For this purpose, one may assume that the R-complex G is K-injective, whence so is the S-complex $\mathrm{Hom}_{\varphi}(S,\mathsf{G})$. Hence the assertion follows from commutativity of the following natural diagram of R-complexes.

$$\begin{array}{ccc}
(\xi\varphi)_*\operatorname{Hom}_\xi(T,\operatorname{Hom}_\varphi(S,\mathsf{G})) & \longrightarrow & (\xi\varphi)_*\operatorname{Hom}_{\xi\varphi}(T,\mathsf{G}) \\
\| & & \| \\
\varphi_*\operatorname{Hom}_S(\xi_*T,\operatorname{Hom}_\varphi(S,\mathsf{G})) & & \operatorname{Hom}_R((\xi\varphi)_*T,\mathsf{G}) \\
\downarrow & & \downarrow \\
\varphi_*\operatorname{Hom}_S(S,\operatorname{Hom}_\varphi(S,\mathsf{G})) & & \\
\simeq\downarrow & & \\
\varphi_*\operatorname{Hom}_\varphi(S,\mathsf{G}) = \operatorname{Hom}_R(\varphi_*S,\mathsf{G}) & \longrightarrow & \operatorname{Hom}_R(R,\mathsf{G}) = \mathsf{G}
\end{array}$$

To verify this commutativity, one checks that that, by traveling around the diagram *either* clockwise *or* counterclockwise from upper left to lower right, an S-homomorphism $\lambda\colon \xi_*T \to \operatorname{Hom}_\varphi(S,\mathsf{G})$ goes to $[\lambda(1_T)](1_S)$. □

Recall, from 3.1.11, the map $\bar\alpha(\mathsf{E},\mathsf{F},\mathsf{G})$ $\big(\mathsf{E},\mathsf{F}\in\mathbf{D}(S),\ \mathsf{G}\in\mathbf{D}(R)\big)$.

Corollary 3.1.24. *It holds that* $\xi_*\pi_{\xi,\varphi}(\mathsf{G}) = \bar\alpha(\xi_*T,S,\mathsf{G})$. □

3.2. This section is devoted to a description of the (concrete) commutative-algebra version of the functorial map β_σ in Theorem 2.6.4.

Let

$$\begin{array}{ccc}
S' & \xleftarrow{\ \nu\ } & S \\
{\scriptstyle\xi}\uparrow & \hat\sigma & \uparrow{\scriptstyle\varphi} \\
R' & \xleftarrow[\ \mu\ ]{} & R
\end{array}
\qquad\qquad
\begin{array}{ccc}
X' & \xrightarrow{\ v\ } & X \\
{\scriptstyle g}\downarrow & \sigma & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\ u\ ]{} & Y
\end{array}$$

be, respectively, a commutative diagram of maps of commutative rings and the corresponding commutative diagram of maps of affine schemes.

The functor

$$\varphi^*(-) := S\otimes_\varphi - \tag{3.2.1}$$

from $\mathcal{A}(R)$ to $\mathcal{A}(S)$ (notation akin to that in §2.2.9) is left-adjoint to φ_*, with unit at $N\in\mathcal{A}(R)$ the natural map

$$\eta_\varphi(N)\colon N\to\varphi_*S\otimes_R N = \varphi_*(S\otimes_\varphi N) = \varphi_*\varphi^*N,$$

and with counit at $M\in\mathcal{A}(S)$ the scalar multiplication map

$$\epsilon_\varphi(M)\colon \varphi^*\varphi_*M = S\otimes_\varphi\varphi_*M\to M.$$

As in the proof of [L09, Proposition 3.2.1], it follows that the derived functor

$$\mathsf{L}\varphi^*(-) := S\otimes^{\mathsf{L}}_\varphi - \tag{3.2.2}$$

from $\mathbf{D}(R)$ to $\mathbf{D}(S)$ is left-adjoint to φ_*, the adjunction isomorphism

$$\alpha\colon\ \operatorname{Hom}_{\mathbf{D}(S)}(\mathsf{L}\varphi^*\mathsf{G},\mathsf{E}) \xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}(R)}(\mathsf{G},\varphi_*\mathsf{E}) \tag{3.2.3}$$

being the unique bifunctorial map making the following natural diagram, where $\mathbf{K}(T)$ is the category of homotopy classes of maps of T-complexes, commute (see [L09, Corollary 3.2.2]):

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{K}(S)}(\varphi^*\mathsf{G},\mathsf{E}) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(S)}(\varphi^*\mathsf{G},\mathsf{E}) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(S)}(\mathsf{L}\varphi^*\mathsf{G},\mathsf{E}) \\ \big\downarrow & & & & \simeq\big\downarrow\alpha \\ \mathrm{Hom}_{\mathbf{K}(R)}(\mathsf{G},\varphi_*\mathsf{E}) & & \longrightarrow & & \mathrm{Hom}_{\mathbf{D}(R)}(\mathsf{G},\varphi_*\mathsf{E}) \end{array}$$

In this adjunction, the unit at G is the natural $\mathbf{D}(R)$-composition

$$\bar\eta_\varphi(\mathsf{G})\colon \mathsf{G} \xrightarrow{\ \sim\ } \mathsf{G}' \xrightarrow{\text{via}\,\eta} \varphi_*\varphi^*\mathsf{G}' \xrightarrow{\ \sim\ } \varphi_*\mathsf{L}\varphi^*\mathsf{G}$$

for any K-flat resolution $\mathsf{G}' \to \mathsf{G}$ (i.e., R-quasi-isomorphism with G' K-flat); and the counit at E is the natural $\mathbf{D}(S)$-composition

$$\bar\epsilon_\varphi(\mathsf{E})\colon \mathsf{L}\varphi^*\varphi_*\mathsf{E} \longrightarrow \varphi^*\varphi_*\mathsf{E} \xrightarrow{\epsilon_\varphi(\mathsf{E})} \mathsf{E}.$$

The functorial $\mathbf{D}(R')$-map

$$\theta_{\hat\sigma}(\mathsf{E})\colon \mathsf{L}\mu^*\varphi_*\mathsf{E} \longrightarrow \xi_*\mathsf{L}\nu^*\mathsf{E} \qquad (\mathsf{E}\in\mathbf{D}(S))$$

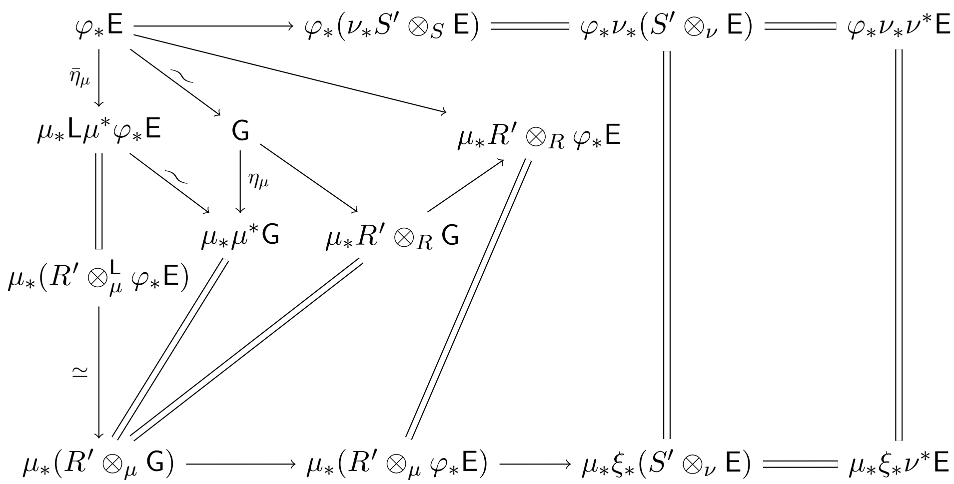
is defined (abstractly) to be the adjoint of the $\mathbf{D}(R)$-map

$$\varphi_*\bar\eta_\nu(\mathsf{E})\colon \varphi_*\mathsf{E} \longrightarrow \varphi_*\nu_*\mathsf{L}\nu^*\mathsf{E} = \mu_*\xi_*\mathsf{L}\nu^*\mathsf{E}. \tag{3.2.4}$$

In terms that are explicit—up to choices of a K-flat S-resolution $\mathsf{E}' \to \mathsf{E}$ and a K-flat R-resolution $\mathsf{G} \to \varphi_*\mathsf{E}'$—one can describe $\theta_{\hat\sigma}(\mathsf{E})$ as being the natural composite $\mathbf{D}(R')$-map

$$\begin{aligned} \mathsf{L}\mu^*\varphi_*\mathsf{E} &= R'\otimes^{\mathsf{L}}_{\mu}\varphi_*\mathsf{E} \xrightarrow{\ \sim\ } R'\otimes^{\mathsf{L}}_{\mu}\varphi_*\mathsf{E}' \\ &\xrightarrow{\ \sim\ } R'\otimes_{\mu}\mathsf{G} \longrightarrow R'\otimes_{\mu}\varphi_*\mathsf{E}' \longrightarrow \xi_*(S'\otimes_{\nu}\mathsf{E}') \\ &\xrightarrow{\ \sim\ } \xi_*\mathsf{L}\nu^*\mathsf{E}. \end{aligned}$$

Indeed, it suffices to consider the case $\mathsf{E}' = \mathsf{E}$, where, with $\mathsf{G} \to \varphi_*\mathsf{E}$ the preceding K-flat resolution, one finds readily that this assertion results from the commutativity (straightforward to verify) of the natural $\mathbf{D}(R)$-diagram



3.2.5. It will be shown next that for any $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and $\mathsf{E} := \mathsf{R}\Gamma_X E$ (so that $E \cong \mathfrak{s}_S\mathsf{E}$, see end of §3.1), *the sheafification* $\mathfrak{s}_{R'}\theta_{\hat\sigma}(\mathsf{E})$ *is naturally isomorphic to the map* $\theta_\sigma(E)\colon \mathsf{L}u^*\mathsf{R}f_*E \to \mathsf{R}g_*\mathsf{L}v^*E$ *in* (2.6.3).

This follows from [L09, Example 3.10.1(a)], but here a somewhat different argument will be given, whose motivation is that sheafification preserves adjointness of maps and that the sheafification of the adjoint (3.2.4) of $\theta_{\hat\sigma}(\mathsf{E})$ is naturally isomorphic to the natural composite map

$$\mathsf{R}f_*E \longrightarrow \mathsf{R}f_*\mathsf{R}v_*\mathsf{L}v^*E \xrightarrow{\,\sim\,} \mathsf{R}u_*\mathsf{R}g_*\mathsf{L}v^*E,$$

which is, by definition, the adjoint of θ_σ.

The precise formulation—Proposition 3.2.10 and its proof—needs some preliminaries.

The standard functorial $\mathcal{O}_X$-isomorphism, for R-complexes G,

$$\varsigma(\mathsf{G}) = \varsigma_\varphi(\mathsf{G})\colon f^*\mathfrak{s}_R\,\mathsf{G} \xrightarrow{\,\sim\,} \mathfrak{s}_S\,\varphi^*\mathsf{G}, \tag{3.2.6}$$

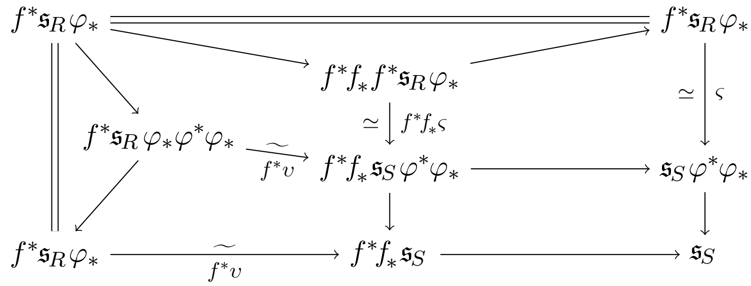
see [GrD71, 1.7.7(i)], is adjoint to the natural composite map

$$\mathfrak{s}_R\,\mathsf{G} \longrightarrow \mathfrak{s}_R\,\varphi_*\varphi^*\mathsf{G} \xrightarrow[\upsilon(\varphi^*\mathsf{G})]{\,\sim\,} f_*\mathfrak{s}_S\,\varphi^*\mathsf{G}$$

with $\upsilon\colon \mathfrak{s}_R\,\varphi_* \xrightarrow{\,\sim\,} f_*\mathfrak{s}_S$ the functorial isomorphism in (3.1.5)′. In other words, the following natural functorial diagram commutes:

$$\begin{array}{ccc} \mathfrak{s}_R & \longrightarrow & f_*f^*\mathfrak{s}_R \\ \big\downarrow & & \simeq\big\downarrow f_*\varsigma \\ \mathfrak{s}_R\,\varphi_*\varphi^* & \xrightarrow[\upsilon]{\,\sim\,} & f_*\mathfrak{s}_S\,\varphi^* \end{array} \tag{3.2.7}$$

There results commutativity of the natural diagram



whose border can be represented as the (commutative) natural diagram

$$\begin{array}{ccc} \mathfrak{s}_S & \longleftarrow & f^*f_*\mathfrak{s}_S \\ \big\uparrow & & \simeq\big\uparrow f^*\upsilon \\ \mathfrak{s}_S\varphi^*\varphi_* & \xleftarrow[\varsigma]{\sim} & f^*\mathfrak{s}_R\varphi_* \end{array} \tag{3.2.8}$$

As for derived versions of the foregoing, upon replacing G by a quasi-isomorphic direct limit of bounded-above flat R-complexes (see the remarks preceding (3.1.8)), one gets a natural functorial $\mathbf{D}(X)$-isomorphism

$$\bar\varsigma(\mathsf{G})\colon \mathsf{L}f^*\mathfrak{s}_R\mathsf{G} \xrightarrow{\sim} \mathfrak{s}_S\mathsf{L}\varphi^*\mathsf{G} \qquad (\mathsf{G}\in\mathbf{D}(R)) \tag{3.2.9}$$

such that the following natural functorial diagram commutes

$$\begin{array}{ccc} \mathfrak{s}_R & \longrightarrow & \mathsf{R}f_*\mathsf{L}f^*\mathfrak{s}_R \\ \big\downarrow & & \simeq\big\downarrow \mathsf{R}f_*\bar\varsigma \\ \mathfrak{s}_R\varphi_*\mathsf{L}\varphi^* & \xrightarrow[(3.1.4)]{\sim} & \mathsf{R}f_*\mathfrak{s}_S\mathsf{L}\varphi^*, \end{array} \tag{3.2.7$'$}$$

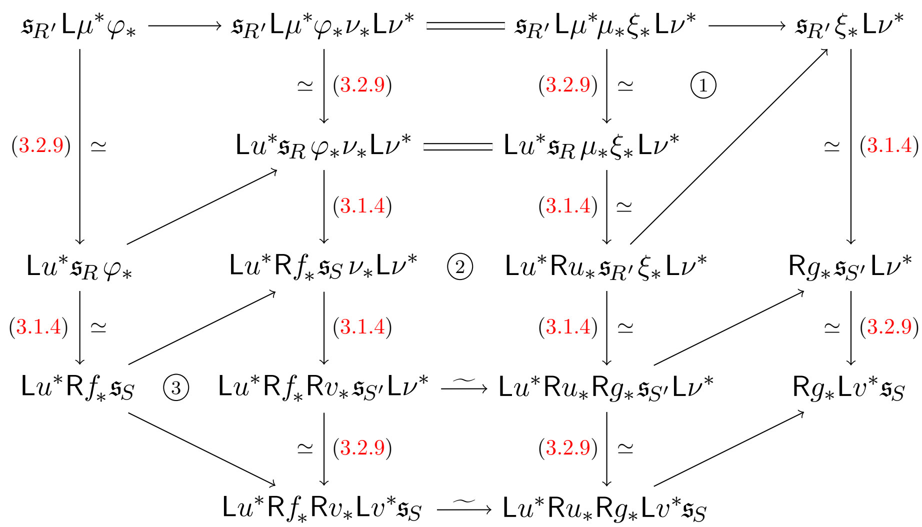
from which one deduces, as above, commutativity of the natural diagram

$$\begin{array}{ccc} \mathfrak{s}_S & \longleftarrow & \mathsf{L}f^*\mathsf{R}f_*\mathfrak{s}_S \\ \big\uparrow & & \simeq\big\uparrow (3.1.4) \\ \mathfrak{s}_S\mathsf{L}\varphi^*\varphi_* & \xleftarrow[\bar\varsigma]{\sim} & \mathsf{L}f^*\mathfrak{s}_R\varphi_* \end{array} \tag{3.2.8$'$}$$

Proposition 3.2.10. *The following diagram, in which* E *is any* S*-complex, commutes.*

$$\begin{array}{ccc} \mathfrak{s}_{R'}\mathsf{L}\mu^*\varphi_*\mathsf{E} & \xrightarrow{\mathfrak{s}_{R'}\theta_{\hat\sigma}(\mathsf{E})} & \mathfrak{s}_{R'}\xi_*\mathsf{L}\nu^*\mathsf{E} \\ (3.2.9)\big\downarrow\simeq & & \simeq\big\downarrow(3.1.4) \\ \mathsf{L}u^*\mathfrak{s}_R\varphi_*\mathsf{E} & & \mathsf{R}g_*\mathfrak{s}_{S'}\mathsf{L}\nu^*\mathsf{E} \\ (3.1.4)\big\downarrow\simeq & & \simeq\big\downarrow(3.2.9) \\ \mathsf{L}u^*\mathsf{R}f_*\mathfrak{s}_S\mathsf{E} & \xrightarrow[\theta_\sigma(\mathfrak{s}_S\mathsf{E})]{} & \mathsf{R}g_*\mathsf{L}v^*\mathfrak{s}_S\mathsf{E} \end{array}$$

Proof. Keeping in mind the adjoint (3.2.4) of $\theta_{\hat\sigma}$, expand the diagram naturally, without "E," thus:



Commutativity of the unlabeled subdiagrams is clear.

Commutativity of ① results from that of diagram (3.2.8)′, with (μ, u) in place of (φ, f).f

Commutativity of ③ results from that of diagram (3.2.7)′, with (ν, v) in place of (φ, f).

Commutativity of ② results from that of the two diagrams obtained from (3.1.6) by making the respective substitutions $(\xi, g) \mapsto (\nu, v)$ and $(\varphi, f) \mapsto (\mu, u)$. □

3.2.11. Now for the commutative-algebra version of the map β_σ in 2.6.4.

Assume that the natural $\mathbf{D}(R)$ composite $R' \otimes^{\mathsf{L}}_R S \to R' \otimes_R S \to S'$ is an isomorphism, i.e., $\mathrm{Tor}_i^R(R', S) = 0$ for all $i > 0$ and the natural map is an isomorphism $R' \otimes_R S \xrightarrow{\ \sim\ } S'$; equivalently, assume σ to be a tor-independent fiber square (see [L09, (3.10.2)(ii)′]). Thus we can, and do, identify the maps ξ and ν with the respective canonical maps $R' \to R' \otimes_R S$ and $S \to R' \otimes_R S$. Then σ is an independent square (see the remarks following (2.6.3)), so for any S-complex E the above map $\theta_\sigma(\mathfrak{s}_S \mathsf{E})$ is an isomorphism, whence, by 3.2.10 and (3.1.2), so is $\theta_{\hat\sigma}(\mathsf{E})$.

As in 2.6.4—but more generally (see 3.1.19), the $\mathbf{D}(X')$-map $\beta_\sigma(G)$ is defined for all $G \in \mathbf{D}(Y)$ to be adjoint to the natural composite map

$$\mathsf{R}g_*\mathsf{L}v^*f^\flat G \xrightarrow[\theta_\sigma^{-1}]{} \mathsf{L}u^*\mathsf{R}f_*f^\flat G \longrightarrow \mathsf{L}u^*G.$$

Let $\beta_{\hat\sigma}(\mathsf{G})$ ($\mathsf{G} \in \mathbf{D}(R)$) be the $\mathbf{D}(S')$-map adjoint to the natural composite

$$\xi_*\mathsf{L}\nu^*\varphi^\#\mathsf{G} \xrightarrow[\theta_{\hat\sigma}^{-1}]{} \mathsf{L}\mu^*\varphi_*\varphi^\#\mathsf{G} \longrightarrow \mathsf{L}\mu^*\mathsf{G},$$

that is, $\beta_{\hat\sigma}(\mathsf{G})$ is the natural composite map

$$\mathsf{L}\nu^*\varphi^{\#}\mathsf{G}\longrightarrow \xi^{\#}\xi_*\mathsf{L}\nu^*\varphi^{\#}\mathsf{G}\xrightarrow{\xi^{\#}\theta_{\hat\sigma}^{-1}}\xi^{\#}\mathsf{L}\mu^*\varphi_*\varphi^{\#}\mathsf{G}\longrightarrow \xi^{\#}\mathsf{L}\mu^*\mathsf{G},$$

a map that is concrete to the extent that up to taking K-flat and K-injective resolutions, it has previously been explicitly described. (See the lines just after (3.1.14) and (3.2.4).)

For $G\in\mathbf{D}_{\mathsf{qc}}(Y)$, $\beta_\sigma(G)$ is isomorphic to the sheafification of $\beta_{\hat\sigma}(\mathsf{R}\Gamma_Y G)$. That follows from the next proposition, with $\mathsf{G}:=\mathsf{R}\Gamma_Y G$.

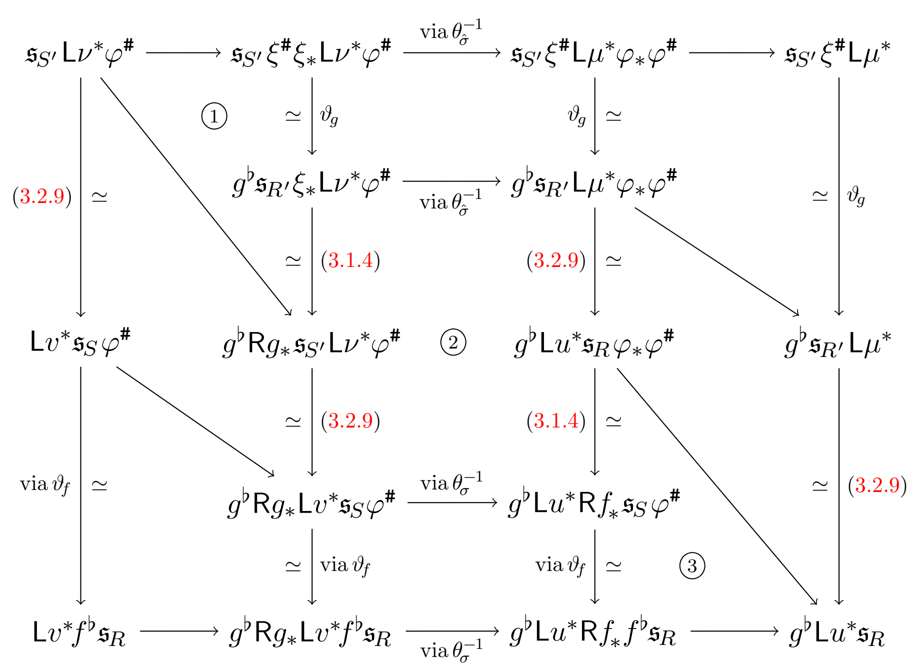
Application of $\mathfrak{s}_S\varphi^{\#}$ to the unit isomorphism (3.1.1) produces a natural functorial isomorphism

$$\vartheta_f(\mathsf{G})\colon \mathfrak{s}_S\varphi^{\#}\mathsf{G}\xrightarrow{\ \sim\ }\mathfrak{s}_S\varphi^{\#}\mathsf{R}\Gamma_Y\mathfrak{s}_R\mathsf{G}=f^\flat\mathfrak{s}_R\mathsf{G}\qquad(\mathsf{G}\in\mathbf{D}(R)).\tag{3.2.12}$$

Proposition 3.2.13. *For any $\mathsf{G}\in\mathbf{D}(R)$, the following diagram commutes.*

$$\begin{array}{ccc}
\mathfrak{s}_{S'}\mathsf{L}\nu^*\varphi^{\#}\mathsf{G} & \xrightarrow{\mathfrak{s}_{S'}\beta_{\hat\sigma}(\mathsf{G})} & \mathfrak{s}_{S'}\xi^{\#}\mathsf{L}\mu^*\mathsf{G}\\
{\scriptstyle(3.2.9)}\Big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\Big\downarrow{\scriptstyle\vartheta_g}\\
\mathsf{L}v^*\mathfrak{s}_S\varphi^{\#}\mathsf{G} & & g^\flat\mathfrak{s}_{R'}\mathsf{L}\mu^*\mathsf{G}\\
{\scriptstyle\mathsf{L}v^*\vartheta_f}\Big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\Big\downarrow{\scriptstyle(3.2.9)}\\
\mathsf{L}v^*f^\flat\mathfrak{s}_R\mathsf{G} & \xrightarrow[\beta_\sigma(\mathfrak{s}_R\mathsf{G})]{} & g^\flat\mathsf{L}u^*\mathfrak{s}_R\mathsf{G}
\end{array}$$

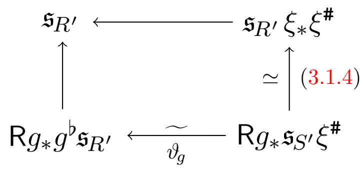
Proof. Expand the diagram naturally, without "G," thus:



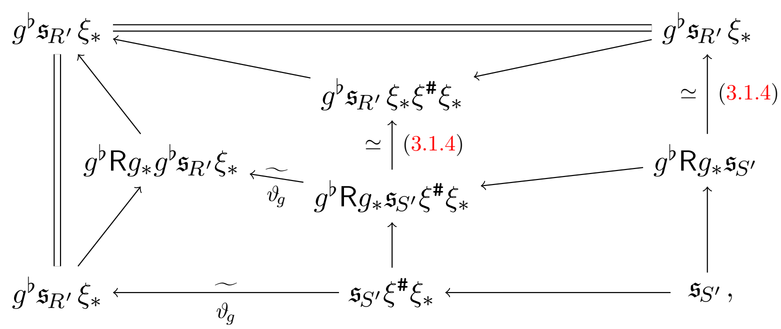
Here, commutativity of the unlabeled subdiagrams is clear.

Commutativity of ② results from Proposition 3.2.10.

Commutativity of ③, verifiable after dropping "$g^\flat \mathsf{L}u^*$," follows easily from the description of the counit map $\mathsf{R}f_*f^\flat \to \mathrm{id}$ in Proposition 3.1.18. Similarly,



commutes, whence so does



giving commutativity of ①. □

Remark 3.2.14. For $\beta_{\hat{\sigma}}(\mathsf{G})$ to be an isomorphism, further conditions are needed—for example, that φ_*S is a pseudo-coherent R-module (whence S' is a pseudo-coherent R'-module, see paragraph preceding 2.3.8), and the R-module μ_*R' has finite tor-dimension (whence the map u has finite tor-dimension), and $\mathsf{G} \in \mathbf{D}^+(R)$.

Remark 3.2.15. Paste the commutative diagrams in Propositions 3.2.13 (without "G") and 2.6.14 (with "G" replaced by "$\mathfrak{s}_R$") along their common edge to get the following natural diagram, the commutativity of whose border expresses the relation between the concrete realizations of β_σ that are indicated in those two places. Conversely, a more concrete proof—avoiding adjoints—that the border commutes would, together with 2.6.14, give 3.2.13.

$$
\begin{array}{ccc}
\mathfrak{s}_{S'}\mathsf{L}\nu^*\varphi^{\#} & \xrightarrow{\ \mathfrak{s}_{S'}\beta_{\tilde\sigma}\ } & \mathfrak{s}_{S'}\xi^{\#}\mathsf{L}\mu^* \\
{\scriptstyle(3.2.9)}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle\vartheta_g} \\
\mathsf{L}v^*\mathfrak{s}_S\varphi^{\#} & & g^\flat\mathfrak{s}_S\mathsf{L}\mu^* \\
{\scriptstyle\text{via }\vartheta_f}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle\text{via }(3.2.9)} \\
\mathsf{L}v^*f^\flat\mathfrak{s}_R & \xrightarrow[\ \beta_\sigma(\mathfrak{s}_R)\ ]{} & g^\flat\mathsf{L}u^*\mathfrak{s}_R \\
\Big\| & & \Big\| \\
\mathsf{L}v^*\bar f^*\phi^\flat\mathfrak{s}_R & & \Big\| \\
\big\downarrow\simeq & & \Big\| \\
\bar g^*\mathsf{L}\bar u^*\phi^\flat\mathfrak{s}_R & \xrightarrow[\ \bar g^*\bar\beta_\sigma\ ]{} & \bar g^*\phi'^\flat\mathsf{L}u^*\mathfrak{s}_R
\end{array}
$$

3.3. This section 3.3 is devoted to proving Proposition 3.3.2, which gives, for maps of affine schemes, a concrete representation of a generalization of the map χ in 2.7.7.

Again, $\varphi\colon R\to S$ is a homomorphism of commutative rings, with corresponding scheme-map $\operatorname{Spec} S =: X \xrightarrow{\ f\ } Y := \operatorname{Spec} R$.

Let $\mathsf F$ and $\mathsf G$ be R-complexes, $F := \mathfrak{s}_R\mathsf F$, $G := \mathfrak{s}_R\mathsf G$. There is a natural bifunctorial $\mathbf{D}(S)$-map

$$
\begin{aligned}
\chi_0(\mathsf F,\mathsf G)\colon \varphi^{\#}\mathsf F\otimes^{\mathsf L}_S \mathsf L\varphi^*\mathsf G &= \mathrm{RHom}_\varphi(S,\mathsf F)\otimes^{\mathsf L}_S (S\otimes^{\mathsf L}_\varphi \mathsf G)\\
&\longrightarrow \mathrm{RHom}_\varphi(S,\mathsf F\otimes^{\mathsf L}_R \mathsf G) = \varphi^{\#}(\mathsf F\otimes^{\mathsf L}_R\mathsf G)
\end{aligned}
\tag{3.3.1}
$$

given by [L09, Corollary 2.6.5], in which, when $\mathsf F$ is K-injective and $\mathsf G$ is K-flat (whence $S\otimes_\varphi\mathsf G$ is K-flat over S), take ζ to be the natural $\mathbf{D}(S)$-isomorphism

$$\mathrm{Hom}_\varphi(S,\mathsf F)\otimes_S(S\otimes_\varphi\mathsf G) \xrightarrow{\ \sim\ } \mathrm{RHom}_\varphi(S,\mathsf F)\otimes^{\mathsf L}_S(S\otimes^{\mathsf L}_\varphi\mathsf G)$$

and β the natural composite $\mathbf{D}(S)$-map

$$
\begin{aligned}
&\mathrm{Hom}_\varphi(S,\mathsf F)\otimes_S(S\otimes_\varphi\mathsf G) \xrightarrow{\ \sim\ } \mathrm{Hom}_\varphi(S,\mathsf F)\otimes_\varphi\mathsf G\\
&\longrightarrow \mathrm{Hom}_\varphi(S,\mathsf F\otimes_R\mathsf G) \longrightarrow \mathrm{RHom}_\varphi(S,\mathsf F\otimes_R\mathsf G) \xrightarrow{\ \sim\ } \mathrm{RHom}_\varphi(S,\mathsf F\otimes^{\mathsf L}_R\mathsf G).
\end{aligned}
$$

As in *loc. cit.*, $\chi_0(\mathsf F,\mathsf G)$ is the unique bifunctorial map that equals $\beta\circ\zeta^{-1}$ whenever $\mathsf F$ is K-injective and $\mathsf G$ is K-flat; to this extent, χ_0 is concrete.

The next proposition gives that the sheafification of $\chi_0(\mathsf F,\mathsf G)$ is naturally isomorphic to the $\mathbf{D}_{\mathsf{qc}}(X)$-map

$$\tilde\chi(F,G)\colon f^\flat F\otimes^{\mathsf L}_X \mathsf L f^*G \longrightarrow f^\flat(F\otimes^{\mathsf L}_Y G)$$

that is adjoint to the natural composite map

$$\mathsf R f_*(f^\flat F\otimes^{\mathsf L}_X \mathsf L f^*G) \underset{p_f}{\xrightarrow{\ \sim\ }} \mathsf R f_*f^\flat F\otimes^{\mathsf L}_Y G \longrightarrow F\otimes^{\mathsf L}_Y G$$

with $p_f := p_f(F,G)$ the projection isomorphism in 2.3.1—thereby making χ_0 a concrete representation of $\tilde\chi$. If S is perfect as an R-module, or if $\mathsf F$ and $\mathsf F\otimes^{\mathsf L}_R\mathsf G$ are in $\mathbf{D}^+(R)$, then $\tilde\chi$ is the map χ in 2.7.7.

Proposition 3.3.2. *The following diagram commutes.*

$$\begin{array}{ccc}
\mathfrak{s}_S\,(\varphi^{\#}\mathsf{F}\otimes^{\mathsf{L}}_S \mathsf{L}\varphi^*\mathsf{G}) & \xrightarrow{\ \mathfrak{s}_S\chi_0\ } & \mathfrak{s}_S\,\varphi^{\#}(\mathsf{F}\otimes^{\mathsf{L}}_R\mathsf{G}) \\
{\scriptstyle(3.1.8)}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle(3.2.12)} \\
\mathfrak{s}_S\,\varphi^{\#}\mathsf{F}\otimes^{\mathsf{L}}_X \mathfrak{s}_S\,\mathsf{L}\varphi^*\mathsf{G} & & f^{\flat}\mathfrak{s}_R\,(\mathsf{F}\otimes^{\mathsf{L}}_R\mathsf{G}) \\
{\scriptstyle(3.2.12)\otimes^{\mathsf{L}}_X(3.2.9)}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle(3.1.8)} \\
f^{\flat}F\otimes^{\mathsf{L}}_X \mathsf{L}f^*G & \xrightarrow[\ \tilde\chi\ ]{} & f^{\flat}(F\otimes^{\mathsf{L}}_Y G)
\end{array}$$

Proof. Lemma 3.3.4 below provides an abstract description of χ_0 resembling that of $\tilde\chi$. The lemma uses a commutative-algebra analog of p_f,

(3.3.3) $p_\varphi(\mathsf{E},\mathsf{G})\colon \varphi_*\mathsf{E}\otimes^{\mathsf{L}}_R\mathsf{G}\xrightarrow{\ \sim\ }\varphi_*(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G})\qquad(\mathsf{E}\in\mathbf{D}(S),\ \mathsf{G}\in\mathbf{D}(R)),$

defined as the unique bifunctorial $\mathbf{D}(R)$-isomorphism that is equal, when G, hence $\varphi^*\mathsf{G}$, is K-flat, to the natural composite isomorphism

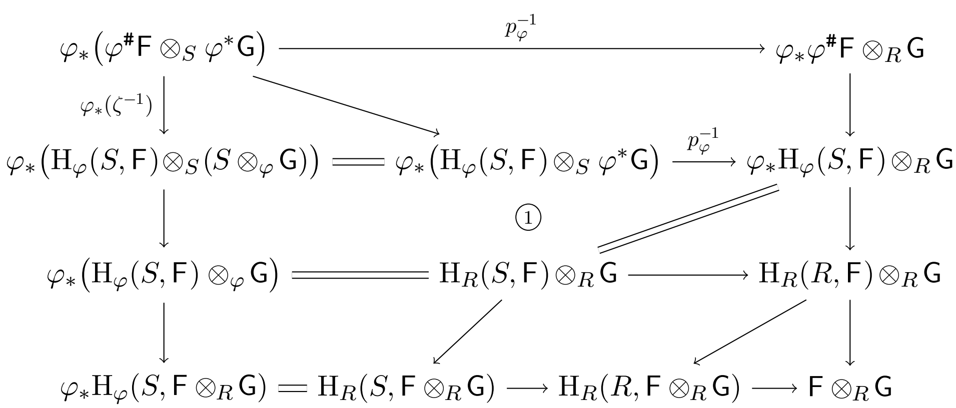
$$\varphi_*\mathsf{E}\otimes^{\mathsf{L}}_R\mathsf{G}\xrightarrow{\ \sim\ }\varphi_*(\mathsf{E}\otimes_S(S\otimes_\varphi\mathsf{G}))=\varphi_*(\mathsf{E}\otimes_S\varphi^*\mathsf{G})\xrightarrow{\ \sim\ }\varphi_*(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G}),$$

see [L09, 2.6.5]. Lemma 3.3.6 will show that the sheafification of p_φ is a concrete realization of p_f. After that, the proof of Proposition 3.3.2 will quickly be concluded.

Lemma 3.3.4. *For all* $\mathsf{F},\mathsf{G}\in\mathbf{D}(R)$, *the map* $\chi_0(\mathsf{F},\mathsf{G})$ *is adjoint to the natural composite map*

$$\varphi_*(\varphi^{\#}\mathsf{F}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G})\xrightarrow{\ p_\varphi^{-1}\ }\varphi_*\varphi^{\#}\mathsf{F}\otimes^{\mathsf{L}}_R\mathsf{G}\longrightarrow\mathsf{F}\otimes^{\mathsf{L}}_R\mathsf{G}.$$

Proof. It is straightforward to reduce to showing that when F is K-injective and G is K-flat (so that $\chi_0=\beta\circ\zeta^{-1}$), the border of the following natural diagram, in which H:= Hom, commutes:

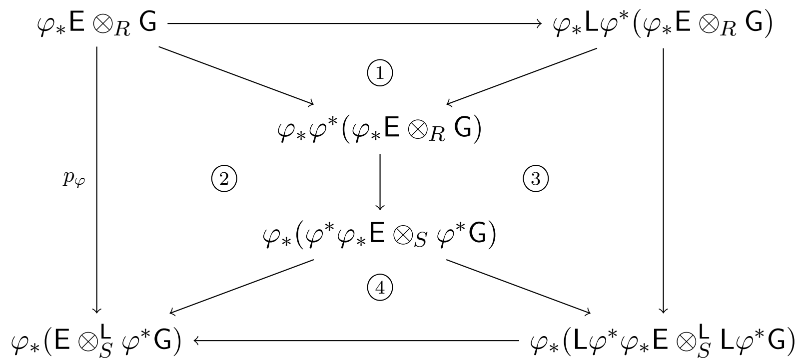


One readily checks commutativity of the unlabeled subdiagrams; and that of ① follows from the above description of p_φ. The assertion results. □

Lemma 3.3.5. *The map* $p_\varphi(\mathsf{E},\mathsf{G})$ *is adjoint to the natural composite map*

$$\mathsf{L}\varphi^*(\varphi_*\mathsf{E}\otimes^{\mathsf{L}}_R\mathsf{G}) \xrightarrow{\ \sim\ } \mathsf{L}\varphi^*\varphi_*\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G} \longrightarrow \mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G}.$$

Proof. One may assume the R-complex G to be K-flat, in which case the assertion amounts to commutativity of the border of the natural diagram



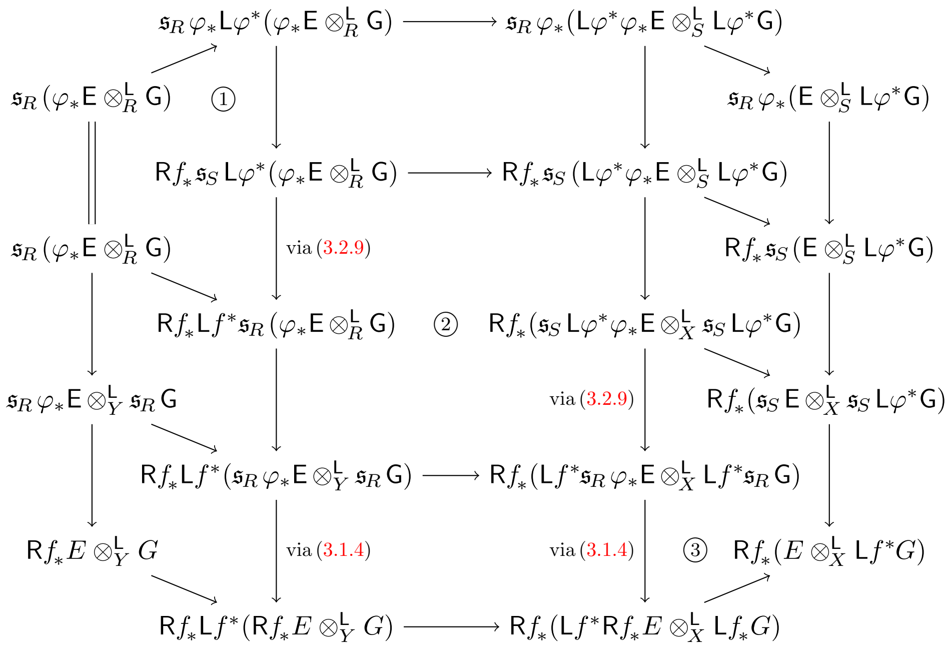
The commutativity of ② is easily checked. For that of ① (resp. ④, resp. ③) cf. [L09, (3.2.1.3), resp. (3.2.1.2), resp. (3.2.4.1)]. The assertion results. □

The sheafification $\mathfrak{s}_R\, p_\varphi$ is naturally isomorphic to the projection map p_f:

Lemma 3.3.6. *The following diagram, with* $E := \mathfrak{s}_S\,\mathsf{E}$, $G := \mathfrak{s}_R\,\mathsf{G}$, *and* p_f *as in* (2.3.1), *commutes.*

$$\begin{array}{ccc}
\mathfrak{s}_R\,(\varphi_*\mathsf{E}\otimes^{\mathsf{L}}_R\mathsf{G}) & \xrightarrow{\ \mathfrak{s}_R\,p_\varphi\ } & \mathfrak{s}_R\,\varphi_*(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G}) \\
{\scriptstyle (3.1.8)}\downarrow\simeq & & \simeq\downarrow{\scriptstyle (3.1.4)} \\
\mathfrak{s}_R\,\varphi_*\mathsf{E}\otimes^{\mathsf{L}}_Y\mathfrak{s}_R\,\mathsf{G} & & \mathsf{R}f_*\mathfrak{s}_S\,(\mathsf{E}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G}) \\
{\scriptstyle (3.1.4)}\downarrow\simeq & & \simeq\downarrow{\scriptstyle (3.1.8)} \\
\mathsf{R}f_*\mathfrak{s}_S\,\mathsf{E}\otimes^{\mathsf{L}}_Y\mathfrak{s}_R\,\mathsf{G} & & \mathsf{R}f_*(\mathfrak{s}_S\,\mathsf{E}\otimes^{\mathsf{L}}_X\mathfrak{s}_S\,\mathsf{L}\varphi^*\mathsf{G}) \\
\Big\| & & \simeq\downarrow{\scriptstyle (3.2.9)} \\
\mathsf{R}f_*E\otimes^{\mathsf{L}}_Y G & \underset{p_f}{\xrightarrow{\ \sim\ }} & \mathsf{R}f_*(E\otimes^{\mathsf{L}}_X\mathsf{L}f^*G)
\end{array}$$

Proof. Keeping in mind 3.3.5 and the analogous property of p_f (a property which defines p_f), expand the diagram, naturally, as follows:

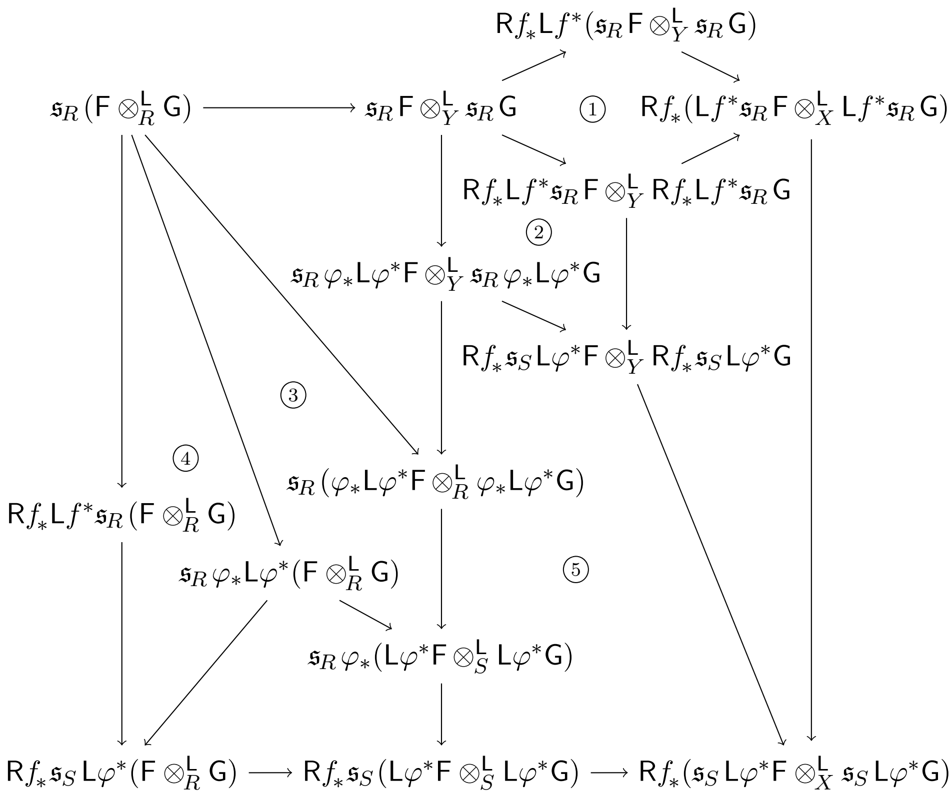


The commutativity of ① follows from that of (3.2.7)′, and that of ③ from (3.2.8)′. The commutativity of ② results from the following formal consequence of Lemma 3.1.9. □

Lemma 3.3.7. *The following diagram of natural isomorphisms commutes*:

$$\begin{array}{ccccc}
\mathsf{L}f^*\mathfrak{s}_R\,(\mathsf{F}\otimes^{\mathsf{L}}_R\mathsf{G}) & \longrightarrow & \mathsf{L}f^*(\mathfrak{s}_R\,\mathsf{F}\otimes^{\mathsf{L}}_Y\mathfrak{s}_R\,\mathsf{G}) & \longrightarrow & \mathsf{L}f^*\mathfrak{s}_R\,\mathsf{F}\otimes^{\mathsf{L}}_X\mathsf{L}f^*\mathfrak{s}_R\,\mathsf{G} \\
{\scriptstyle (3.2.9)}\downarrow & & & & \downarrow{\scriptstyle \text{via } (3.2.9)} \\
\mathfrak{s}_S\,\mathsf{L}\varphi^*(\mathsf{F}\otimes^{\mathsf{L}}_R\mathsf{G}) & \longrightarrow & \mathfrak{s}_S\,(\mathsf{L}\varphi^*\mathsf{F}\otimes^{\mathsf{L}}_S\mathsf{L}\varphi^*\mathsf{G}) & \longrightarrow & \mathfrak{s}_S\,\mathsf{L}\varphi^*\mathsf{F}\otimes^{\mathsf{L}}_X\mathfrak{s}_S\,\mathsf{L}\varphi^*\mathsf{G}
\end{array}$$

Proof. It is enough to show commutativity of the adjoint diagram, which expands naturally as:



Commutativity of the unlabeled subdiagrams is clear.

Commutativity of ① follows from the fact that the natural map

$$\mathsf{L}f^*(\mathfrak{s}_R\,\mathsf{F}\otimes^{\mathsf{L}}_{Y}\mathfrak{s}_R\,\mathsf{G})\to\mathsf{L}f^*\mathfrak{s}_R\,\mathsf{F}\otimes^{\mathsf{L}}_{X}\mathsf{L}f^*\mathfrak{s}_R\,\mathsf{G}$$

is adjoint to the natural composite map

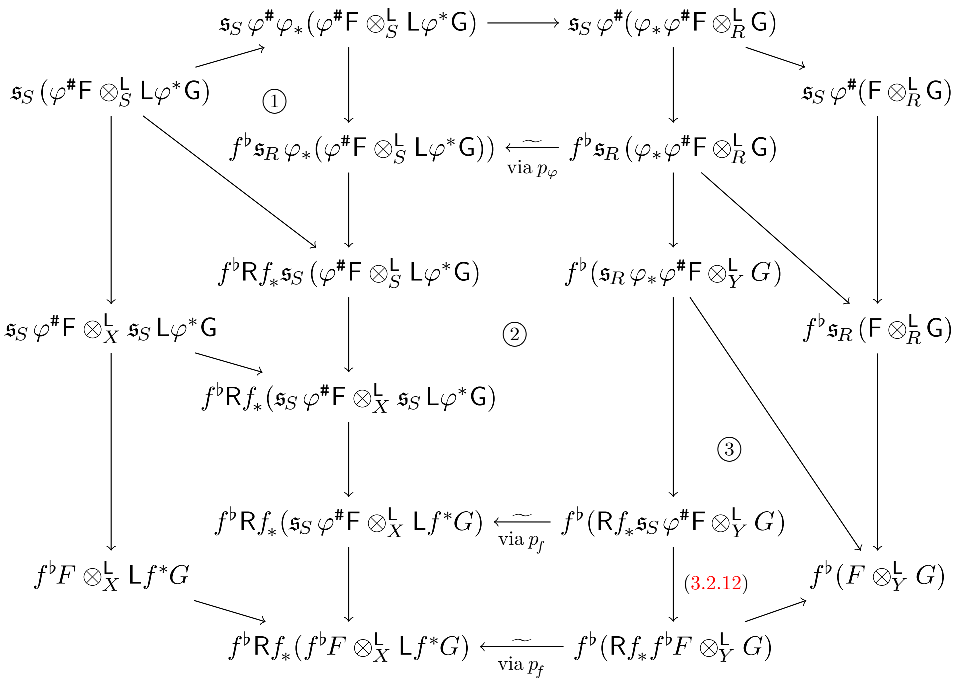
$$\mathfrak{s}_R\,\mathsf{F}\otimes^{\mathsf{L}}_{Y}\mathfrak{s}_R\,\mathsf{G}\to\mathsf{R}f_*\mathsf{L}f^*\mathfrak{s}_R\,\mathsf{F}\otimes^{\mathsf{L}}_{Y}\mathsf{R}f_*\mathsf{L}f^*\mathfrak{s}_R\,\mathsf{G}\to\mathsf{R}f_*(\mathsf{L}f^*\mathfrak{s}_R\,\mathsf{F}\otimes^{\mathsf{L}}_{Y}\mathsf{L}f^*\mathfrak{s}_R\,\mathsf{G}),$$

cf. [L09, beginning of §3.4.5]. That of ③ holds for analogous reasons.

Commutativity of ② and of ④ follows from (3.2.7)′.

Commutativity of ⑤ is given by Lemma 3.1.9. □

Now, to prove Proposition 3.3.2, use Lemma 3.3.4 to expand the diagram in question, naturally, as follows:



Commutativity of ③ (resp. ①) is essentially the same as commutativity of ③ (resp. ①) in the proof of Proposition 3.2.13.

The commutativity of ② is given by Lemma 3.3.6 (with $\mathsf{E} := \varphi^\#\mathsf{F}$). □

3.4. This section is devoted to proving Proposition 3.4.1, which gives, for maps of affine schemes, a concrete representation of a generalization of the map ζ in 2.8.2(iii).

Once again, $\varphi\colon R \to S$ is a homomorphism of commutative rings, with corresponding scheme-map $\operatorname{Spec} S =: X \xrightarrow{f} Y := \operatorname{Spec} R$. The global-section functor from $\mathcal{A}(Y)$ to $\mathcal{A}(R)$ is denoted by Γ_Y, and the sheafification functor from $\mathcal{A}(R)$ to $\mathcal{A}_{\mathsf{qc}}(Y)$ by $\bar{\mathfrak{s}}_R$.

The composite *quasi-coherator* functor

$$Q_Y\colon \mathcal{A}(Y) \xrightarrow{\Gamma_Y} \mathcal{A}(R) \xrightarrow{\bar{\mathfrak{s}}_R} \mathcal{A}_{\mathsf{qc}}(Y)$$

is right-adjoint to the inclusion functor $j_Y\colon \mathcal{A}_{\mathsf{qc}}(Y) \hookrightarrow \mathcal{A}(Y)$. Since j_Y is fully faithful, the unit map for this adjunction is an isomorphism

$$A \xrightarrow{\sim} Q_Y j_Y A = Q_Y A \qquad (A \in \mathcal{A}_{\mathsf{qc}}(Y)).$$

The counit is the natural map

$$j_Y Q_Y B = \mathfrak{s}_R \Gamma_Y B \longrightarrow B \qquad (B \in \mathcal{A}(Y)).$$

Since j_Y is exact, therefore Q_Y preserves K-injectivity, and there results an adjunction $\boldsymbol{j}_Y \dashv \mathsf{R}Q_Y$ between the derived functors. In fact, $\boldsymbol{j}_Y$ gives an equivalence $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(Y)) \xrightarrow{\approx} \mathbf{D}_{\mathsf{qc}}(Y) \subset \mathbf{D}(Y)$ [BN93, p. 225, 5.1], with quasi-inverse the restriction to $\mathbf{D}_{\mathsf{qc}}(Y)$ of the right adjoint $\mathsf{R}Q_Y$. So if $C \in \mathbf{D}_{\mathsf{qc}}(Y)$ then the counit map is an isomorphism $\mathsf{R}Q_Y C = \boldsymbol{j}_Y \mathsf{R}Q_Y C \xrightarrow{\sim} C$.

Set

$$\mathcal{H}om^{\mathsf{qc}}_Y := Q_Y \mathcal{H}om_Y \colon \mathcal{A}(Y)^{\mathsf{op}} \times \mathcal{A}(Y) \to \mathcal{A}_{\mathsf{qc}}(Y),$$
$$\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y := \mathsf{R}Q_Y \mathsf{R}\mathcal{H}om_Y \colon \mathbf{D}(Y)^{\mathsf{op}} \times \mathbf{D}(Y) \to \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(Y)).$$

For $\mathcal{O}_Y$-complexes F and G, the natural bifunctorial map

$$\mathcal{H}om^{\mathsf{qc}}_Y(F,G) \to \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y(F,G)$$

is an isomorphism if F is K-flat and G is K-injective (so that $\mathcal{H}om_Y(F,G)$ is K-injective).

For any R-complexes F and G, and $F := \mathfrak{s}_R \mathsf{F}$, $G := \mathfrak{s}_R \mathsf{G}$, there is a natural $\mathcal{A}_{\mathsf{qc}}(Y)$-isomorphism

$$\kappa(\mathsf{F},\mathsf{G}) \colon \mathfrak{s}_R \operatorname{Hom}_R(\mathsf{F},\mathsf{G}) \xrightarrow{\sim} \mathfrak{s}_R \Gamma_Y \mathcal{H}om_Y(F,G) = \mathcal{H}om^{\mathsf{qc}}_Y(F,G).$$

Furthermore, there is a natural composite $\mathbf{D}(\mathcal{A}_{\mathsf{qc}}(Y))$-isomorphism

$$\boldsymbol{\kappa}(\mathsf{F},\mathsf{G}) \colon \mathfrak{s}_R \mathsf{R}\operatorname{Hom}_R(\mathsf{F},\mathsf{G}) \xrightarrow{\sim} \mathfrak{s}_R \mathsf{R}\Gamma_Y \mathsf{R}\mathcal{H}om_Y(F,G) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y(F,G).$$

whose first component is an isomorphism because the natural equivalences

$$\mathbf{D}(R) \xrightarrow[\bar{\mathfrak{s}}_R]{\approx} \mathbf{D}(\mathcal{A}_{\mathsf{qc}}(Y)) \xrightarrow[\boldsymbol{j}_Y]{\approx} \mathbf{D}_{\mathsf{qc}}(Y).$$

give rise, for any $n \in \mathbb{Z}$, to natural isomorphisms

$$\begin{aligned} H^n \mathsf{R}\operatorname{Hom}_R(\mathsf{F},\mathsf{G}) \cong \operatorname{Hom}_{\mathbf{D}(R)}(\mathsf{F},\mathsf{G}[n]) &\cong \operatorname{Hom}_{\mathbf{D}(Y)}(F,G[n]) \\ &\cong H^n \mathsf{R}\operatorname{Hom}_Y(F,G) \\ &\cong H^n \mathsf{R}\Gamma_Y \mathsf{R}\mathcal{H}om_Y(F,G). \end{aligned}$$

Next, there is a natural bifunctorial isomorphism of S-complexes

$$\operatorname{Hom}_S(S \otimes_\varphi \mathsf{F}, \operatorname{Hom}_\varphi(S,\mathsf{G})) \xrightarrow{\sim} \operatorname{Hom}_\varphi(S \otimes_\varphi \mathsf{F}, \mathsf{G}) \xrightarrow{\sim} \operatorname{Hom}_\varphi(S, \operatorname{Hom}_R(\mathsf{F},\mathsf{G})).$$

From this, with F K-flat and G K-injective, one gets (via, e.g., [L09, 2.6.5]) a natural bifunctorial $\mathbf{D}(S)$-isomophism, concrete up to choice of K-flat or K-injective resolutions,

$$\begin{aligned} \zeta_0 \colon \mathsf{R}\operatorname{Hom}_S(\mathsf{L}\varphi^* \mathsf{F}, \varphi^{\#}\mathsf{G}) &= \mathsf{R}\operatorname{Hom}_S(S \otimes^{\mathsf{L}}_\varphi \mathsf{F}, \mathsf{R}\operatorname{Hom}_\varphi(S,\mathsf{G})) \\ &\xrightarrow{\sim} \mathsf{R}\operatorname{Hom}_\varphi(S, \mathsf{R}\operatorname{Hom}_R(\mathsf{F},\mathsf{G})) = \varphi^{\#} \mathsf{R}\operatorname{Hom}_R(\mathsf{F},\mathsf{G}). \end{aligned}$$

Proposition 3.4.1. *The sheafification $\mathfrak{s}_S \zeta_0$ is naturally isomorphic to the map* (*that must then be an* isomorphism)

$$\zeta \colon \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(\mathsf{L}f^*F, f^\flat G) \longrightarrow f^\flat \mathsf{R}\mathcal{H}om_Y(F,G)$$

which is adjoint to the natural composite map

$$\begin{aligned} \mathsf{R}f_* \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(\mathsf{L}f^*F, f^\flat G) &\longrightarrow \mathsf{R}f_* \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) \\ &\longrightarrow \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathsf{L}f^*F, \mathsf{R}f_* f^\flat G) \longrightarrow \mathsf{R}\mathcal{H}om_Y(F,G). \end{aligned}$$

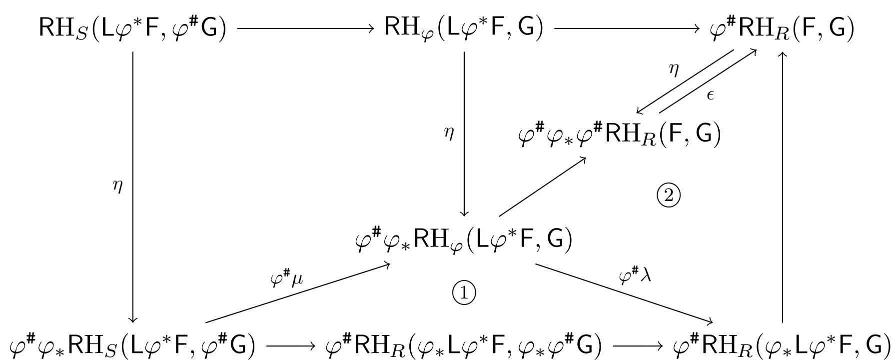
Remark. If $C := \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) \in \mathbf{D}_{\mathsf{qc}}(Y)$ (which is so under the hypotheses of 2.8.2(iii)) then the map $\mathsf{R}f_*\mathsf{R}Q_Y C \longrightarrow \mathsf{R}f_*C$ in 3.4.1 is an isomorphism (see above). Keeping in mind [L09, 3.2.4(i)] and the paragraph preceding [L09, 3.5.5], one finds then that the map ζ in 2.8.2(iii) is a special case of the map ζ in 3.4.1.

Proof. The strategy is, to relate the sheafification of the adjoint of ζ_0 to the adjoint of ζ.

Lemma 3.4.2. *The map ζ_0 is adjoint to the natural composite map*

$$\varphi_*\mathrm{RHom}_S(\mathsf{L}\varphi^*\mathsf{F}, \varphi^\#\mathsf{G}) \longrightarrow \mathrm{RHom}_R(\varphi_*\mathsf{L}\varphi^*\mathsf{F}, \varphi_*\varphi^\#\mathsf{G}) \longrightarrow \mathrm{RHom}_R(\mathsf{F}, \mathsf{G}).$$

Proof. The assertion means that the border of the following natural diagram, in which $\mathrm{H} = \mathrm{Hom}$, and whose top row composes to ζ_0, commutes.



The unlabeled subdiagrams obviously commute, and $\epsilon\circ\eta = \mathrm{id}$; so it's enough to show that ① and ② commute.

For ① to commute, it suffices, by Proposition 3.1.16, that $\lambda\mu$ be inverse to the isomorphism $\bar{\alpha}_\varphi(\mathsf{L}\varphi^*\mathsf{F}, S, \mathsf{G})$ from Proposition 3.1.13—which it is: one may assume that G is K-injective, so that all Rs can be dropped, making the assertion into an easily verified one in commutative algebra.

As for ②, one may assume G to be K-injective and F to be K-flat, thereby reducing to verifying commutativity of the natural diagram

$$\begin{array}{ccc} \varphi_*\mathrm{H}_\varphi(S\otimes_R \mathsf{F}, \mathsf{G}) & \longrightarrow & \varphi_*\mathrm{H}_\varphi(S, \mathrm{H}_R(\mathsf{F},\mathsf{G})) \\ \big\downarrow & & \big\downarrow \\ \mathrm{H}_\varphi(\varphi_*(S\otimes_R \mathsf{F}), \mathsf{G}) & \longrightarrow & \mathrm{H}_R(\mathsf{F},\mathsf{G}) \end{array}$$

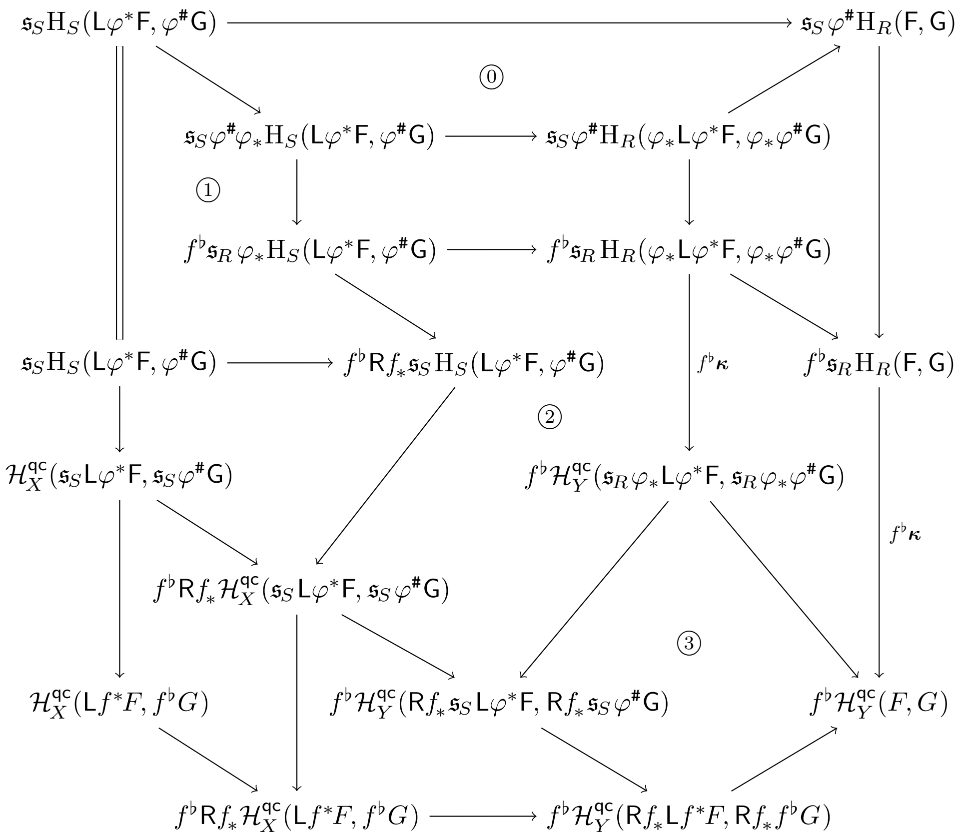
To check this commutativity, send any map $\lambda\colon S\otimes_R \mathsf{F} \to \mathsf{G}$ from the upper left to the lower right corner, both clockwise and counterclockwise, and note that either way produces the map taking $x \in \mathsf{F}$ to $\lambda(1\otimes x)$. $\square$

Now, the adjunction $\boldsymbol{j}_Y \dashv \mathsf{R}Q_Y$ entails a natural commutative diagram, with vertical arrows induced by counit maps,

$$\begin{array}{ccccc}
\mathsf{R}f_*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(\mathsf{L}f^*F, f^\flat G) & \longrightarrow & \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y(\mathsf{R}f_*\mathsf{L}f^*F, \mathsf{R}f_*f^\flat G) & \longrightarrow & \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y(F,G) \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{\boldsymbol{j},\mathsf{R}Q}} \\
\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*F, f^\flat G) & \longrightarrow & \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathsf{L}f^*F, \mathsf{R}f_*f^\flat G) & \longrightarrow & \mathsf{R}\mathcal{H}om_Y(F,G)
\end{array}$$

Also, $\mathrm{Hom}_{\mathbf{D}(Y)}(E, \epsilon_{\boldsymbol{j},\mathsf{R}Q})$ is an isomorphism for any $E \in \mathbf{D}_{\mathsf{qc}}(X)$, and so $\epsilon_{\boldsymbol{j},\mathsf{R}Q}$ itself is an isomorphism.

Hence, Proposition 3.4.1 results from the commutativity of the border of the following natural diagram, in which $\mathrm{H} := \mathsf{R}\mathrm{Hom}$ and $\mathcal{H} := \mathsf{R}\mathcal{H}om$:



Commutativity of the unlabeled subdiagrams is simple to check.

Commutativity of ⓪ is given by Lemma 3.4.2.

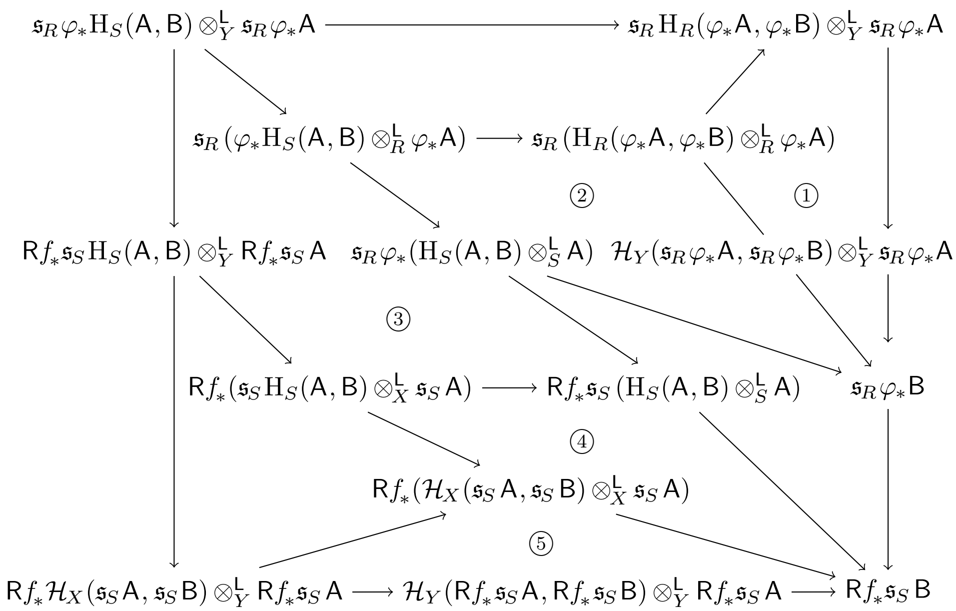
Commutativity of ① is essentially the same as that of subdiagram ① in the proof of Proposition 3.2.13.

Commutativity of ③ follows easily from that of the diagram in (3.2.7)′ and of subdiagram ③ in the proof of Proposition 3.2.13.

As for the remaining subdiagram ②, let us show, more generally, that for any S-complexes A and B, the following natural $\mathbf{D}_{\mathsf{qc}}(Y)$-diagram commutes.

$$\begin{array}{ccc}
\mathfrak{s}_R\varphi_*\mathrm{H}_S(\mathsf{A},\mathsf{B}) & \longrightarrow & \mathfrak{s}_R\,\mathrm{H}_R(\varphi_*\mathsf{A},\varphi_*\mathsf{B}) \\
\downarrow & & \downarrow \\
\mathsf{R}f_*\mathfrak{s}_S\mathrm{H}_S(\mathsf{A},\mathsf{B}) & & \mathcal{H}^{\mathsf{qc}}_Y(\mathfrak{s}_R\varphi_*\mathsf{A},\mathfrak{s}_R\varphi_*\mathsf{B}) \\
\downarrow & & \downarrow \\
\mathsf{R}f_*\mathcal{H}^{\mathsf{qc}}_X(\mathfrak{s}_S\mathsf{A},\mathfrak{s}_S\mathsf{B}) & \longrightarrow & \mathcal{H}^{\mathsf{qc}}_Y(\mathsf{R}f_*\mathfrak{s}_S\mathsf{A},\mathsf{R}f_*\mathfrak{s}_S\mathsf{B})
\end{array}$$

The natural isomorphism $\mathrm{Hom}(\mathfrak{s}_R\mathsf{E},\mathcal{H}^{\mathsf{qc}}_Y(F,G)) \xrightarrow{\sim} \mathrm{Hom}(\mathfrak{s}_R\mathsf{E},\mathcal{H}_Y(F,G))$ ($\mathsf{E}\in\mathbf{D}(R)$; $F,G\in\mathbf{D}(Y)$), allows one to replace $\mathcal{H}^{\mathsf{qc}}$ in the preceding diagram by $\mathcal{H}$, whereupon it suffices to show commutativity of the adjoint diagram, which is, up to obvious isomorphisms, the border of the following natural diagram:



The commutativity of the unlabeled subdiagram is easily checked.

The commutativity of ② and ⑤ follows at once the abstract definition of the map [L09, (3.5.4.1)], a map whose concrete realization is the usual one—see, e.g., the two lines preceding [L09, 3.1.9].

The commutativity of ③ is an instance of Lemma 3.1.9.

Finally, the commutativity of subdiagrams ① and ④ results from the next lemma, which expresses compatibility of (the counit map for) hom-⊗ adjunction with sheafification.

Lemma 3.4.3. *For any R-complexes $\mathsf E$ and $\mathsf F$, the following natural diagram commutes.*

$$\begin{array}{ccc} \mathfrak s_R \mathrm H_S(\mathsf E,\mathsf F)\otimes^{\mathsf L}_Y \mathfrak s_R\mathsf E & \longrightarrow & \mathcal H_Y(\mathfrak s_R\mathsf E,\mathfrak s_R\mathsf F)\otimes^{\mathsf L}_Y \mathfrak s_R\mathsf E \\ \Big\downarrow & & \Big\downarrow \\ \mathfrak s_R(\mathrm H_S(\mathsf E,\mathsf F)\otimes^{\mathsf L}_R\mathsf E) & \longrightarrow & \mathfrak s_R\mathsf F \end{array}$$

Proof. (Sketch.) It may be assumed that $\mathsf E$ is a direct limit of bounded-above flat complexes, and that $\mathsf F$ is K-injective. Then the diagram can be replaced by its non-derived counterpart, whose commutativity can be verified elementwise, via the definitions of the maps involved. □

This completes the proof of Proposition 3.4.1. □

Remark 3.4.4. With assumptions and notation as in Proposition 2.8.2, that proposition and 3.4.1 show that the following natural diagram commutes:

$$\begin{array}{ccc} \mathfrak s_S\,\mathsf R\mathrm{Hom}_S(\mathsf L\varphi^*\mathsf F,\varphi^{\#}\mathsf G) & \xrightarrow{\ \mathfrak s_R\zeta_0\ } & \mathfrak s_S\,\varphi^{\#}\mathsf R\mathrm{Hom}_R(\mathsf F,\mathsf G) \\ \simeq\Big\downarrow & & \Big\downarrow\simeq \\ \mathsf R\mathcal Hom_X(\mathsf Lf^*F,f^\flat G) & & f^\flat\mathsf R\mathcal Hom_Y(F,G) \\ \simeq\Big\downarrow(2.8.1) & & \Big\uparrow\simeq \\ \bar f^*\mathsf R\mathcal Hom_{\overline Y}(\mathsf L\phi^*F,\phi^\flat G)G) & \xrightarrow[\ \bar f^*\bar\zeta\ ]{} & \bar f^*\phi^\flat\mathsf R\mathcal Hom_Y(F,G) \end{array}$$

Conversely, a more concrete proof—eschewing adjoints—of this commutativity would, together with 2.8.2, give 3.4.1.

Exercise 3.4.5. Prove Proposition 3.4.1 along the following lines:
Show how sheafification respects conjugacy of functorial maps. Then show that ζ_0 and ζ are right-conjugate, respectively, to the projection maps

$$p_\varphi\colon \varphi_*\mathsf E\otimes^{\mathsf L}_R\mathsf F\longrightarrow\varphi_*(\mathsf E\otimes^{\mathsf L}_S\mathsf L\varphi^*\mathsf F),$$
$$p_f\colon \mathsf Rf_*E\otimes^{\mathsf L}_Y F\longrightarrow\mathsf Rf_*(E\otimes^{\mathsf L}_X\mathsf Lf^*F).$$

(Cf. [L09, 4.2.3(e)].) Finally, apply Lemma 3.3.6.

3.5 (Completion of proof of 2.10.23)**.** Recall that 2.10.23 reduces via 2.10.24 to the situation where the closed immersions $X\xrightarrow{f}Y\xrightarrow{g}Z$ correspond to surjective ring homomorphisms $R\overset{\varphi}{\twoheadrightarrow}S\overset{\xi}{\twoheadrightarrow}T$ whose kernels J and I are generated by Koszul-regular sequences $\mathbf r=(r_1,\dots,r_e)$ and $\bar{\mathbf s}=(\bar s_1,\dots,\bar s_d)$ respectively, and, if $s_i\in R$ is such that $\bar s_i=\xi(s_i)$ $(1\le i\le d)$ then the R-sequence $(\mathbf r,\mathbf s)=(r_1,\dots,r_e,s_1,\dots,s_d)$ is Koszul-regular and it generates the kernel L of $\xi\varphi$. Thus 2.10.23 results from its commutative-algebra analog, Proposition 3.5.1 below.

Proposition 3.5.1. involves commutative-algebra versions of some of the maps in Lemma 2.10.13. Set $\bar{\mathbf{s}} := (\bar{s}_1, \dots, \bar{s}_d)$, let $K_S(\bar{\mathbf{s}})$ be the associated Koszul complex, let $\vartheta_{\bar{\mathbf{s}}}\colon T^d \xrightarrow{\sim} I/I^2$ be the T-isomorphism sending the i-th canonical basis element of T^d to $(\bar{s}_i + I^2) \in I/I^2$ $(1 \le i \le d)$, and let $c^{\#}_{\bar{\mathbf{s}}}$ be the natural composite $\mathbf{D}(T)$-isomorphism

$$\begin{aligned} N_\xi := \mathrm{Hom}_T\big(\wedge^d_T(I/I^2), T\big)[-d] &\xrightarrow{\text{via } \vartheta_{\bar{\mathbf{s}}}} \mathrm{Hom}_T\big(\wedge^d_T(T^d), T\big)[-d] \\ &\xrightarrow{\sim} \xi^*\big(H^d\mathrm{Hom}_S(K_S(\bar{\mathbf{s}}), S)\big)[-d] \\ &\xrightarrow{\sim} \xi^*\big(H^d\mathsf{R}\mathrm{Hom}_S(\xi_*T, S)\big)[-d] \\ &\xrightarrow{\sim} \big(\xi^*H^d\xi_*\mathsf{R}\mathrm{Hom}_\xi(T, S)\big)[-d] \\ &\xrightarrow{\sim} \big(H^d\mathsf{R}\mathrm{Hom}_\xi(T, S)\big)[-d] \\ &\xrightarrow{\sim} \mathsf{R}\mathrm{Hom}_\xi(T, S) =: \xi^{\#}S. \end{aligned}$$

(The last isomorphism holds since $H^i\mathsf{R}\mathrm{Hom}_\xi(T, S) = H^i\mathrm{Hom}_S(K_S(\bar{\mathbf{s}}), S) = 0$ if $i \ne d$.) In view of the relevant details of the equivalence described in section 3.1, Lemma 2.10.13 implies that $c^{\#}_{\bar{\mathbf{s}}}$ is obtained (up to canonical isomorphism) by applying the global section functor to $c^\flat_f\colon \omega_f \xrightarrow{\sim} f^\flat\mathcal{O}_Y$ in (2.10.12.1). It follows (or can be checked directly) that $c^{\#}_{\bar{\mathbf{s}}}$ does not depend on the choice of the Koszul-regular generating sequence $\bar{\mathbf{s}}$ of I, so that it can—and will—be denoted by $c^{\#}_\xi$.

The $\mathbf{D}(S)$-isomorphism

$$c^{\#}_\varphi\colon N_\varphi := \mathrm{Hom}_S\big(\wedge^e_S(J/J^2), S\big)[-e] \xrightarrow{\sim} \varphi^{\#}R$$

and the $\mathbf{D}(T)$-isomorphism

$$c^{\#}_{\xi\varphi}\colon N_{\xi\varphi} := \mathrm{Hom}_T\big(\wedge^{d+e}_T(L/L^2), T\big)[-d-e] \xrightarrow{\sim} (\xi\varphi)^{\#}R$$

are defined analogously.

Note that N_ξ (resp. N_φ, $N_{\xi\varphi}$) is flat over T (resp. S, T).

Proposition 3.5.1. *The next* $\mathbf{D}(T)$*-diagram, with* $\mathrm{H} := \mathrm{Hom}$, *commutes, and its sheafification is isomorphic to the* $\mathbf{D}(X)$*-diagram in* 2.10.23.

$$\begin{array}{ccc} N_\xi \otimes_T \xi^* N_\varphi & \xrightarrow[\text{(2.10.21)}]{h} & N_{\xi\varphi} \\ {\scriptstyle c^{\#}_\xi \otimes^{\mathsf{L}}_T \mathsf{L}\xi^* c^{\#}_\varphi}\big\downarrow & & \\ \xi^{\#}S \otimes^{\mathsf{L}}_T \mathsf{L}\xi^*\varphi^{\#}R & & \big\downarrow{\scriptstyle c^{\#}_{\xi\varphi}} \\ {\scriptstyle \chi_0(S,\varphi^{\#}R)}\big\downarrow{\scriptstyle (3.3.1)} & & \\ \xi^{\#}\varphi^{\#}R & \xrightarrow[\text{(3.1.21)}]{\sim} & (\xi\varphi)^{\#}R \end{array}$$

Proof. As regards sheafification, there are evident natural isomorphisms

$$\begin{aligned}\mathfrak{s}_T N_\xi &\xrightarrow{\sim} \big(\wedge_X^d f^*(\mathcal{I}/\mathcal{I}^2)\big)^\vee[-d] =: \omega_f\\ \mathfrak{s}_T N_\varphi &\xrightarrow{\sim} f^*\big(\wedge_Y^e g^*(\mathcal{J}/\mathcal{J}^2)\big)^\vee[-e] =: \omega_g\\ \mathfrak{s}_T N_{\xi\varphi} &\xrightarrow{\sim} \big(\wedge_X^{d+e}(gf)^*(\mathcal{L}/\mathcal{L}^2)\big)^\vee[-d-e] =: \omega_{gf}\,,\end{aligned}$$

via which the sheafification of the top row in 3.5.1 is naturally isomorphic to the top row in 2.10.23.

Lemma 2.10.13, *mutatis mutandis,* shows the sheafification of the right column in 3.5.1 to be isomorphic to the right column in 2.10.23. The top arrows in the respective left columns can be treated similarly.

That χ_0 in 3.5.1 sheafifies to the map in 2.10.23 labeled 2.7.7 is given by Proposition 3.3.2.

That the sheafification of the bottom row in 3.5.1 is naturally isomorphic to the bottom row in 2.10.23 is given by (3.1.22), *mutatis mutandis.*

Next, to show the commutativity of the diagram in 3.5.1, it suffices to show the commutativity of its adjoint, as follows.

Let p_φ and p_ξ be as in (3.3.3) *ff.*

For a sequence $\mathbf{q}$ in a ring Q, let $K_Q^\bullet(\mathbf{q})$ be the complex $\mathrm{Hom}_Q(K_Q(\mathbf{q}), Q)$. Let $\mathfrak{t}_\mathbf{q}\colon K_Q^\bullet((\mathbf{q})) \to K_Q^0(\mathbf{q}) = \mathrm{Hom}_Q(Q,Q) = Q$ be the natural map.

One has the composite natural isomorphism (with $\tilde\gamma_0$ an isomorphism because the complex $K_R(\mathbf{r})$ is strictly perfect)

$$\begin{aligned}\mathfrak{g}_{\mathbf{r},\mathbf{s}}\colon K_R^\bullet(\mathbf{r})\otimes_R K_R^\bullet(\mathbf{s}) &= \mathrm{Hom}_R(K_R(\mathbf{r}),R)\otimes_R \mathrm{Hom}_R(K_R(\mathbf{s}),R)\\ &\underset{\tilde\gamma_0}{\xrightarrow{\sim}} \mathrm{Hom}_R(K_R(\mathbf{r}),\mathrm{Hom}_R(K_R(\mathbf{s}),R))\\ &\xrightarrow{\sim} \mathrm{Hom}_R(K_R(\mathbf{r})\otimes_R K_R(\mathbf{s}),R) \xrightarrow{\sim} K_R^\bullet((\mathbf{r},\mathbf{s})).\end{aligned}$$

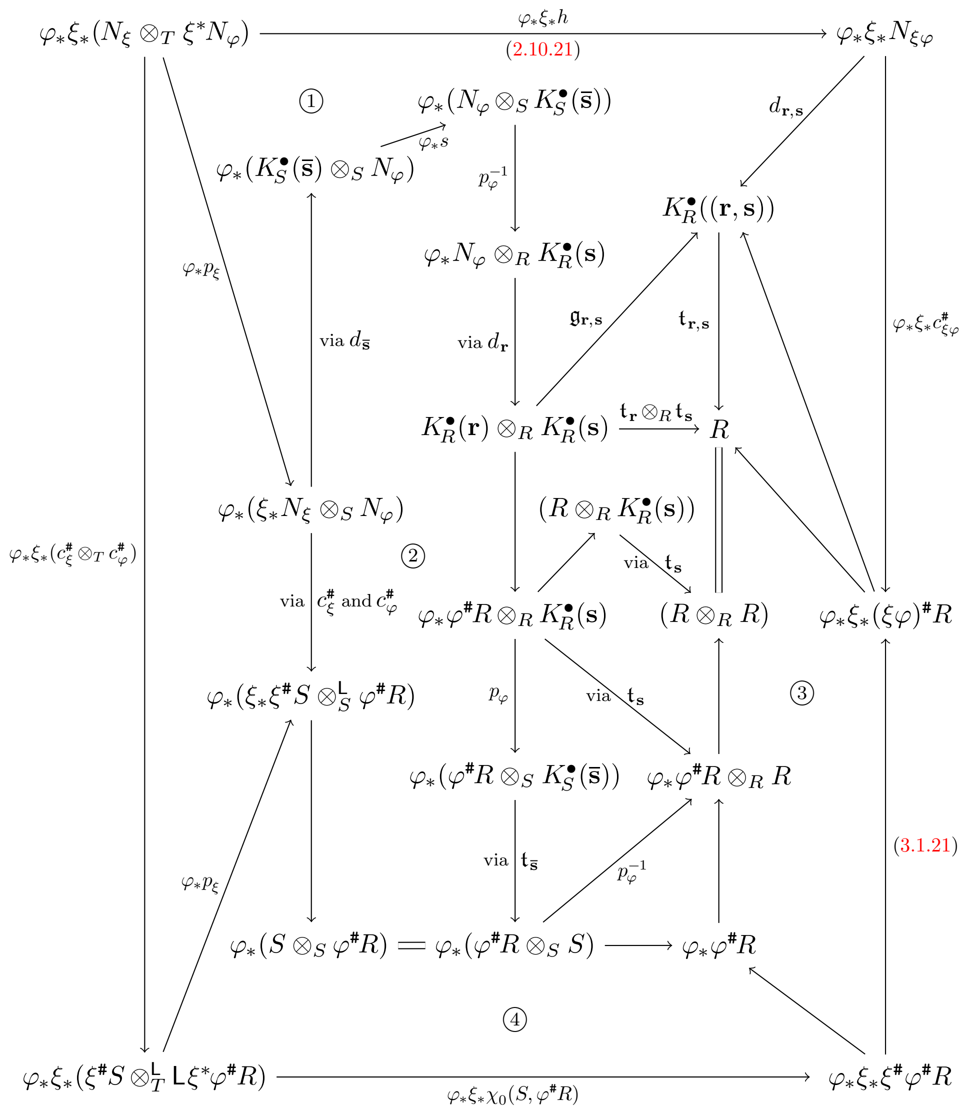
Let $d_{\bar{\mathbf{s}}}$ be the natural composite $\mathbf{D}(S)$-isomorphism

$$\xi_*N_\xi \xrightarrow{\xi_* c_\xi^\#} \xi_*\xi^\# S \xrightarrow{\sim} \mathsf{R}\mathrm{Hom}_S(\xi_*T,S) \xrightarrow{\sim} K_S^\bullet(\bar{\mathbf{s}}),$$

and define analogously

$$d_\mathbf{r}\colon \varphi_*N_\varphi \xrightarrow{\sim} K_R^\bullet(\mathbf{r}),\qquad d_{\mathbf{r},\mathbf{s}}\colon \varphi_*\xi_*N_{\xi\varphi} \xrightarrow{\sim} K_R^\bullet((\mathbf{r},\mathbf{s})).$$

Now expand the adjoint diagram naturally as in diagram (3.5.2) below. Diagram-chasing shows that proving commutativity of all the subdiagrams of this expanded diagram gives the desired commutativity of the adjoint diagram itself.



(3.5.2)

For each of the unlabeled subdiagrams, commutativity is either obvious or straightforward to verify.

For the commutativity of ②, use the commutativity (resulting directly from the definition of $d_{\mathbf{r}}$) of

$$\begin{array}{ccc}
\varphi_*(N_\varphi \otimes_S K_S^\bullet(\bar{\mathbf{s}})) & \xleftarrow{\;p_\varphi\;} & \varphi_* N_\varphi \otimes_R K_R^\bullet(\mathbf{s}) \\
 & & \big\downarrow{\scriptstyle \text{via } d_{\mathbf{r}}} \\
\big\downarrow{\scriptstyle \text{via } c_\varphi^\#} & & K_R^\bullet(\mathbf{r}) \otimes_R K_R^\bullet(\mathbf{s}) \\
 & & \big\downarrow{\scriptstyle \text{natural}} \\
\varphi_*(\varphi^\# R \otimes_S K_S^\bullet(\bar{\mathbf{s}})) & \xleftarrow[\;p_\varphi\;]{} & \varphi_*\varphi^\# R \otimes_R K_R^\bullet(\mathbf{s}),
\end{array}$$

plus the definition of $d_{\bar{\mathbf{s}}}$, to reduce to noting the obvious commutativity of

$$\begin{array}{ccc}
\varphi_*(K_S^\bullet(\bar{\mathbf{s}}) \otimes_S N_\varphi) & \xleftarrow{\text{natural}} & \varphi_*(N_\varphi \otimes_S K_S^\bullet(\bar{\mathbf{s}})) \\
{\scriptstyle \text{via } c_\varphi^\#}\big\downarrow & & \big\downarrow{\scriptstyle \text{via } c_\varphi^\#} \\
\varphi_*(K_S^\bullet(\bar{\mathbf{s}}) \otimes_S \varphi^\# R) & \xleftarrow{\text{natural}} & \varphi_*(\varphi^\# R \otimes_S K_S^\bullet(\bar{\mathbf{s}})) \\
{\scriptstyle \text{via } \mathfrak{t}_{\bar{\mathbf{s}}}}\big\downarrow & & \big\downarrow{\scriptstyle \text{via } \mathfrak{t}_{\bar{\mathbf{s}}}} \\
\varphi_*(S \otimes_S \varphi^\# R) & \Longrightarrow\!= & \varphi_*(\varphi^\# R \otimes_S S)
\end{array}$$

The commutativity of ③ follows from the definition of the map (3.1.21).
The commutativity of ④ is given by Lemma 3.3.4, *mutatis mutandis.*.
It remains to verify the commutativity of ①, or equivalently, of the natural diagram

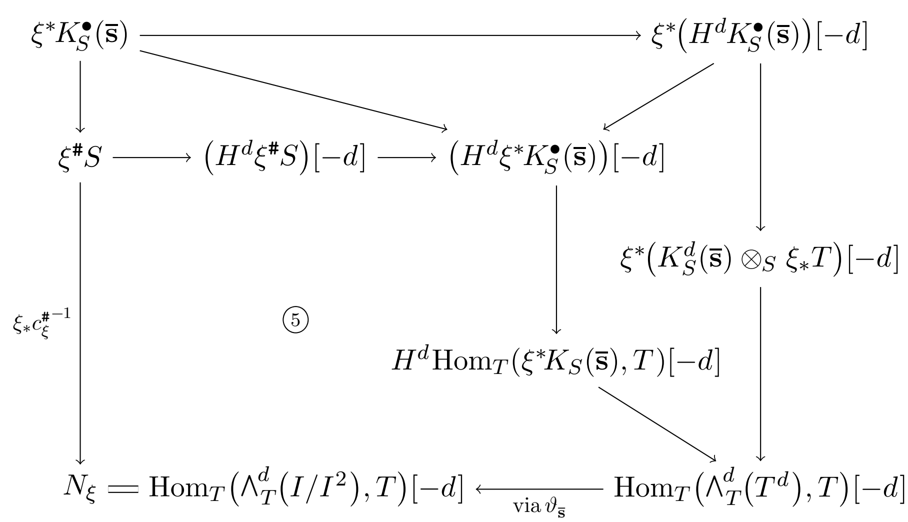
$$\begin{array}{ccccc}
\varphi_*\xi_*(N_\xi \otimes_T \xi^* N_\varphi) & \xleftarrow{\;\varphi_*\xi_* h^{-1}\;} & \varphi_*\xi_* N_{\xi\varphi} & \xleftarrow{\;d_{\mathbf{r},\mathbf{s}}^{-1}\;} & K_R^\bullet((\mathbf{r},\mathbf{s})) \\
{\scriptstyle \varphi_* p_\xi}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathfrak{g}_{\mathbf{r},\mathbf{s}}^{-1}} \\
\varphi_*(\xi_* N_\xi \otimes_S N_\varphi) & & ①' & & K_R^\bullet(\mathbf{r}) \otimes_R K_R^\bullet(\mathbf{s}) \\
{\scriptstyle \text{via } d_{\bar{\mathbf{s}}}^{-1}}\big\uparrow & & & & \big\downarrow{\scriptstyle \text{via } d_{\mathbf{r}}^{-1}} \\
\varphi_*(K_S^\bullet(\bar{\mathbf{s}}) \otimes_S N_\varphi) & \xleftarrow[\;\varphi_* s\;]{} & \varphi_*(N_\varphi \otimes_S K_S^\bullet(\bar{\mathbf{s}})) & \xleftarrow[\;p_\varphi\;]{} & \varphi_* N_\varphi \otimes_R K_R^\bullet(\mathbf{s})
\end{array}$$

This diagram is the canonical image in $\mathbf{D}(R)$ of an explicitly describable diagram in the category of R-complexes. To see this, represent $d_{\bar{\mathbf{s}}}^{-1}$ as the image in $\mathbf{D}(S)$ of a map of complexes, as follows (and analogously for $d_{\mathbf{r}}^{-1}$ and $d_{\mathbf{r},\mathbf{s}}^{-1}$):

Lemma 3.5.3. *The map $d_{\bar{\mathbf{s}}}^{-1}$ is the canonical image in $\mathbf{D}(S)$ of the natural composite map of S-complexes*

$$\begin{aligned}
d'_{\bar{\mathbf{s}}}\colon K_S^\bullet(\bar{\mathbf{s}}) \longrightarrow \big(H^d K_S^\bullet(\bar{\mathbf{s}})\big)[-d] &\xrightarrow{\ \sim\ } \big(K_S^d(\bar{\mathbf{s}}) \otimes_S \xi_* T\big)[-d] \\
&\xrightarrow{\ \sim\ } \xi_* \operatorname{Hom}_T\big(\wedge_T^d(T^d), T\big)[-d] \\
&\xrightarrow[\text{via } \vartheta_{\bar{\mathbf{s}}}]{\ \sim\ } \xi_* \operatorname{Hom}_T\big(\wedge_T^d(I/I^2), T\big)[-d] = \xi_* N_\xi.
\end{aligned}$$

Proof. It suffices to prove equality of the adjoint maps, that is, that the border of the following natural diagram commutes:



Showing commutativity of subdiagram ⑤ is a minor variant of showing that $c^{\#}_{\xi}$ is identifiable with $\Gamma(X, c^{\flat}_{f})$ (see the third paragraph before 3.5.1). Details are left to the reader.

The commutativity of the unlabeled subdiagrams is easy to check.

The desired conclusion results. □

In continuation of the proof of 3.5.1, to describe $d'_{\bar{\mathbf{s}}}$ more explicitly, the following abbreviations are helpful.

With $r_i^L := (r_i + L^2) \in L/L^2$, and so on, one has the generators

$$\begin{aligned} \mathbf{r}^J &:= r_1^J \wedge \cdots \wedge r_e^J && \text{of } \textstyle\bigwedge_S^e(J/J^2),\\ \bar{\mathbf{s}}^I &:= \bar{s}_1^I \wedge \cdots \wedge \bar{s}_d^I && \text{of } \textstyle\bigwedge_T^d(I/I^2),\\ (\mathbf{r},\mathbf{s})^L &:= r_1^L \wedge \cdots \wedge r_e^L \wedge s_1^L \wedge \cdots \wedge s_d^L && \text{of } \textstyle\bigwedge_T^{d+e}(L/L^2). \end{aligned}$$

With $(v_1, \dots, v_{d+e})$ the standard basis of $R^{d+e} = R^d \oplus R^e$, and $(w_1, \dots, w_d)$ the standard basis of S^d, one has the generators

$$\begin{aligned} \mathbf{v}^{d+e} &:= v_1 \wedge \cdots \wedge v_{d+e} && \text{of } \textstyle\bigwedge_R^{d+e}(R^{d+e}),\\ \mathbf{v}^{d} &:= v_1 \wedge \cdots \wedge v_{d} && \text{of } \textstyle\bigwedge_R^{d}(R^{d}),\\ \mathbf{v}^{d,e} &:= v_{d+1} \wedge \cdots \wedge v_{d+e} && \text{of } \textstyle\bigwedge_R^{e}(R^{e}),\\ \mathbf{w}^{d} &:= w_1 \wedge \cdots \wedge w_{d} && \text{of } \textstyle\bigwedge_R^{d}(S^{d}). \end{aligned}$$

For a ring Q and a generator $\mathbf{g}$ of a rank-one free Q-module G, denote by $\mathbf{1}_Q/\mathbf{g}$ the map in $\mathrm{Hom}_Q(G, Q)$ that takes $\mathbf{g}$ to the identity 1_Q of Q.

One checks that in degree $d + e$,

$$\mathfrak{g}_{\mathbf{r},\mathbf{s}}\big((1_R/\mathbf{v}^{d,e}) \otimes_R (1_R/\mathbf{v}^{d})\big) = (-1)^{de}(1_R/\mathbf{v}^{d+e}).$$

Further, in degree d,

$$d'_{\bar{\mathbf{s}}}(1_S/\mathbf{w}^d) = 1_T/\bar{\mathbf{s}}^I,$$

and analogously,

$$d'_{\mathbf{r}}(1_R/\mathbf{v}^{d,e}) = 1_S/\mathbf{r}^J \qquad \text{(in degree } e\text{)},$$
$$d'_{\mathbf{r},\mathbf{s}}(1_R/\mathbf{v}^{d+e}) = 1_T/(\mathbf{r},\mathbf{s})^L \qquad \text{(in degree } d+e\text{)}.$$

Now one need only verify commutativity of the diagram of R-complexes represented by ①$'$ with d^{-1} replaced by d'; and for this, one need only look at what happens to the generator $1_R/\mathbf{v}^{d+e}$ of the R-module $K_R^{d+e}((\mathbf{r},\mathbf{s}))$.

Recalling (2.10.21), one checks that moving around counterclockwise from $K_R^{d+e}((\mathbf{r},\mathbf{s}))$ to $\varphi_*(\xi_*N_\xi \otimes_S N_\varphi)^{d+e}$ acts successively on $1_R/\mathbf{v}^{d+e}$ as:

$$\begin{aligned} 1_R/\mathbf{v}^{d+e} &\longmapsto 1_T/(\mathbf{r},\mathbf{s})^L \\ &\longmapsto (1_T/\bar{\mathbf{s}}^I)\otimes_T (1_T\otimes_S 1_S/\mathbf{r}^{\mathbf{J}}) \longmapsto (1_T/\bar{\mathbf{s}}^I)\otimes_S (1_S/\mathbf{r}^J). \end{aligned}$$

On the other hand, moving clockwise acts successively on $1_R/\mathbf{v}^{d+e}$ as:

$$\begin{aligned} 1_R/\mathbf{v}^{d+e} &\longmapsto (-1)^{de}(1_R/\mathbf{v}^{d,e}\otimes_R (1_R/\mathbf{v}^d) \\ &\longmapsto (-1)^{de}(1_S/\mathbf{r}^J)\otimes_R (1_R/\mathbf{v}^d) \\ &\longmapsto (-1)^{de}(1_S/\mathbf{r}^J)\otimes_S (1_S/\mathbf{w}^d) \qquad \text{(see (3.3.3) \textit{ff.})} \\ &\longmapsto (1_S/\mathbf{w}^d)\otimes_S (1_S/\mathbf{r}^J) \longmapsto (1_T/\bar{\mathbf{s}}^I)\otimes_S (1_S/\mathbf{r}^J). \end{aligned}$$

This completes the proof. □

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