BIVARIANCE, GROTHENDIECK DUALITY AND HOCHSCHILD HOMOLOGY, II: THE FUNDAMENTAL CLASS OF A FLAT SCHEME-MAP

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Abstract. Fix a noetherian scheme S. For any flat map $f\colon X \to Y$ of separated essentially-finite-type perfect S-schemes we define a canonical derived-category map $\mathbf{c}_f\colon \mathcal{H}_X \to f^!\mathcal{H}_Y$, the *fundamental class of* f, where $\mathcal{H}_Z$ is the (pre-)Hochschild complex of an S-scheme Z and $f^!$ is the twisted inverse image coming from Grothendieck duality theory. When $Y = S$ and f is essentially smooth of relative dimension n, this gives an isomorphism $\Omega^n_f[n] = H^{-n}(\mathcal{H}_X)[n] \xrightarrow{\sim} f^!\mathcal{O}_S$. We focus mainly on *transitivity* of $\mathbf{c}$ vis-à-vis compositions $X \to Y \to Z$, and on the compatibility of $\mathbf{c}$ with *flat base change.* These properties imply that $\mathbf{c}$ orients the flat maps in the bivariant theory of part I [AJL], compatibly with essentially étale base change. Furthermore, $\mathbf{c}$ leads to a *dual* oriented bivariant theory, whose homology is the classical Hochschild homology of flat S-schemes. When $Y = S$, $\mathbf{c}$ is used to define a duality map $\mathfrak{d}_X\colon \mathcal{H}_X \to \mathsf{R}\mathcal{H}om(\mathcal{H}_X, f^!\mathcal{O}_S)$, an isomorphism if f is essentially smooth. These results apply in particular to flat essentially-finite-type maps of noetherian rings.

Contents

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Introduction

0.1. In the prequel [AJL] we developed a *bivariant theory* on the category of separated, essentially finite-type, perfect (i.e., finite tor-dimension) schemes $x\colon X \to S$ over a fixed noetherian base scheme S. The theory is based on properties of the (pre-)Hochschild complex $\mathcal{H}_x := \mathsf{L}\delta_x^*\mathsf{R}\delta_{x*}\mathcal{O}_X$ where the map $\delta_x\colon X \to X \times_S X$ is the diagonal (§1 below), and of the twisted inverse image pseudofunctor $(-)^!$ from Grothendieck duality theory. It associates to a morphism $f\colon (X \xrightarrow{x} S) \to (Y \xrightarrow{y} S)$ of such S-schemes the graded group

$$\mathrm{HH}^*(f) := \oplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{H}_x, f^!\mathcal{H}_y) = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}(X)}\big(\mathcal{H}_x, f^!\mathcal{H}_y[i]\big),$$

so that the associated cohomology groups are

$$\mathrm{HH}^i(X|S) := \mathrm{HH}^i(\mathrm{id}_X) = \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{H}_x, \mathcal{H}_x)$$

and the associated homology groups are

$$\mathrm{HH}_i(X|S) := \mathrm{HH}^{-i}(x) = \operatorname{Ext}^{-i}_{\mathcal{O}_X}(\mathcal{H}_x, x^!\mathcal{O}_S).$$

Before proceeding, let us emphasize that being able (thanks to [Nk]) to work with *essentially* finite-type maps, one sees, upon consideration of affine schemes, that *the preceding results, and those that follow, hold, in particular, in the context of local commutative algebra.*

0.2. In this paper we prove some basic properties of the *fundamental class of a flat map* $f\colon X \to Y$ of S-schemes $x\colon X \to S$, $y\colon Y \to S$ as above. Having fixed S, we'll often set $\delta_X := \delta_x$, $\mathcal{H}_X := \mathcal{H}_x$. With such notation, the fundamental class of f is a natural functorial map, defined in §2,

$$\boxed{\mathbf{c}_f\colon \mathsf{L}\delta_X^*\mathsf{R}\delta_{X*}\mathsf{L}f^* \to f^!\mathsf{L}\delta_Y^*\mathsf{R}\delta_{Y*}\,.} \tag{0.2.1}$$

The map $\mathbf{c}_f$ entails a natural map $\mathbf{c}_f(\mathcal{O}_Y)\colon \mathcal{H}_X \to f^!\mathcal{H}_Y$. Thus one has the canonical element

$$c_f := \mathbf{c}_f(\mathcal{O}_Y) \in \mathrm{HH}^0(f).$$

In particular, when $y = \mathrm{id}_S$, one gets a map in $\mathrm{HH}^0(x) = \mathrm{HH}_0(X|S)$:

$$c_x\colon \mathcal{H}_X \to x^!\mathcal{O}_S. \tag{0.2.2}$$

In this case there is a natural (up to sign) $\mathcal{O}_X$-isomorphism

$$\Omega^1_{X|S} \xrightarrow{\ \sim\ } \mathcal{T}or_1^{X\times_S X}(\mathcal{O}_X, \mathcal{O}_X) = H^{-1}\mathcal{H}_X,$$

whence, by means of the standard alternating graded $\mathcal{O}_X$-algebra structure on $\oplus_{i\ge 0}\mathcal{T}or_i^{X\times_S X}(\mathcal{O}_X, \mathcal{O}_X)$, the universal property of exterior algebras gives rise to natural maps

$$\Omega^i_{X|S} \to \mathcal{T}or_i^{X\times_S X}(\mathcal{O}_X, \mathcal{O}_X) = H^{-i}\mathcal{H}_X \qquad (i \ge 0).$$

Composing these with the maps $H^{-i}\mathcal{H}_X \to H^{-i}x^!\mathcal{O}_S$ induced by (0.2.2), one gets natural maps of coherent sheaves

$$\Omega^i_{X|S} \to H^{-i}x^!\mathcal{O}_S \qquad (i \ge 0).$$

In particular, if x is equidimensional of relative dimension n, one gets a map

$$\Omega^n_{X|S} \to \omega_{X|S} := H^{-n}x^!\mathcal{O}_S,$$

where $\omega_{X|S}$ is the relative dualizing (or canonical) sheaf associated to x; or equivalently, a derived-category map

$$\boxed{C_{X|S}\colon \Omega^n_{X|S}[n] \to x^!\mathcal{O}_S.} \tag{0.2.3}$$

This byproduct of (0.2.1) is what has usually been regarded in the literature as the fundamental class.

0.3. The central concern of this paper is with Theorem 3.1, which asserts *transitivity* of the fundamental class vis-à-vis composition of flat S-maps $X \xrightarrow{u} Y \xrightarrow{v} Z$; that is,

$$\boxed{\mathbf{c}_{vu} = u^!\mathbf{c}_v \circ \mathbf{c}_u v^*.}$$

Transitivity gives in particular that $\mathbf{c}_{vu}(\mathcal{O}_Z) = u^!\mathbf{c}_v(\mathcal{O}_Z) \circ \mathbf{c}_u(\mathcal{O}_Y)$. In terms of the bivariant product $\mathrm{HH}^0(u) \times \mathrm{HH}^0(v) \to \mathrm{HH}^0(vu)$, this says:

$$c_{vu} = c_u \cdot c_v.$$

Thus the family c_f is a family of *canonical orientations* for the flat maps in our bivariant theory [FM, p. 28, 2.6.2].

When f is essentially étale, so that $f^! = f^*$, $\mathbf{c}_f$ turns out, nontrivially (see Proposition 2.5), to be inverse to the "Hochschild localization isomorphism" of Theorem 1.7. From this, and transitivity, it follows that *the above orientations are compatible with essentially étale base change,* see Corollary 3.3.

With all this in hand, one can apply the general considerations in [FM] to obtain, for example, *Gysin morphisms,* that provide "wrong-way" functorialities for homology and cohomology (see §3.4).

0.4. Some other applications of the fundamental class are given in §4. We construct an oriented bivariant theory *dual* to the one mentioned above, having the same associated cohomology groups, but whose associated homology groups are the classical Hochschild homology groups—given by the homology of the derived global sections of the Hochschild complex. Also, combining c_x with the usual product $\mathcal{H}_X \otimes^{\mathsf{L}}_X \mathcal{H}_X \to \mathcal{H}_X$ leads to a map $\mathcal{H}_X \to \mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S)$. This is an *isomorphism* whenever x is essentially smooth; and if, moreover, the base scheme $S = \operatorname{Spec} H$ with H a Gorenstein artinian ring, there results a nonsingular pairing of the classical homology groups into H. Presumably (though we have no proof) this pairing is closely related to the Mukai pairing of Căldăraru.

0.5. The proof of the transitivity property, Theorem 3.1, is given in §6, which occupies more than one third of this paper. The reason for the length is that the fundamental class is defined to be the composition of a dozen or so maps, some of which are themselves composed of more elementary maps. Transitivity means roughly that a juxtaposition of two such sequences of maps can be transformed into another such sequence; and this is shown by justifying and combining *many* transformations of subsequences.

Put differently, the Theorem asserts commutativity of a square whose sides are composed of a dozen or so maps; and the strategy for proving this is to decompose this large diagram into smaller ones, and then decompose the smaller ones into still smaller ones, and so on, until the original diagram is decomposed into *many* tiny ones, whose commutativity holds for elementary reasons. We found carrying this process to completion extremely tedious, and not at all straightforward, as any reader who sets out to check the details in §6 will soon see. (And, preliminaries aside, not all the details appear there: some of the easier ones are left to the reader, and for quite a few others, reference is made to [L3].)

Can the proof be made more palatable? No doubt some technical and organizational improvements are possible; but we suspect such improvements would not have a major effect. Some kind of *coherence theorem*—beyond those presently available—might guarantee the commutativity of numerous diagrams in the proof, making much of the minute examination superfluous. Unearthing such a theorem, or a different conceptual approach, remains a challenge.

0.6. In any case, why bother? To respond, let us give the fundamental class some historical context, and mention a number of problems and potential applications for further study.

The fundamental class links the concrete and abstract approaches to Grothendieck duality (see, respectively, [Co] and [L3]). The correspondences between these two approaches are generally taken for granted; but full justifications are not readily available in the literature. For example, for *smooth* morphisms of noetherian schemes, in the concrete approach, the map (0.2.3), an isomorphism in this case, exists more or less by definition; the point is then to show that the top-degree differentials satisfy a suitable generalization of Serre duality, see [H, Chapter VII, §4]. In the abstract approach of Verdier and Deligne, where duality is proved directly, such an isomorphism comes out of the flat base-change theorem and the fundamental local isomorphism for complete intersections, see [V, p. 397, Theorem 3]. Does Verdier's construction, when interpreted in concrete terms, yield the concretely-defined isomorphism? And does his isomorphism behave pseudofunctorially with respect to smooth maps?

If $x\colon X \to S$ is essentially smooth of relative dimension n, then *using Verdier's isomorphism*, we show in Proposition 2.4.2 that (0.2.3) is also an isomorphism. But we don't know whether these two isomorphisms are the same, even up to sign.

More generally (at least in characteristic zero), in [EZ] and [AnZ] El Zein and Angéniol associate to any noetherian $\mathbb{Q}$-scheme S and any morphism $x\colon X \to S$ that is as above, and also equidimensional of relative dimension n, a derived-category map $\gamma_{X|S}\colon \Omega^n_{X|S}[n] \to x^!\mathcal{O}_S$. In [An], Angéniol uses this map for his treatment of Chow schemes.

When $S = \mathrm{Spec}(k)$ with k a perfect field, and X is an integral algebraic scheme over k, a map like $\gamma_{X|k}$ is realized in [L1] as a globalization of the local residue maps at the points of X, leading to explicit versions of local and global duality and the relation between them. These results are generalized to certain maps of noetherian schemes in [HS].

How is $\gamma_{X|S}$ related to (0.2.3)? For this, one will have to explicate the relation between (0.2.3) and the characterizing "trace property" of γ (cf. [An, p. 114, 7.1.3], [AnL, p. 50, 5.2.8 and p. 55, 6.3.1].) A small step toward this is taken in Example 2.6 below.

In all these treatments, an important role is played—via factorizations of x as smooth$\,\circ\,$finite—by the case $n = 0$, where the notion of fundamental class is equivalent to that of traces of differential forms. This leads to a concrete realization of the fundamental class in terms of *regular differential forms*, an algebraic treatment of which is given in [KW].

For more recent developments, in the context of complex spaces, see [Kd].)

Finally, some vague remarks about possible future projects. One should clarify the connection between the fundamental class and Verdier's isomorphism (see above). More generally, one should explicate some concrete aspects of the fundamental class in terms of differential modules, or perhaps cotangent complexes, via their relation to Hochschild complexes, especially in characteristic zero (see, e.g., [BF2]).

As indicated above, there is a close relation between the fundamental class and *residues*. This becomes clearer over formal schemes, where local and global duality merge into a single duality theory, of which fundamental classes and residues are adjoint aspects. From this viewpoint, the transitivity theorem for smooth (resp. finite) maps should be closely related to the properties (R4) and (R10) of residues given in [H, pp. 198–199].

If the theory of the fundamental class could be extended from flat maps to perfect maps, then (R3) could be added to this list. More importantly, such an extension would be desirable for dealing bivariantly with arbitrary finite-type maps between smooth S-schemes. It may involve differential-graded and simplicial methods, as in [BF1], or perhaps cotriples.

1. The (pre-)Hochschild functor

Let $f\colon X \to Y$ be any scheme-map, with associated diagonal map

$$\delta = \delta_f : X \to X \times_Y X.$$

The *pre-Hochschild functor* of f is

$$\boldsymbol{\mathcal{H}}_f := \mathsf{L}\delta^*\mathsf{R}\delta_*\colon \mathbf{D}(X) \to \mathbf{D}(X),$$

where $\mathbf{D}(X)$ is the derived category of (sheaves of) $\mathcal{O}_X$-modules.

The *pre-Hochschild complex* of f is

$$\mathcal{H}_f := \mathsf{L}\delta^*\mathsf{R}\delta_*\mathcal{O}_X.$$

When f is flat, the prefix “pre-” is omitted, see [BF1, p. 222, 2.3.1].

We'll often use the less precise notations $\boldsymbol{\mathcal{H}}_{X|Y}$ for $\boldsymbol{\mathcal{H}}_f$ and $\mathcal{H}_{X|Y}$ for $\mathcal{H}_f$.

This section contains some basic facts about $\boldsymbol{\mathcal{H}}_f$ and $\mathcal{H}_f$ that are needed in the subsequent treatment of fundamental class maps. The key points are Corollary 1.6.3 (transitivity for $\boldsymbol{\mathcal{H}}_f$) and Theorem 1.7 (essentially étale localization for $\boldsymbol{\mathcal{H}}_f$, generalizing to the present setting a result of Geller and Weibel [GeW, Theorem (0.1)]).

In §§1.1–1.5 we review some necessary preliminaries.

Then in §1.6 we discuss the variance of $\boldsymbol{\mathcal{H}}_f$ with f, and in particular, its compatibility with flat base-change (Corollary 1.6.2) and its transitivity. As special cases of variance one has, for scheme-diagrams $X \xrightarrow{f} Y \xrightarrow{g} Z$, homomorphisms

$$\mathcal{H}_{X|Z} \to \mathcal{H}_{X|Y}\,, \qquad \mathsf{L}f^*\mathcal{H}_{Y|Z} \to \mathcal{H}_{X|Z}\,, \qquad \mathcal{H}_{Y|Z} \to \mathsf{R}f_*\mathcal{H}_{X|Z}\,,$$

the third being adjoint to the second (Example 1.6.4).

1.1. The term *qcqs*, adjectivally modifying “scheme” or “map,” will be used as an abbreviation for *quasi-compact and quasi-separated* (see [Gr1, §6.1]). (In the oft-to-be-used reference [L3], qcqs is called *concentrated*.)

Any scheme-map with noetherian source is qcqs.

A scheme-map $f\colon X \to Y$ is *essentially of finite presentation* (efp) if it is qcqs and if for all $\xi \in X$ there exist affine open neighborhoods $\operatorname{Spec} L$ of ξ and $\operatorname{Spec} K$ of $f(\xi)$ such that L is a ring of fractions of a finitely-presentable K-algebra. If f is qcqs and for each ξ there are such K and L with L a ring of fractions of K itself, then f is said to be *localizing*.

When X and Y are noetherian, one can use for “finitely-presentable” the equivalent term “finite-type.”

The map $f\colon X \to Y$ is *essentially smooth* (resp. *essentially étale*, resp. *essentially unramified*) if f is efp and formally smooth (resp. formally étale, resp. formally unramified), see [Gr4, §17.1].

When f is essentially smooth, the module of relative differentials Ω^1_f is locally free of finite rank, say, n_f, where n_f, the *relative dimension* of f, is a function from X to $\mathbb{Z}$, constant on connected components. (For local projectivity, see [Gr4, (16.10.2)]; and for finiteness, see, e.g., the proof of 1.7 below.) Moreover, if Y is noetherian then the diagonal map $X \to X \times_Y X$ is a *regular immersion*: each $\xi \in X \subset X \times_Y X$ has an open neighborhood $U \subset X \times_Y X$ such that $\Gamma(U \cap X, \mathcal{O}_X)$ is a quotient of $\Gamma(U, \mathcal{O}_U)$ by a regular sequence of length $n_f(\xi)$, see [Gr4, §16.9].

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be scheme-maps, where g is qcqs (resp. efp). One verifies then, via [Gr1, §6.1], that f is qcqs (resp. efp) if and only if so is gf; and if $Z' \to Z$ is any scheme-map then the projection $Z' \times_Z Y \to Z'$ is qcqs (resp. efp). It follows that the fiber product, in the category of schemes, of any two qcqs (resp. efp) maps with the same target, is also a fiber product in the subcategory of qcqs (resp. efp) maps.

Thus if f and g are qcqs (resp. efp) then so is the graph $\Gamma_f\colon X \to X\times_Z Y$.

Similar assertions hold with "separated" (resp. "essentially étale") in place of "efp", see [Gr1, p. 279, (5.3.1)] (resp. [Gr4, (17.1.3)(ii) and (iii), (17.1.4)]).

From [Gr4, Theorem (17.5.1)] it follows that any essentially smooth map—in particular, any essentially étale map—is flat.

1.2. For any scheme W let $\mathbf{D}(W)$ be the derived category of $\mathcal{O}_W$-modules; and let

$$\mathbf{D}_W := \mathbf{D}_{\mathsf{qc}}(W) \subset \mathbf{D}(W)$$

be the full subcategory with objects the complexes whose homology sheaves are all quasi-coherent. When W is qcqs, the natural functor into $\mathbf{D}_W$ from the derived category of quasi-coherent $\mathcal{O}_W$-modules is an equivalence of categories [BN, p. 230, 5.5].

As we will deal almost exclusively with derived functors, we will usually lighten notation by omitting the symbols L and R: given a scheme-map $f\colon X \to Y$, we'll write $f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ for $\mathsf{R}f_*$, $f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X)$ for $\mathsf{L}f^*$, and $\otimes_X\colon \mathbf{D}(X)\times\mathbf{D}(X) \to \mathbf{D}(X)$ for the left-derived functor $\otimes^{\mathsf{L}}_X$. In the presence of such abbreviations, it should not be forgotten that we are working with derived functors, unless otherwise indicated.

Remarks. • Derived functors are determined up to canonical isomorphism, by universal properties. We assume throughout that some specific choice of such functors has been made. As we make use only of the characteristic universal properties, our results do not depend on the choice.

In this vein, we always assume that id^*_X and $(\mathcal{O}_X \otimes_X -)$ are identity functors, and that for $f\colon X \to Y$ that $f^*\mathcal{O}_Y = \mathcal{O}_X$.

• For any scheme-map $f\colon X \to Y$, one has $f^*\mathbf{D}_Y \subset \mathbf{D}_X$ [L3, 3.9.1], and if f is qcqs (§1.1) then $f_*\mathbf{D}_X \subset \mathbf{D}_Y$ [L3, 3.9.2]. Hence if f is qcqs then $\boldsymbol{\mathcal{H}}_f\mathbf{D}_X \subset \mathbf{D}_X$.

• For qcqs f, there is a canonical functorial isomorphism (cf. (2.2.6)):

$$\zeta(G)\colon \mathcal{H}_f \otimes^{\mathsf{L}}_X G \xrightarrow{\ \sim\ } \boldsymbol{\mathcal{H}}_f(G) \qquad (G\in\mathbf{D}_X).$$

As this will not be used in what follows, we'll say no more about it.

• In this paper, the functors and functorial maps that appear respect the usual triangulated and graded structures on $\mathbf{D}_{\mathsf{qc}}$ (see, e.g., [AJL, §§5.2, 5.7]). That fact will play no role.

1.3. On the category of schemes there are *adjoint monoidal pseudofunctors* $(-)^*$ and $(-)_*$ (the first contravariant and the second covariant) assigning to any map $f\colon X \to Y$ the functors f^* and f_* in §1.2 (see [L3, 3.6.10]). Adjointness means there are functorial unit and counit maps

$$\eta = \eta_f\colon \mathrm{id} \to f_*f^* \quad\text{ and }\quad \epsilon = \epsilon_f\colon f^*f_* \to \mathrm{id} \tag{1.3.1}$$

such that for $A\in\mathbf{D}_X$ and $C\in\mathbf{D}_Y$ the corresponding compositions

$$f_*A \xrightarrow{\eta_{f_*A}} f_*f^*f_*A \xrightarrow{f_*\epsilon_A} f_*A, \qquad f^*C \xrightarrow{f^*\eta_C} f^*f_*f^*C \xrightarrow{\epsilon_{f^*C}} f^*C$$

are identity maps. Pseudofunctoriality of $(-)^*$ and $(-)_*$ entails, for any scheme-maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, isomorphisms

$$\mathsf{ps}^*\colon f^*g^* \xrightarrow{\ \sim\ } (gf)^*, \qquad \mathsf{ps}_*\colon (gf)_* \xrightarrow{\ \sim\ } g_*f_*, \tag{1.3.2}$$

satisfying a kind of associativity vis-à-vis $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ (see, e.g., [L3, p. 120]); and pseudofunctoriality of the foregoing adjunction is expressed by commutativity, for any $X \xrightarrow{f} Y \xrightarrow{g} Z$, of the diagram

$$\begin{tikzcd}
\mathrm{id} \arrow[rr, "\eta_g"] \arrow[d, "\eta_{gf}"'] & & g_*g^* \arrow[r, "\text{via } \eta_f"] & g_*(f_*f^*g^*) \arrow[d, equal] \\
(gf)_*(gf)^* \arrow[r, equal, "\mathsf{ps}_*"'] & g_*f_*(gf)^* \arrow[rr, equal, "\text{via } \mathsf{ps}^*"'] & & g_*f_*f^*g^*
\end{tikzcd}$$

For details, work backwards from [L3, p. 124, 3.6.10].

These pseudofunctors interact with the left-derived tensor product $\otimes$ via a natural isomorphism

$$\nu_{f,E,F}\colon f^*(E \otimes_Y F) \xrightarrow{\ \sim\ } f^*E \otimes_X f^*F \qquad \big(E, F \in \mathbf{D}(Y)\big), \tag{1.3.3}$$

see [L3, 3.2.4]; via the functorial map

$$f_*G \otimes_Y f_*H \to f_*(G \otimes_X H) \qquad \big(G, H \in \mathbf{D}(X)\big) \tag{1.3.4}$$

adjoint to the natural composite map

$$f^*(f_*G \otimes_Y f_*H) \underset{(1.3.3)}{\xrightarrow{\ \sim\ }} f^*f_*G \otimes_X f^*f_*H \longrightarrow G \otimes_X H;$$

and via the functorial *projection isomorphisms,* for $F \in \mathbf{D}_Y,\ G \in \mathbf{D}_X$,

$$f_*G \otimes_Y F \xrightarrow{\ \sim\ } f_*(G \otimes_X f^*F), \qquad F \otimes_Y f_*G \xrightarrow{\ \sim\ } f_*(f^*F \otimes_X G), \tag{1.3.5}$$

the first being defined qua map to be the natural composition

$$f_*G \otimes_Y F \longrightarrow f_*G \otimes_Y f_*f^*F \underset{(1.3.4)}{\longrightarrow} f_*(G \otimes_X f^*F),$$

and similarly for the second, see [L3, 3.9.4].

The pseudofunctorially adjoint pair $\big((-)^*, (-)_*\big)$ is ultimately determined, by categorical properties, only up to unique isomorphism. The pair can be so chosen that for any scheme-map $f\colon X \to Y$, one has that $f^*\mathcal{O}_Y = \mathcal{O}_X$, that the map η_f in (1.3.1) is the natural composition $\mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathsf{R}f_*\mathcal{O}_X$ (where for just this moment, f_* is the *nonderived* direct image functor), that the map $\mathsf{ps}^*(\mathcal{O}_Z)$ in (1.3.2) is the identity map of $\mathcal{O}_X$, that the map (1.3.3) is the obvious one when either E or F is $\mathcal{O}_Y$, and that the map (1.3.5) is the obvious one when $F = \mathcal{O}_Y$ (cf. e.g., [L3, 3.4.7(iii)]).

1.4. In a category, an *orientation* of a relation $f \circ v = u \circ g$ among maps is an ordered pair (right arrow, bottom arrow) whose members are f and u.

This can be represented by one of two *oriented commutative squares,* namely $\mathbf{d}$ with bottom arrow u, and its transpose $\mathbf{d}'$ with bottom arrow f.

$$
\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathbf{d}"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\mathbf{d}'"] & \bullet \arrow[d, "u"] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}
$$

For any oriented commutative square of scheme-maps

$$
\begin{tikzcd}
X' \arrow[r, "1"] \arrow[d, "2"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "4"] \\
Y' \arrow[r, "3"'] & Y
\end{tikzcd}
$$

the natural map of functors

$$\theta_{\mathbf{d}} : 3^*4_* \to 2_*1^*, \tag{1.4.1}$$

is defined to be the composition of the following chain of maps of functors (from $\mathbf{D}(X)$ to $\mathbf{D}(Y')$):

$$3^*4_* \xrightarrow{\text{via } \eta_2} 2_*2^*3^*4_* \overset{\text{via } \mathsf{ps}^*}{=\!=\!=} 2_*1^*4^*4_* \xrightarrow{\text{via } \epsilon_4} 2_*1^*, \tag{1.4.2}$$

or equivalently (see [L3, p. 127, 3.7.2]),

$$3^*4_* \xrightarrow{\text{via } \eta_1} 3^*4_*1_*1^* \overset{\text{via } \mathsf{ps}_*}{=\!=\!=} 3^*3_*2_*1^* \xrightarrow{\text{via } \epsilon_3} 2_*1^*. \tag{1.4.3}$$

1.4.4. If $\mathbf{d}$ is a fiber square (i.e., the naturally associated map is an isomorphism $X' \xrightarrow{\sim} X \times_Y Y'$) with 4 qcqs (§1.1) and 3 flat, then $\theta_{\mathbf{d}}(G)$ is an isomorphism for all $G \in \mathbf{D}_X$ (see [L3, p. 142, Proposition 3.9.5]).

1.5. Let there be given an oriented commutative square of scheme-maps

$$
\begin{tikzcd}
X' \arrow[r, "1"] \arrow[d, "2"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "4"] \\
Y' \arrow[r, "3"'] & Y
\end{tikzcd}
$$

With ps^* the natural isomorphism ([L3, p. 118, (3.6.1)*]) and $\theta_{\mathbf{d}}$ as in 1.4, define

$$\phi_{\mathbf{d}} \colon 1^*4^*4_* \longrightarrow 2^*2_*1^* \tag{1.5.1}$$

to be the following composition of functorial maps:

$$1^*4^*4_* \overset{\mathsf{ps}^*}{=\!=} 2^*3^*4_* \xrightarrow{2^*\theta_{\mathbf{d}}} 2^*2_*1^*. \tag{1.5.2}$$

Proposition 1.5.3. *If $\mathbf{d}$ is a fiber square in which the map 4 is qcqs and 3 is flat, then $\phi_{\mathbf{d}}(G)$ is an isomorphism for all $G \in \mathbf{D}_X$.*

Proof. This holds because $\theta_{\mathbf{d}}(G)$ is an isomorphism (see 1.4.4). $\square$

Here is a transitivity property of ϕ.

Proposition 1.5.4. *Let* $\mathbf{d} = \mathbf{u}\circ\mathbf{v}$ *be the composite oriented commutative square*

$$\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "2"'] \arrow[dr, phantom, "\mathbf{v}"] & \bullet \arrow[r, "5"] \arrow[d, "4"'] \arrow[dr, phantom, "\mathbf{u}"] & \bullet \arrow[d, "6"] \\
\bullet \arrow[r, "3"'] & \bullet \arrow[r, "7"'] & \bullet
\end{tikzcd}$$

With $\phi_{\mathbf{d}}$, $\phi_{\mathbf{v}}$ *and* $\phi_{\mathbf{u}}$ *as in* (1.5.1), *the following diagram commutes.*

$$\begin{tikzcd}
(5\circ 1)^*6^*6_* \arrow[rr, "\phi_{\mathbf{d}}"] \arrow[d, equal, "\mathsf{ps}^*"'] & & 2^*2_*(5\circ 1)^* \arrow[d, equal, "2^*2_*\mathsf{ps}^*"] \\
1^*5^*6^*6_* \arrow[r, "1^*\phi_{\mathbf{u}}"'] & 1^*4^*4_*5^* \arrow[r, "\phi_{\mathbf{v}}"'] & 2^*2_*1^*5^*
\end{tikzcd}$$

Proof. Expand the diagram in question, as follows:

$$\begin{tikzcd}
(5\circ 1)^*6^*6_* \arrow[rr, equal, "\mathsf{ps}^*"] \arrow[dd, equal, "\mathsf{ps}^*"'] & & 2^*(7\circ 3)^*6_* \arrow[rr, "2^*\theta_{\mathbf{d}}"] \arrow[d, equal, "2^*\mathsf{ps}^*"] & & 2^*2_*(5\circ 1)^* \arrow[dd, equal, "2^*2_*\mathsf{ps}^*"] \\
 & \textcircled{1} & 2^*3^*7^*6_* \arrow[dl, equal, "\mathsf{ps}^*"'] \arrow[dr, "2^*3^*\theta_{\mathbf{u}}"] & \textcircled{2} & \\
1^*5^*6^*6_* \arrow[r, equal, "1^*\mathsf{ps}^*"'] & 1^*4^*7^*6_* \arrow[dr, "1^*4^*\theta_{\mathbf{u}}"'] & \textcircled{3} & 2^*3^*4_*5^* \arrow[r, "2^*\theta_{\mathbf{v}}"'] \arrow[dl, equal, "\mathsf{ps}^*"] & 2^*2_*1^*5^* \\
 & & 1^*4^*4_*5^* & &
\end{tikzcd}$$

Commutativity of ① follows from associativity of pseudofunctoriality, of ② follows from transitivity for θ (see [L3, p. 128, Proposition 3.7.2(iii)]), and of ③ is obvious, whence the conclusion. $\square$

1.6. We examine the variance of $\boldsymbol{\mathcal{H}}_f$ with respect to f.

Given an oriented commutative square of scheme-maps

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd} \tag{1.6.0}$$

let $\mathbf{d}_\times$ be the oriented commutative square

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "\delta_{f'}"'] \arrow[dr, phantom, "\mathbf{d}_\times"] & X \arrow[d, "\delta_f"] \\ X'\times_{Y'} X' \arrow[r, "h\times h"'] & X\times_Y X \end{tikzcd}$$

and let $\mathbf{d}^\sharp\colon h^*\boldsymbol{\mathcal{H}}_{X|Y} \to \boldsymbol{\mathcal{H}}_{X'|Y'}h^*$ be the functorial morphism $\phi_{\mathbf{d}_\times}$:

(1.6.1) $$\mathbf{d}^\sharp\colon h^*\boldsymbol{\mathcal{H}}_{X|Y} = h^*\delta_f^*\delta_{f*} \xrightarrow{\phi_{\mathbf{d}_\times}} \delta_{f'}^*\delta_{f'*}h^* = \boldsymbol{\mathcal{H}}_{X'|Y'}h^*.$$

Also, let

$$\mathbf{d}_\sharp\colon \boldsymbol{\mathcal{H}}_{X|Y} \to h_*\boldsymbol{\mathcal{H}}_{X'|Y'}h^*$$

be the adjoint of $\mathbf{d}^\sharp$, that is, the composition (with η as in (1.3.1))

$$\boldsymbol{\mathcal{H}}_{X|Y} \xrightarrow{\eta_h} h_*h^*\boldsymbol{\mathcal{H}}_{X|Y} \xrightarrow{h_*\mathbf{d}^\sharp} h_*\boldsymbol{\mathcal{H}}_{X'|Y'}h^*.$$

We define the map

$$\mathsf{d}^\sharp\colon h^*\mathcal{H}_{X|Y} \to \mathcal{H}_{X'|Y'}$$

to be the composition

$$h^*\mathcal{H}_{X|Y} = h^*\boldsymbol{\mathcal{H}}_{X|Y}\mathcal{O}_X \xrightarrow{\mathbf{d}^\sharp(\mathcal{O}_X)} \boldsymbol{\mathcal{H}}_{X'|Y'}h^*\mathcal{O}_X \underset{\text{can}}{\xrightarrow{\;\sim\;}} \boldsymbol{\mathcal{H}}_{X'|Y'}\mathcal{O}_{X'} = \mathcal{H}_{X'|Y'},$$

and let

$$\mathsf{d}_\sharp\colon \mathcal{H}_{X|Y} \to h_*\mathcal{H}_{X'|Y'}$$

be the corresponding adjoint map.

Corollary 1.6.2. *If $\mathbf{d}$ in (1.6.0) is an oriented fiber square in which g is flat and f is qcqs, then for all $G\in\mathbf{D}_X$, $\mathbf{d}^\sharp$ is an isomorphism*

$$h^*\boldsymbol{\mathcal{H}}_{X|Y}G \xrightarrow{\;\sim\;} \boldsymbol{\mathcal{H}}_{X'|Y'}h^*G.$$

In particular, $\mathsf{d}^\sharp : h^*\mathcal{H}_{X|Y} \to \mathcal{H}_{X'|Y'}$ is an isomorphism.

Proof. Since $\mathbf{d}$ is a fiber square, therefore so is $\mathbf{d}_\times$.

Since f is qcqs therefore so is δ_f [Gr1, p. 294, (6.1.9)(i), (iii), and p. 291, (6.1.5)(v)].

The projection $(X\times_Y X)\times_Y Y' \to X\times_Y X$ is flat (since g is), and its composition with the natural isomorphism

$$X'\times_{Y'} X' \xrightarrow{\;\sim\;} (X\times_Y X)\times_Y Y'$$

is the bottom arrow of $\mathbf{d}_\times$, which is therefore flat.

So the assertion results from Proposition 1.5.3. $\square$

(A more general result is given in Corollary 1.7.1 below.)

Here is a transitivity property of $\mathsf{d}^\sharp$.

Corollary 1.6.3. *Let* d *be the composite oriented commutative square*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\mathsf{v}"] & Y \arrow[r, "g"] \arrow[d, "y"'] \arrow[dr, phantom, "\mathsf{u}"] & Z \arrow[d, "z"] \\
S'' \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

The following diagram commutes.

$$\begin{tikzcd}
(gf)^*\boldsymbol{\mathcal{H}}_{Z|S} \arrow[rr, "\mathsf{d}^\sharp"] \arrow[d, equal, "\mathsf{ps}^*"'] & & \boldsymbol{\mathcal{H}}_{X|S''}(gf)^* \arrow[d, equal, "\text{via }\mathsf{ps}^*"] \\
f^*g^*\boldsymbol{\mathcal{H}}_{Z|S} \arrow[r, "f^*\mathsf{u}^\sharp"'] & f^*\boldsymbol{\mathcal{H}}_{Y|S'}g^* \arrow[r, "\mathsf{v}^\sharp"'] & \boldsymbol{\mathcal{H}}_{X|S''}f^*g^*
\end{tikzcd}$$

In particular, the following diagram commutes.

$$\begin{tikzcd}
(gf)^*\mathcal{H}_{Z|S} \arrow[r, "\mathsf{d}^\sharp"] \arrow[d, equal, "\mathsf{ps}^*"'] & \mathcal{H}_{X|S''} \\
f^*g^*\mathcal{H}_{Z|S} \arrow[r, "f^*\mathsf{u}^\sharp"'] & f^*\mathcal{H}_{Y|S'} \arrow[u, "\mathsf{v}^\sharp"']
\end{tikzcd}$$

Proof. This is just Proposition 1.5.4 applied to the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\delta_x"'] \arrow[dr, phantom, "\mathsf{v}_\times"] & Y \arrow[r, "g"] \arrow[d, "\delta_y"'] \arrow[dr, phantom, "\mathsf{u}_\times"] & Z \arrow[d, "\delta_z"] \\
X \times_{S''} X \arrow[r] & Y \times_{S'} Y \arrow[r] & Z \times_S Z
\end{tikzcd}$$

where the arrows in the bottom row are the obvious ones. □

Examples 1.6.4. Given scheme-maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, one has, as special cases of the above constructions, canonical associated morphisms

$$\boldsymbol{\mathcal{H}}_{X|Z} \to \boldsymbol{\mathcal{H}}_{X|Y}\,, \qquad f^*\boldsymbol{\mathcal{H}}_{Y|Z} \to \boldsymbol{\mathcal{H}}_{X|Z}f^*, \qquad \boldsymbol{\mathcal{H}}_{Y|Z} \to f_*\boldsymbol{\mathcal{H}}_{X|Z}f^*,$$

the last two adjoint to each other.

Evaluation at $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) yields canonical homomorphisms

$$\mathcal{H}_{X|Z} \to \mathcal{H}_{X|Y}, \qquad f^*\mathcal{H}_{Y|Z} \to \mathcal{H}_{X|Z}, \qquad \mathcal{H}_{Y|Z} \to f_*\mathcal{H}_{X|Z}.$$

Here are the details. Let $i : X \times_Y X \to X \times_Z X$ be the canonical immersion. One has then the oriented commutative squares

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathbf{d}_{f,g}"] & X \arrow[d, "gf"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\delta_f"'] \arrow[dr, phantom, "(\mathbf{d}_{f,g})_\times"] & X \arrow[d, "\delta_{gf}"] \\ X \times_Y X \arrow[r, "i"'] & X \times_Z X \end{tikzcd}$$

and one checks that $(\mathbf{d}_{f,g})^\sharp = (\mathbf{d}_{f,g})_\sharp \colon \boldsymbol{\mathcal{H}}_{X|Z} \to \boldsymbol{\mathcal{H}}_{X|Y}$ is the composition

$$\delta_{gf}^*\delta_{gf*} \xrightarrow[\quad]{\delta_{gf}^*\,\mathsf{ps}_*} \delta_{gf}^*\, i_*\delta_{f*} \xrightarrow[\quad]{\mathsf{ps}^*} \delta_f^* i^* i_*\delta_{f*} \xrightarrow{\delta_f^*\epsilon_i} \delta_f^*\delta_{f*}.$$

If, for example, g is *essentially unramified* (§1.1) then, since

(1.6.4.1)
$$\begin{tikzcd} X \times_Y X \arrow[r, "i"] \arrow[d, "\text{natural}"'] & X \times_Z X \arrow[d, "f \times f"] \\ Y \arrow[r, "\delta_g"'] & Y \times_Z Y; \end{tikzcd}$$

is a fiber square and, as follows from [Gr4, 17.4.1], δ_g is a local isomorphism, therefore i is a local isomorphism, and so $\varepsilon_i \colon i^*i_* \to \mathrm{id}$ is an isomorphism. Thus $(\mathbf{d}_{f,g})^\sharp$ is a functorial isomorphism

$$\boldsymbol{\mathcal{H}}_{X|Z} \xrightarrow{\ \sim\ } \boldsymbol{\mathcal{H}}_{X|Y}.$$

For example, if $h \colon X \to Z$ is a qcqs map such that the kernel I of the associated map $\mathcal{O}_Z \to h_*\mathcal{O}_X$ is of finite type, then for $Y \subset Z$ the schematic image (defined by I, see [Gr1, (6.10.5)]) one has, canonically, $\boldsymbol{\mathcal{H}}_{X|Z} \cong \boldsymbol{\mathcal{H}}_{X|Y}$.

One also has oriented commutative squares

(1.6.4.2)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "gf"'] \arrow[dr, phantom, "\mathbf{d}^{f,g}"] & Y \arrow[d, "g"] \\ Z \arrow[r, "\mathrm{id}"'] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\delta_{gf}"'] \arrow[dr, phantom, "(\mathbf{d}^{f,g})_\times"] & Y \arrow[d, "\delta_g"] \\ X \times_Z X \arrow[r, "f \times f"'] & Y \times_Z Y \end{tikzcd}$$

whence the associated morphism

(1.6.4.3)
$$\boldsymbol{(f,g)}^\sharp := (\mathbf{d}^{f,g})^\sharp \colon f^*\boldsymbol{\mathcal{H}}_{Y|Z} \to \boldsymbol{\mathcal{H}}_{X|Z}f^*$$

and its adjoint

$$(\mathbf{d}^{f,g})_\sharp \colon \boldsymbol{\mathcal{H}}_{Y|Z} \to f_*\boldsymbol{\mathcal{H}}_{X|Z}f^*.$$

If, for example, f is a *flat monomorphism,* then $(\mathbf{d}^{f,g})_\times$ is an oriented fiber square with flat bottom arrow. If, in addition, g is qcqs, then so is δ_g (see proof of 1.6.2), and so by Proposition 1.5.3, if $F \in \mathbf{D}_Y$ then $(\mathbf{d}^{f,g})^\sharp(F)$ *is an isomorphism* $f^*\boldsymbol{\mathcal{H}}_{Y|Z}F \xrightarrow{\sim} \boldsymbol{\mathcal{H}}_{X|Z}f^*F$.

(See also Theorem 1.7 below).

Corollary 1.6.5. *For any* $W \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z$ *the next diagram commutes.*

$$\begin{tikzcd}
(fe)^*\boldsymbol{\mathcal{H}}_{Y|Z} \arrow[rr, "(\boldsymbol{fe},\boldsymbol{g})^\sharp"] \arrow[d, equal, "\mathsf{ps}^*"'] & & \boldsymbol{\mathcal{H}}_{W|Z}(fe)^* \arrow[d, equal, "\text{via } \mathsf{ps}^*"] \\
e^*f^*\boldsymbol{\mathcal{H}}_{Y|Z} \arrow[r, "e^*(\boldsymbol{f},\boldsymbol{g})^\sharp"'] & e^*\boldsymbol{\mathcal{H}}_{X|Z}f^* \arrow[r, "(\boldsymbol{e},\boldsymbol{gf})^\sharp"'] & \boldsymbol{\mathcal{H}}_{W|Z}e^*f^*
\end{tikzcd}$$

Proof. Apply 1.6.3 to

$$\begin{tikzcd}
W \arrow[r, "e"] \arrow[d, "gfe"'] \arrow[dr, phantom, "\mathbf{d}^{e,gf}"] & X \arrow[r, "f"] \arrow[d, "gf"'] \arrow[dr, phantom, "\mathbf{d}^{f,g}"] & Y \arrow[d, "g"] \\
Z \arrow[r, equal] & Z \arrow[r, equal] & Z
\end{tikzcd}$$

Theorem 1.7. *For scheme-maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *with* f *essentially étale, and* $F \in \mathbf{D}_Y$, *the map*

$$(\boldsymbol{f},\boldsymbol{g})^\sharp(F)\colon f^*\boldsymbol{\mathcal{H}}_{Y|Z}F \to \boldsymbol{\mathcal{H}}_{X|Z}f^*F$$

is an isomorphism.

Proof. It suffices to show that for every open immersion $W \xrightarrow{e} X$ with W affine, $e^*(\boldsymbol{f},\boldsymbol{g})^\sharp(F)$ is an isomorphism. It results therefore from Corollary 1.6.5 that it's enough to prove Theorem 1.7 when X is affine or when f is an open immersion. The latter case was disposed of at the end of §1.6.4.

We may in fact assume that f factors as $X \xrightarrow{f_0} Y_0 \xrightarrow{i} Y$ where Y_0 is affine and i is an open immersion. Then an argument like the preceding one, applied to $X \xrightarrow{f_0} Y_0 \xrightarrow{i} Y \xrightarrow{g} Z$, shows that we can replace f by f_0. Thus we may assume that Y as well as X is affine, and so since f is efp, we can set $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} M^{-1}B$ where B is a finitely presentable A-algebra and $M \subset B$ is a multiplicatively closed subset.

Furthermore, Y and X being affine, the maps f and g are both separated, and so the canonical immersions

$$\delta := \delta_f \colon X \hookrightarrow X \times_Y X =: V, \qquad j\colon X \times_Y X \hookrightarrow X \times_Z X$$

are both closed.

The closed immersion δ is essentially étale, hence flat (§1.1); so with $\mathcal{I}$ the kernel of the natural surjection $\mathcal{O}_V \to \delta_*\mathcal{O}_X$, $\mathcal{O}_V/\mathcal{I}$ is flat over $\mathcal{O}_V$, and

$$\mathcal{I}/\mathcal{I}^2 \cong \mathcal{T}or_1^{\mathcal{O}_V}(\mathcal{O}_V/\mathcal{I}, \mathcal{O}_V/\mathcal{I}) = 0.$$

Moreover, $\mathcal{I}$ is a *finite-type* $\mathcal{O}_V$-ideal. For, with A, M and B as before, and

$$N := \{\, m_1 \otimes m_2 \mid m_1, m_2 \in M \,\} \subset B \otimes_A B,$$

the kernel of the multiplication map $\mu\colon B \otimes_A B \to B$ is finitely generated ([Gr1, p. 301, 6.2.6.2]), as is the kernel of $N^{-1}\mu\colon M^{-1}B \otimes_A M^{-1}B \to M^{-1}B$, giving the assertion.

So by Nakayama's lemma, at any $v \in V$ the stalk $\mathcal{I}_v$, being a finitely-generated idempotent $\mathcal{O}_{V,v}$-ideal, is either (0) or $\mathcal{O}_{V,v}$; and it follows that *δ induces an isomorphism of X onto an open-and-closed subscheme of V.*

Setting $V' := j(V \setminus \delta(X))$, one has then the diagram

$$\begin{tikzcd}
\delta(X) \arrow[r, "\delta^{-1}"] \arrow[d, "j"'] \arrow[dr, phantom, "\mathbf{v}"] & X \arrow[r, "f"] \arrow[d, "\delta_{gf}"'] \arrow[dr, phantom, "\mathbf{u} = (\mathbf{d}^{f,g})_\times"] & Y \arrow[d, "\delta_g"] \\
(X \times_Z X) \setminus V' \arrow[r, "k"'] & X \times_Z X \arrow[r, "f \times f"'] & Y \times_Z Y
\end{tikzcd}$$

where k is an open immersion and each of $\mathbf{d} = \mathbf{u} \circ \mathbf{v}$ and $\mathbf{v}$ is a fiber square with flat bottom arrow. Since X is affine, δ_g and δ_{gf} are qcqs (use [Gr1, p. 291ff, 6.1.4, 6.1.9(iv) and 6.1.9(v)]). By 1.5.3, $\phi_{\mathbf{d}}(F)$ and $\phi_{\mathbf{v}}(f^*F)$ are both isomorphisms, whence, by 1.5.4, so is $\phi_{\mathbf{u}}(F)$, which is exactly what Theorem 1.7 asserts. □

From 1.7, 1.6.2 and 1.6.3 (with X replaced by X', Y by $X \times_Y Y'$, Z by X, S by Y and $S'' \to S'$ by the identity map of Y'), one gets:

Corollary 1.7.1. (Cf. [GeW, (0.1)].) *For an oriented commutative square of scheme-maps*

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

with g flat and f qcqs, whose associated map $X' \to X \times_Y Y'$ is essentially étale, *and for $G \in \mathbf{D}_X$, $\mathbf{d}^\sharp$ is an isomorphism*

$$h^*\boldsymbol{\mathcal{H}}_{X|Y}G \xrightarrow{\sim} \boldsymbol{\mathcal{H}}_{X'|Y'}h^*G.$$

In particular, $\mathsf{d}^\sharp : h^*\mathcal{H}_{X|Y} \to \mathcal{H}_{X'|Y'}$ is an isomorphism.

2. The fundamental class of a flat map

Fix a noetherian scheme S. Let S be the category of separated, efp (hence noetherian) S-schemes (see §1.1). All maps in S are separated and efp. The fiber product, in the category of all schemes, of two S-maps with the same target is a fiber product in S.

In this section we describe how to assign to every *flat* S-map, $f : X \to Y$, with twisted inverse-image functor $f^!\colon \mathbf{D}_Y \to \mathbf{D}_X$ as in §2.1.2, and with $\boldsymbol{\mathcal{H}}_{X|S}$, $\boldsymbol{\mathcal{H}}_{Y|S}$ as in §1, a canonical functorial map

$$\mathsf{c}_f\colon \boldsymbol{\mathcal{H}}_{X|S}f^* \longrightarrow f^!\boldsymbol{\mathcal{H}}_{Y|S},$$

called the *fundamental class of* f.

When f is essentially étale, so that $f^! = f^*$, c_f turns out, nontrivially, to be inverse to the isomorphism in Theorem 1.7, see Proposition 2.5.

As in 0.2, we set

$$\mathcal{H}_X := \boldsymbol{\mathcal{H}}_{X|S}\mathcal{O}_X\,.$$

If $x\colon X \to S$ is essentially smooth of relative dimension n, then c_x induces an *isomorphism*

$$\Omega^n_f[n] \cong (H^{-n}\mathcal{H}_X)[n] \xrightarrow{\ \sim\ } (H^{-n}x^!\mathcal{O}_S)[n] \cong x^!\mathcal{O}_S,$$

that should be closely related to the well-known one of Verdier [V, p. 397, Thm. 3], see Remark 2.4.4.

If $x\colon X \to S$ is finite and flat, then $\mathsf{c}_x(\mathcal{O}_X)$ is closely related to the *trace map* $x_*\mathcal{O}_X \to \mathcal{O}_S$, see Example 2.6.

2.1. We first review a few preliminary considerations.

Let S and S be as above. As is customary, we usually denote an object $w\colon W \to S$ in S simply by W, with the understanding that W is equipped with a separated efp "structure map" w. We set

$$\mathbf{D}_W := \mathbf{D}_{\mathsf{qc}}(W) \qquad \text{(see §1.2).}$$

For W_1, $W_2 \in \mathsf{S}$, we denote $W_1 \times_S W_2$ by $W_1 \times W_2$. The diagonal map $W \to W \times W$ will be denoted by δ_W. We set

$$\boldsymbol{\mathcal{H}}_W := \boldsymbol{\mathcal{H}}_{W|S} = \boldsymbol{\mathcal{H}}_w := (\delta_W)^*(\delta_W)_* \qquad (:= \mathsf{L}\delta_W^*(\delta_W)_*,\ \text{see §1.2}).$$

2.1.1. For $f\colon X \to Y$ in S one has, with the notational convention of §1.2, $f^*\mathbf{D}_Y \subset \mathbf{D}_X$ [L3, 3.9.1] and $f_*\mathbf{D}_X \subset \mathbf{D}_Y$ [L3, 3.9.2]; so the adjoint pseudofunctors $(-)^*$ and $(-)_*$ in §1.3 can be restricted to take values in the categories $\mathbf{D}_W$. *It is assumed henceforth that they are so restricted.*

2.1.2. For any scheme W, let $\mathbf{D}^+_{\mathsf{qc}}(W) \subset \mathbf{D}_W$ be the full subcategory with objects those complexes $G \in \mathbf{D}_W$ such that $H^n(G) = 0$ for all $n \ll 0$.

According to [Nk, 5.3], there is a contravariant $\mathbf{D}^+_{\mathsf{qc}}$-valued pseudofunctor $(-)^!_+$ over S, uniquely determined up to isomorphism by the properties:

(i) When restricted to proper maps, $(-)^!_+$ is pseudofunctorially right-adjoint to the right-derived direct-image pseudofunctor $(-)_*$.

(ii) When restricted to essentially étale maps, $(-)^!_+$ is equal to the usual inverse-image pseudofunctor (derived or not).

(iii) For each oriented fiber square $\mathbf{d}$ in S,

$$\begin{tikzcd} X' \arrow[r, "1"] \arrow[d, "2"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "4"] \\ Y' \arrow[r, "3"'] & Y \end{tikzcd}$$

with 4 (hence 2) proper and 3 (hence 1) essentially étale, and with $\theta_{\mathbf{d}}$ as in 1.4.4, the natural composite isomorphism

$$1^*4^!_+ = 1^!_+4^!_+ \xrightarrow{\sim} (4\circ 1)^!_+ = (3\circ 2)^!_+ \xrightarrow{\sim} 2^!_+3^!_+ = 2^!_+3^*$$

is adjoint to the composition (with $\int_+$ the counit map coming from (i) above):

$$2_*1^*4^!_+ \underset{\theta_{\mathbf{d}}^{-1}}{\xrightarrow{\sim}} 3^*4_*4^!_+ \underset{\int_+}{\longrightarrow} 3^*.$$

As in the first Remark in §1.2, we fix once and for all a specific such pseudofunctor such that $(\mathrm{id}_X)^!_+$ is the identity functor. The point of what follows is to extend this to a $\mathbf{D}_{\mathsf{qc}}$-valued pseudofunctor, at least over *essentially perfect* (i.e., finite tor-dimension) S-maps.

(Henceforth we will abuse terminology by calling S-maps of finite tor-dimension "perfect" instead of "essentially perfect." For the purposes of this paper, "flat" may be substituted throughout for "perfect.")

Let $\mathsf{S_p}$ be the subcategory of perfect maps in S. (Perfection is preserved by composition, see, e.g., [Il, p. 243, Cor. 3.4].) As in [AJL, §5.7], there is over $\mathsf{S_p}$ a contravariant *twisted inverse-image* pseudofunctor $(-)^!$, taking values in the categories $\mathbf{D}_W$, such that

$$f^!F = (f^!_+\mathcal{O}_Y \otimes_X f^*F) \in \mathbf{D}_X \qquad (f\colon X \to Y \text{ in } \mathsf{S_p};\ F \in \mathbf{D}_Y).$$

From the assumptions in the first Remark in §1.2, one gets then that $f^!\mathcal{O}_Y = f^!_+\mathcal{O}_Y$—so that

(2.1.2.1) $$f^!F = (f^!\mathcal{O}_Y \otimes_X f^*F) \in \mathbf{D}_X \qquad (f\colon X \to Y \text{ in } \mathsf{S_p};\ F \in \mathbf{D}_Y).$$

When $X = Y$ and $f = \mathrm{id}_X$ then $f^!$ is the identity functor on $\mathbf{D}_X$.

For $F \in \mathbf{D}^+_{\mathsf{qc}}(Y)$, there is, as in [Nk, 5.9], a natural isomorphism

(2.1.2.2) $$f^!F \cong f^!_+F.$$

When $F = \mathcal{O}_Y$, this is the identity map of $f^!\mathcal{O}_Y = f^!_+\mathcal{O}_Y$.

When f is essentially étale, (2.1.2.2) is the identity map (cf. [L3, 4.9.2.3], with $E := \mathcal{O}_X$).

Further, for $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathsf{S_p}$, the associated isomorphism

$$\mathsf{ps}^!\colon f^!g^! \xrightarrow{\;\sim\;} (gf)^!$$

is the natural composition

$$\begin{aligned} f_+^!\mathcal{O}_Y \otimes_X f^*(g_+^!\mathcal{O}_Z \otimes_Y g^*) &\xrightarrow{\;\sim\;} (f_+^!\mathcal{O}_Y \otimes_X f^*g_+^!\mathcal{O}_Z) \otimes_X f^*g^* \\ &=\!= f^!g_+^!\mathcal{O}_Z \otimes_X f^*g^* \\ &\xrightarrow{\;\sim\;} f_+^!g_+^!\mathcal{O}_Z \otimes_X f^*g^* \\ &\xrightarrow{\;\sim\;} (gf)_+^!\mathcal{O}_Z \otimes_X (gf)^*. \end{aligned} \tag{2.1.2.3}$$

In view of (ii) above, one finds that the restrictions of $(-)^!$ and $(-)^*$ to essentially étale S-maps are *identical pseudofunctors*.

Over $\mathsf{S_p}$ the isomorphism (2.1.2.2) is *pseudofunctorial*. Thus $(-)^!$ may be viewed as an extension of $(-)_+^!$ to a $\mathbf{D}_{\mathsf{qc}}$-valued pseudofunctor.

2.1.3. To each *proper* $\mathsf{S_p}$-map $f\colon X \to Y$ is associated, as in [AJL, §5.9], a functorial map (with id the identity functor of $\mathbf{D}_Y$):

$$\textstyle\int_f\colon f_*f^! \to \mathrm{id},$$

such that for any $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathsf{S_p}$ with f and g proper, the following diagram commutes

$$\begin{tikzcd} (gf)_*(gf)^! \arrow[rr, equal, "\text{via } \mathsf{ps}_* \text{ and } \mathsf{ps}^!"] \arrow[d, "\int_{gf}"'] & & g_*f_*f^!g^! \arrow[d, "g_*\int_f"] \\ \mathrm{id} & & g_*g^! \arrow[ll, "\int_g"] \end{tikzcd} \tag{2.1.3.1}$$

This $\int_f$ is given by the natural functorial composition, with $F \in \mathbf{D}_Y$,

$$f_*f^!F = f_*(f_+^!\mathcal{O}_Y \otimes_X f^*F) \underset{(1.3.5)}{\longrightarrow} f_*f_+^!\mathcal{O}_Y \otimes_Y F \xrightarrow{\int_+ \otimes\, \mathrm{id}} \mathcal{O}_Y \otimes_Y F = F,$$

where $\int_+$ arises from (i) in §2.1.2.

If f is the identity map of X then $\int_f$ can be identified with the identity transformation.

More generally, let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be S-maps with f proper and both g and gf perfect. The functorial map

$$\textstyle\int_f^g\colon f_*(gf)^! \to g^! \tag{2.1.3.2}$$

is defined to be the natural composition

$$
\begin{aligned}
f_*\big((gf)^!_+\mathcal{O}_Z \otimes_X (gf)^*\big) &\xrightarrow{\,\sim\,} f_*\big(f^!_+ g^!_+\mathcal{O}_Z \otimes_X f^*g^*\big) \\
&\underset{(1.3.5)}{\xrightarrow{\,\sim\,}} f_*f^!_+g^!_+\mathcal{O}_Z \otimes_Y g^* \xrightarrow{\int_+\otimes\,\mathrm{id}} g^!_+\mathcal{O}_Z \otimes_Y g^*.
\end{aligned}
$$

The next Lemma will be used in the proof of Proposition 2.5.

Lemma 2.1.4. *For* $\mathsf{S_p}$*-maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *with* f *proper,* $\int_f^g$ *factors as*

$$
f_*(gf)^! \overset{f_*\mathsf{ps}^!}{=\!=} f_*f^!g^! \xrightarrow{\int_f} g^!.
$$

Proof. The assertion is that the border of the following natural diagram commutes:

$$
\begin{tikzcd}
f_*f^!g^! \arrow[r, equal, "f_*\mathsf{ps}^!"] \arrow[d, equal] & f_*(gf)^! \arrow[rr, equal] \arrow[d, phantom, "①"] & & f_*\big((gf)^!_+\mathcal{O}_Z \otimes_X (gf)^*\big) \arrow[d, "\simeq"] \\
f_*(f^!_+\mathcal{O}_Y \otimes_X f^*g^!) \arrow[r, equal] \arrow[d, "(1.3.5)"'] & f_*\big(f^!_+\mathcal{O}_Y \otimes_X f^*(g^!_+\mathcal{O}_Z \otimes_Y g^*)\big) \arrow[rr, "\sim"] \arrow[d, "(1.3.5)"'] & & f_*\big(f^!_+g^!_+\mathcal{O}_Z \otimes_X f^*g^*\big) \arrow[d, "(1.3.5)"] \\
f_*f^!_+\mathcal{O}_Y \otimes_Y g^! \arrow[r, equal] \arrow[dd, "\int_+\otimes\,\mathrm{id}"'] & f_*f^!_+\mathcal{O}_Y \otimes_Y g^!_+\mathcal{O}_Z \otimes_Y g^* \arrow[dd, "\int_+\otimes\,\mathrm{id}"'] \arrow[rr, phantom, "②"] & & f_*f^!_+g^!_+\mathcal{O}_Z \otimes_Y g^* \arrow[dl, "\simeq"'] \arrow[ddd, "\int_+\otimes\,\mathrm{id}"] \\
& & f_*(f^!_+\mathcal{O}_Y \otimes_X f^*g^!_+\mathcal{O}_Z) \otimes_Y g^* \arrow[ul, "(1.3.5)"'] & \\
\mathcal{O}_Y \otimes_Y g^! \arrow[r, equal] \arrow[d, equal] & \mathcal{O}_Y \otimes_Y g^!_+\mathcal{O}_Z \otimes_Y g^* \arrow[drr, equal] \arrow[r, phantom, "③"] & & \\
g^! \arrow[rrr, equal] & & & g^!_+\mathcal{O}_Z \otimes_Y g^*
\end{tikzcd}
$$

Commutativity of subdiagram ① results from the definition (2.1.2.3) of $\mathsf{ps}^!$, of ② from [L3, 3.4.7(iv)] with $(A,B,C) := (g^*, g^!\mathcal{O}_Z, f^!_+)$, *mutatis mutandis*, of ③ from the definition of the isomorphism (2.1.2.2) for proper f, [Nk, 5.7], and of the unlabeled subdiagrams is clear, whence the conclusion. □

The following "transitivity" property of $\int_f^g$—that in view of Lemma 2.1.4 generalizes commutativity of (2.1.3.1)—will be needed in §6.

Proposition 2.1.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be S-maps with f, g proper and h, hg, hgf perfect. The following diagram commutes.*

$$
\begin{tikzcd}
(gf)_*(hgf)^! \arrow[r, equal, "\mathsf{ps}_*"] \arrow[d, "\int^h_{gf}"'] & g_*f_*(hgf)^! \arrow[d, "g_*\int^{hg}_f"] \\
h^! & g_*(hg)^! \arrow[l, "\int^h_g"]
\end{tikzcd}
$$

Proof. The diagram expands naturally as

$$
\begin{tikzcd}[column sep=tiny]
(gf)_*\big((hgf)^!_+\mathcal{O}_W \otimes (hgf)^*\big) \arrow[rrrr, equal, "\mathsf{ps}_*"] \arrow[d, equal, "\mathsf{ps}^!_+"', "\mathsf{ps}^*"] & & & & g_*f_*\big((hgf)^!_+\mathcal{O}_W \otimes (hgf)^*\big) \arrow[d, equal, "\mathsf{ps}^!_+"', "\mathsf{ps}^*"] \\
(gf)_*\big((gf)^!_+h^!_+\mathcal{O}_W \otimes (gf)^*h^*\big) \arrow[drr, equal, "\mathsf{ps}^*", "\mathsf{ps}_*"'] \arrow[ddddd] & & \textcircled{\scriptsize 1} & & g_*f_*\big(f^!_+(hg)^!_+\mathcal{O}_W \otimes f^*(hg)^*\big) \arrow[ddl, equal, "\mathsf{ps}^!_+"', "\mathsf{ps}^*"] \arrow[ddd] \\
 & & g_*f_*\big((gf)^!_+h^!_+\mathcal{O}_W \otimes f^*g^*h^*\big) \arrow[dl] \arrow[dr, equal, "\mathsf{ps}^!_+"] & & \\
 & g_*\big(f_*(gf)^!_+h^!_+\mathcal{O}_W \otimes g^*h^*\big) \arrow[dd] & & g_*f_*\big(f^!_+g^!_+h^!_+\mathcal{O}_W \otimes f^*g^*h^*\big) \arrow[dd] & \\
 & \textcircled{\scriptsize 2} & & & g_*\big(f_*f^!_+(hg)^!_+\mathcal{O}_W \otimes (hg)^*\big) \arrow[dd] \\
 & g_*f_*(gf)^!_+h^!_+\mathcal{O}_W \otimes h^* \arrow[dl, equal, "\mathsf{ps}_*"'] \arrow[dr, equal, "\mathsf{ps}^!_+"] & & g_*\big(f_*f^!_+g^!_+h^!_+\mathcal{O}_W \otimes g^*h^*\big) \arrow[dl] & \\
(gf)_*(gf)^!_+h^!_+\mathcal{O}_W \otimes h^* \arrow[rr, equal, "\mathsf{ps}^!_+", "\mathsf{ps}_*"'] \arrow[dd, "\int_+"'] & & g_*f_*f^!_+g^!_+h^!_+\mathcal{O}_W \otimes h^* \arrow[dd] & & g_*\big((hg)^!_+\mathcal{O}_W \otimes (hg)^*\big) \arrow[dd, equal, "\mathsf{ps}^!_+"] \\
 & \textcircled{\scriptsize 3} & & & \\
h^!_+\mathcal{O}_W \otimes h^* & & g_*g^!_+h^!_+\mathcal{O}_W \otimes h^* \arrow[ll] & & g_*\big(g^!_+h^!_+\mathcal{O}_W \otimes g^*h^*\big) \arrow[ll]
\end{tikzcd}
$$

Commutativity of subdiagram ① follows from pseudofunctoriality of $(-)^!_+$, $(-)^*$ and $(-)_*$. That of ② is given by [L3, 3.7.1], *mutatis mutandis.* As the pseudofunctors $(-)^!_+$ and $(-)^!$ agree on $\mathbf{D}^+_{\mathsf{qc}}$ (see §2.1.2), that of ③ is given by that of (2.1.3.1). That of the unlabeled subdiagrams is clear (by functoriality). Proposition 2.1.5 results. $\square$

2.1.6. To each oriented fiber square in S_p

$$
\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}
$$

with u, v flat, there is associated a functorial *isomorphism*

$$\mathsf{B}_{\mathbf{d}}\colon v^*f^! \xrightarrow{\sim} g^!u^*, \tag{2.1.6.1}$$

satisfying "transitivity" with respect to vertical and horizontal composition of such squares (see [AJL, §5.8]).

With the "relative dualizing complexes" $\mathcal{D}_f := f_+^!\mathcal{O}_Y$, $\mathcal{D}_g := g_+^!\mathcal{O}_{Y'}$, and with

$$\bar{\mathsf{B}}_{\mathbf{d}}\colon v^*\mathcal{D}_f = v^*f_+^!\mathcal{O}_Y \xrightarrow{\sim} g_+^!u^*\mathcal{O}_Y = \mathcal{D}_g$$

as in [AJL, 5.8.3], $\mathsf{B}_{\mathbf{d}}$ is the natural composition

$$v^*f^! = v^*(\mathcal{D}_f \otimes f^*) \xrightarrow{\sim} v^*\mathcal{D}_f \otimes v^*f^* \xrightarrow{\bar{\mathsf{B}}_{\mathbf{d}}\otimes \mathsf{ps}^*} \mathcal{D}_g \otimes g^*u^* = g^!u^*.$$

The next Lemma will be used in the proof of Theorem 5.1.

Lemma 2.1.7. *For any fiber square diagram in* S:

$$\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "j"'] \arrow[dr, phantom, "\mathbf{e}"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathbf{d}"] & \bullet \arrow[d, "g"] \\
X \arrow[r, "u"'] & Y
\end{tikzcd}$$

with u (hence h and v) flat, f (hence j) proper, and g, gf, k and kj perfect, the following diagram commutes.

$$\begin{tikzcd}
j_*v^*(gf)^! \arrow[r, "\mathsf{B}_{\mathbf{de}}"] & j_*(kj)^!u^* \arrow[dd, "\int_j^k"] \\
h^*f_*(gf)^! \arrow[u, "\theta_{\mathbf{e}}"] \arrow[d, "\int_f^g"'] & \\
h^*g^! \arrow[r, "\mathsf{B}_{\mathbf{d}}"'] & k^!u^*
\end{tikzcd}$$

Proof. The assertion is that the border of the following diagram commutes—where $\mathcal{O} := \mathcal{O}_Y$ and $\mathcal{O}^* := u^*\mathcal{O}_Y = \mathcal{O}_X$, and where each map is induced by the natural transformation(s) specified in its label.

$$
\begin{tikzcd}
j_*v^*((gf)^!_+\mathcal{O}\otimes(gf)^*) \arrow[r, "(1.3.3)"] & j_*(v^*(gf)^!_+\mathcal{O}\otimes v^*(gf)^*) \arrow[rr, "\mathsf{ps}^*", "\bar{\mathsf{B}}_{\mathbf{de}}"'] & & j_*((kj)^!_+\mathcal{O}^*\otimes(kj)^*u^*) \\
h^*f_*((gf)^!_+\mathcal{O}\otimes(gf)^*) \arrow[u, "\theta_{\mathbf{e}}"] & j_*(v^*f^!_+g^!_+\mathcal{O}\otimes j^*h^*g^*) \arrow[u, equal, "\mathsf{ps}^!_+", "\mathsf{ps}^*"'] \arrow[d, "(1.3.5)"'] \arrow[rr, phantom, "②"] & & j_*(j^!_+k^!_+\mathcal{O}^*\otimes j^*k^*u^*) \arrow[u, equal, "\mathsf{ps}^!_+", "\mathsf{ps}^*"'] \arrow[d, "(1.3.5)"] \\
h^*f_*(f^!_+g^!_+\mathcal{O}\otimes f^*g^*) \arrow[u, equal, "\mathsf{ps}^!_+", "\mathsf{ps}^*"'] \arrow[dd, "(1.3.5)"'] \arrow[r, phantom, "①"] & j_*v^*f^!_+g^!_+\mathcal{O}\otimes h^*g^* \arrow[dr, "\bar{\mathsf{B}}_{\mathbf{e}}"] & & j_*j^!_+k^!_+\mathcal{O}^*\otimes h^*g^* \arrow[ddd, "\textstyle\int_+"] \\
& & j_*j^!_+h^*g^!_+\mathcal{O}\otimes h^*g^* \arrow[ur, "\bar{\mathsf{B}}_{\mathbf{d}}"] \arrow[dd, "\textstyle\int_+"] \arrow[dl, phantom, "③"] & \\
h^*(f_*f^!_+g^!_+\mathcal{O}\otimes g^*) \arrow[r, "(1.3.3)"] \arrow[d, "\textstyle\int_+"'] & h^*f_*f^!_+g^!_+\mathcal{O}\otimes h^*g^* \arrow[uu, "\theta_{\mathbf{e}}"] \arrow[dr, "\textstyle\int_+"'] & & \\
h^*(g^!_+\mathcal{O}\otimes g^*) \arrow[rr, "(1.3.3)"'] & & h^*g^!_+\mathcal{O}\otimes h^*g^* \arrow[r, "\bar{\mathsf{B}}_{\mathbf{d}}"'] & k^!_+\mathcal{O}^*\otimes h^*g^*
\end{tikzcd}
$$

Commutativity of subdiagram ① is a consequence of the mirror image of [L3, 3.7.3], with $(f, f', g, g', P, Q) := (f, j, h, v, g^*, f^!_+g^!_+\mathcal{O})$.

Commutativity of ② follows from transitivity of $\bar{\mathsf{B}}$ (see [AJL, §5.8.4]).

Commutativity of ③ is immediate from the definition of $\bar{\mathsf{B}}_{\mathbf{e}}$ (see second paragraph in [AJL, §5.8.2]).

The rest is clear. $\square$

2.2. The *fundamental class* of a flat S-map $f\colon X \to Y$,

$$\mathbf{c}_f\colon \boldsymbol{\mathcal{H}}_X f^* \longrightarrow f^!\boldsymbol{\mathcal{H}}_Y,$$

is the composition of two functorial maps, with $\Gamma = \Gamma_f\colon X \to X\times Y$ the graph of f (a map in S):

$$\boldsymbol{\mathcal{H}}_X f^* = \delta_X^*\delta_{X*}f^* \xrightarrow[\mathbf{a}_f]{} \Gamma^*\Gamma_*f^! \xrightarrow[\mathbf{b}_f]{\sim} f^!\delta_Y^*\delta_{Y*} = f^!\boldsymbol{\mathcal{H}}_Y, \tag{2.2.1}$$

specified as follows.

To define $\mathbf{a}_f\colon \delta_X^*\delta_{X*}f^* \to \Gamma^*\Gamma_*f^!$ consider the commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "\delta := \delta_f"] & X\times_Y X \arrow[r, "i"] \arrow[d, "p_1"'] & X\times X \arrow[r, "p_X"] \arrow[d, "\mathrm{id}_X\times f"] & X \arrow[d, "f"] \\
& X \arrow[r, "\Gamma"'] & X\times Y \arrow[r, "p_Y"'] & Y
\end{tikzcd}
\tag{2.2.2}
$$

where p_X and p_Y are the projections onto the second factor, p_1 onto the first, and i is the natural map.

More generally, consider any commutative S-diagram $\mathbf{d}$

$$
\begin{tikzcd}
\bullet \arrow[r, "1"] & \bullet \arrow[r, "2"] \arrow[d, "6"'] & \bullet \arrow[r, "3"] \arrow[d, "7"'] & \bullet \arrow[d, "8"] \\
& \bullet \arrow[r, "4"'] & \bullet \arrow[r, "5"'] & \bullet
\end{tikzcd}
\tag{2.2.3}
$$

in which the squares are fiber squares, the map 8 is flat, 1 is proper and both $3\circ 2\circ 1$ and $5\circ 4$ are perfect. To such a $\mathbf{d}$ associate the map $\lambda_{\mathbf{d}}$ given by the composition

$$
\begin{array}{lrl}
(\mathsf{ps}_*) & (2\circ 1)_*(3\circ 2\circ 1)^!8^* & \xrightarrow{\sim} 2_*1_*(3\circ 2\circ 1)^!8^* \\
(2_*\textstyle\int_1^{3\circ 2}) & & \longrightarrow 2_*(3\circ 2)^!8^* \\
(2_*\mathsf{B}^{-1}) & & \xrightarrow{\sim} 2_*6^*(5\circ 4)^! \\
(\theta^{-1}\ (\S 1.4.4)) & & \xrightarrow{\sim} 7^*4_*(5\circ 4)^!
\end{array}
$$

In (2.2.2), $i\circ\delta=\delta_X$, $p_X\circ i\circ\delta=\mathrm{id}_X$, and $p_Y\circ\Gamma=f$. Thus one has a map

$$\delta_{X*}f^*=\delta_{X*}\mathrm{id}_X^!\,f^*\longrightarrow(\mathrm{id}_X\times f)^*\Gamma_*f^!, \tag{2.2.4}$$

to which one applies $\delta_X^*\cong\delta^*i^*$ to produce the natural composite map

$$\mathbf{a}_f\colon\delta_X^*\delta_{X*}f^*\longrightarrow\delta^*i^*(\mathrm{id}_X\times f)^*\Gamma_*f^!\xrightarrow{\sim}\delta^*p_1^*\Gamma^*\Gamma_*f^!\xrightarrow{\sim}\Gamma^*\Gamma_*f^!.$$

To define the isomorphism $\mathbf{b}_f\colon\Gamma^*\Gamma_*f^!\xrightarrow{\sim}f^!\delta_Y^*\delta_{Y*}$ in (2.2.1), consider the fiber square

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "\mathbf{h}"] & Y \arrow[d, "\delta_Y"] \\
X\times Y \arrow[r, "f\times\mathrm{id}_Y"'] & Y\times Y
\end{tikzcd}
\tag{2.2.5}
$$

Let $p\colon X\times Y\to X$ be the projection, so that $p\Gamma=\mathrm{id}$, and Γ^*p^* is isomorphic to the identity functor of $\mathbf{D}_X$. One has then the functorial isomorphism

$$\mu_f=\mu_{\Gamma,p}\colon\Gamma^*\Gamma_*(A\otimes B)\xrightarrow{\sim}A\otimes\Gamma^*\Gamma_*B\qquad(A,\,B\in\mathbf{D}_X) \tag{2.2.6}$$

that is defined to be the natural composite isomorphism

$$
\begin{aligned}
\Gamma^*\Gamma_*(A\otimes B)\underset{\text{via }\mathsf{ps}^*}{=\!=\!=}\Gamma^*\Gamma_*(\Gamma^*p^*A\otimes B)&\underset{(1.3.5)^{-1}}{\xrightarrow{\sim}}\Gamma^*(p^*A\otimes\Gamma_*B)\\
&\underset{(1.3.3)}{\xrightarrow{\sim}}\Gamma^*p^*A\otimes\Gamma^*\Gamma_*B\underset{\text{via }\mathsf{ps}^*}{=\!=\!=}A\otimes\Gamma^*\Gamma_*B.
\end{aligned}
$$

The map $\mathbf{b}_f$ is the composite isomorphism (with $\phi_{\mathbf{h}}$ as in 1.5.3)

$$
\begin{aligned}
\Gamma^*\Gamma_*f^!=\Gamma^*\Gamma_*(f^!\mathcal{O}_Y\otimes f^*)&\underset{\mu_f}{\xrightarrow{\sim}}f^!\mathcal{O}_Y\otimes\Gamma^*\Gamma_*f^*\\
&\underset{1\otimes\phi_{\mathbf{h}}^{-1}}{\xrightarrow{\sim}}f^!\mathcal{O}_Y\otimes f^*\delta_Y^*\delta_{Y*}=f^!\delta_Y^*\delta_{Y*}.
\end{aligned}
\tag{2.2.7}
$$

This completes the definition of the fundamental class $\mathbf{c}_f$.

Example 2.3. In the "absolute" case, when $Y = S$ and f is the structure map $x\colon X \to S$ (assumed flat), the map $\mathbf{b}_x$ is the identity. Diagram (2.2.2) collapses to

$$\begin{tikzcd} X \arrow[r, "\delta := \delta_X"] & X \times X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "x"] \\ & X \arrow[r, "x"'] & S \end{tikzcd}$$

with p_1 and p_2 the projections onto the first and second factors, respectively. The fundamental class $\mathbf{c}_x = \mathbf{a}_x$ is then the composite map

$$\boxed{\delta^*\delta_*x^* = \delta^*\delta_*(p_2\circ\delta)^!x^* \underset{\delta^*\int_\delta^{p_2}}{\longrightarrow} \delta^*p_2^!x^* \underset{\delta^*\mathsf{B}^{-1}}{\xrightarrow{\sim}} \delta^*p_1^*x^! \underset{\mathsf{ps}^*}{\xrightarrow{\sim}} \mathrm{id}^*x^! = x^!.}$$

Remark 2.3.1. What happens to $\mathbf{c}_x$ when p_1 and p_2 in its definition are interchanged? Denoting the resulting map by $\mathbf{c}'_x$, one can show that $\mathbf{c}'_x = \mathbf{c}_x \circ e_x$, where, $\sigma\colon X\times X \to X\times X$ being the symmetry isomorphism (that is, $p_1\sigma = p_2$ and $p_2\sigma = p_1$), $e_x\colon \boldsymbol{\mathcal{H}}_X \to \boldsymbol{\mathcal{H}}_X$ is the automorphism given by the composition

$$\delta^*\delta_* = (\sigma\delta)^*(\sigma\delta)_* \underset{\mathsf{ps}_*}{\overset{\mathsf{ps}^*}{=\!=}} \delta^*\sigma^*\sigma_*\delta_* \xrightarrow{\delta^*\epsilon_\sigma} \delta^*\delta_*.$$

The square of this automorphism is easily seen to be the identity, but the automorphism itself need not be. For example, if x is smooth, then working locally with a Koszul resolution of $\delta_*\mathcal{O}_X$, one finds that $e_x(\mathcal{O}_X)$ induces multiplication by $(-1)^i$ on $H^i(\mathcal{H}_X)$. (See also [L3, Exercise 3.4.4.1].)

The map $\mathbf{c}_x$ is canonical in that though the pair $(\boldsymbol{\mathcal{H}}_X(E), \mathbf{c}_x(E))$ $(E \in \mathbf{D}_X)$ depends on a choice of flat resolution for the $\mathcal{O}_{X\times X}$-complex δ_*E, for two such choices there is a canonical isomorphism between the resulting pairs. What this example illustrates (for instance when x is the natural map $\operatorname{Spec} R[T] \to \operatorname{Spec} R$ with R a ring and T an indeterminate) is that the canonical isomorphism can induce the identity on $\boldsymbol{\mathcal{H}}_X(\mathcal{O}_X)$ while not inducing the identity on $\mathbf{c}_x(\mathcal{O}_X)$.

Example 2.4. If $V := X \times X$ and $\mathcal{I}$ is the kernel of the natural surjection $\mathcal{O}_V \to \delta_*\mathcal{O}_X$, then using any flat resolution of $\delta_*\mathcal{O}_X$ one gets a "natural"[1] isomorphism of $\mathcal{O}_X$-modules

$$\Omega^1_x = \mathcal{I}/\mathcal{I}^2 \cong \mathcal{T}or_1^{\mathcal{O}_V}(\mathcal{O}_V/\mathcal{I}, \mathcal{O}_V/\mathcal{I}) = H^{-1}\mathcal{H}_x,$$

whence a map of graded-commutative $\mathcal{O}_X$-algebras, with $\Omega^i_x := \wedge^i\Omega^1_x$,

$$\oplus_{i\ge 0}\, \Omega^i_x \to \oplus_{i\ge 0}\, \mathcal{T}or_i^{\mathcal{O}_V}(\mathcal{O}_V/\mathcal{I}, \mathcal{O}_V/\mathcal{I}) = \oplus_{i\ge 0}\, H^{-i}\mathcal{H}_x. \tag{2.4.1}$$

Proposition 2.4.2. *With notation as in 2.3, if $x\colon X \to S$ is essentially smooth of relative dimension n (see §1.1), then there is a natural composite $\mathbf{D}_X$-isomorphism*

$$\Omega^n_x[n] \xrightarrow{\sim} (H^{-n}\mathcal{H}_X)[n] \underset{\text{via } \mathbf{c}_x}{\xrightarrow{\sim}} (H^{-n}x^!\mathcal{O}_S)[n] \xrightarrow{\sim} x^!\mathcal{O}_S.$$

[1] The negative of this isomorphism is equally natural. This leads to some sign issues, which we will not get into here.

Proof. Since x is essentially smooth the map (2.4.1) is an *isomorphism*, as can be checked locally, over affine open sets in V where $\mathcal{I}$ is generated by a regular sequence of length n, whose associated Koszul complex provides a flat resolution of $\mathcal{O}_V/\mathcal{I}$ (see §1.1).

The complex $x^!\mathcal{O}_S$ is concentrated in degree $-n$: there exists an isomorphism $\Omega^n_x[n] \cong x^!\mathcal{O}_S$ (the proof of [V, p. 397, Thm. 3] holds for essentially smooth maps), whence a natural isomorphism $(H^{-n}x^!\mathcal{O}_S)[n] \xrightarrow{\sim} x^!\mathcal{O}_S$. Likewise for the complex $p_2^!\mathcal{O}_X$, since p_2 is also essentially smooth of relative dimension n (or by the flat base-change isomorphism $p_2^!\mathcal{O}_X \cong p_1^*x^!\mathcal{O}_S$).

In view of Example 2.3 and the definition of (2.1.3.2), the problem is readily reduced to showing that *the natural map*

$$H^{-n}\delta^*\delta_*\delta_+^!p_2^!\mathcal{O}_X \to H^{-n}\delta^*p_2^!\mathcal{O}_X. \tag{2.4.3}$$

is an isomorphism.

Using that $\delta_+^!$ is right-adjoint to δ_*, one can identify $\delta_*\delta_+^!p_2^!\mathcal{O}_X \to p_2^!\mathcal{O}_X$ with the map $\mathsf{R}\mathcal{H}om(\delta_*\mathcal{O}_X, p_2^!\mathcal{O}_X) \to \mathsf{R}\mathcal{H}om(\mathcal{O}_V, p_2^!\mathcal{O}_X)$ induced by the natural map $\mathcal{O}_V \to \delta_*\mathcal{O}_X$, and then check (2.4.3) locally, where, again, one can replace $\delta_*\mathcal{O}_X \cong \mathcal{O}_V/\mathcal{I}$ by the Koszul complex of a regular sequence. $\square$

Remark 2.4.4. As of this writing, the authors do not know whether the natural isomorphism in 2.4.2 coincides (up to sign?) with that of Verdier. Nor do we know whether *either* of these isomorphisms becomes $\pm$identity when all the data are interpreted as in [H] or [Co].

The answers might well emerge from the relation of these maps to traces and residues, a relation to be explored in detail elsewhere.

Proposition 2.5. *If the* S*-map* $f\colon X \to Y$ *is* essentially étale *then with* $y\colon Y \to S$ *the structure map and*

$$\boldsymbol{f}^\sharp\colon f^!\boldsymbol{\mathcal{H}}_Y = f^*\boldsymbol{\mathcal{H}}_Y \xrightarrow{(\boldsymbol{f},\boldsymbol{y})^\sharp} \boldsymbol{\mathcal{H}}_X f^*$$

the isomorphism from Theorem 1.7, it holds that

$$\mathbf{c}_f = (\boldsymbol{f}^\sharp)^{-1}.$$

Proof. Let us see what $\mathbf{c}_f = \mathbf{b}_f \circ \mathbf{a}_f$ looks like when the pseudofunctorial identification of $(-)^!$ with $(-)^*$ for essentially étale maps is implemented.

More specifically, with reference to (1.6.4.2), and notation as in (2.2.2), consider the following decomposition $(\mathbf{d}^{f,y})_\times = \mathbf{uvw}$:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\delta"'] \arrow[dr, phantom, "\mathbf{w}"] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[dd, "\delta_Y"] \\
X\times_Y X \arrow[r, "p_1"] \arrow[d, "i"'] \arrow[dr, phantom, "\mathbf{v}"] & X \arrow[d, "\Gamma"] \arrow[r, phantom, "\mathbf{u}"] & {} \\
X\times X \arrow[r, "\mathrm{id}_X\times f"'] & X\times Y \arrow[r, "f\times \mathrm{id}_Y"'] & Y\times Y
\end{tikzcd} \tag{2.5.1}$$

By the definition of $(\boldsymbol{f}, \boldsymbol{y})^\sharp$ (see (1.6.4.3) and (1.6.1)), and Proposition 1.5.4,

$$\boldsymbol{f}^\sharp := \phi_{\mathbf{uvw}} = \phi_{\mathbf{vw}} f^* \circ \phi_{\mathbf{u}}.$$

Since $\phi_{\mathbf{u}}$ is an isomorphism (see Proposition 1.5.3), therefore so is $\phi_{\mathbf{vw}} f^*$. To prove Proposition 2.5 it will suffice then to show that

$$\mathbf{a}_f = (\phi_{\mathbf{vw}} f^*)^{-1} \tag{2.5.2}$$

and

$$\mathbf{b}_f = \phi_{\mathbf{u}}^{-1}. \tag{2.5.3}$$

We first treat some constituent parts of the definition of $\mathbf{a}_f$.

2.5.4. Since f is essentially étale, therefore so too are the diagonal map $\delta\colon X \to X \times_Y X$ and the projection $p_j\colon X \times_Y X \to X$ to the j-th factor $(j = 1, 2)$, see §1.1. Since the identification $(-)^! = (-)^*$ for essentially étale maps is pseudofunctorial, therefore *for each j, the two isomorphisms*

$$\mathrm{id} = (p_j\delta)^! \overset{\mathsf{ps}^!}{=\!=} \delta^! p_j^! \quad\text{and}\quad \mathrm{id} = (p_j\delta)^* \overset{\mathsf{ps}^*}{=\!=} \delta^* p_j^*$$

are identical.

2.5.5. In (2.1.6.1), *if f is essentially étale* (whence so is g, see §1.1) *then the following diagram commutes.*

$$\begin{tikzcd}
v^*f^! \arrow[r, "\mathsf{B}_{\mathbf{d}}"] \arrow[d, equal] & g^!u^* \arrow[d, equal] \\
v^*f^* \arrow[r, equal, "\mathsf{ps}^*"'] & g^*u^*
\end{tikzcd}$$

To see this, one uses (2.1.2.1), monoidality of the pseudofunctor $(-)^*$ (see [L3, p. 121, 3.6.7(b)]) and the *dual* (see [L3, p. 105, (3.4.5)]) of diagram ② in [AJL, p. 109, 3.4.7.1] to reduce to showing commutativity of the diagram *after* it is applied to $\mathcal{O}_Y$; and that follows from [L3, p. 208, Theorem 4.8.3(ii)], by a straightforward extension of [L3, Remark 4.8.5.2] with "finitely presentable" (resp. "étale") replaced by "efp" (resp. "essentially étale"). Details—routine, but somewhat tedious—are left to the reader.

2.5.6. Using Lemma 2.1.4, and applying 2.5.4 and 2.5.5, one checks now that the map $\mathbf{a}_f$ is the composition

$$\begin{aligned}
\delta_X^*\delta_{X*}f^* &\overset{\mathsf{ps}_*}{=\!=} \delta_X^* i_*\delta_* f^* \overset{\mathsf{ps}^*}{=\!=} \delta_X^* i_*\delta_*\delta^* p_1^* f^* =\!= \delta_X^* i_*\delta_*\delta^! p_1^* f^* \\
&\xrightarrow{\text{via } \textstyle\int_\delta} \delta_X^* i_* p_1^* f^* \xrightarrow{\delta_X^*\theta_{\mathbf{v}}^{-1}} \delta_X^*(\mathrm{id}_X \times f)^*\Gamma_* f^* \overset{\mathsf{ps}^*}{=\!=} \delta^* p_1^*\Gamma^*\Gamma_* f^* \overset{\mathsf{ps}^*}{=\!=} \Gamma^*\Gamma_* f^*.
\end{aligned}$$

It follows that $\phi_{\mathbf{vw}} f^* \circ \mathbf{a}_f$ is obtained by going around the following diagram clockwise from $\delta_X^*\delta_{X*}f^*$ back to itself.

$$
\begin{tikzcd}
\delta_X^*\delta_{X*}f^* \arrow[r, equal, "\mathsf{ps}_*", "1"'] & \delta_X^* i_*\delta_* f^* \arrow[rr, equal, "\mathsf{ps}^*", "2"'] \arrow[dr, phantom, "ⓑ"] & & \delta_X^* i_*\delta_*\delta^* p_1^* f^* \arrow[d, "3", "="'] \\
\delta_X^*(\mathrm{id}_X\times f)^*\Gamma_* f^* \arrow[u, "\delta_X^*\theta_{\mathbf{vw}}", "6"'] \arrow[ur, phantom, "ⓐ"] & \delta_X^* i_* p_1^* f^* \arrow[l, "5", "\delta_X^*\theta_{\mathbf{v}}^{-1}"'] \arrow[u, "\delta_X^* i_*\theta_{\mathbf{w}}", "7"'] \arrow[urr, "\delta_X^* i_*\eta_\delta", "8"'] & ⓒ & \delta_X^* i_*\delta_*\delta^! p_1^* f^* \arrow[ll, "4", "\delta_X^* i_*\int_\delta"']
\end{tikzcd}
$$

Commutativity of subdiagram ⓐ results from transitivity of θ (see [L3, p. 128, (iii)]); that of ⓑ is the definition (1.4.2) of $\theta_{\mathbf{w}}$; and that of ⓒ is given by the next Lemma. Thus $\phi_{\mathbf{vw}} f^*\circ \mathbf{a}_f = 6\circ 5\circ 4\circ 3\circ 2\circ 1$ is the composition

$$1^{-1}\circ 7\circ 4\circ 3\circ 2\circ 1 = 1^{-1}\circ 2^{-1}\circ 8\circ 4\circ 3\circ 2\circ 1 = 1^{-1}\circ 2^{-1}\circ 3^{-1}\circ 3\circ 2\circ 1,$$

which is the identity map of $\delta_X^*\delta_{X*}f^*$; and this proves (2.5.2).

Lemma 2.5.7. *The following composite map is the identity.*

$$\delta_X^* i_* \xrightarrow{\delta_X^* i_*\eta_\delta} \delta_X^* i_*\delta_*\delta^* = \delta_X^* i_*\delta_*\delta^! \xrightarrow{\delta_X^* i_*\int_\delta} \delta_X^* i_*.$$

Proof. As in the proof of Theorem 1.7, δ is an isomorphism of X onto an open-and-closed subscheme of $V := X\times_Y X$. Let $i\colon V\to X\times X$ be the natural map, and set $V' := i(V\setminus \delta(X))$. We have then a fiber square, with k an open immersion,

$$
\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "j"'] & X\times_Y X \arrow[d, "i"] \\
(X\times X)\setminus V' \arrow[r, "k"'] & X\times X
\end{tikzcd}
$$

and hence isomorphisms (see 1.5.3), with $\delta_X := i\delta$,

$$\delta_X^* i_* \underset{\mathsf{ps}^*}{=\!=} \delta^* i^* i_* \underset{(1.5.1)}{\xrightarrow{\ \sim\ }} j^* j_*\delta^*. \tag{2.5.8}$$

So it's enough to show that *the following composite map is the identity.*

$$\delta^* \xrightarrow{\delta^*\eta_\delta} \delta^*\delta_*\delta^* = \delta^*\delta_*\delta^! \xrightarrow{\delta^*\int_\delta} \delta^*. \tag{2.5.9}$$

But $\delta^*\int_\delta$ is the same map as

$$\delta^*\delta_*\delta^! \xrightarrow{\epsilon_\delta} \delta^! = \delta^*.$$

To see this, one can use the isomorphisms (2.1.2.1) and (2.1.2.2), and the description of $\int_\delta$ in [AJL, §5.9], to reduce to showing that the diagram commutes *after* it is applied to $\mathcal{O}_{X\times_Y X}$—which follows from [L3, p. 168, Exercise 4.9.1(c), and p. 204, Theorem 4.8.1(iii)]. Details are left to the reader.

Thus the composite map (2.5.9) is the same as

$$\delta^* \xrightarrow{\delta^*\eta_\delta} \delta^*\delta_*\delta^* = \delta^*\delta_*\delta^! \xrightarrow{\epsilon_\delta} \delta^! = \delta^*,$$

that is,

$$\delta^* \xrightarrow{\delta^*\eta_\delta} \delta^*\delta_*\delta^* = \delta^*\delta_*\delta^! = \delta^*\delta_*\delta^* \xrightarrow{\epsilon_\delta} \delta^*,$$

which is indeed the identity map. □

2.5.10. As for (2.5.3), recall from (2.1.2.1) and the first Remark in §1.2 the equalities

$$\begin{tikzcd}
f^! \arrow[r, equal] \arrow[d, equal] & f^* \arrow[d, equal] \\
f^!\mathcal{O}_Y \otimes f^* \arrow[r, equal] & f^*\mathcal{O}_Y \otimes f^*
\end{tikzcd}$$

Using this, one checks that (2.5.3) asserts commutativity of the outer border of the following diagram of natural maps, where $\mathcal{O} := f^*\mathcal{O}_Y = f^!\mathcal{O}_Y$, and $\mathcal{O}_\times := \mathcal{O}_{X\times Y}$, so that $\mathcal{O} = \Gamma^*\mathcal{O}_\times$:

$$\begin{tikzcd}
\Gamma^*\Gamma_*(\mathcal{O}\otimes\Gamma^*\Gamma_*f^*) \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "④"] & & \Gamma^*\Gamma_*(\Gamma^*\mathcal{O}_\times\otimes\Gamma^*\Gamma_*f^*) \\
\Gamma^*\Gamma_*\Gamma^*\Gamma_*f^* \arrow[rr, equal] \arrow[d, shift right, "\Gamma^*\Gamma_*\epsilon_\Gamma"'] & & \Gamma^*\Gamma_*\Gamma^*(\mathcal{O}_\times\otimes\Gamma_*f^*) \arrow[u, "(1.3.3)", "\simeq"'] \\
\Gamma^*\Gamma_*f^* \arrow[rr, equal] \arrow[u, shift right, "\Gamma^*\eta_\Gamma"'] \arrow[d, equal] \arrow[drr, phantom, "⑤"] & & \Gamma^*(\mathcal{O}_\times\otimes\Gamma_*f^*) \arrow[u, "\Gamma^*\eta_\Gamma"'] \\
\Gamma^*\Gamma_*f^* \arrow[r, equal] & \mathcal{O}\otimes\Gamma^*\Gamma_*f^* \arrow[r, equal] & \Gamma^*\mathcal{O}_\times\otimes\Gamma^*\Gamma_*f^* \arrow[u, "(1.3.3)", "\simeq"'] \\
\Gamma^*(f\times\mathrm{id}_Y)^*\delta_{Y*} \arrow[r, equal] \arrow[u, "\Gamma^*\theta_{\mathbf{u}}"] \arrow[d, equal, "\mathsf{ps}^*"'] & \mathcal{O}\otimes\Gamma^*(f\times\mathrm{id}_Y)^*\delta_{Y*} \arrow[u, "1\otimes\Gamma^*\theta_{\mathbf{u}}"'] \arrow[d, equal, "1\otimes\mathsf{ps}^*"] & \\
f^*\delta_Y^*\delta_{Y*} \arrow[r, equal] & \mathcal{O}\otimes f^*\delta_Y^*\delta_{Y*} &
\end{tikzcd}$$

Commutativity of the unlabeled subdiagrams is clear. That of subdiagrams ④ and ⑤ is straightforward to check, either directly or by *dualizing* the first commutative square in [L3, p. 103, (3.4.2)] (see [L3, §(3.4.5)]).

Since $\Gamma_*\epsilon_\Gamma\circ\eta_\Gamma$ is the identity map, one finds then that the outer border does indeed commute.

This completes the proof of (2.5.3) and of Proposition 2.5. □

Here is one more concrete illustration.

Example 2.6. Let A be a noetherian ring, and B a finite-rank projective A-algebra, with corresponding scheme-map $x\colon X = \operatorname{Spec} B \to \operatorname{Spec} A = S$. From adjointness of the functors $x^!$ and x_* one gets a canonical isomorphism

$x^!\mathcal{O}_S \cong \mathrm{Hom}_A(B,A)^\sim$, via which $\mathbf{c}_x(\mathcal{O}_S)$ can be identified with an $\mathcal{O}_X$-homomorphism

$$H^0(\delta^*\delta_*\mathcal{O}_X) = \mathcal{O}_X \to \mathrm{Hom}_A(B,A)^\sim,$$

the sheafification of a B-homomorphism $c\colon B \to \mathrm{Hom}_A(B,A)$.

This identification being made, one finds, following through definitions, that c factors as

$$\begin{aligned} B \longrightarrow \mathrm{Hom}_A(B,B)\otimes_{B\otimes_A B} B &\xrightarrow{\sim} (\mathrm{Hom}_A(B,A)\otimes_A B)\otimes_{B\otimes_A B} B\\ &\xrightarrow{\sim} \mathrm{Hom}_A(B,A)\otimes_B B\\ &\xrightarrow{\sim} \mathrm{Hom}_A(B,A), \end{aligned}$$

where the first map takes $b \in B$ to $\mathrm{id}\otimes b$, and the isomorphisms are the natural ones. Hence $c(1)$ is the *trace map* $B \to A$.

We won't use this example further, so details are left to the reader.

This is a simple case of a fundamental relation, to be treated elsewhere, between $f_*\mathbf{c}_f$ (f any finite S-map) and a certain trace map for Hochschild complexes (cf. [L2, §§4.5–4.6], [AnL, p. 55, Proposition 6.3.1]).

3. Transitivity of the fundamental class; bivariant interpretation

After stating the central "transitivity" result of this paper, Theorem 3.1—whose proof will be given in §6—we interpret it in terms of orientations for flat maps in a bivariant Hochschild theory (§3.2), orientations that are compatible with essentially étale base change (Corollary 3.3); and, in §3.4, illustrate by a brief discussion of the resulting Gysin maps for bivariant homology and cohomology.

Notation remains as in §2.1. For any W in S, $\delta_W\colon W \to W\times W$ denotes the diagonal. For any flat S-map f, the fundamental class $\mathbf{c}_f$ is as in §2.2.

Theorem 3.1. *Let $X \xrightarrow{u} Y \xrightarrow{v} Z$ be flat S-maps. The following functorial diagram commutes.*

$$\begin{tikzcd} \delta_X^*\delta_{X*}u^*v^* \arrow[r, "\mathbf{c}_u v^*"] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & u^!\delta_Y^*\delta_{Y*}v^* \arrow[r, "u^!\mathbf{c}_v"] & u^!v^!\delta_Z^*\delta_{Z*} \arrow[d, equal, "\text{via } \mathsf{ps}^!"] \\ \delta_X^*\delta_{X*}(vu)^* \arrow[rr, "\mathbf{c}_{vu}"'] & & (vu)^!\delta_Z^*\delta_{Z*} \end{tikzcd}$$

Remarks 3.1.1. (A) When u and v are both essentially étale, the assertion results, in view of Proposition 2.5, from Corollary 1.6.3.

(B) If u (but not necessarily v) is essentially étale, then Proposition 2.5 and Theorem 3.1 provide a canonical identification of $\mathbf{c}_{vu}$ with $u^*\mathbf{c}_v$.

3.2. In view of Theorem 3.1, fundamental classes are *orientations* for the flat maps in a suitable *bivariant Hochschild theory,* as follows.

The setup for the theory is constructed in [AJL, §5]. The underlying category is $\mathsf{S}_\mathsf{p} \subset \mathsf{S}$, the category of *perfect* S-maps, the *confined maps* being the proper S_p-maps, and the *independent squares* being the oriented fiber squares with essentially étale bottom arrow, cf. [AJL, §5.1.5(a)]. Coefficients are provided by the *pre-Hochschild complexes* $\mathcal{H}_X$ ($X \in \mathsf{S}$) in §1 above. That these $\mathcal{H}_X$ satisfy the conditions at the beginning of [AJL, §3.2] is seen as follows.

First, for any S-map $f\colon X \to Y$, letting $y\colon Y \to S$ be the structure map define $f^\sharp\colon f^*\mathcal{H}_Y \to \mathcal{H}_X$ to be the map $(\boldsymbol{f},\boldsymbol{y})^\sharp(\mathcal{O}_Y)$ (see (1.6.4.3)).

If $Y = X$ and f is the identity map, then $f^\sharp$ is the identity map of $\mathcal{H}_X$.

Next, in Corollary 1.6.3, suppose $S'' = S' = S$, both maps $S'' \to S' \to S$ being the identity. With the preceding notation, the conclusion is that the following diagram commutes:

$$
\begin{tikzcd}
(gf)^*\mathcal{H}_Z \arrow[r, "(gf)^\sharp"] \arrow[d, equal, "\mathsf{ps}^*"'] & \mathcal{H}_X \\
f^*g^*\mathcal{H}_Z \arrow[r, "f^*g^\sharp"'] & f^*\mathcal{H}_Y \arrow[u, "f^\sharp"']
\end{tikzcd}
\tag{3.2.1}
$$

This is the diagram (3.2.1) in [AJL], a diagram whose commutativity is required for the bivariant theory constructed there.

The remaining requirement in [AJL, §3.2], that $f^\sharp$ be an isomorphism when f is essentially étale, is given by Theorem 1.7.

To any S_p-map $f\colon X \to Y$ this bivariant theory assigns the graded group

$$\mathrm{HH}^*(f) := \oplus_{j\in\mathbb{Z}} \operatorname{Ext}^j(\mathcal{H}_X, f^!\mathcal{H}_Y).$$

So for flat f the fundamental class $\mathbf{c}_f$ induces a canonical element

$$c_f := \mathbf{c}_f(\mathcal{O}_Y) \in \mathrm{HH}^0(f); \tag{3.2.2}$$

and in terms of the *bivariant product* $\mathrm{HH}^0(u) \times \mathrm{HH}^0(v) \to \mathrm{HH}^0(vu)$ [AJL, 3.3.2], Theorem 3.1 says:

$$\boxed{c_{vu} = c_u \cdot c_v\,.} \tag{3.2.3}$$

Together with the easily-checked fact that if $X = Y$ and u is the identity map, then c_u is the identity map of $\mathcal{H}_X$, this shows that the family c_f is a family of *canonical orientations* for the flat maps in our bivariant theory, see [FM, p. 28, 2.6.2].

Remark 3.3.1(A) below shows that these orientations are compatible with essentially étale base change.

The next Corollary is also a special case of Theorem 5.1.

Corollary 3.3. *If in the oriented fiber square of flat* S*-maps*

$$
\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}
$$

u (hence v) is essentially étale, then, with notation as in Proposition 2.5, $\mathbf{c}_g u^*$ factors as

$$\boldsymbol{\mathcal{H}}_{X'} g^* u^* \underset{\mathsf{ps}^*}{=\!=} \boldsymbol{\mathcal{H}}_{X'} v^* f^* \xrightarrow[(\boldsymbol{v}^\sharp)^{-1}]{\sim} v^* \boldsymbol{\mathcal{H}}_X f^* \xrightarrow[v^*\mathbf{c}_f]{} v^* f^! \boldsymbol{\mathcal{H}}_Y \xrightarrow[\mathsf{B}_{\mathbf{d}}]{\sim} g^! u^* \boldsymbol{\mathcal{H}}_Y \xrightarrow[g^!\boldsymbol{u}^\sharp]{\sim} g^! \boldsymbol{\mathcal{H}}_{Y'} u^*.$$

Proof. We have $u^* = u^!$, $v^* = v^!$; and the following diagram commutes:

$$
\begin{tikzcd}
v^* f^! \arrow[r, "\mathsf{B}_{\mathbf{d}}"] \arrow[d, equal] & g^! u^* \arrow[d, equal] \\
v^! f^! \arrow[r, equal, "\mathsf{ps}^!"'] & g^! u^!
\end{tikzcd}
$$

This looks like [L3, p. 208, Theorem 4.8.3(iii)]; but that theorem applies only to the full subcategory $\mathbf{D}^+_{\mathsf{qc}} \subset \mathbf{D}_{\mathsf{qc}}$ of homologically bounded-below complexes. To treat all of $\mathbf{D}_{\mathsf{qc}}$ one must expand the diagram according to the definitions of $\mathsf{B}_{\mathbf{d}}$ and $\mathsf{ps}^!$ (see (2.1.2.3) and (2.1.6.1), which agree on $\mathbf{D}^+_{\mathsf{qc}}$ with the usual definitions), and then check that the expanded diagram commutes. The cited Theorem 4.8.3(iii) enters into this verification, but only as applied to $\mathcal{O}_Y$. Details—routine, though tedious—are left to the reader.

In view of Proposition 2.5, we need then to show that subdiagram ① in the next diagram commutes.

$$
\begin{tikzcd}
\boldsymbol{\mathcal{H}}_{X'} g^* u^* \arrow[rr, equal, "\mathsf{ps}^*"] \arrow[dr, equal, "\mathsf{ps}^*"] \arrow[ddd, "\mathbf{c}_g u^*"'] & & \boldsymbol{\mathcal{H}}_{X'} (ug)^* \arrow[rr, "\mathbf{c}_{ug}"] \arrow[d, equal] & & (ug)^! \boldsymbol{\mathcal{H}}_Y \arrow[ddd, equal, "\mathsf{ps}^!"] \\
& \boldsymbol{\mathcal{H}}_{X'} v^* f^* \arrow[r, equal, "\mathsf{ps}^*"] \arrow[d, "\mathbf{c}_v"'] & \boldsymbol{\mathcal{H}}_{X'} (fv)^* \arrow[r, "\mathbf{c}_{fv}"] \arrow[d, phantom, "②"] & (fv)^! \boldsymbol{\mathcal{H}}_Y \arrow[ur, equal] \arrow[d, equal, "\mathsf{ps}^!"'] & \\
& v^! \boldsymbol{\mathcal{H}}_X f^* \arrow[rr, "v^!\mathbf{c}_f"'] \arrow[d, phantom, "①"] & & v^! f^! \boldsymbol{\mathcal{H}}_Y \arrow[dr, equal, "\mathsf{ps}^!"'] & \\
g^! \boldsymbol{\mathcal{H}}_{Y'} u^* \arrow[rrrr, "g^!\mathbf{c}_u"'] & & & & g^! u^! \boldsymbol{\mathcal{H}}_Y
\end{tikzcd}
$$

It is clear that the unlabeled subdiagrams commute. By Theorem 3.1, the outer border and subdiagram ② both commute. The conclusion follows. $\square$

Remarks 3.3.1. (See Remark following [FM, p. 28, 2.6.2].)

(A) When applied to $\mathcal{O}_Y$, Corollary 3.3 says, in bivariant terms, that for any independent square d as above,

$$\boxed{c_g = u^\star c_f}$$

where $u^\star\colon \mathrm{HH}^0(f) \to \mathrm{HH}^0(g)$ is the *pullback,* see [AJL, 3.3.4].

(B) As for *pushforward* (see [AJL, 3.3.3]), for $\mathsf{S_p}$-maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f proper, it holds that

$$\boxed{f_\star c_{gf} = f_\star(c_f \cdot c_g) = (f_\star c_f)\cdot c_g\,.}$$

where the first equality is given by (3.2.3), and the second by [AJL, 4.4]. Of course this doesn't convey much without further information on $f_\star c_f$.

Note that by definition, $f_\star c_{gf}$ factors as

$$\mathcal{H}_Y \xrightarrow{f_\sharp} f_*\mathcal{H}_X \xrightarrow{\widetilde{c}_{gf}} g^!\mathcal{H}_Z$$

where $f_\sharp$ is adjoint to $f^\sharp$ (see §3.2) and $\widetilde{c}_{gf}$ corresponds under duality to c_{gf}.

As indicated at the end of Example 2.6, further study of $f_\star c_f$ will be carried out elsewhere.

3.4. For any S-scheme W and $i \in \mathbb{Z}$, set $\mathrm{HH}_i(W) := \mathrm{HH}_i(W|S)$ and $\mathrm{HH}^i(W) := \mathrm{HH}^i(W|S)$ (see §0.1).

The orientations c_f of flat S-maps $f\colon X \to Y$ give rise to "wrong way" *Gysin homomorphisms*

$$f^{\mathsf{c}}\colon\ \mathrm{HH}_j(Y) \to \mathrm{HH}_j(X)\ \ (j \in \mathbb{Z})$$

and, if f is also proper,

$$f_{\mathsf{c}}\colon\ \mathrm{HH}^j(X) \to \mathrm{HH}^j(Y).$$

As in [FM, §§2.5, 2.6.2], these homomorphisms are defined by

$$f^{\mathsf{c}}(\beta) = c_f\cdot\beta, \qquad f_{\mathsf{c}}(\alpha) = f_\star(\alpha\cdot c_f).$$

More explicitly, if $x\colon X \to S$ and $y\colon Y \to S$ are the structure maps, $\beta\colon \mathcal{H}_Y \to y^!\mathcal{O}_S[-j]$ is in $\mathrm{HH}_j(Y)$, and $\alpha\colon \mathcal{H}_X \to \mathcal{H}_X[j]$ is in $\mathrm{HH}^j(X)$, then $f^{\mathsf{c}}(\beta)$ and $f_{\mathsf{c}}(\alpha)$ are given, respectively, by the compositions

$$\mathcal{H}_X \xrightarrow{c_f} f^!\mathcal{H}_Y \xrightarrow{f^!\beta} f^!y^!\mathcal{O}_S[-j] \overset{\mathsf{ps}^!}{=\!=} x^!\mathcal{O}_S[-j]$$

and (with $f_\sharp$ adjoint to $f^\sharp$, see §3.2)

$$\mathcal{H}_Y \xrightarrow{f_\sharp} f_*\mathcal{H}_X \xrightarrow{f_*\alpha} f_*\mathcal{H}_X[j] \xrightarrow{f_*c_f} f_*f^!\mathcal{H}_Y[j] \xrightarrow{\int_f} \mathcal{H}_Y[j].$$

The basic properties of Gysin homomorphisms are listed in [FM, p. 26]. As noted there, they are all immediate consequences of the bivariant axioms. Let us briefly review the interpretation of these properties and their derivations in the present context, for which purpose we will need the transitivity of the fundamental class (Theorem 3.1) and the base-change Corollary 3.3.

For the remainder of this section, when a pushforward like $h_\star$ appears, the S_p-map h is assumed to be proper; and when a pullback like $h^\star$ appears, the S_p-map h is assumed to be essentially étale.

First, Gysin maps are functorial:

Proposition 3.4.1. *For flat* S*-maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$, *one has*

$$(gf)^{\mathsf{c}} = f^{\mathsf{c}} g^{\mathsf{c}} \quad \text{and} \quad (gf)_{\mathsf{c}} = g_{\mathsf{c}} f_{\mathsf{c}}.$$

Proof. In view of Theorem 3.1, the first equality results from associativity of the $\cdot$ product [AJL, Proposition 4.1], and the second from that associativity plus functoriality of pushforward [AJL, Proposition 4.2] plus commutativity of pushforward with product [AJL, Proposition 4.4]. $\square$

Gysin maps behave well with respect to essentially-étale base change:

Proposition 3.4.2. *For any oriented fiber square in* S

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

with f *(hence* f'*) flat and* g *(hence* g'*) essentially étale, one has*

$$f^{\mathsf{c}} g_\star = g'_\star f'^{\mathsf{c}};$$

and if in addition f *(hence* f'*) is proper, then*

$$g^\star f_{\mathsf{c}} = f'_{\mathsf{c}} g'^\star.$$

Remark 3.4.3. In Proposition 5.2, we will prove that if f and g are flat and f is proper, then $g^{\mathsf{c}} f_\star = f'_\star g'^{\mathsf{c}}$.

Proof. By Remark 3.3.1(A), $\mathsf{c}_{f'} = g^\star \mathsf{c}_f$. Hence the first equality results from the projection formula [AJL, Proposition 4.7], and the second from commutativity of pullback with product, see [AJL, Proposition 4.5] as applied to the following diagram (where $\alpha \in \mathrm{HH}^*(X)$):

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, equal] & X \arrow[d, equal] \arrow[dl, phantom, "\alpha"] \\ X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \arrow[dl, phantom, "c_f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

□

Notation: for an essentially étale S-map $f\colon X \to Y$ and $\alpha \in \mathrm{HH}^*(Y)$, $f^\star\alpha$ is the pullback of α by f, through the independent square

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ \| & & \| \\ X & \xrightarrow[f]{} & Y \end{array}$$

The relation of Gysin maps and pushforward is shown in the next result.

Proposition 3.4.4. *For flat* S*-maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$, *with* f *proper, one has*

(1) $$f_\star\big((gf)^{\mathsf{c}}\beta\big) = (f_\star \mathsf{c}_{gf})\cdot\beta = (f_\star c_f)\cdot g^{\mathsf{c}}\beta \qquad (\beta \in \mathrm{HH}_*(Z)),$$

and if, moreover, f *is essentially étale,*

(2) $$(gf)_{\mathsf{c}}(f^\star\alpha) = g_\star(\alpha\cdot f_\star \mathsf{c}_{gf}) = g_{\mathsf{c}}(\alpha\cdot f_\star c_f) \qquad (\alpha \in \mathrm{HH}^*(Y)).$$

Proof. The first equality in (1) follows at once from commutativity of pushforward with product [AJL, Proposition 4.4]; and in (2) from functoriality of pushforward [AJL, Proposition 4.2] and the projection formula [AJL, Proposition 4.7]. The second equality in both (1) and (2) results, in view of associativity of the $\cdot$ product [AJL, Proposition 4.1], from 3.3.1(B). □

Finally, there are two projection-like properties.

Proposition 3.4.5. *For a flat* S*-map* $X \xrightarrow{f} Y$, $\alpha \in \mathrm{HH}^*(Y)$, $\beta \in \mathrm{HH}_*(Y)$ *and* $\alpha' \in \mathrm{HH}^*(X)$, *one has*

(1) $$f_\star\big(\alpha'\cdot(f^{\mathsf{c}}\beta)\big) = \big(f_{\mathsf{c}}\alpha'\big)\cdot\beta,$$

and if, moreover, f *is essentially étale,*

(2) $$f_{\mathsf{c}}\big((f^\star\alpha)\cdot\alpha'\big) = \alpha\cdot\big(f_{\mathsf{c}}\alpha'\big).$$

Remark. The products in (1) and (2) can be interpreted, respectively, as cap and cup, see [AJL, §3.6].

Proof. The equality (1) follows at once from commutativity of product and pushforward [AJL, Proposition 4.4], and (2) from the projection formula [AJL, Proposition 4.7]. □

4. The dual oriented bivariant theory.

Together with Theorem 3.1, the constructions in [AJL] give, at least for flat maps, an oriented bivariant theory $\mathcal{B}$, see §3.2. In this context there is an order-2 symmetry taking $\mathcal{B}$ to a *dual* oriented bivariant theory $\underline{\mathcal{B}}$, as detailed (in a more abstract situation) in Theorem 4.2 below. While the cohomology groups $\underline{\mathrm{HH}}^i(X)$ $(i \in \mathbb{Z})$ associated to a flat S-scheme X by $\underline{\mathcal{B}}$ are isomorphic to those coming from $\mathcal{B}$, the homology groups $\underline{\mathrm{HH}}_i(X)$ are the classical Hochschild homology groups $\mathrm{H}^{-i}(X, \mathcal{H}_X)$.

In the specialization of Theorem 4.2 to the just-mentioned context, the fundamental class of a flat map plays a key role, illustrated in Example 4.6.

After proving Theorem 4.2, we define (for any flat S-map $x\colon X \to S$) a pairing

$$\mathfrak{p}_X = \mathfrak{p}_x \colon \mathcal{H}_X \otimes_X \mathcal{H}_X \to x^!\mathcal{O}_S$$

by composing the fundamental class $c_x\colon \mathcal{H}_X \to x^!\mathcal{O}_S$ with a natural product map $\mathcal{H}_X \otimes \mathcal{H}_X \to \mathcal{H}_X$. Correspondingly, there is a *duality map*

$$\mathfrak{d}_X \colon \mathcal{H}_X \to \mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S)$$

that turns out to be compatible with étale localization. Whenever x is essentially smooth, $\mathfrak{d}_X$ is an *isomorphism* (Theorem 4.8). There results an isomorphism between the bivariant groups associated to any flat map of essentially smooth S-schemes by $\mathcal{B}$ and by $\underline{\mathcal{B}}$. In particular, one gets (again, cf. [AJL, §§6.4, 6.5]) that the bivariant Hochschild homology groups $\mathrm{HH}_i(X) := \mathrm{HH}_i(X|S)$ of an essentially smooth S-scheme X (see §0.1) are isomorphic to the classical Hochschild homology groups $\mathrm{H}^{-i}(X, \mathcal{H}_X)$.

One deduces directly from Theorem 4.8 that if $S = \operatorname{Spec} H$ with H a Gorenstein artinian ring and if $x\colon X \to S$ is proper and smooth, then there is a *non-singular* pairing on classical Hochschild homology

$$\mathrm{H}^{-i}(X, \mathcal{H}_X) \otimes_H \mathrm{H}^i(X, \mathcal{H}_X) \to H,$$

see Corollary 4.8.4. We haven't figured out the precise relation of this pairing to the Mukai pairing of [CaW, §5].

Also left open is the relation of $\mathfrak{d}_X$ to some other duality isomorphisms that appear in the literature in connection e.g., with proving Riemann-Roch theorems via Hochschild homology (see §4.9).

4.1. Let there be given a setup

$$\Sigma := \big(\mathsf{S}, H, (\mathbf{D}_W)_{W\in\mathsf{S}}, (-)^*, (-)^!, \dots\big)$$

as in [AJL, §3.1.1], but modified slightly as specified in §4.3 below; and a family of degree-0 $\mathbf{D}_X$-maps

$$(f^\sharp\colon f^*\mathcal{H}_Y \to \mathcal{H}_X)_{f\colon X\to Y\,\in\,\mathsf{S}}$$

as in [AJL, §3.2]. Let there also be given a family of degree-0 $\mathbf{D}_X$-maps

$$(c_f\colon \mathcal{H}_X \to f^!\mathcal{H}_Y)_{f\colon X\to Y\,\in\,\mathsf{S}}$$

such that for any S-maps $X \xrightarrow{u} Y \xrightarrow{v} Z$, c_{vu} factors as

$$\mathcal{H}_X \xrightarrow{c_u} u^!\mathcal{H}_Y \xrightarrow{u^!c_v} u^!v^!\mathcal{H}_Z \overset{\mathsf{ps}^!}{=\!=} (vu)^!\mathcal{H}_Z,$$

and such that c_f is an isomorphism whenever f is the bottom or top arrow of an independent square.

Example 4.1.1. In the bivariant theory of §3.2, restricting to flat maps—a restriction which we hope eventually to eliminate—one gets such data from Proposition 2.5 and Theorem 3.1.

Theorem 4.2. *Under the preceding conditions, there is a bivariant theory* $\underline{\mathcal{B}}$ *assigning to an* S*-map* $f\colon X \to Y$ *the symmetric graded* H*-module*

$$\underline{\mathrm{HH}}^*(X \xrightarrow{f} Y) := \oplus_{i\in\mathbb{Z}}\,\mathbf{D}^i_X(f^*\mathcal{H}_Y, \mathcal{H}_X)$$

(*so that the family* $(f^\sharp)$ *orients* $\underline{\mathcal{B}}$), *and having the following operations.*

4.2.1. *Product.* Let $f : X \to Y$ and $g : Y \to Z$ be in S.

For $i, j \in \mathbb{Z}$ and $\alpha \in \underline{\mathrm{HH}}^i(X \xrightarrow{f} Y)$, $\beta \in \underline{\mathrm{HH}}^j(Y \xrightarrow{g} Z)$, the product

$$\alpha\cdot\beta \in \underline{\mathrm{HH}}^{i+j}(X \xrightarrow{gf} Z)$$

is $(-1)^{ij}$ times the composite $\mathbf{D}_X$-map

$$(gf)^*\mathcal{H}_Z \overset{\mathsf{ps}^*}{=\!=} f^*g^*\mathcal{H}_Z \xrightarrow{f^*\beta} f^*\mathcal{H}_Y \xrightarrow{\alpha} \mathcal{H}_X.$$

4.2.2. *Pushforward.* Let $f\colon X \to Y$ and $g\colon Y \to Z$ be S-maps, f confined. The pushforward by f

$$f_{\underline{\star}}\colon \underline{\mathrm{HH}}^*(X \xrightarrow{gf} Z) \to \underline{\mathrm{HH}}^*(Y \xrightarrow{g} Z)$$

is the graded H-linear $\mathbf{D}_Y$-map such that for $i \in \mathbb{Z}$ and $\alpha \in \underline{\mathrm{HH}}^i(X \xrightarrow{gf} Z)$, the image $f_{\underline{\star}}\alpha \in \underline{\mathrm{HH}}^i(Y \xrightarrow{g} Z)$ is the $\mathbf{D}_Y$-composition

$$g^*\mathcal{H}_Z \xrightarrow{\eta_f} f_*f^*g^*\mathcal{H}_Z \overset{f_*\mathsf{ps}^*}{=\!=\!=} f_*(gf)^*\mathcal{H}_Z \xrightarrow{f_*\alpha} f_*\mathcal{H}_X \xrightarrow{f_*c_f} f_*f^!\mathcal{H}_Y \xrightarrow{\int_f} \mathcal{H}_Y.$$

4.2.3. *Pullback.* Given an independent square in S

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathbf{d}"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

let $\pi\colon (-)^* \xrightarrow{\,\sim\,} (-)^!$ be the pseudofunctorial isomorphism in (4.3.3) below, and let $\mathsf{B}'_{\mathbf{d}}$ be the composed isomorphism

$$g^*u^! \xrightarrow{g^*\pi^{-1}} g^*u^* \overset{\mathsf{ps}^*}{=\!=} v^*f^* \xrightarrow{\pi} v^!f^*.$$

The pullback by u, through $\mathbf{d}$,

$$u^{\underline{\star}}\colon \underline{\mathrm{HH}}^*(X \xrightarrow{f} Y) \longrightarrow \underline{\mathrm{HH}}^*(X' \xrightarrow{g} Y')$$

is the graded H-linear $\mathbf{D}_Y$-map such that for $i \in \mathbb{Z}$ and $\alpha \in \underline{\mathrm{HH}}^i(X \xrightarrow{f} Y)$, the image $u^{\underline{\star}}\alpha \in \underline{\mathrm{HH}}^i(X' \xrightarrow{g} Y')$ is the $\mathbf{D}_Y$-composition

$$g^*\mathcal{H}_{Y'} \xrightarrow{g^*c_u} g^*u^!\mathcal{H}_Y \overset{\mathsf{B}'_{\mathbf{d}}}{=\!=\!=} v^!f^*\mathcal{H}_Y \xrightarrow{v^!\alpha} v^!\mathcal{H}_X \xrightarrow{c_v^{-1}} \mathcal{H}_{X'}.$$

Remark 4.2.4. In the specific circumstances dealt with in §2, in particular Proposition 2.5, the pullback of $\alpha \in \underline{\mathrm{HH}}^i(X \xrightarrow{f} Y)$ is the $\mathbf{D}_Y$-composition

$$g^*\mathcal{H}_{Y'} \xrightarrow{g^*u^{\sharp-1}} g^*u^*\mathcal{H}_Y \overset{\mathsf{ps}^*}{=\!=\!=} v^*f^*\mathcal{H}_Y \xrightarrow{v^*\alpha} v^*\mathcal{H}_X \xrightarrow{v^\sharp} \mathcal{H}_{X'}.$$

Remark 4.2.5. The homology groups associated to $\underline{\mathcal{B}}$ are

$$\underline{\mathrm{HH}}_i(X) := \mathbf{D}_X^{-i}(\mathcal{O}_X, \mathcal{H}_X) \qquad (i \in \mathbb{Z}).$$

For the example just before Theorem 4.2, these are the classical Hochschild (hyper)homology groups $\mathrm{H}^{-i}(X, \mathcal{H}_X)$ of the flat S-scheme X.

These groups are covariant for confined S-maps, via pushforward (take $Z = S$ in 4.2.2); and contravariant for all $f\colon X \to Y$, via Gysin maps

$$f^{\underline{\mathsf{c}}}\colon \underline{\mathrm{HH}}_j(Y) \to \underline{\mathrm{HH}}_j(X) \ (j \in \mathbb{Z})$$

such that for any $\beta\colon \mathcal{O}_Y \to \mathcal{H}_Y[-j]$, $f^{\underline{\mathsf{c}}}\beta$ is the composition

$$\mathcal{O}_X = f^*\mathcal{O}_Y \xrightarrow{f^*\beta} f^*\mathcal{H}_Y[-j] \xrightarrow{f^\sharp} \mathcal{H}_X[-j].$$

The cohomology H-algebras

$$\underline{\mathrm{HH}}^i(X) := \mathbf{D}_X^i(\mathcal{H}_X, \mathcal{H}_X),$$

are contravariant via pullback for co-confined maps. (This pullback functor, associated with $\underline{\mathcal{B}}$, actually coincides with the one in [AJL, 3.6.1].) They are also covariant, as groups, for confined maps $f\colon X \to Y$, via Gysin maps

$$f_{\underline{\mathsf{c}}}\colon \underline{\mathrm{HH}}^j(X) \to \underline{\mathrm{HH}}^j(Y)$$

that are such that for any $\alpha\colon \mathcal{H}_X \to \mathcal{H}_X[j]$, $f_{\underline{\mathsf{c}}}\alpha$ is the composition

$$\mathcal{H}_Y \xrightarrow{\eta_f} f_*f^*\mathcal{H}_Y \xrightarrow{f_*f^\sharp} f_*\mathcal{H}_X \xrightarrow{f_*\alpha} f_*\mathcal{H}_X[j] \xrightarrow{f_*c_f} f_*f^!\mathcal{H}_Y[j] \xrightarrow{\int_f} \mathcal{H}_Y[j].$$

This $f_{\underline{\mathsf{c}}}$ coincides with the Gysin map f_{c} in §3.4.

Remark 4.2.6. For essentially smooth maps, where sometimes the Hochschild homology groups can be interpreted in terms of Hodge homology (via so-called Hochschild-Kostant-Rosenberg isomorphisms), one would like to know more about the relation between the preceding operations on Hochschild homology groups and those on Hodge groups that play an important role in [ChR]. For this, our constructions are at present too limited, in that they do not apply to arbitrary perfect maps, such as local complete intersection maps, nor to homology with supports.

4.3. *Proof of Theorem* 4.2*.* The following modification of the given setup does not compromise the validity of [AJL, Theorem 3.4]. Accordingly, there exists a bivariant theory $\mathcal{B}$ oriented by the family c_f (cf. §3). In this situation we are going to *dualize* all the given data, and then find that in the dualized situation all the assumptions needed for [AJL, Theorem 3.4] are satisfied. The resulting bivariant theory $\underline{\mathcal{B}}$ is the one referred to in Theorem 4.2.

Furthermore, $\underline{\mathcal{B}}$ is oriented by the family of maps $f^\sharp$, and its setup conforms to the following modifications. So $\underline{\mathcal{B}}$ itself can be dualized; and it will result easily from the detailed description of $\underline{\mathcal{B}}$ that $\underline{\underline{\mathcal{B}}} = \mathcal{B}$.

To begin with, let us weaken slightly the conditions defining a "setup" upon which bivariant theories are built (see [AJL, §3.1]).

To wit, in [AJL], require only that in §2.4 the pseudofunctor $(-)_*$ and the pseudofunctorial adjunction $(-)^* \dashv (-)_*$ exist over the subcategory of *confined* S-maps; that in §2.5, $\theta_{\mathbf{d}}\colon u^*f_* \xrightarrow{\ \sim\ } g_*v^*$ be defined only for those independent squares

$$
\begin{tikzcd}
\bullet \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathbf{d}"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "u"'] & \bullet
\end{tikzcd}
\tag{4.3.1}
$$

in which f and g are *confined,* to be the following composition:[2]

$$
u^*f_* \xrightarrow{\eta_g} g_*g^*u^*f_* \overset{\mathsf{ps}^*}{=\!=} g_*v^*f^*f_* \xrightarrow{\epsilon_f} g_*v^*;
\tag{4.3.2}
$$

and that in §3.2, $f_\sharp$ be defined when f is confined. It is then straightforward to check that the definitions in [AJL, §3.3] of product, pushforward and pullback, and the verification in [AJL, §4] of the bivariant axioms remain valid without change.

Also, for the sake of the symmetry about to be described we assume that the map $\int_f\colon f_*f^! \to \mathrm{id}$ in [AJL, (2.4.3)] is the counit for a pseudofunctorial adjunction $(-)_* \dashv (-)^!$ holding over confined maps. (*Pseudofunctoriality* of the adjunction means that the diagram [AJL, (2.4.4)] commutes.) This condition holds for the setup constructed in [AJL, §5] (see [AJL], proof of 5.9.3).

From these assumptions it follows that commutativity of diagram (2.6.2) in [AJL] is implied by that of (2.6.1). Indeed, for an independent square $\mathbf{d}$ as above, with f and g confined, the latter commutativity means that the map $\mathsf{B}_{\mathbf{d}}\colon v^*f^! \to g^!u^*$ is adjoint to

$$
g_*v^*f^! \xrightarrow{\theta_{\mathbf{d}}^{-1}} u^*f_*f^! \xrightarrow{\int_f} u^*,
$$

so that $\mathsf{B}_{\mathbf{d}}$ is the composite map

$$
v^*f^! \to g^!g_*v^*f^! \xrightarrow{\theta_{\mathbf{d}}^{-1}} g^!u^*f_*f^! \xrightarrow{\int_f} g^!u^*.
$$

Knowing this, one proves commutativity of (2.6.2) just as in [AJL, §5.10.2].

One further assumption: there is a category $\mathsf{CC} \subset \mathsf{S}$ that contains the top and bottom arrows of every independent square, and an isomorphism π from the restriction to CC of the pseudofunctor $(-)^*$ to that of $(-)^!$, such

[2] This definition, rather than [AJL, (2.5.1)] was used in [AJL, (2.6.1)], where the equivalence of the two definitions should have been noted—see [L3, 3.7.2(i)].

that for any independent $\mathbf{d}$ as above (f and g not necessarily confined), the next diagram commutes:

$$
\begin{tikzcd}
v^*f^! \arrow[r, "\mathsf{B}_{\mathbf{d}}"] \arrow[d, "\pi"', "\simeq"] & g^!u^* \arrow[d, "\simeq"', "g^!\pi"] \\
v^!f^! \arrow[r, equal, "\mathsf{ps}^!"'] & g^!u^!
\end{tikzcd}
\tag{4.3.3}
$$

This assumption too holds for the setup in [AJL, §5]—see [Nk, p. 541] and (2.1.2.1) above.

4.4. Let there be given a setup

$$\Sigma := \big(\mathsf{S}, H, (\mathbf{D}_W)_{W\in\mathsf{S}}, (-)^*, (-)^!, \dots\big),$$

modified as in §4.3. Referring to [AJL, §2], we now construct a *dual setup*

$$\underline{\Sigma} = \big(\mathsf{S}, H, (\underline{\mathbf{D}}_W)_{W\in\mathsf{S}}, (-)^{\underline{*}}, (-)^{\underline{!}}, \dots\big).$$

First, the category S and its confined maps and independent squares, as well as the graded-commutative ring H, remain the same. (See [AJL, §2.1].)

Next, to each object $W \in \mathsf{S}$ associate the category $\underline{\mathbf{D}}_W$ *dual to* the H-graded category $\mathbf{D}_W$ originally associated to W. By definition, the dual $\underline{\mathbf{E}}$ of an H-graded category $\mathbf{E}$ has the same objects as $\mathbf{E}$; for each object $A \in \mathbf{E}$ let $\underline{A}$ be A considered as an object of $\underline{\mathbf{E}}$. For any $A,\ B \in \mathbf{E}$, let $\underline{\mathbf{E}}(\underline{A}, \underline{B})$ be the graded H-module $\mathbf{E}(B, A)$; for each $\alpha \in \mathbf{E}(B, A)$ let $\underline{\alpha}$ be α considered as an element of $\underline{\mathbf{E}}(\underline{A}, \underline{B})$. Finally, define composition in $\underline{\mathbf{E}}$,

$$\underline{\mathbf{E}}(\underline{B}, \underline{C}) \times \underline{\mathbf{E}}(\underline{A}, \underline{B}) \xrightarrow{\ \underline{\circ}\ } \underline{\mathbf{E}}(\underline{A}, \underline{C}),$$

to be the unique $\mathbb{Z}$-bilinear map taking $(\underline{\beta}, \underline{\alpha}) \in \underline{\mathbf{E}}^p(\underline{B}, \underline{C}) \times \underline{\mathbf{E}}^q(\underline{A}, \underline{B})$ to

$$\underline{\beta} \,\underline{\circ}\, \underline{\alpha} := (-1)^{pq}\, \alpha \circ \beta \in \mathbf{E}^{p+q}(C, A) = \underline{\mathbf{E}}^{p+q}(\underline{A}, \underline{C}).$$

One checks that this $\underline{\mathbf{E}}$ is an H-graded category. (See [AJL, §1.1].)

Any H-graded functor F between H-graded categories can be regarded in the obvious way as an H-graded functor, denoted $\underline{F}$, between the dual H-graded categories. A functorial map $\xi\colon F \to G$ of degree n is then the same as a functorial map $\underline{\xi}\colon \underline{G} \to \underline{F}$ of degree n. (See [AJL, §1.2].)

For any S-map $f\colon X \to Y$, define the functors

$$f^{\underline{*}} := \underline{f^!}\colon \underline{\mathbf{D}}_Y \to \underline{\mathbf{D}}_X, \qquad f^{\underline{!}} := \underline{f^*}\colon \underline{\mathbf{D}}_Y \to \underline{\mathbf{D}}_X, \qquad f_{\underline{*}} := \underline{f_*}\colon \underline{\mathbf{D}}_X \to \underline{\mathbf{D}}_Y.$$

There result H-graded pseudofunctors with, for a second S-map $Y \xrightarrow{g} Z$,

$$\mathsf{ps}^{\underline{*}} = (\underline{\mathsf{ps}^!})^{-1}\colon f^{\underline{*}} g^{\underline{*}} \xrightarrow{\ \sim\ } (gf)^{\underline{*}},$$
$$\mathsf{ps}^{\underline{!}} = (\underline{\mathsf{ps}^*})^{-1}\colon f^{\underline{!}} g^{\underline{!}} \xrightarrow{\ \sim\ } (gf)^{\underline{!}},$$
$$\mathsf{ps}_{\underline{*}} = (\underline{\mathsf{ps}_*})^{-1}\colon (gf)_{\underline{*}} \xrightarrow{\ \sim\ } g_{\underline{*}} f_{\underline{*}}.$$

It is easily seen that over confined maps, the pseudofunctorial adjunctions $(-)^* \dashv (-)_*$ and $(-)_* \dashv (-)^!$ (§4.3) give rise to pseudofunctorial adjunctions $(-)_{\underline{*}} \dashv (-)^{\underline{!}}$ and $(-)^{\underline{*}} \dashv (-)_{\underline{*}}$ respectively. For confined f, let $\underline{\int}_f \colon f_{\underline{*}} f^{\underline{!}} \to \mathrm{id}$ be the counit map associated with the first of these adjunctions.

Over co-confined maps, one has the pseudofunctorial isomorphism

$$\underline{\pi}\colon (-)^{\underline{*}} \xrightarrow{\ \sim\ } (-)^{\underline{!}}.$$

For an independent $\mathbf{d}$ as in (4.3.1), let $\mathsf{B}'_{\mathbf{d}}$ be the composite isomorphism

$$g^*u^! \xrightarrow{g^*\pi^{-1}} g^*u^* \overset{\mathsf{ps}^*}{=\!=} v^*f^* \xrightarrow{\ \pi\ } v^!f^*;$$

and set

$$\underline{\mathsf{B}}_{\mathbf{d}} := \underline{\mathsf{B}'_{\mathbf{d}}}\colon v^{\underline{*}}f^{\underline{!}} \xrightarrow{\ \sim\ } g^{\underline{!}}u^{\underline{*}}. \tag{4.4.1}$$

Horizontal and vertical transitivity of $\underline{\mathsf{B}}_{\mathbf{d}}$ (cf. [AJL, (2.3.1) and (2.3.2)]) are straightforward consequences of $\underline{\pi}$ being an isomorphism of pseudofunctors. The required commutativity of the diagram (cf. (4.3.3) (dualized))

$$\begin{array}{ccc}
v^{\underline{*}}f^{\underline{!}} & \xrightarrow{\ \underline{\mathsf{B}}_{\mathbf{d}}\ } & g^{\underline{!}}u^{\underline{*}} \\
{\scriptstyle \underline{\pi}}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle g^{\underline{!}}\underline{\pi}} \\
v^{\underline{!}}f^{\underline{!}} & \underset{\mathsf{ps}^{\underline{!}}}{=\!=\!=} & g^{\underline{!}}u^{\underline{!}}
\end{array}$$

follows easily from (4.3.3) (dualized) and the definition of $\underline{\mathsf{B}}_{\mathbf{d}}$.

Also, when f and g in $\mathbf{d}$ are confined, let $\theta'_{\mathbf{d}}$ be the natural composition

$$g_*v^! \to g_*v^!f^!f_* \overset{\mathsf{ps}^!}{=\!=} g_*g^!u^!f_* \to u^!f_*;$$

and set

$$\underline{\theta}_{\mathbf{d}} := \underline{\theta'_{\mathbf{d}}}\colon u^{\underline{*}}f_{\underline{*}} \to g_{\underline{*}}v^{\underline{*}}.$$

In other words, $\underline{\theta}_{\mathbf{d}}$ is the natural composition (cf. (4.3.2))

$$u^{\underline{*}}f_{\underline{*}} \to g_{\underline{*}}g^{\underline{*}}u^{\underline{*}}f_{\underline{*}} \overset{\mathsf{ps}^{\underline{*}}}{=\!=} g_{\underline{*}}v^{\underline{*}}f^{\underline{*}}f_{\underline{*}} \to g_{\underline{*}}v^{\underline{*}}.$$

It needs to be shown that $\underline{\theta}_{\mathbf{d}}$ is an isomorphism, or equivalently, that $\theta'_{\mathbf{d}}$ is an isomorphism. Since $\theta_{\mathbf{d}}$ and π are isomorphisms, this results from:

Lemma 4.4.2. *The following diagram commutes.*

$$\begin{array}{ccc}
g_*v^* & \xrightarrow[g_*\pi]{\ \sim\ } & g_*v^! \\
{\scriptstyle \theta_{\mathbf{d}}^{-1}}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \theta'_{\mathbf{d}}} \\
u^*f_* & \xrightarrow[\pi]{\ \sim\ } & u^!f_*
\end{array}$$

Proof. Embed the diagram in question as subdiagram ① of the following diagram, where $\varpi\colon \mathrm{id} \to f^!f_*$ is the unit map of the adjunction $f_* \dashv f^!$ (see paragraph a few lines after (4.3.2)).

$$
\begin{tikzcd}
g_*v^* \arrow[r, "g_*\pi"] \arrow[d, "\theta_{\mathbf{d}}^{-1}"'] \arrow[dr, phantom, "①"] & g_*v^! \arrow[rr, "g_*v^!\varpi"] \arrow[d, "\theta'_{\mathbf{d}}"] \arrow[drr, phantom, "②"] & & g_*v^!f^!f_* \arrow[dddd, "g_*\pi^{-1}"] \arrow[dl, equal, "\mathsf{ps}^!"'] \\
u^*f_* \arrow[r, "\pi"'] \arrow[ddd, shift right, "u^*f_*\varpi"'] & u^!f_* \arrow[dd, phantom, "③"] & g_*g^!u^!f_* \arrow[l, "\int_g"] \arrow[dl, "g_*g^!\pi^{-1}"] \arrow[dd, phantom, "④"] & \\
 & g_*g^!u^*f_* \arrow[ul, "\int_g"] & & \\
 & ⑤ & & \\
u^*f_*f^!f_* \arrow[uuu, shift right, "u^*\int_f"'] \arrow[rrr, "\theta_{\mathbf{d}}"'] & & & g_*v^*f^!f_* \arrow[uul, "g_*\mathsf{B}_{\mathbf{d}}"']
\end{tikzcd}
$$

The outer border of this diagram commutes: going clockwise around the border from g_*v^* to $g_*v^*f^!f_*$ clearly gives the map $g_*v^*\varpi$, as does going around counterclockwise.

Commutativity of subdiagram ② is the definition of $\theta'_{\mathbf{d}}$; that of ③ is clear; that of ④ results from that of diagram (4.3.3); and that of ⑤ from that of [AJL, (2.6.1)].

With these commutativities in view, and since $\int_f \circ f_*\varpi$ is the identity map of f_*, diagram chasing yields the desired commutativity of ①. □

The remaining nontrivial condition for $\underline{\Sigma}$ to be a setup (as in §4.3) is commutativity of the analog of [AJL, (2.6.1)].

Lemma 4.4.3. *The following diagram commutes.*

$$
\begin{tikzcd}
u^{\underline{*}}f_{\underline{*}}f^{\underline{!}} \arrow[r, "\underline{\theta}_{\mathbf{d}}"] \arrow[d, "u^{\underline{*}}\underline{\int}_f"'] & g_{\underline{*}}v^{\underline{*}}f^{\underline{!}} \arrow[d, "g_{\underline{*}}\underline{\mathsf{B}}_{\mathbf{d}}"] \\
u^* & g_{\underline{*}}g^{\underline{!}}u^{\underline{*}} \arrow[l, "\underline{\int}_g"]
\end{tikzcd}
$$

Proof. After expansion of the "dualized" diagram, the assertion becomes that the border of the following natural diagram commutes.

$$
\begin{tikzcd}
u^! f_* f^* & g_* g^! u^! f_* f^* \arrow[l] \arrow[r, equal, "\text{via } \mathsf{ps}^!"] \arrow[dr, phantom, "\textcircled{1}"] & g_* v^! f^! f_* f^* & g_* v^! f^! \arrow[l] \\
u^* f_* f^* \arrow[u, "\pi"] & g_* g^! u^* f_* f^* \arrow[l] \arrow[u, "\text{via } \pi"'] \arrow[dr, phantom, "\textcircled{2}"] & g_* v^* f^! f_* f^* \arrow[l, "\text{via } \mathsf{B_d}"] \arrow[u, "\text{via } \pi"'] & g_* v^* f^* \arrow[l] \arrow[u, "\text{via } \pi"'] \arrow[d, equal, "\mathsf{ps}^*"] \\
u^* \arrow[u] \arrow[rrr] & & & g_* g^* u^* \\
u^! \arrow[u, "\pi^{-1}"] \arrow[rrr] & & & g_* g^! u^! \arrow[u, "\text{via } \pi^{-1}"']
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams is clear; and that of ① is given by (4.3.3). Subdiagram ② expands as

$$
\begin{tikzcd}
g_* g^! u^* f_* f^* \arrow[dd] \arrow[drr, phantom, "\textcircled{3}"] & & & & g_* v^* f^! f_* f^* \arrow[llll, "\text{via } \mathsf{B_d}"'] \\
& & u^* f_* f^! f_* f^* \arrow[urr, "\theta"] \arrow[dll, shift left] & & \\
u^* f_* f^* \arrow[urr, shift left] \arrow[rrrr, "\theta"] \arrow[dr] & & & & g_* v^* f^* \arrow[uu] \arrow[dl, shift left] \arrow[dd, equal, "\mathsf{ps}^*"] \\
& g_* g^* u^* f_* f^* \arrow[rr, equal, "\mathsf{ps}^*"] & & g_* v^* f^* f_* f^* \arrow[ur, shift left] & \\
u^* \arrow[uu] \arrow[rrrr] & & & & g_* g^* u^* \arrow[ulll]
\end{tikzcd}
$$

Here, commutativity of the unlabeled subdiagrams is clear. The oblique arrow-pairs compose to the identity maps of $u^*f_*f^*$ and $g_*v^*f^*$, respectively. Further, subdiagram ③ commutes by [AJL, (2.6.1)]. It follows then by diagram chasing that the outer border commutes, proving the Lemma. $\square$

In conclusion. To each setup Σ as in §4.3 we have associated a setup $\underline{\Sigma}$ satisfying the same conditions. Moreover, one verifies that $\underline{\underline{\Sigma}} = \Sigma$.

4.5. With reference to the situation described just before 4.2, and denoting the image of $\mathcal{H}_-$ in the dual category $\underline{\mathbf{D}}_-$ by $\underline{\mathcal{H}}_-$, assign to $f\colon X \to Y$ in S the $\underline{\mathbf{D}}_X$-map

$$f^\sharp = \underline{c_f}\colon f^*\underline{\mathcal{H}}_Y \to \underline{\mathcal{H}}_X. \tag{4.5.1}$$

The adjoint map $f_{\underline{\sharp}}\colon \underline{\mathcal{H}}_Y \to f_{\underline{*}}\underline{\mathcal{H}}_X$ is defined whenever f is confined.

Transitivity for the family $(f^{\underline{\sharp}})$, the property that $f^{\underline{\sharp}}$ is an isomorphism if f is the bottom or top arrow of some independent square, and that $f^{\underline{\sharp}}$ is an identity map if f is, all result from the corresponding properties of the family (c_f).

By [AJL, Theorem 3.4], we get a bivariant theory $\underline{\mathcal{B}}$ over S, assigning to each S-map $f\colon X \to Y$ the symmetric graded H-module

$$\begin{aligned}\underline{\mathrm{HH}}^*(X \xrightarrow{f} Y) := \underline{\mathbf{D}}_X(\underline{\mathcal{H}}_X, f^! \underline{\mathcal{H}}_Y) &= \oplus_{i\in\mathbb{Z}}\, \underline{\mathbf{D}}^i_X(\underline{\mathcal{H}}_X, f^! \underline{\mathcal{H}}_Y)\\ &= \oplus_{i\in\mathbb{Z}}\, \mathbf{D}^i_X(f^*\mathcal{H}_Y, \mathcal{H}_X).\end{aligned}$$

This is the theory referred to in Theorem 4.2.

The descriptions in Theorem 4.2 of product, pushforward and pullback for $\underline{\mathcal{B}}$ are obtained by dualizing those in [AJL, §3.3]—as applied to $\underline{\mathcal{B}}$, taking into account the relations $(-)^{\underline{\sharp}} = \underline{c_{(-)}}$ (see 4.5.1) and $\underline{\mathsf{B}}_{\mathbf{d}} := \underline{\mathsf{B}'_{\mathbf{d}}}$ (see 4.4.1).

This completes the proof of Theorem 4.2.

Example 4.6. Assume S has a dualizing complex $\mathcal{D}$. Then for $x\colon X \to S$ in S, $\mathcal{D}_x := x^!_+\mathcal{D}$ (see 2.1.2) is a dualizing complex on X: localization over noetherian rings preserves injectivity of modules, and hence for a localizing immersion f the functor $f^! = f^*$ preserves finiteness of injective dimension, so that the proof of [L3, Proposition 4.10.1(i)] extends to the efp context. As before, we abuse notation by writing $\mathcal{D}_X$ for $\mathcal{D}_x$.

Note in particular the case where x is essentially smooth of relative dimension n, so that by Proposition 2.4.2, or otherwise, $\mathcal{D}_X \cong \Omega^n_x[n] \otimes x^*\mathcal{D}$.

Let $\mathbf{D}^+_{\mathsf{c}}(X)$ (resp. $\mathbf{D}^-_{\mathsf{c}}(X)$) be that full subcategory of $\mathbf{D}(X)$ whose objects are the $\mathcal{O}_X$-complexes E such that the cohomology sheaf $H^n(E)$ is coherent for all $n \in \mathbb{Z}$ and vanishes for $n \ll 0$ (resp. $n \gg 0$). The functor

$$\boldsymbol{\mathcal{D}}_X(-) := \mathsf{R}\mathcal{H}om(-, \mathcal{D}_X)$$

induces quasi-inverse anti-equivalences between $\mathbf{D}^+_{\mathsf{c}}(X)$ and $\mathbf{D}^-_{\mathsf{c}}(X)$, and so corresponds to an equivalence of each of $\mathbf{D}^+_{\mathsf{c}}(X)$ and $\mathbf{D}^-_{\mathsf{c}}(X)$ with the dual category of the other—even as graded H-categories, see the beginning of §4.4.

Suppose now that $f\colon X \to Y$ is an S_{p}-map, i.e., a perfect S-map. Then $f^*\mathbf{D}^+_{\mathsf{c}}(Y) \subset \mathbf{D}^+_{\mathsf{c}}(X)$ and $f^*\mathbf{D}^-_{\mathsf{c}}(Y) \subset \mathbf{D}^-_{\mathsf{c}}(X)$ (cf. [AIL, Remark 2.1.5]).

Moreover, $f^!\mathbf{D}^+_{\mathsf{c}}(Y) \subset \mathbf{D}^+_{\mathsf{c}}(X)$: indeed, for $F \in \mathbf{D}^+_{\mathsf{c}}(Y)$, the property that $f^!F \in \mathbf{D}^+_{\mathsf{c}}(X)$ is local on X, so to verify that property one may assume that $f = pi$ where $p\colon Z \to Y$ is essentially smooth of relative dimension n (say) and $i\colon X \to Z$ is a closed immersion in S_{p}, with $i_*\mathcal{O}_X$ perfect over $\mathcal{O}_Z$ ([AIL, Remark 2.1.2]); and then $p^!F \cong \Omega^n_p[n] \otimes_Z p^*F \in \mathbf{D}^+_{\mathsf{c}}(Z)$ (see 2.4.2), so it suffices to note that since the $\mathcal{O}_Z$-complex $i_*i^!\mathcal{O}_Z \cong \mathsf{R}\mathcal{H}om_Z(i_*\mathcal{O}_X, \mathcal{O}_Z)$ is perfect, therefore

$$i_*f^!F \cong i_*i^!p^!F = i_*(i^!\mathcal{O}_Z \otimes_X i^*p^!F) \underset{(1.3.5)}{\cong} i_*i^!\mathcal{O}_Z \otimes_Z p^!F \in \mathbf{D}^+_{\mathsf{c}}(Z).$$

Similarly, $f^!\mathbf{D}_{\mathsf{c}}^-(Y) \subset \mathbf{D}_{\mathsf{c}}^-(X)$.

If f is also proper, then $f_*\mathbf{D}_{\mathsf{c}}^+(X) \subset \mathbf{D}_{\mathsf{c}}^+(Y)$ and $f_*\mathbf{D}_{\mathsf{c}}^-(X) \subset \mathbf{D}_{\mathsf{c}}^-(Y)$ (see, e.g., [L3, 3.9.2 and 4.3.3.2]); and as $\mathcal{D}_X \cong f_+^!\mathcal{D}_Y = f^!\mathcal{D}_Y$, Grothendieck duality gives an isomorphism of functors

$$f_*\boldsymbol{\mathcal{D}}_X \cong \boldsymbol{\mathcal{D}}_Y f_*.$$

Finally, there are functorial isomorphisms

$$\begin{aligned} f^*\boldsymbol{\mathcal{D}}_Y M &\cong \boldsymbol{\mathcal{D}}_X f^! M \qquad (M \in \mathbf{D}_{\mathsf{c}}^-(Y)),\\ f^!\boldsymbol{\mathcal{D}}_Y N &\cong \boldsymbol{\mathcal{D}}_X f^* N \qquad (N \in \mathbf{D}_{\mathsf{c}}^+(Y)). \end{aligned}$$

Indeed, for $M \in \mathbf{D}_{\mathsf{c}}^-(Y)$, [L3, Proposition 4.6.7] and [AIL, Lemma 2.1.10]—in which, using the present definition of $f^!$ (see (2.1.2.1)), one need only assume $M \in \mathbf{D}_{\mathsf{c}}^-(Y)$—give functorial isomorphisms

$$\begin{aligned} f^*\boldsymbol{\mathcal{D}}_Y M = f^*\mathsf{R}\mathcal{H}om_Y(M, \mathcal{D}_Y) &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(f^*M, f^*\mathcal{D}_Y)\\ &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(f^!M, f^!\mathcal{D}_Y) \cong \boldsymbol{\mathcal{D}}_X f^! M; \end{aligned}$$

and there results a sequence of functorial isomorphisms

$$f^!\boldsymbol{\mathcal{D}}_Y N \xrightarrow{\ \sim\ } \boldsymbol{\mathcal{D}}_X\boldsymbol{\mathcal{D}}_X f^!\boldsymbol{\mathcal{D}}_Y N \xrightarrow{\ \sim\ } \boldsymbol{\mathcal{D}}_X f^*\boldsymbol{\mathcal{D}}_Y\boldsymbol{\mathcal{D}}_Y N \xrightarrow{\ \sim\ } \boldsymbol{\mathcal{D}}_X f^* N.$$

(One could also imitate the proof of [L3, Proposition 4.10.1(ii)]—without ignoring, as that proof does, the question of dependence of the constructed isomorphism on the choice of the implicitly used compactification.)

It follows (details left to the reader) that for the bivariant theory described in §3.2, but restricted to flat S-maps and to complexes in $\mathbf{D}_{\mathsf{c}}^-$, one can regard the dual bivariant theory as arising from the same setup, except that $\mathbf{D}_{\mathsf{c}}^-$ is replaced throughout by $\mathbf{D}_{\mathsf{c}}^+$, and $\mathcal{H}$ by $\mathcal{H}'_W := \boldsymbol{\mathcal{D}}_W\mathcal{H}_W$, and for any flat S-map $f\colon X \to Y$, $f^{\sharp\prime}\colon f^*\mathcal{H}'_Y \to \mathcal{H}'_X$ is defined to be the dual of the fundamental class c_f, i.e., the natural composition

$$f^*\mathcal{H}'_Y = f^*\boldsymbol{\mathcal{D}}_Y\mathcal{H}_Y \xrightarrow{\ \sim\ } \boldsymbol{\mathcal{D}}_X f^!\mathcal{H}_Y \xrightarrow{\boldsymbol{\mathcal{D}}_X c_f} \boldsymbol{\mathcal{D}}_X\mathcal{H}_X = \mathcal{H}'_X.$$

4.7. Next, for a flat $x\colon X \to S$ in S, we discuss a product $\mathcal{H}_X \otimes \mathcal{H}_X \to \mathcal{H}_X$, and then combine it with the fundamental class c_x to define the duality map (4.7.6)

$$\mathfrak{d}_X\colon \mathcal{H}_X \to \mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S).$$

As mentioned at the beginning of §4, Theorem 4.8 says that when x is essentially smooth, $\mathfrak{d}_X$ is an *isomorphism*.

Let $f\colon X \to Y$ be a scheme-map, with diagonal $\delta\colon X \to X \times_Y X$ and pre-Hochschild functor

$$\boldsymbol{\mathcal{H}}_f := \delta^*\delta_*\colon \mathbf{D}(X) \to \mathbf{D}(X),$$

see beginning of §1 (with, as usual, the notational convention of §1.2).

Define the bifunctorial map

$$\mathbf{t}_f(A,B)\colon \boldsymbol{\mathcal{H}}_f A\otimes_X \boldsymbol{\mathcal{H}}_f B \longrightarrow \boldsymbol{\mathcal{H}}_f(A\otimes_X B) \qquad (A,B\in \mathbf{D}(X)) \tag{4.7.1}$$

to be the natural composition

$$\delta^*\delta_*A\otimes_X \delta^*\delta_*B \xrightarrow{\ \sim\ } \delta^*(\delta_*A\otimes_{X\times_Y X}\delta_*B) \longrightarrow \delta^*\delta_*(A\otimes_X B).$$

In particular, for $x\colon X\to S$ in S one has the map

$$\mathbf{t}_x(\mathcal{O}_X,\mathcal{O}_X)\colon \mathcal{H}_X\otimes_X \mathcal{H}_X\to \mathcal{H}_X. \tag{4.7.2}$$

Corresponding to $\mathbf{t}_x(\mathcal{O}_X,\mathcal{O}_X)$ under hom$-\otimes$ adjunction, there is a $\mathbf{D}_X$-map $\mathcal{H}_X\to \mathsf{R}\mathcal{H}om_X(\mathcal{H}_X,\mathcal{H}_X)$, whence for each $i\in\mathbb{Z}$ a natural map

$$\mathrm{H}^{-i}(X,\mathcal{H}_X)\to \mathrm{Ext}^{-i}_{\mathcal{O}_X}(\mathcal{H}_X,\mathcal{H}_X)=\mathrm{HH}^{-i}(X|S) \tag{4.7.3}$$

from the i-th classical Hochschild homology of X to the $(-i)$-th bivariant cohomology (see §0.1).

The map $\mathbf{t}$ is also functorial on the category of schemes over a fixed Z:

Lemma 4.7.4. *For scheme-maps $X\xrightarrow{f}Y\xrightarrow{g}Z$ and $E,F\in\mathbf{D}(Y)$, the following natural diagram commutes.*

$$\begin{tikzcd}
f^*(\boldsymbol{\mathcal{H}}_g E\otimes_Y \boldsymbol{\mathcal{H}}_g F) \arrow[rr, "f^*\mathbf{t}_g"] \arrow[d, "\simeq"'] & & f^*\boldsymbol{\mathcal{H}}_g(E\otimes_Y F) \arrow[d, "(1.6.4.3)"] \\
f^*\boldsymbol{\mathcal{H}}_g E\otimes_X f^*\boldsymbol{\mathcal{H}}_g F \arrow[d, "(1.6.4.3)"'] & & \boldsymbol{\mathcal{H}}_{gf} f^*(E\otimes_Y F) \arrow[d, "\simeq"] \\
\boldsymbol{\mathcal{H}}_{gf} f^*E\otimes_X \boldsymbol{\mathcal{H}}_{gf} f^*F \arrow[rr, "\mathbf{t}_{gf}"'] & & \boldsymbol{\mathcal{H}}_{gf}(f^*E\otimes_X f^*F)
\end{tikzcd}$$

Proof. Let $\delta\colon X\to X\times_Z X$ and $\bar\delta\colon Y\to Y\times_Z Y$ be the diagonal maps, and $h:=f\times_Z f\colon X\times_Z X\to Y\times_Z Y$, so that $\bar\delta f=h\delta$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\delta"'] & Y \arrow[d, "\bar\delta"] \\
X\times_Z X \arrow[r, "h"'] & Y\times_Z Y
\end{tikzcd}$$

The diagram in 4.7.4 expands naturally, via definitions and multiple uses of (1.3.3), as follows (where the obvious subscripts for $\otimes$ have been omitted).

$$
\begin{tikzcd}
f^*(\bar\delta^*\bar\delta_*E\otimes\bar\delta^*\bar\delta_*F) \arrow[r] \arrow[d] & f^*\bar\delta^*(\bar\delta_*E\otimes\bar\delta_*F) \arrow[r] \arrow[d, equal, "\mathsf{ps}^*"'] & f^*\bar\delta^*\bar\delta_*(E\otimes F) \arrow[d, equal, "\mathsf{ps}^*"] \\
f^*\bar\delta^*\bar\delta_*E\otimes f^*\bar\delta^*\bar\delta_*F \arrow[d, equal, "\mathsf{ps}^*"'] \arrow[r, phantom, "\textcircled{1}"] & \delta^*h^*(\bar\delta_*E\otimes\bar\delta_*F) \arrow[r] \arrow[d] & \delta^*h^*\bar\delta_*(E\otimes F) \arrow[d, "(1.4.1)"] \\
\delta^*h^*\bar\delta_*E\otimes\delta^*h^*\bar\delta_*F \arrow[d, "(1.4.1)"'] & \delta^*(h^*\bar\delta_*E\otimes h^*\bar\delta_*F) \arrow[l] \arrow[d, "(1.4.1)"] \arrow[r, phantom, "\textcircled{2}"] & \delta^*\delta_*f^*(E\otimes F) \arrow[d] \\
\delta^*\delta_*f^*E\otimes\delta^*\delta_*f^*F \arrow[r] & \delta^*(\delta_*f^*E\otimes\delta_*f^*F) \arrow[r] & \delta^*\delta_*(f^*E\otimes f^*F)
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams is obvious. For commutativity of ① argue as in the last half of [L3, §3.6.10]. For commutativity of ②, note that since δ^* and δ_* are adjoint, therefore for any $\mathbf{D}(X\times_Z X)$-maps $\alpha,\beta\colon C\to\delta_*D$ the $\mathbf{D}(X)$-maps $\delta^*\alpha$ and $\delta^*\beta$ are equal if the same holds after composition with the natural map $\delta^*\delta_*D\to D$—so that it suffices to show commutativity of the next natural diagram:

$$
\begin{tikzcd}
\delta^*h^*(\bar\delta_*E\otimes\bar\delta_*F) \arrow[rrr] \arrow[d] \arrow[drr, equal, "\mathsf{ps}^*"'] & & & \delta^*h^*\bar\delta_*(E\otimes F) \arrow[ddddd, "(1.4.1)"] \\
\delta^*(h^*\bar\delta_*E\otimes h^*\bar\delta_*F) \arrow[dd] & \textcircled{1} & f^*\bar\delta^*(\bar\delta_*E\otimes\bar\delta_*F) \arrow[dl] \arrow[dd] & \\
 & f^*(\bar\delta^*\bar\delta_*E\otimes\bar\delta^*\bar\delta_*F) \arrow[d] \arrow[dddr] & \textcircled{3} & \\
\delta^*h^*\bar\delta_*E\otimes\delta^*h^*\bar\delta_*F \arrow[dr, equal, "\mathsf{ps}^*"'] \arrow[ddd, "(1.4.1)"'] & & f^*\bar\delta^*\bar\delta_*(E\otimes F) \arrow[dd] \arrow[uuur, equal, "\mathsf{ps}^*"'] & \textcircled{4} \\
 & f^*\bar\delta^*\bar\delta_*E\otimes f^*\bar\delta^*\bar\delta_*F \arrow[ddr] & & \\
 & \textcircled{5} & f^*(E\otimes F) \arrow[d] & \delta^*\delta_*f^*(E\otimes F) \arrow[l] \arrow[dd] \\
\delta^*\delta_*f^*E\otimes\delta^*\delta_*f^*F \arrow[rr] \arrow[d] & \arrow[d, phantom, "\textcircled{6}"] & f^*E\otimes f^*F & \\
\delta^*(\delta_*f^*E\otimes\delta_*f^*F) \arrow[rrr] & & & \delta^*\delta_*(f^*E\otimes f^*F) \arrow[ul]
\end{tikzcd}
$$

Subdiagram ① is the same as the commutative subdiagram ① in the preceding diagram.

Commutativity of ③ and ⑥ is given by [L3, 3.6.7d(iv)]—as realized in [L3, 3.6.10].

Commutativity of ④ and ⑤ results from the definition in (1.4.2) of the map (1.4.1).

Commutativity of the remaining three subdiagrams is obvious. □

For any flat map $x\colon X \to S$ in S, one has then the map

$$\mathfrak{p}_X = \mathfrak{p}_x \colon \mathcal{H}_X \otimes_X \mathcal{H}_X \xrightarrow{\mathbf{t}_x(\mathcal{O}_X, \mathcal{O}_X)} \mathcal{H}_X \xrightarrow[(3.2.2)]{c_x} x^!\mathcal{O}_S.$$

From Proposition 2.5, Theorem 3.1 and Lemma 4.7.4, one deduces:

Corollary 4.7.5. *Let $x\colon X \to S$ and $y\colon Y \to S$ be flat maps in S, and let $f\colon X \to Y$ be an essentially étale S-map. The following diagram commutes.*

$$\begin{tikzcd}
f^*(\mathcal{H}_Y \otimes_Y \mathcal{H}_Y) \arrow[rr, "f^*\mathfrak{p}_Y"] \arrow[d, "\simeq"'] & & f^*y^!\mathcal{O}_S \arrow[d, equal] \\
f^*\mathcal{H}_Y \otimes_X f^*\mathcal{H}_Y \arrow[d, "(1.6.4.3)"'] & & f^!y^!\mathcal{O}_S \arrow[d, equal, "\mathsf{ps}^!"] \\
\mathcal{H}_X \otimes_X \mathcal{H}_X \arrow[rr, "\mathfrak{p}_X"'] & & x^!\mathcal{O}_S
\end{tikzcd}$$

Corresponding to $\mathfrak{p}_X$ under hom$-\otimes$ adjunction, there is a *duality map*

$$\mathfrak{d}_X \colon \mathcal{H}_X \to \mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S). \tag{4.7.6}$$

whence for each $i \in \mathbb{Z}$ a natural map

$$\mathrm{H}^{-i}(X, \mathcal{H}_X) \to \mathrm{Ext}^{-i}_{\mathcal{O}_X}(\mathcal{H}_x, x^!\mathcal{O}_S) = \mathrm{HH}_i(X|S) \tag{4.7.7}$$

from the i-th classical Hochschild homology of X to the i-th bivariant homology (see §0.1).

Also, for any scheme-map $f\colon X \to Y$ there is a bifunctorial map

$$f^*\mathsf{R}\mathcal{H}om_Y(E, F) \to \mathsf{R}\mathcal{H}om_X(f^*E, f^*F) \qquad (E, F \in \mathbf{D}(Y)), \tag{4.7.8}$$

corresponding under hom$-\otimes$ adjunction to the natural composition

$$f^*\mathsf{R}\mathcal{H}om_Y(E, F) \otimes_X f^*E \to f^*(\mathsf{R}\mathcal{H}om_Y(E, F) \otimes_X E) \to f^*F,$$

see [L3, 3.5.6(a)], or [L3, 3.5.6(g)], with $(C, D, E) := (\mathsf{R}\mathcal{H}om_Y(E, F), E, F)$.

If X is noetherian, f is perfect, E is cohomologically bounded-above, with coherent homology, and F is cohomologically bounded below, then the map (4.7.8) is an *isomorphism* [L3, 4.6.7].

The duality map $\mathfrak{d}$ is compatible with essentially étale localization:

Proposition 4.7.9. *Let $x\colon X \to S$ and $y\colon Y \to S$ be flat S-maps and let $f\colon X \to Y$ be an essentially étale S-map. The following diagram commutes.*

$$
\begin{tikzcd}
f^*\mathcal{H}_Y \arrow[r, "f^*\mathfrak{d}_Y"] \arrow[dd, "(1.7)"', "\simeq"] & f^*\mathsf{R}\mathcal{H}om_Y(\mathcal{H}_Y, y^!\mathcal{O}_S) \arrow[r, "\sim", "(4.7.8)"'] & \mathsf{R}\mathcal{H}om_Y(f^*\mathcal{H}_Y, f^*y^!\mathcal{O}_S) \arrow[d, equal] \\
& & \mathsf{R}\mathcal{H}om_Y(f^*\mathcal{H}_Y, f^!y^!\mathcal{O}_S) \arrow[d, equal, "\mathsf{ps}^!"] \\
\mathcal{H}_X \arrow[r, "\mathfrak{d}_X"'] & \mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S) \arrow[r, "\sim", "(1.7)"'] & \mathsf{R}\mathcal{H}om_Y(f^*\mathcal{H}_Y, x^!\mathcal{O}_S)
\end{tikzcd}
$$

Proof. It suffices to show that the adjoint diagram—that is, the border of the following natural diagram, where $\mathsf{R}\mathcal{H} := \mathsf{R}\mathcal{H}om$, and the obvious subscripts for $\otimes$ are omitted—commutes:

$$
\begin{tikzcd}
f^*\mathsf{R}\mathcal{H}_Y(\mathcal{H}_Y, y^!\mathcal{O}_S)\otimes f^*\mathcal{H}_Y \arrow[rr] \arrow[dr] & & \mathsf{R}\mathcal{H}_Y(f^*\mathcal{H}_Y, f^*y^!\mathcal{O}_S)\otimes f^*\mathcal{H}_Y \arrow[dd] \\
& f^*(\mathsf{R}\mathcal{H}om_Y(\mathcal{H}_Y, y^!\mathcal{O}_S)\otimes\mathcal{H}_Y) \arrow[dr] \arrow[u, phantom, "\textcircled{1}"] & \\
f^*\mathcal{H}_Y\otimes f^*\mathcal{H}_Y \arrow[uu, "\text{via }\mathfrak{d}_Y"] \arrow[r, "\sim"] \arrow[d] & f^*(\mathcal{H}_Y\otimes\mathcal{H}_Y) \arrow[u, "\text{via }\mathfrak{d}_Y"] \arrow[r, "f^*\mathfrak{p}_Y"'] \arrow[d, phantom, "\textcircled{2}"] & f^*y^!\mathcal{O}_S \arrow[d, equal] \\
\mathcal{H}_X\otimes f^*\mathcal{H}_Y \arrow[r] \arrow[d, "\text{via }\mathfrak{d}_X"'] & \mathcal{H}_X\otimes\mathcal{H}_X \arrow[d, "\text{via }\mathfrak{d}_X"'] \arrow[dr, "\mathfrak{p}_X"] & f^!y^!\mathcal{O}_S \arrow[d, equal, "\mathsf{ps}^!"] \\
\mathsf{R}\mathcal{H}_X(\mathcal{H}_X, x^!\mathcal{O}_S)\otimes f^*\mathcal{H}_Y \arrow[r] \arrow[dr] & \mathsf{R}\mathcal{H}_X(\mathcal{H}_X, x^!\mathcal{O}_S)\otimes\mathcal{H}_X \arrow[r] & x^!\mathcal{O}_S \\
& \mathsf{R}\mathcal{H}_X(f^*\mathcal{H}_Y, x^!\mathcal{O}_S)\otimes f^*\mathcal{H}_Y \arrow[ur] \arrow[u, phantom, "\textcircled{3}"] &
\end{tikzcd}
$$

Commutativity of subdiagram ① is just the definition following (4.7.8); that of ② is given by Corollary 4.7.5; that of ③ is given by [L3, 3.5.3(h)]; and that of the unlabeled subdiagrams is obvious. □

Theorem 4.8. *If $x\colon X \to S$ is essentially smooth then the duality map $\mathfrak{d}_X$ in (4.7.6) is an isomorphism.*

Proof. Corollary 4.7.5 shows the assertion is local on X; so we may assume that X and S are affine, say $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} R$, with R an essentially smooth A-algebra such that the kernel of the multiplication map $T := R\otimes_A R \to R$ is generated by a regular sequence $\mathbf{t} = (t_1, \dots, t_n)$ in T (see §1.1). Then $\mathbf{t}/\mathbf{t}^2 = \Omega^1_{R|A}$, and the Koszul complex $\mathcal{K}_\bullet(\mathbf{t})$ is a flat resolution of the T-module $R = T/\mathbf{t}T$.

Omitting the obvious subscripts for $\otimes$, we have that the map

$$\delta_*\mathcal{O}_X \otimes \delta_*\mathcal{O}_X \to \delta_*(\mathcal{O}_X \otimes \mathcal{O}_X) = \delta_*\mathcal{O}_X$$

that forms part of the definition of $\mathbf{t}_x(\mathcal{O}_X, \mathcal{O}_X)$ (see (4.7.2)) is, by [L3, 3.4.5.2], the unique ξ such that the following natural diagram commutes:

$$\begin{tikzcd}
\delta^*(\delta_*\mathcal{O}_X \otimes \delta_*\mathcal{O}_X) \arrow[r, "\delta^*\xi"] \arrow[d] & \delta^*\delta_*\mathcal{O}_X \arrow[d, "\epsilon"] \\
\delta^*\delta_*\mathcal{O}_X \otimes \delta^*\delta_*\mathcal{O}_X \arrow[r, "\epsilon\otimes\epsilon"'] & \mathcal{O}_X
\end{tikzcd}$$

The counit map $\epsilon\colon \delta^*\delta_*\mathcal{O}_X \to \mathcal{O}_X$ can be identified with the sheafification of the natural map of complexes (concentrated in negative degrees, and with vanishing differentials)

$$\mathcal{K}_\bullet(\mathbf{t}) \otimes_T R = \textstyle\bigwedge_\bullet \mathbf{t}/\mathbf{t}^2 = \oplus_{i=-n}^{0} \bigwedge^{-i} \mathbf{t}/\mathbf{t}^2 \to \bigwedge^0 \mathbf{t}/\mathbf{t}^2 = R.$$

It results that the map ξ can be identified with the usual multiplication map $\mathcal{K}_\bullet(\mathbf{t}) \otimes_T \mathcal{K}_\bullet(\mathbf{t}) \to \mathcal{K}_\bullet(\mathbf{t})$ (a map of complexes, $\mathcal{K}_\bullet(\mathbf{t})$ being a differential graded algebra). Hence the map $\mathbf{t}_x(\mathcal{O}_X, \mathcal{O}_X)$ can be identified with the exterior multiplication map

$$\textstyle\bigwedge_\bullet \Omega_x^1 \otimes \bigwedge_\bullet \Omega_x^1 \longrightarrow \bigwedge_\bullet \Omega_x^1.$$

Furthermore, Proposition 2.4.2 gives an identification of $c_x\colon \mathcal{H}_X \to x^!\mathcal{O}_X$ with the natural map of complexes

$$\textstyle\bigwedge_\bullet \Omega_x^1 \to \Omega_x^n[n]. \tag{4.8.1}$$

The assertion results now from the well-known isomorphisms

$$\Omega^i_{R|A} \xrightarrow{\ \sim\ } \mathrm{Hom}_R(\Omega^{n-i}_{R|A}, \Omega^n_{R|A}).$$

arising from exterior multiplication followed by (4.8.1). $\square$

Recall from [AJL, §6.5] that for essentially smooth x the Căldăraru-Willerton version of Hochschild homology, $\mathrm{HH}_i^{\mathrm{cl}}(X)$, is isomorphic to the bivariant homology

$$\mathrm{HH}_i(X|S) := \mathrm{Ext}^{-i}_{\mathcal{O}_X}(\mathcal{H}_X, x^!\mathcal{O}_S) = \mathrm{Hom}_{\mathsf{D}(X)}\big(\mathcal{O}_X[i], \mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S)\big).$$

Corollary 4.8.2. *If $x\colon X \to S$ is an essentially smooth S-map then for each $i \in \mathbb{Z}$ the map (4.7.7) is an isomorphism. Hence there is a natural isomorphism*

$$\mathrm{H}^{-i}(X, \mathcal{H}_X) \xrightarrow{\ \sim\ } \mathrm{HH}_i^{\mathrm{cl}}(X).$$

More generally:

Corollary 4.8.3. *Let $f\colon X \to Y$ be a flat map of essentially smooth S-schemes. The bivariant groups associated to f by $\underline{\mathcal{B}}$ and $\mathcal{B}$ are isomorphic.*

Proof. One has, for $i\in\mathbb Z$, natural isomorphisms (the last one induced by $\boldsymbol{\mathfrak d}_X$ and $\boldsymbol{\mathfrak d}_Y$):

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal O_X}(f^*\mathcal H_Y,\mathcal H_X)&\cong \operatorname{Ext}^i_{\mathcal O_X}(\boldsymbol{\mathcal D}_X\mathcal H_X,\boldsymbol{\mathcal D}_X f^*\mathcal H_Y)\\ &\cong \operatorname{Ext}^i_{\mathcal O_X}(\boldsymbol{\mathcal D}_X\mathcal H_X,f^!\boldsymbol{\mathcal D}_Y\mathcal H_Y)\cong \operatorname{Ext}^i_{\mathcal O_X}(\mathcal H_X,f^!\mathcal H_Y).\end{aligned}$$

□

When $Y=S$, one gets from 4.8.3 homology isomorphisms

$$\underline{\mathrm{HH}}_i(X)=\operatorname{Ext}^{-i}_X(\mathcal O_X,\mathcal H_X)\cong \mathrm H^{-i}(X,\mathcal H_X)\xrightarrow{\ \sim\ }\operatorname{Ext}^{-i}_X(\mathcal H_X,x^!\mathcal O_S)=\mathrm{HH}_i(X)$$

that, one checks, coincide with those in (4.7.7).

When f is the identity map of $X=Y$, one gets cohomology isomorphisms

$$\underline{\mathrm{HH}}^i(X)=\operatorname{Ext}^i_X(\mathcal H_X,\mathcal H_X)\xrightarrow{\ \sim\ }\operatorname{Ext}^i_X(\mathcal H_X,\mathcal H_X)=\mathrm{HH}^i(X)$$

that are in fact identity maps.

For proper x there is a natural pairing on classical Hochschild homology, with $H:=\mathrm H^0(S,\mathcal O_S)$:

$$\begin{aligned}\mathrm H^{-i}(X,\mathcal H_X)\otimes_H \mathrm H^i(X,\mathcal H_X)&\to \mathrm H^0(X,\mathcal H_X\otimes_X\mathcal H_X)\\ &\xrightarrow{\text{via }\mathfrak p_X}\mathrm H^0(X,x^!\mathcal O_S)\cong \mathrm H^0(S,x_*x^!\mathcal O_S)\to \mathrm H^0(S,\mathcal O_S)=H,\end{aligned}$$

where the first map is the case $j=-i$, $k=i$ of the map

$$\operatorname{Ext}^j(\mathcal O_X,\mathcal H_X)\otimes_H\operatorname{Ext}^k(\mathcal O_X,\mathcal H_X)\to\operatorname{Ext}^{j+k}(\mathcal O_X,\mathcal H_X\otimes_X\mathcal H_X)\qquad (j,k\in\mathbb Z)$$

that takes $\alpha\otimes_H\beta$ to the $\mathbf D(X)$-map

$$\mathcal O_X=\mathcal O_X\otimes_X\mathcal O_X\xrightarrow{\alpha\otimes\beta}\mathcal H_X[j]\otimes_X\mathcal H_X[k]\cong\big(\mathcal H_X\otimes_X\mathcal H_X\big)[j+k].$$

Corollary 4.8.4. *If $S=\operatorname{Spec}H$ with H a self-injective (i.e., Gorenstein) artinian ring, and $x\colon X\to S$ is proper and smooth, then the above pairing is* non-singular, *that is, the associated H-linear map is an isomorphism*

$$\mathrm H^{-i}(X,\mathcal H_X)\xrightarrow{\ \sim\ }\operatorname{Hom}_H(\mathrm H^i(X,\mathcal H_X),H).$$

Proof. Theorem 4.8, the assumption on H, and the 0-dimensionality of S entail natural isomorphisms

$$\begin{aligned}\mathrm H^{-i}(X,\mathcal H_X)&\xrightarrow{\ \sim\ }\mathrm H^{-i}(X,\mathsf R\mathcal Hom_X(\mathcal H_X,x^!\mathcal O_S))\\ &\xrightarrow{\ \sim\ }\mathrm H^{-i}(S,x_*\mathsf R\mathcal Hom_X(\mathcal H_X,x^!\mathcal O_S))\\ &\xrightarrow{\ \sim\ }\mathrm H^{-i}(S,\mathsf R\mathcal Hom_S(x_*\mathcal H_X,\mathcal O_S))\\ &\xrightarrow{\ \sim\ }\mathrm H^{-i}\,\mathsf R\operatorname{Hom}_S(x_*\mathcal H_X,\mathcal O_S)\\ &\xrightarrow{\ \sim\ }\mathrm H^{-i}\operatorname{Hom}_S(x_*\mathcal H_X,\mathcal O_S)\\ &\xrightarrow{\ \sim\ }\mathrm H^{-i}\operatorname{Hom}_H(\Gamma(S,x_*\mathcal H_X),H)\\ &\xrightarrow{\ \sim\ }\operatorname{Hom}_H(\mathrm H^i\,\Gamma(S,x_*\mathcal H_X),H)\xrightarrow{\ \sim\ }\operatorname{Hom}_H(\mathrm H^i(X,\mathcal H_X),H).\end{aligned}$$

□

4.9. (Unfinished business.)

• See Remark 4.2.6.

• The duality isomorphism $\mathfrak{d}$ calls to mind other maps in the literature. For one, when $x\colon X \to S$ is essentially smooth there is an isomorphism, attributed to Căldăraru,

$$\delta_*\mathcal{H}_X \xrightarrow{\sim} \delta_*\mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S),$$

described in [R, p. 648]. For another, there is an isomorphism first defined by Kashiwara

$$\mathrm{td}\colon \mathcal{H}_X \xrightarrow{\sim} \delta^!\delta_*x^!\mathcal{O}_S,$$

see [KS, p. 122, (5.2.2)] (which with $\omega_X := x^!\mathcal{O}_S$ makes sense, as a map, for any flat S-map x), whence an isomorphism

$$\delta_*\mathcal{H}_X \xrightarrow{\delta_*\mathrm{td}} \delta_*\delta^!\delta_*x^!\mathcal{O}_S \cong \delta_*\mathsf{R}\mathcal{H}om_X(\mathcal{H}_X, x^!\mathcal{O}_S).$$

These two isomorphisms have interesting connections to Todd classes and Riemann-Roch theorems. Are they the same? How do they relate to $\delta_*\mathfrak{d}_X$? How is the isomorphism (4.8.2) related to those in [CaW, §§4.2, 5]?

• How does the pairing in 4.8.4 relate to the Mukai pairing of [CaW, §5]?

• One might ask whether the isomorphisms in 4.8.3 respect the orientations and the bivariant operations in $\underline{\mathcal{B}}$ and $\mathcal{B}$. For this, one needs commutativity—which we haven't yet been able to prove or disprove—of the diagram

$$\begin{tikzcd}
\mathcal{H}_X \arrow[rr, "c_f"] \arrow[d, "\mathfrak{d}_X"'] & & f^!\mathcal{H}_Y \arrow[d, "f^!\mathfrak{d}_Y"] \\
\boldsymbol{\mathcal{D}}_X\mathcal{H}_X \arrow[r, "\boldsymbol{\mathcal{D}}_X f^\sharp"'] & \boldsymbol{\mathcal{D}}_X f^*\mathcal{H}_Y \arrow[r, "\sim"] & f^!\boldsymbol{\mathcal{D}}_Y\mathcal{H}_Y
\end{tikzcd}$$

5. Fundamental class and base-change

The fundamental class $\mathbf{c}_f = \mathbf{b}_f \circ \mathbf{a}_f$ of a flat S-map $f\colon X \to Y$,

$$\boldsymbol{\mathcal{H}}_X f^* = \delta_X^*\delta_{X*}f^* \xrightarrow[\mathbf{a}_f]{} \Gamma^*\Gamma_* f^! \xrightarrow[\mathbf{b}_f]{\sim} f^!\delta_Y^*\delta_{Y*} = f^!\boldsymbol{\mathcal{H}}_Y,$$

is as in §2.2. The next Theorem describes its behavior under flat base change. There results a flat-base-change property for contravariant Gysin maps (Proposition 5.2).

Theorem 5.1. *For any oriented fiber square of flat* S*-maps*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

the following diagram, with B *as in* (2.1.6.1), *commutes:*

(5.1.1)
$$
\begin{tikzcd}
g'^*\boldsymbol{\mathcal{H}}_X f^* \arrow[r, "g'^*\mathsf{c}_f"] \arrow[d, "(1.6.4.3)"'] & g'^*f^!\boldsymbol{\mathcal{H}}_Y \arrow[d, "\mathsf{B}", "\simeq"'] \\
\boldsymbol{\mathcal{H}}_{X'}g'^*f^* \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & f'^!g^*\boldsymbol{\mathcal{H}}_Y \arrow[d, "(1.6.4.3)"] \\
\boldsymbol{\mathcal{H}}_{X'}f'^*g^* \arrow[r, "\mathsf{c}_{f'}g^*"'] & f'^!\boldsymbol{\mathcal{H}}_{Y'}g^*
\end{tikzcd}
$$

Proof. Notation will be as in the following commutative diagram, in which $\nu := \delta_f$, $\nu' := \delta_{f'}$, $\delta := \delta_X = i\nu$ and $\delta' := \delta_{X'} = i'\nu'$ are diagonal maps, $\Gamma = \Gamma_f$ (resp. $\Gamma' = \Gamma_{f'}$) is the graph of f (resp. f'), i and i' are the natural immersions, t and t' are the projections onto the first factor, p, p', q and q' are the projections onto the second factor, and h is the composite map

$$X' \times_{Y'} X' \xrightarrow{\text{natural}} X' \times_Y X' \xrightarrow{g'\times_Y g'} X \times_Y X.$$

$$
\begin{tikzcd}
 & & X' \arrow[rr, "g'"] \arrow[d, "\nu' := \delta_{f'}"'] & & X \arrow[d, "\nu := \delta_f"] \\
 & & X'\times_{Y'}X' \arrow[rr, "h"] \arrow[dll, "t'"'] \arrow[dr, "j"] \arrow[dd, "i'"'] & & X\times_Y X \arrow[dl, "t"] \arrow[dd, "i"] \\
X' \arrow[rrr, "g'"] \arrow[drr, "\delta'"] \arrow[dd, "\Gamma'"'] & & & X \arrow[dr, "\delta"] \arrow[dd, "\Gamma"] & \\
 & & X'\times X' \arrow[rr, "g'\times g'"] \arrow[dll, "\mathrm{id}_{X'}\times f'"'] \arrow[dr, "k"] \arrow[dd, "p'"'] & & X\times X \arrow[dl, "\mathrm{id}_X\times f"] \arrow[dd, "p"] \\
X'\times Y' \arrow[rrr, "g'\times g"] \arrow[dd, "q'"'] & & & X\times Y \arrow[dd, "q"] & \\
 & & X' \arrow[rr, "g'"] \arrow[dll, "f'"'] \arrow[dr, "\ell"] & & X \arrow[dl, "f"] \\
Y' \arrow[rrr, "g"'] & & & Y &
\end{tikzcd}
$$

Diagram (5.1.1) (transposed) expands as follows, with ϕ as in (1.5.1):

$$\begin{tikzcd}
g'^*\delta^*\delta_*f^* \arrow[r, "\phi"] \arrow[d, "g'^*\mathsf{a}_f"'] & \delta'^*\delta'_*g'^*f^* \arrow[r, equal, "\text{via }\mathsf{ps}^*"] \arrow[d, phantom, "(\#_{\mathsf{a}})"] & \delta'^*\delta'_*f'^*g^* \arrow[d, "\mathsf{a}_{f'}"] \\
g'^*\Gamma^*\Gamma_*f^! \arrow[r, "\phi"] \arrow[d, "g'^*\mathsf{b}_f"'] & \Gamma'^*\Gamma'_*g'^*f^! \arrow[r, "\text{via }\mathsf{B}"] \arrow[d, phantom, "(\#_{\mathsf{b}})"] & \Gamma'^*\Gamma'_*f'^!g^* \arrow[d, "\mathsf{b}_{f'}"] \\
g'^*f^!\delta_Y^*\delta_{Y*} \arrow[r, "\mathsf{B}"'] & f'^!g^*\delta_Y^*\delta_{Y*} \arrow[r, "\text{via }\phi"'] & f'^!\delta_{Y'}^*\delta_{Y'*}g^*
\end{tikzcd}$$

We first prove commutativity of subdiagram $(\#_{\mathsf{a}})$, expanded as follows, with $s := pi$, $s' := p'i'$ (see §2.2, and also, recall that both $(s\nu)^! = \mathrm{id}_X^!$ and $(s'\nu')^! = \mathrm{id}_{X'}^!$ are identity functors). Each map in this diagram is induced by the natural transformation specified in its label. The commutative S-square to which any label B, B^{-1} or θ^{-1} is associated is in each case easily verified to be an oriented fiber square with flat bottom arrow. In particular, one sees from the fiber square

$$\begin{tikzcd}
X'\times_{Y'}X' \arrow[r, "h"] \arrow[d, "s'"'] & X\times_Y X \arrow[d, "s"] \\
X' \arrow[r, "g'"'] & X
\end{tikzcd}$$

that the map h is flat.

$$
\begin{tikzcd}
g'^*\delta^*\delta_*f^* \arrow[rr, equal, "\mathsf{ps}^*"] \arrow[dd, equal, "\mathsf{ps}_*"'] & & \delta'^*(g'\times g')^*\delta_*f^* \arrow[rr, "\theta"] \arrow[dl, equal, "\mathsf{ps}_*"] & & \delta'^*\delta'_*g'^*f^* \arrow[rr, equal, "\mathsf{ps}^*"] \arrow[dl, equal, "\mathsf{ps}_*"] \arrow[dr, equal, "\mathsf{ps}_*"] & & \delta'^*\delta'_*f'^*g^* \arrow[dd, equal, "\mathsf{ps}_*"] \\
& \delta'^*(g'\times g')^*i_*\nu_*(s\nu)^!f^* \arrow[dl, equal, "\mathsf{ps}^*"] \arrow[dd, "\int_\nu^s"'] \arrow[drr, "\theta"] & \textcircled{1} & \delta'^*i'_*\nu'_*g'^*(s\nu)^!f^* \arrow[rr, equal, "\mathsf{B}"] & \textcircled{2} & \delta'^*i'_*\nu'_*(s'\nu')^!g'^*f^* \arrow[dd, "\int_{\nu'}^{s'}"] \arrow[dr, equal, "\mathsf{ps}_*"] & \\
g'^*\delta^*i_*\nu_*(s\nu)^!f^* \arrow[dd, "\int_\nu^s"'] & & & \delta'^*i'_*h^*\nu_*(s\nu)^!f^* \arrow[u, "\theta"'] \arrow[d, "\int_\nu^s"'] & \textcircled{3} & & \delta'^*i'_*\nu'_*(s'\nu')^!f'^*g^* \arrow[dd, "\int_{\nu'}^{s'}"] \\
& \delta'^*(g'\times g')^*i_*s^!f^* \arrow[r, "\theta"] \arrow[dl, equal, "\mathsf{ps}^*"] \arrow[dd, "\mathsf{B}^{-1}"'] & & \delta'^*i'_*h^*s^!f^* \arrow[rr, "\mathsf{B}"] \arrow[dd, "\mathsf{B}^{-1}"] & & \delta'^*i'_*s'^!g'^*f^* \arrow[dd, equal, "\mathsf{ps}_*"] \arrow[dr, equal, "\mathsf{ps}_*"] & \\
g'^*\delta^*i_*s^!f^* \arrow[dd, "\mathsf{B}^{-1}"'] & & & & \textcircled{4}_1 & & \delta'^*i'_*s'^!f'^*g^* \arrow[dl, equal, "\mathsf{ps}^*"] \arrow[ddd, "\mathsf{B}^{-1}"] \\
& \delta'^*(g'\times g')^*i_*t^*f^! \arrow[rr, "\theta"] \arrow[dl, equal, "\mathsf{ps}^*"] \arrow[dd, "\theta^{-1}"'] & & \delta'^*i'_*h^*t^*f^! \arrow[dr, equal, "\mathsf{ps}_*"] & & \delta'^*i'_*s'^!\ell^* & \\
g'^*\delta^*i_*t^*f^! \arrow[dd, "\theta^{-1}"'] & & \textcircled{5}_1 & & \delta'^*i'_*j^*f^! \arrow[ur, "\mathsf{B}"'] \arrow[d, equal, "\mathsf{ps}^*"] & \textcircled{4}_2 & \\
& \delta'^*(g'\times g')^*(\mathrm{id}_X\times f)^*\Gamma_*f^! \arrow[dl, equal, "\mathsf{ps}^*"] \arrow[dd, equal, "\mathsf{ps}^*"'] \arrow[dr, equal, "\mathsf{ps}_*"] & & & \delta'^*i'_*t'^*g'^*f^! \arrow[rr, "\mathsf{B}"'] \arrow[dd, "\theta^{-1}"] & & \delta'^*i'_*t'^*f'^!g^* \arrow[dd, "\theta^{-1}"] \\
g'^*\delta^*(\mathrm{id}_X\times f)^*\Gamma_*f^! \arrow[dd, equal, "\mathsf{ps}^*"'] & & \delta'^*k^*\Gamma_*f^! \arrow[uurr, "\theta"'] \arrow[dl, equal, "\mathsf{ps}^*"] & \textcircled{5}_2 & & & \\
& \delta'^*(\mathrm{id}_{X'}\times f')^*(g'\times g)^*\Gamma_*f^! \arrow[rrr, "\theta"'] \arrow[d, equal, "\mathsf{ps}^*"'] & & & \delta'^*(\mathrm{id}_{X'}\times f')^*\Gamma'_*g'^*f^! \arrow[rr, "\mathsf{B}"] \arrow[d, equal, "\mathsf{ps}^*"] & & \delta'^*(\mathrm{id}_{X'}\times f')^*\Gamma'_*f'^!g^* \arrow[d, equal, "\mathsf{ps}^*"] \\
g'^*\Gamma^*\Gamma_*f^! \arrow[r, equal, "\mathsf{ps}^*"'] & \Gamma'^*(g'\times g)^*\Gamma_*f^! \arrow[rrr, "\theta"'] & & & \Gamma'^*\Gamma'_*g'^*f^! \arrow[rr, "\mathsf{B}"'] & & \Gamma'^*\Gamma'_*f'^!g^*
\end{tikzcd}
$$

In this diagram, commutativity of the unlabeled subdiagrams is easy to check, via (pseudo)functoriality of the maps involved. So it suffices to show commutativity of the labeled subdiagrams.

Commutativity of ① results from vertical transitivity of θ [L3, 3.7.2(ii)]; and commutativity of $⑤_1$ and $⑤_2$ from horizontal transitivity [L3, 3.7.2(iii)].

For ②, it's enough to note that the map $\mathsf{B}\colon g'^*(s\nu)^! \to (s'\nu')^!g'^*$ is the identity, see [L3, 4.8.1(iii)].

Commutativity of ③ is given by Lemma 2.1.7, with $(f,g,h,j,k,u,v) := (\nu, s, h, \nu', s', g', g')$.

Commutativity of $④_1$ and $④_2$ follows from horizontal transitivity of B [AJL, §5.8.4].

Next, we prove commutativity of diagram ($\#_{\mathbf{b}}$). The morphisms used to define $\mathbf{b}_f$ and $\mathbf{b}_{f'}$ fit into the commutative cube

$$
\begin{tikzcd}
 & X' \arrow[rr, "g'"] \arrow[dl, "\Gamma'"'] \arrow[dd, "f'"' near end] \arrow[drr, phantom, "\mathbf{v}"] & & X \arrow[dl, "\Gamma"'] \arrow[dd, "f"] \\
X' \times Y' \arrow[rr, "g' \times g" near start] \arrow[dd, "f' \times \mathrm{id}_{Y'}"'] \arrow[dr, phantom, "\mathbf{v}'"] & & X \times Y \arrow[dd, "f \times \mathrm{id}_Y" near start] \arrow[r, phantom, "\mathbf{u}"] & \\
 & Y' \arrow[rr, "g"' near end] \arrow[dl, "\delta' := \delta_{Y'}"] \arrow[dr, phantom, "\mathbf{u}'"] & & Y \arrow[dl, "\delta := \delta_Y"] \\
Y' \times Y' \arrow[rr, "g \times g"'] & & Y \times Y &
\end{tikzcd}
$$

Diagram ($\#_{\mathbf{b}}$) expands as follows, where $\mathcal{O} := \mathcal{O}_Y$, $\mathcal{O}^* := g^*\mathcal{O}_Y = \mathcal{O}_{Y'}$, μ is as in (2.2.6), and each arrow is labeled with the natural transformation that induces it.

$$
\begin{tikzcd}[column sep=small]
g'^*\Gamma^*\Gamma_*(f^!\mathcal{O} \otimes f^*) \arrow[dr, "\phi"] \arrow[dd, "\mu"'] & & \Gamma'^*\Gamma'_*(g'^*f^!\mathcal{O} \otimes g'^*f^*) \arrow[dr, equal, "\mathsf{ps}^*"] \arrow[dd, "\mu"'] & & \Gamma'^*\Gamma'_*(f'^!\mathcal{O}^* \otimes f'^*g^*) \arrow[dd, "\mu"] \\
 & \Gamma'^*\Gamma'_*g'^*(f^!\mathcal{O} \otimes f^*) \arrow[ur, "(1.3.3)"] \arrow[d, phantom, "\textcircled{1}"] & & \Gamma'^*\Gamma'_*(g'^*f^!\mathcal{O} \otimes f'^*g^*) \arrow[ur, "\mathsf{B}"] \arrow[dd, "\mu"'] & \\
g'^*(f^!\mathcal{O} \otimes \Gamma^*\Gamma_*f^*) \arrow[dr, "(1.3.3)"] \arrow[ddd, "\phi^{-1}"'] & {} & g'^*f^!\mathcal{O} \otimes \Gamma'^*\Gamma'_*g'^*f^* \arrow[dr, equal, "\mathsf{ps}^*"'] & & f'^!\mathcal{O}^* \otimes \Gamma'^*\Gamma'_*f'^*g^* \arrow[ddd, "\phi^{-1}"] \\
 & g'^*f^!\mathcal{O} \otimes g'^*\Gamma^*\Gamma_*f^* \arrow[ur, "\phi"'] \arrow[d, "\phi^{-1}"'] \arrow[drr, phantom, "\textcircled{2}"] & & g'^*f^!\mathcal{O} \otimes \Gamma'^*\Gamma'_*f'^*g^* \arrow[ur, "\mathsf{B}"] \arrow[d, "\phi^{-1}"'] & \\
 & g'^*f^!\mathcal{O} \otimes g'^*f^*\delta^*\delta_* \arrow[dr, equal, "\mathsf{ps}^*"] & & g'^*f^!\mathcal{O} \otimes f'^*\delta'^*\delta'_*g^* \arrow[dr, "\mathsf{B}"] & \\
g'^*(f^!\mathcal{O} \otimes f^*\delta^*\delta_*) \arrow[ur, "(1.3.3)"] & & g'^*f^!\mathcal{O} \otimes f'^*g^*\delta^*\delta_* \arrow[ur, "\phi"] & & f'^!\mathcal{O}^* \otimes f'^*\delta'^*\delta'_*g^*
\end{tikzcd}
$$

Subdiagram ① commutes by Lemma 6.4.2 (with $u := g'$, etc.).

Commutativity of ② is given by Lemma 1.5.4 applied to each of the two decompositions $\mathbf{uv}$ and $\mathbf{u'v'}$ of the diagram

$$\begin{tikzcd} X' \arrow[r, "fg' = gf'"] \arrow[d, "\Gamma'"'] & Y \arrow[d, "\delta"] \\ X' \times Y' \arrow[r, "(f\times \mathrm{id}_Y)(g'\times g) = (g\times g)(f'\times \mathrm{id}_{Y'})"'] & Y\times Y \end{tikzcd}$$

Commutativity of the remaining subdiagrams is clear.

Thus $(\#_{\mathbf{b}})$ commutes, as well as $(\#_{\mathbf{a}})$, and the proof of Theorem 5.1 is complete. $\square$

For an S-map $h\colon V \to W$, the Gysin map h^{c} is as in §3.4. If h is proper, and $v\colon V \to S$, $w\colon W \to S$ are the structure maps, then for any $j \in \mathbb{Z}$, the pushforward

$$h_\star\colon \mathrm{HH}_j(V|S) = \mathrm{Ext}^{-j}_{\mathcal{O}_V}(\mathcal{H}_V, v^!\mathcal{O}_S) \to \mathrm{Ext}^{-j}_{\mathcal{O}_W}(\mathcal{H}_W, w^!\mathcal{O}_S) = \mathrm{HH}_j(W|S)$$

is as in §3.3.1(B): it takes $\beta\colon \mathcal{H}_V \to v^!\mathcal{O}_S[-j]$ to the composite map

$$\mathcal{H}_W \xrightarrow{h_\sharp} h_*\mathcal{H}_V \xrightarrow{h_*\beta} h_*v^!\mathcal{O}_S[-j] \overset{\mathsf{ps}^!}{=\!=} h_*h^!w^!\mathcal{O}_S[-j] \xrightarrow{\int_h} w^!\mathcal{O}_S[-j]$$

where $h_\sharp$ is adjoint to $h^\sharp$ (see §3.2).

Proposition 5.2. *For any oriented fiber square of flat* S*-maps*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

with f (hence f') proper, one has

$$g^{\mathsf{c}} f_\star = f'_\star g'^{\mathsf{c}}.$$

Proof. Let $x\colon X \to S$ and $y\colon Y \to S$ be the structure maps. By definition of $(-)^{\mathsf{c}}$ and $(-)_\star$, the assertion is that for any

$$\alpha\colon \mathcal{H}_X \to x^!\mathcal{O}_S[i] \overset{\mathsf{ps}^!}{=\!=} f^!y^!\mathcal{O}_S[i] \qquad (i \in \mathbb{Z}),$$

the outer border of the following diagram—where each arrow is labeled with the natural transformation that induces it—commutes.

$$
\begin{tikzcd}
\mathcal{H}_{Y'} \arrow[r, "\eta_{f'}"] \arrow[d, "\mathbf{c}_g"'] & f'_*f'^*\mathcal{H}_{Y'} \arrow[r, "f'^\sharp"] \arrow[d, "\mathbf{c}_g"'] \arrow[dr, phantom, "①"] & f'_*\mathcal{H}_{X'} \arrow[r, "\mathbf{c}_{g'}"] & f'_*g'^!\mathcal{H}_X \arrow[ddd, equal] \\
g^!\mathcal{H}_Y \arrow[r, "\eta_{f'}"'] \arrow[d, "\eta_f"'] & f'_*f'^*g^!\mathcal{H}_Y \arrow[r, "\mathsf{B}"'] \arrow[d, "\eta_f"'] & f'_*g'^!f^*\mathcal{H}_Y \arrow[ur, "f^\sharp"'] \arrow[d, shift right, "\eta_f"'] & \\
g^!f_*f^*\mathcal{H}_Y \arrow[r, "\eta_{f'}"'] \arrow[d, "f^\sharp"'] & f'_*f'^*g^!f_*f^*\mathcal{H}_Y \arrow[r, "\mathsf{B}"'] \arrow[d, "f^\sharp"'] & f'_*g'^!f^*f_*f^*\mathcal{H}_Y \arrow[u, shift right, "\epsilon_f"'] \arrow[d, "f^\sharp"] & \\
g^!f_*\mathcal{H}_X \arrow[r, "\eta_{f'}"'] \arrow[d, "\alpha"'] & f'_*f'^*g^!f_*\mathcal{H}_X \arrow[r, "\mathsf{B}"'] \arrow[d, "\alpha"'] & f'_*g'^!f^*f_*\mathcal{H}_X \arrow[r, "\epsilon_f"] \arrow[d, "\alpha"] & f'_*g'^!\mathcal{H}_X \arrow[d, "\alpha"] \\
g^!f_*f^!y^!\mathcal{O}_S[i] \arrow[r, "\eta_{f'}"] \arrow[d, "\int_f"'] & f'_*f'^*g^!f_*f^!y^!\mathcal{O}_S[i] \arrow[r, "\mathsf{B}"] \arrow[d, phantom, "②"] & f'_*g'^!f^*f_*f^!y^!\mathcal{O}_S[i] \arrow[r, "\epsilon_f"] & f'_*g'^!f^!y^!\mathcal{O}_S[i] \arrow[d, equal, "\mathsf{ps}^!"] \\
g^!y^!\mathcal{O}_S[i] & & & f'_*f'^!g^!y^!\mathcal{O}_S[i] \arrow[lll, "\int_{f'}"]
\end{tikzcd}
$$

Since commutativity of the unlabeled subdiagrams is clear, and $\epsilon_f\circ\eta_f$ is the identity map, it's enough to prove commutativity of subdiagrams ① and ②.

Commutativity of ① is given by application of the commutative functorial diagram in Theorem 5.1 to $\mathcal{O}_Y$, after transposition of the fiber square in that theorem (i.e., make the interchange $(f, f', X) \leftrightarrow (g, g', Y')$.)

As for ②, we can replace $f^!$ by $f^!_+$, and similarly for f', g and g', interpreting $\int$ as the counit map given by 2.1.2(i). This is because by definition the isomorphism (2.1.2.2) corresponds via 2.1.2(i) to $\int_f$, and because that isomorphism is pseudofunctorial (last paragraph in §2.1.2) and compatible with the base-change map B (see [L3, Exercise 4.9.3(c)]). We will now show that the resulting diagram commutes, *even when g and g' are not flat.*

By [Nk, 2.8.1 and Theorem 4.1], there exists a fiber square diagram $\bar{\mathbf{d}}\circ\mathbf{d}$,

$$
\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathbf{d}"] & \overline{X}' \arrow[r, "\bar g'"] \arrow[d, "\bar f"] \arrow[dr, phantom, "\bar{\mathbf{d}}"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & \overline{Y}' \arrow[r, "\bar g"'] & Y
\end{tikzcd}
$$

where u (hence v) is a localizing immersion, $\bar g$ (hence $\bar g'$) is proper, $g = \bar g u$ and $g' = \bar g' v$, cf. [AJL, §5.8.2]. Among other things, [Nk, 5.3] gives $u^!_+ = u^*$, $v^!_+ = v^*$, and $\mathsf{B}_{\mathbf{d}} = \mathsf{ps}^!\colon v^*\bar f^!_+ \xrightarrow{\;\sim\;} f'^!_+u^*$. So subdiagram ②, without $y^!_+\mathcal{O}_S[i]$, expands as follows, with $\phi\colon \bar f_*\bar g'^!_+ \xrightarrow{\;\sim\;} \bar g^!_+ f_*$ as in [L3, 3.10.4] (see 2.1.2(i)):

$$
\begin{tikzcd}
u^*\bar g^!_+ f_* f^!_+ \arrow[rr, "\eta_{f'}"] \arrow[dddr, "\phi^{-1}"] \arrow[dddd, "\int_f"'] & & f'_* f'^* u^* \bar g^!_+ f_* f^!_+ \arrow[rr, "\mathsf{B}_{\bar{\mathbf{d}}\circ\mathbf{d}}"] \arrow[d, "\phi^{-1}"] \arrow[dr, equal, "\mathsf{ps}^*"] & & f'_* v^* \bar g'^!_+ f^* f_* f^!_+ \arrow[dd, "\epsilon_f"] \\
& & f'_* f'^* u^* \bar f_* \bar g'^!_+ f^!_+ \arrow[d, equal, "\mathsf{ps}^*"] & f'_* v^* \bar f^* \bar g^!_+ f_* f^!_+ \arrow[ur, "\mathsf{B}_{\bar{\mathbf{d}}}"'] \arrow[dl, "\phi^{-1}"] \arrow[r, phantom, "④"] \arrow[u, phantom, "③"] & \\
& & f'_* v^* \bar f^* \bar f_* \bar g'^!_+ f^!_+ \arrow[rr, "\epsilon_{\bar f}"] & & f'_* v^* \bar g'^!_+ f^!_+ \arrow[d, equal, "\mathsf{ps}^!"] \\
& u^* \bar f_* \bar g'^!_+ f^!_+ \arrow[uur, "\eta_{f'}"] \arrow[urrr, "\theta"] \arrow[r, equal, "\mathsf{ps}^!"'] \arrow[u, phantom, "⑤"] \arrow[d, phantom, "⑥"] & u^* \bar f_* \bar f^!_+ \bar g^!_+ \arrow[rr, "\theta"] \arrow[d, "\int_{\bar f}"] \arrow[drr, phantom, "⑦"] & & f'_* v^* \bar f^!_+ \bar g^!_+ \arrow[d, "\mathsf{B}_{\mathbf{d}}"] \\
u^* \bar g^!_+ \arrow[rr, equal] & & u^* \bar g^!_+ & & f'_* f'^!_+ u^* \bar g^!_+ \arrow[ll, "\int_{f'}"]
\end{tikzcd}
$$

Using [L3, p. 208, Theorem 4.8.3(ii) and Remark 4.8.5.2] as in 2.5.5, with the replacement $(f, g, u, v) \mapsto (u, v, \bar f, f')$, one gets $\mathsf{B}_{\mathbf{d}} = \mathsf{ps}^* \colon f'^* u^* \xrightarrow{\ \sim\ } v^* \bar f^*$. Consequently, commutativity of ③ results from horizontal transitivity of B.

Commutativity of ④ is given by [L3, 3.10.4(b)], with the replacement $(f, g, u, v) \mapsto (\bar g, \bar g', f, \bar f)$.

Commutativity of ⑤ is given by [L3, 3.7.2(i)(c)].

From the adjunction $\bar f_* \dashv \bar f^!_+$ in 2.1.2(i), with unit ϖ_f, one deduces that commutativity of ⑥ results from that of ⑨ below, with $\tilde\phi$ the adjoint of ϕ.

$$
\begin{tikzcd}
\bar g'^!_+ f^!_+ f_* f^!_+ \arrow[r, shift right, "\int_f"'] \arrow[d, equal, "\mathsf{ps}^!_+"'] \arrow[dr, phantom, "⑧"] & \bar g'^!_+ f^!_+ \arrow[l, shift right, "\varpi_f"'] \arrow[dl, "\tilde\phi"] \arrow[d, equal, "\mathsf{ps}^!_+"] \\
\bar f^!_+ \bar g^!_+ f_* f^!_+ \arrow[r, "\int_f"'] \arrow[ur, phantom, "⑨"'] & \bar f^!_+ \bar g^!_+
\end{tikzcd}
$$

Subdiagram ⑧ commutes, by [L3, 3.10.4(c)] applied to the above diagram $\bar{\mathbf{d}}$; and since $\int_f \circ \varpi_f$ is the identity map, it follows that ⑨, hence ⑥, commutes.

Commutativity of ⑦ is given by the definition of $\mathsf{B}_{\mathbf{d}}$ [AJL, (5.7.2), (5.8.5)].

Thus ② commutes, and the proof is complete. □

6. Proof of transitivity

6.1. Referring to the statement of transitivity, Theorem 3.1, let Γ_u, Γ_v and Γ_{vu} be the graphs of u, v and vu respectively. According to (2.2.1) we can expand the diagram in the statement as follows:

(6.1.1)

$$\begin{tikzcd}
\delta_X^*\delta_{X*}u^*v^* \arrow[r, "\mathsf{a}_u"] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] \arrow[dr, phantom, "(\#)"] & \Gamma_u^*\Gamma_{u*}u^!v^* \arrow[r, "\mathsf{b}_u"] \arrow[d, "?"'] \arrow[drr, phantom, "(\#\#)"] & u^!\delta_Y^*\delta_{Y*}v^* \arrow[r, "u^!\mathsf{c}_v"] & u^!v^!\delta_Z^*\delta_{Z*} \arrow[d, equal, "\text{via } \mathsf{ps}^!"] \\
\delta_X^*\delta_{X*}(vu)^* \arrow[r, "\mathsf{a}_{vu}"'] & \Gamma_{vu}^*\Gamma_{vu*}(vu)^! \arrow[rr, "\mathsf{b}_{vu}"'] & & (vu)^!\delta_Z^*\delta_{Z*}
\end{tikzcd}$$

where the map labeled ? is defined just below. It suffices then to show that the two subdiagrams (#) and (##) are commutative.

To define the map ? in (6.1.1), consider the diagram of fiber squares

(6.1.2)

$$\begin{tikzcd}
X \arrow[r, "\delta_u"] & X\times_Y X \arrow[r, "j"] \arrow[d, "s_1"'] & X\times_Z X \arrow[r, "l"] \arrow[d, "\mathrm{id}\times_Z u"'] & X\times X \arrow[r, "p_X"] \arrow[d, "\mathrm{id}\times u"'] & X \arrow[d, "u"] \\
& X \arrow[r, "g"] & X\times_Z Y \arrow[r, "k"] \arrow[d, "r"'] & X\times Y \arrow[r, "p_Y"] \arrow[d, "\mathrm{id}\times v"'] & Y \arrow[d, "v"] \\
& & X \arrow[r, "\Gamma_{vu}"'] & X\times Z \arrow[r, "p_Z"'] & Z
\end{tikzcd}$$

where j, k and l are the natural closed immersions, s_1 and r are the projections onto the first factor, p_X, p_Y and p_Z are the projections onto the second factor, and g is the unique closed immersion such that $k\circ g=\Gamma_u$.

The subdiagram e formed by the bottom two rows is an instance of the diagram (2.2.3), and so has associated to it the map

$$\lambda_{\mathsf{e}}\colon \Gamma_{u*}u^!v^*\to(\mathrm{id}\times v)^*\Gamma_{vu*}(vu)^!,$$

from which we get the map ? in (6.1.1) as the composition

$$\begin{aligned}\Gamma_u^*\Gamma_{u*}u^!v^* \xrightarrow{\Gamma_u^*\lambda_{\mathsf{e}}} \Gamma_u^*(\mathrm{id}\times v)^*\Gamma_{vu*}(vu)^! &\overset{\mathsf{ps}^*}{=\!=} g^*k^*(\mathrm{id}\times v)^*\Gamma_{vu*}(vu)^!\\ &\overset{\mathsf{ps}^*}{=\!=} g^*r^*\Gamma_{vu}^*\Gamma_{vu*}(vu)^! \overset{\mathsf{ps}^*}{=\!=} \Gamma_{vu}^*\Gamma_{vu*}(vu)^!.\end{aligned}$$

6.2 (*Step I*)**.** For showing that (#) commutes consider more generally a diagram of fiber squares in the category S

(6.2.1)

$$\begin{tikzcd}
\bullet \arrow[r, "16"] & \bullet \arrow[r, "0"] \arrow[d, "15"'] & \bullet \arrow[r, "2"] \arrow[d, "6"'] & \bullet \arrow[r, "3"] \arrow[d, "7"'] & \bullet \arrow[d, "8"] \\
& \bullet \arrow[r, "14"] & \bullet \arrow[r, "4"] \arrow[d, "9"'] & \bullet \arrow[r, "5"] \arrow[d, "10"'] & \bullet \arrow[d, "11"] \\
& & \bullet \arrow[r, "12"'] & \bullet \arrow[r, "13"'] & \bullet
\end{tikzcd}$$

where 14 (hence 0) and 16 are proper, 8 and 11 are flat (whence so are 7, 6, 15, 10 and 9), and $3\circ 2\circ 0\circ 16$, $5\circ 4\circ 14$ and $13\circ 12$ are perfect (whence so are $3\circ 2\circ 0$, $5\circ 4$ and $3\circ 2$, see [Il, p. 245, Cor. 3.5.2]).

From this we extract the following five subdiagrams, all of which satisfy the conditions imposed on the diagram $\mathbf{d}$ in (2.2.3), and thus have associated λ maps:

($\mathbf{d}^+$)
$$
\begin{tikzcd}
\bullet \arrow[r, "0\circ 16"] & \bullet \arrow[r, "2"] \arrow[d, "6"'] & \bullet \arrow[r, "3"] \arrow[d, "7"'] & \bullet \arrow[d, "8"] \\
& \bullet \arrow[r, "4"'] & \bullet \arrow[r, "5"'] & \bullet
\end{tikzcd}
$$

($\mathbf{d}''$)
$$
\begin{tikzcd}
\bullet \arrow[r, "0\circ 16"] & \bullet \arrow[r, "2"] \arrow[d, "9\circ 6"'] & \bullet \arrow[r, "3"] \arrow[d, "10\circ 7"'] & \bullet \arrow[d, "11\circ 8"] \\
& \bullet \arrow[r, "12"'] & \bullet \arrow[r, "13"'] & \bullet
\end{tikzcd}
$$

($\mathbf{d}'$)
$$
\begin{tikzcd}
\bullet \arrow[r, "16"] & \bullet \arrow[r, "2\circ 0"] \arrow[d, "15"'] & \bullet \arrow[r, "3"] \arrow[d, "7"'] & \bullet \arrow[d, "8"] \\
& \bullet \arrow[r, "4\circ 14"'] & \bullet \arrow[r, "5"'] & \bullet
\end{tikzcd}
$$

($\mathbf{e}$)
$$
\begin{tikzcd}
\bullet \arrow[r, "14"] & \bullet \arrow[r, "4"] \arrow[d, "9"'] & \bullet \arrow[r, "5"] \arrow[d, "10"'] & \bullet \arrow[d, "11"] \\
& \bullet \arrow[r, "12"'] & \bullet \arrow[r, "13"'] & \bullet
\end{tikzcd}
$$

($\mathbf{e}^-$)
$$
\begin{tikzcd}
\bullet \arrow[r, equal] & \bullet \arrow[r, "4"] \arrow[d, "9"'] & \bullet \arrow[r, "5"] \arrow[d, "10"'] & \bullet \arrow[d, "11"] \\
& \bullet \arrow[r, "12"'] & \bullet \arrow[r, "13"'] & \bullet
\end{tikzcd}
$$

It is straightforward, if demanding of patience, to verify that commutativity of (#) in (6.1.1) is obtained, upon specialization of (6.2.1) to (6.1.2), by application of the functor $(2\circ 0\circ 16)^*$ to the diagram in the next lemma.

Lemma 6.2.2. *The following diagram of* $\mathbf{D_{qc}}$*-valued functors commutes.*

$$
\begin{tikzcd}
(2\circ 0\circ 16)_*(3\circ 2\circ 0\circ 16)^! 8^* 11^* \arrow[rrr, "\lambda_{\mathsf{d}'}"] \arrow[dddd, equal, "\text{via } \mathsf{ps}^*"'] \arrow[rrdd, "\lambda_{\mathsf{d}^+}"] & & & 7^*(4\circ 14)_*(5\circ 4\circ 14)^! 11^* \arrow[dddd, "7^*\lambda_{\mathsf{e}}"] \\
& & \text{①} & \\
& & 7^* 4_* 14_*(5\circ 4\circ 14)^! 11^* \arrow[uur, equal, "7^*\mathsf{ps}_*"] \arrow[d, "\text{via } \int_{14}^{5\circ 4}"] \arrow[r, phantom, "\text{③}"] & {} \\
& \text{②} & 7^* 4_*(5\circ 4)^! 11^* \arrow[dr, "7^*\lambda_{\mathsf{e}^-}"] & \\
(2\circ 0\circ 16)_*(3\circ 2\circ 0\circ 16)^!(11\circ 8)^* \arrow[r, "\lambda_{\mathsf{d}''}"'] & (10\circ 7)^* 12_*(13\circ 12)^! \arrow[rr, equal, "\mathsf{ps}^*"'] & & 7^* 10^* 12_*(13\circ 12)^!
\end{tikzcd}
$$

Proof. Subdiagram ① without 11^* expands, by the definition of λ, to

$$
\begin{tikzcd}
2_*0_*16_*(3\circ 2\circ 0\circ 16)^! 8^* \arrow[r, "\text{via } \int_{16}^{3\circ 2\circ 0}"] \arrow[d, equal, "\mathsf{ps}_*"'] & 2_*0_*(3\circ 2\circ 0)^! 8^* \arrow[r, "\text{via } \mathsf{B}^{-1}"] \arrow[ddd, "\text{via } \int_0^{3\circ 2}"'] & 2_*0_*15^*(5\circ 4\circ 14)^! \arrow[r, equal, "\mathsf{ps}_*"] \arrow[dd, "2_*\theta^{-1}"'] & (2\circ 0)_*15^*(5\circ 4\circ 14)^! \arrow[d, "\theta^{-1}"] \\
(2\circ 0)_*16_*(3\circ 2\circ 0\circ 16)^! 8^* \arrow[d, equal, "\mathsf{ps}_*"'] \arrow[r, phantom, "\text{④}"] & {} \arrow[r, phantom, "\text{⑤}"] & {} \arrow[r, phantom, "\text{⑥}"] & 7^*(4\circ 14)_*(5\circ 4\circ 14)^! \arrow[d, equal, "7^*\mathsf{ps}_*"] \\
(2\circ 0\circ 16)_*(3\circ 2\circ 0\circ 16)^! 8^* \arrow[d, equal, "\mathsf{ps}_*"'] & & 2_*6^*14_*(5\circ 4\circ 14)^! \arrow[r, "\theta^{-1}"] \arrow[d, "\text{via } \int_{14}^{5\circ 4}"'] \arrow[dr, phantom, "\text{⑦}"] & 7^*4_*14_*(5\circ 4\circ 14)^! \arrow[d, "\text{via } \int_{14}^{5\circ 4}"] \\
2_*(0\circ 16)_*(3\circ 2\circ 0\circ 16)^! 8^* \arrow[r, "\text{via } \int_{0\circ 16}^{3\circ 2}"'] & 2_*(3\circ 2)^! 8^* \arrow[r, "\text{via } \mathsf{B}^{-1}"'] & 2_*6^*(5\circ 4)^! \arrow[r, "\theta^{-1}"'] & 7^*4_*(5\circ 4)^!
\end{tikzcd}
$$

The commutativity of ⑦ is obvious, of ⑥ follows from transitivity of θ (see [L3, Proposition 3.7.2(iii)]), and of ④ is given by Proposition 2.1.5.

As for commutativity of ⑤, with regard to the fiber square S-diagram $\mathbf{uv}$:

$$
\begin{tikzcd}
\bullet \arrow[r, "a := 0"] \arrow[d, "e := 15"'] \arrow[dr, phantom, "\mathbf{v}"] & \bullet \arrow[r, "c := 3\circ 2"] \arrow[d, "f := 6"'] \arrow[dr, phantom, "\mathbf{u}"] & X \arrow[d, "g := 8"] \\
\bullet \arrow[r, "b := 14"'] & \bullet \arrow[r, "d := 5\circ 4"'] & Y
\end{tikzcd}
$$

(where b and a are proper, g, f, e are flat, and d, db, c, ca are perfect), it's enough to show commutativity of the next diagram, in which π stands for projection maps as in (1.3.5), and the unlabeled maps are the obvious ones.

$$
\begin{tikzcd}[column sep=small]
a_*(ca)^!g^* \arrow[ddd, equal] & & a_*e^*(db)^! \arrow[ll, "a_*\mathsf{B}_{\mathsf{uv}}"'] \arrow[d, equal] & & f^*b_*(db)^! \arrow[ll, "\theta_{\mathsf{v}}"'] \arrow[d, equal] \\
& \textcircled{5}_1 & a_*e^*\big((db)^!_+\mathcal{O}_Y \otimes (db)^*\big) \arrow[dl] \arrow[dr, equal, "\mathsf{ps}^!_+"', "\mathsf{ps}^*"] & & f^*b_*\big((db)^!_+\mathcal{O}_Y \otimes (db)^*\big) \arrow[ll, "\theta_{\mathsf{v}}"'] \arrow[dd, equal, "\mathsf{ps}^!_+"', "\mathsf{ps}^*"] \\
& a_*\big(e^*(db)^!_+\mathcal{O}_Y \otimes e^*(db)^*\big) \arrow[dl, "\bar{\mathsf{B}}_{\mathsf{uv}}"', "\mathsf{ps}^*"] \arrow[dr, equal, "\mathsf{ps}^!_+"', "\mathsf{ps}^*"] & & a_*e^*\big(b^!_+d^!_+\mathcal{O}_Y \otimes b^*d^*\big) \arrow[dl] & \\
a_*\big((ca)^!_+\mathcal{O}_X \otimes (ca)^*g^*\big) \arrow[dd, equal, "\mathsf{ps}^!_+"', "\mathsf{ps}^*"] & & a_*\big(e^*b^!_+d^!_+\mathcal{O}_Y \otimes e^*b^*d^*\big) \arrow[dd, equal, "\mathsf{ps}^*"] & & f^*b_*\big(b^!_+d^!_+\mathcal{O}_Y \otimes b^*d^*\big) \arrow[ul, "\theta_{\mathsf{v}}"'] \arrow[dddd, "\pi"] \\
& \textcircled{5}_2 & & & \\
a_*\big(a^!_+c^!_+\mathcal{O}_X \otimes a^*c^*g^*\big) \arrow[dd, "\pi"'] & & a_*\big(e^*b^!_+d^!_+\mathcal{O}_Y \otimes a^*f^*d^*\big) \arrow[dl, "\bar{\mathsf{B}}_{\mathsf{v}}"] \arrow[dd, "\pi"] & & \\
& a_*\big(a^!_+f^*d^!_+\mathcal{O}_Y \otimes a^*f^*d^*\big) \arrow[ul, "\bar{\mathsf{B}}_{\mathsf{u}}", "\mathsf{ps}^*"'] \arrow[dd, "\pi"'] & & \textcircled{5}_3 & \\
a_*a^!_+c^!_+\mathcal{O}_X \otimes c^*g^* \arrow[dd] & & a_*e^*b^!_+d^!_+\mathcal{O}_Y \otimes f^*d^* \arrow[dl, "\bar{\mathsf{B}}_{\mathsf{v}}"] & & f^*\big(b_*b^!_+d^!_+\mathcal{O}_Y \otimes d^*\big) \arrow[dl] \arrow[dd] \\
& a_*a^!_+f^*d^!_+\mathcal{O}_Y \otimes f^*d^* \arrow[ul, "\bar{\mathsf{B}}_{\mathsf{u}}", "\mathsf{ps}^*"'] \arrow[dr] & \textcircled{5}_4 & f^*b_*b^!_+d^!_+\mathcal{O}_Y \otimes f^*d^* \arrow[ul, "\theta_{\mathsf{v}}"'] \arrow[dl] & \\
c^!_+\mathcal{O}_X \otimes c^*g^* & & f^*d^!_+\mathcal{O}_Y \otimes f^*d^* \arrow[ll, "\mathsf{ps}^*"', "\mathsf{B}_{\mathsf{u}}"] & & f^*\big(d^!_+\mathcal{O}_Y \otimes d^*\big) \arrow[ll]
\end{tikzcd}
$$

Commutativity of ⑤$_1$ results directly from the definition of B_{uv} (§2.1.6). Commutativity of ⑤$_2$ is given by pseudofunctoriality of $(-)^*$ and transitivity of $\bar{\mathsf{B}}$ (see [AJL, §5.8.4]). Commutativity of ⑤$_3$ is given by [L3, 3.7.3], in which one makes the substitution $(f, g, f', g', P, Q) \mapsto (b, f, a, e, d^*, b^!_+d^!_+\mathcal{O}_Y)$ (and harmlessly reverses the order of the factors in the tensor products). Since b and a are proper, commutativity of ⑤$_4$ holds by the definition of $\bar{\mathsf{B}}_{\mathsf{v}}$ [AJL, §5.8.2]. Commutativity of the unlabeled subdiagrams is clear.

Thus ① does indeed commute.

We deal next with ②, which expands, by the definition of λ, to

$$
\begin{tikzcd}
(2\circ 0\circ 16)_*(3\circ 2\circ 0\circ 16)^!8^*11^* \arrow[r] \arrow[dd, equal, "\text{via } \mathsf{ps}^*"'] & 2_*(3\circ 2)^!8^*11^* \arrow[r, "2_*\mathsf{B}^{-1}"] \arrow[dd, equal, "\text{via } \mathsf{ps}^*"'] & 2_*6^*(5\circ 4)^!11^* \arrow[r, "\theta^{-1}"] \arrow[d, "\text{via } \mathsf{B}^{-1}"'] & 7^*4_*(5\circ 4)^!11^* \arrow[d, "\text{via } \mathsf{B}^{-1}"] \\
 & \arrow[r, phantom, "⑧"] & 2_*9^*(13\circ 12)^! \arrow[r, "\theta^{-1}"] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & 7^*4_*9^*(13\circ 12)^! \arrow[dd, "7^*\theta^{-1}"] \\
(2\circ 0\circ 16)_*(3\circ 2\circ 0\circ 16)^!(11\circ 8)^* \arrow[r] & 2_*(3\circ 2)^!(11\circ 8)^* \arrow[r, "2_*\mathsf{B}^{-1}"] & 2_*(9\circ 6)^*(13\circ 12)^! \arrow[d, "\text{via } \theta^{-1}"'] \arrow[r, phantom, "⑨"] & \\
 & & (10\circ 7)^*12_*(13\circ 12)^! \arrow[r, equal, "\mathsf{ps}^*"'] & 7^*10^*12_*(13\circ 12)^!
\end{tikzcd}
$$

The unlabeled maps are induced by ps_* and $\int_{3\circ 2}^{0\circ 16}$. Commutativity of the two unlabeled diagrams is obvious, that of ⑧ follows from transitivity of B (§2.1.6), and that of ⑨ from transitivity of θ [L3, Proposition 3.7.2(ii)]. Thus ② commutes.

Commutativity of ③ results directly from the definitions of $\lambda_{\mathbf{e}}$ and $\lambda_{\mathbf{e}^-}$.

This completes the proof of Lemma 6.2.2, and of Step I (commutativity of subdiagram (#) in (6.1.1)). $\square$

6.3 (*Step II*)**.** Let us now check that diagram (##) in (6.1.1) commutes. With $\mathbf{a}$ and $\mathbf{b}$ as in §2.2, and with χ given by the composite isomorphism

$$
u^!v^! \xlongequal{\mathsf{ps}^!} (vu)^! \xlongequal{\quad} (vu)^!\mathcal{O}_Z \otimes (vu)^* \xlongequal{\mathsf{ps}^! \otimes \mathsf{ps}^*} u^!v^!\mathcal{O}_Z \otimes u^*v^*,
$$

the diagram expands as

(6.3.1)

$$
\begin{tikzcd}
\Gamma_u^*\Gamma_{u*}u^!v^* \arrow[r, "4"] \arrow[d, "?"', "\text{in } (6.1.1)"] & u^!\mathcal{O}_Y \otimes u^*\delta_Y^*\delta_{Y*}v^* \arrow[r, equal, "5"] \arrow[dd, "\text{via } \mathbf{a}_v"'] & u^!\delta_Y^*\delta_{Y*}v^* \arrow[dd, "u^!\mathbf{a}_v"', "10"] \\
\Gamma_{vu}^*\Gamma_{vu*}(vu)^! \arrow[d, equal, "0"', "\text{via } \mathsf{ps}^!"] & \mathbf{A} & \\
\Gamma_{vu}^*\Gamma_{vu*}u^!v^! \arrow[r, "6"'] \arrow[d, "2"', "\text{via } \chi"] & u^!\mathcal{O}_Y \otimes u^*\Gamma_v^*\Gamma_{v*}v^! \arrow[r, equal, "7"'] \arrow[dd, "\text{via } \mathbf{b}_v"'] & u^!\Gamma_v^*\Gamma_{v*}v^! \arrow[dd, "u^!\mathbf{b}_v"', "11"] \\
\Gamma_{vu}^*\Gamma_{vu*}(u^!v^!\mathcal{O}_Z \otimes u^*v^*) \arrow[d, "3"'] & \mathbf{B} & \\
u^!v^!\mathcal{O}_Z \otimes \Gamma_{vu}^*\Gamma_{vu*}u^*v^* \arrow[r, "8"'] & u^!\mathcal{O}_Y \otimes u^*v^!\delta_Z^*\delta_{Z*} \arrow[r, equal, "9"'] & u^!v^!\delta_Z^*\delta_{Z*}
\end{tikzcd}
$$

Here 3 is an instance of the isomorphism μ_{vu} (see 2.2.6); 4 is the composition of the first three maps in (2.2.7) (with u in place of f), so that $11\circ 10\circ 5\circ 4$ is the composition of the two arrows in the first row of (##); 6 is the composite isomorphism

$$\begin{aligned}\Gamma_{vu}^*\Gamma_{vu*}u^!v^! = \Gamma_{vu}^*\Gamma_{vu*}(u^!\mathcal{O}_Y\otimes u^*v^!) &\xrightarrow{\mu_{vu}} u^!\mathcal{O}_Y\otimes\Gamma_{vu}^*\Gamma_{vu*}u^*v^!\\ &\xrightarrow{\text{via }\phi_{\mathsf{u}}^{-1}} u^!\mathcal{O}_Y\otimes u^*\Gamma_v^*\Gamma_{v*}v^!,\end{aligned}$$

where, relative to the next diagram, ϕ is as in (Proposition 1.5.3)

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\Gamma_{vu}"'] \arrow[dr, phantom, "\mathsf{u}"] & Y \arrow[r, "v"] \arrow[d, "\Gamma_v"'] \arrow[dr, phantom, "\mathsf{v}"] & Z \arrow[d, "\delta_Z"] \\ X\times Z \arrow[r, "u\times\mathrm{id}_Z"'] & Y\times Z \arrow[r, "v\times\mathrm{id}_Z"'] & Z\times Z; \end{tikzcd} \tag{6.3.2}$$

and with $\mathsf{ps}^!$ and $\mathbf{b}_{vu}$ as in (##), $8 := 9^{-1}(\mathsf{ps}^!\circ\mathbf{b}_{vu})0^{-1}2^{-1}3^{-1}$, so that

$$9\circ 8\circ 3\circ 2\circ 0 = \mathsf{ps}^!\circ\mathbf{b}_{vu}.$$

It is clear that the unlabeled subdiagrams in (6.3.1) commute; so it suffices to show that the subdiagrams **A** and **B** commute.

6.4 (*Step IIB*)**.** We deal first with **B**. Let $\bar\phi$ be the composite isomorphism

$$\Gamma_{vu}^*\Gamma_{vu*}u^*v^* \xrightarrow{\phi_{\mathsf{u}}^{-1}} u^*\Gamma_v^*\Gamma_{v*}v^* \xrightarrow{u^*\phi_{\mathsf{v}}^{-1}} u^*v^*\delta_Z^*\delta_{Z*} \qquad (\mathsf{u},\mathsf{v}\text{ as above}).$$

The map $8 := 9^{-1}(\mathsf{ps}^!\circ\mathbf{b}_{vu})0^{-1}2^{-1}3^{-1}$ in **B** factors as

$$\begin{aligned}u^!v^!\mathcal{O}_Z\otimes\Gamma_{vu}^*\Gamma_{vu*}u^*v^* &\xrightarrow{\text{via }\bar\phi} u^!v^!\mathcal{O}_Z\otimes u^*v^*\delta_Z^*\delta_{Z*}\\ &\xrightarrow{\chi^{-1}} u^!v^!\delta_Z^*\delta_{Z*}\\ &=\!= u^!\mathcal{O}_Y\otimes u^*v^!\delta_Z^*\delta_{Z*}.\end{aligned}$$

that is, as

$$\begin{aligned}u^!v^!\mathcal{O}_Z\otimes\Gamma_{vu}^*\Gamma_{vu*}u^*v^* &\xrightarrow{\text{via }\bar\phi} u^!v^!\mathcal{O}_Z\otimes u^*v^*\delta_Z^*\delta_{Z*}\\ &\overset{\text{via }\mathsf{ps}^!,\,\mathsf{ps}^*}{=\!=\!=} (vu)^!\mathcal{O}_Z\otimes(vu)^*\delta_Z^*\delta_{Z*}\\ &=\!= (vu)^!\delta_Z^*\delta_{Z*} \overset{\mathsf{ps}^!}{=\!=} u^!v^!\delta_Z^*\delta_{Z*}\\ &=\!= u^!\mathcal{O}_Y\otimes u^*v^!\delta_Z^*\delta_{Z*}.\end{aligned}$$

This results from commutativity of all the subdiagrams of the following diagram, where the subdiagram 0 commutes by Proposition 1.5.4, and the rest by the definitions of the maps involved.

$$
\begin{tikzcd}
 & & u^!v^!\mathcal{O}_Z\otimes u^*v^*\delta_Z^*\delta_{Z*} \arrow[rr, equal, "\text{via }\mathsf{ps}^!,\,\mathsf{ps}^*"] & & \\
u^!v^!\mathcal{O}_Z\otimes \Gamma^*_{vu}\,\Gamma_{vu*}u^*v^* \arrow[urr, "\text{via }\bar\phi"] \arrow[drr, equal, "\text{via }\mathsf{ps}^!,\,\mathsf{ps}^*"'] \arrow[ddddd, "3^{-1}"'] & & ⓪ & & (vu)^!\mathcal{O}_Z\otimes (vu)^*\delta_Z^*\delta_{Z*} \arrow[dd, equal] \\
 & & (vu)^!\mathcal{O}_Z\otimes \Gamma^*_{vu}\,\Gamma_{vu*}(vu)^* \arrow[urr, "\text{via }\phi_{\mathbf{vu}}^{-1}"'] & & \\
 & & & & (vu)^!\delta_Z^*\delta_{Z*} \\
 & & \Gamma^*_{vu}\,\Gamma_{vu*}((vu)^!\mathcal{O}_Z\otimes (vu)^*) \arrow[uu, "(2.2.6)"] & & \\
\Gamma^*_{vu}\,\Gamma_{vu*}(u^!v^!\mathcal{O}_Z\otimes u^*v^*) \arrow[urr, equal] \arrow[drr, "2^{-1}"'] & & & & \Gamma^*_{vu}\,\Gamma_{vu*}(vu)^! \arrow[uu, "\mathbf{b_{vu}}"'] \arrow[ull, equal] \\
 & & \Gamma^*_{vu}\,\Gamma_{vu*}u^!v^! \arrow[urr, equal, "0^{-1}"'] & &
\end{tikzcd}
$$

So $\mathbf{B}$ expands as follows, with $\Gamma=\Gamma_v$, $\Gamma'=\Gamma_{vu}$, $\delta:=\delta_Z$, and ν standing for natural isomorphisms of the form $u^*(E\otimes F)\xrightarrow{\,\sim\,}u^*E\otimes u^*F$ (see (1.3.3)).

$$
\begin{tikzcd}
\Gamma'^*\Gamma'_*u^!v^! \arrow[r] \arrow[ddddd, "\text{via }\chi"'] & \Gamma'^*\Gamma'_*(u^!\mathcal{O}_Y\otimes u^*v^!) \arrow[dd, equal] \arrow[dr, "\mu_{vu}"] & & u^!\mathcal{O}_Y\otimes u^*\Gamma^*\Gamma_*v^! \arrow[dd, equal] \\
 & & u^!\mathcal{O}_Y\otimes \Gamma'^*\Gamma'_*u^*v^! \arrow[ur, "\text{via }\phi_{\mathbf{u}}^{-1}"] \arrow[dd, equal] & \\
 & \Gamma'^*\Gamma'_*(u^!\mathcal{O}_Y\otimes u^*(v^!\mathcal{O}_Z\otimes v^*)) \arrow[dr, "\mu_{vu}"] \arrow[dd, "\text{via }\nu"'] & & u^!\mathcal{O}_Y\otimes u^*\Gamma^*\Gamma_*(v^!\mathcal{O}_Z\otimes v^*) \arrow[dd, "\text{via }\mu_v"] \\
 {} \arrow[r, phantom, "\mathbf{B_1}"] & {} & u^!\mathcal{O}_Y\otimes \Gamma'^*\Gamma'_*u^*(v^!\mathcal{O}_Z\otimes v^*) \arrow[ur, "\text{via }\phi_{\mathbf{u}}^{-1}"] \arrow[dd, "\text{via }\nu"] & \\
 & \Gamma'^*\Gamma'_*(u^!\mathcal{O}_Y\otimes (u^*v^!\mathcal{O}_Z\otimes u^*v^*)) \arrow[dr, "\mu_{vu}"] \arrow[dd, equal] & \mathbf{B_3} & u^!\mathcal{O}_Y\otimes u^*(v^!\mathcal{O}_Z\otimes \Gamma^*\Gamma_*v^*) \arrow[dd, "\text{via }\nu"] \\
\Gamma'^*\Gamma'_*(u^!v^!\mathcal{O}_Z\otimes u^*v^*) \arrow[dr, equal] \arrow[dd, "3"'] & & u^!\mathcal{O}_Y\otimes \Gamma'^*\Gamma'_*(u^*v^!\mathcal{O}_Z\otimes u^*v^*) \arrow[dd, "\text{via }\mu_{vu}"] & \\
 & \Gamma'^*\Gamma'_*((u^!\mathcal{O}_Y\otimes u^*v^!\mathcal{O}_Z)\otimes u^*v^*) \arrow[dr, "\mu_{vu}"] & \mathbf{B_2} & u^!\mathcal{O}_Y\otimes u^*v^!\mathcal{O}_Z\otimes u^*\Gamma^*\Gamma_*v^* \arrow[dd, "\text{via }\phi_{\mathbf{v}}^{-1}"] \\
u^!v^!\mathcal{O}_Z\otimes \Gamma'^*\Gamma'_*u^*v^* \arrow[rr, equal] \arrow[d, "\text{via }\bar\phi"'] & & u^!\mathcal{O}_Y\otimes u^*v^!\mathcal{O}_Z\otimes \Gamma'^*\Gamma'_*u^*v^* \arrow[ur, "\text{via }\phi_{\mathbf{u}}^{-1}"'] \arrow[dr, "\text{via }\bar\phi"] & \\
u^!v^!\mathcal{O}_Z\otimes u^*v^*\delta^*\delta_* \arrow[rrr, equal] \arrow[d, "\chi^{-1}"'] & \mathbf{B_4} & & u^!\mathcal{O}_Y\otimes u^*v^!\mathcal{O}_Z\otimes u^*v^*\delta^*\delta_* \arrow[d, "\text{via }\nu^{-1}"] \\
u^!v^!\delta^*\delta_* \arrow[r, equal] & u^!\mathcal{O}_Y\otimes u^*v^!\delta^*\delta_* \arrow[rr, equal] & & u^!\mathcal{O}_Y\otimes u^*(v^!\mathcal{O}_Z\otimes v^*\delta^*\delta_*)
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams is easily verified. That of $\mathbf{B_1}$ (without $\Gamma'^*\Gamma'_*$) is essentially the definition of the isomorphism $u^!v^! \overset{\mathsf{ps}^!}{=\!=} (vu)^!$, see [AJL, (5.7.5)]; and similarly for $\mathbf{B_4}$ (without $\delta^*\delta_*$).

Commutativity of $\mathbf{B_2}$ is contained in the next Lemma.

Lemma 6.4.1. *Let* $X \xrightarrow{\Gamma} W \xrightarrow{p} X$ *be qcqs maps with* $p\Gamma = \mathrm{id}_X$, *and let*

$$\mu = \mu_{\Gamma,\,p}\colon \Gamma^*\Gamma_*(E \otimes F) \xrightarrow{\,\sim\,} E \otimes \Gamma^*\Gamma_* F \qquad (E,\, F \in \mathbf{D}_X)$$

be the functorial isomorphism defined as in 2.2.6. Then for any E, F *and* G *in* $\mathbf{D}_X$ *the following diagram commutes:*

$$\begin{tikzcd}
\Gamma^*\Gamma_*(E \otimes (F \otimes G)) \arrow[r, "\text{via}\,\mu"] \arrow[d, equal] & E \otimes \Gamma^*\Gamma_*(F \otimes G) \arrow[d, "\text{via}\,\mu"] \\
\Gamma^*\Gamma_*((E \otimes F) \otimes G) \arrow[r, "\text{via}\,\mu"'] & E \otimes F \otimes \Gamma^*\Gamma_* G
\end{tikzcd}$$

Proof. Referring to the definition of μ, expand the diagram to the following natural one, where the isomorphism ps^* is denoted by an equality.

$$\begin{tikzcd}[column sep=tiny]
\Gamma^*\Gamma_*(E\otimes(F\otimes G)) \arrow[dd, equal] \arrow[dr, equal] & & \Gamma^*(p^*E\otimes\Gamma_*(F\otimes G)) \arrow[dd, equal] \arrow[dr] & & E\otimes\Gamma^*\Gamma_*(F\otimes G) \arrow[dl, equal] \arrow[dd, equal] \\
& \Gamma^*\Gamma_*(\Gamma^*p^*E\otimes(F\otimes G)) \arrow[ur] \arrow[dd, equal] & & \Gamma^*p^*E\otimes\Gamma^*\Gamma_*(F\otimes G) \arrow[dd, equal] & \\
\Gamma^*\Gamma_*(E\otimes(\Gamma^*p^*F\otimes G)) \arrow[dr, equal] \arrow[dd] & & \Gamma^*(p^*E\otimes\Gamma_*(\Gamma^*p^*F\otimes G)) \arrow[dr] \arrow[dd] & & E\otimes\Gamma^*\Gamma_*(\Gamma^*p^*F\otimes G) \arrow[dl, equal] \arrow[dd] \\
& \Gamma^*\Gamma_*(\Gamma^*p^*E\otimes(\Gamma^*p^*F\otimes G)) \arrow[ur] \arrow[dd] & & \Gamma^*p^*E\otimes\Gamma^*\Gamma_*(\Gamma^*p^*F\otimes G) \arrow[dd] & \\
\Gamma^*\Gamma_*((E\otimes\Gamma^*p^*F)\otimes G) \arrow[dr, equal] \arrow[ddddd, equal] & & \Gamma^*(p^*E\otimes(p^*F\otimes\Gamma_*G)) \arrow[dr] \arrow[ddd] & & E\otimes\Gamma^*(p^*F\otimes\Gamma_*G) \arrow[dl, equal] \arrow[ddd] \\
& \Gamma^*\Gamma_*((\Gamma^*p^*E\otimes\Gamma^*p^*F)\otimes G) \arrow[ddd] & & \Gamma^*p^*E\otimes\Gamma^*(p^*F\otimes\Gamma_*G) \arrow[d] & \\
{} \arrow[dr, phantom, "\mathbf{B_{21}}"] & {} \arrow[r, phantom, "\mathbf{B_{22}}"] & {} \arrow[dr, phantom, "\mathbf{B_{23}}"] & \Gamma^*p^*E\otimes\Gamma^*p^*F\otimes\Gamma^*\Gamma_*G \arrow[dr, equal] \arrow[dd] & \\
& & \Gamma^*((p^*E\otimes p^*F)\otimes\Gamma_*G) \arrow[dr] \arrow[dd] & {} \arrow[dr, phantom, "\mathbf{B_{24}}"] & E\otimes\Gamma^*p^*F\otimes\Gamma^*\Gamma_*G \arrow[dd, equal] \\
& \Gamma^*\Gamma_*(\Gamma^*(p^*E\otimes p^*F)\otimes G) \arrow[ur] \arrow[dd] & & \Gamma^*(p^*E\otimes p^*F)\otimes\Gamma^*\Gamma_*G \arrow[dd] & \\
\Gamma^*\Gamma_*((E\otimes F)\otimes G) \arrow[dr, equal] & & \Gamma^*(p^*(E\otimes F)\otimes\Gamma_*G) \arrow[dr] & & E\otimes F\otimes\Gamma^*\Gamma_*G \arrow[dl, equal] \\
& \Gamma^*\Gamma_*(\Gamma^*p^*(E\otimes F)\otimes G) \arrow[ur] & & \Gamma^*p^*(E\otimes F)\otimes\Gamma^*\Gamma_*G &
\end{tikzcd}$$

The commutativity of the unlabeled diagrams is obvious.

Commutativity of $\mathbf{B_{21}}$ and of $\mathbf{B_{24}}$ results directly from the fact that the contravariant pseudofunctor $(-)^*$ is *monoidal* (see [L3, 3.6.7(b)], a proof of which is outlined in [L3, (3.6.10)]).

Commutativity of $\mathbf{B_{22}}$ is given by [L3, 3.4.7(iv)] (with $f = \Gamma$, $A = p^*E$, $B = p^*F$ and $C = G$).

Finally, $\mathbf{B_{23}}$ is *dual* (see [L3, 3.4.5]) to the largest commutative diagram in [L3, (3.4.2.2)], *mutatis mutandis*, and so is itself commutative. $\square$

To complete Step IIB, it remains to show that subdiagram $\mathbf{B_3}$ in (6.4) commutes. For this, it suffices to apply the next Lemma, with $E := v^!\mathcal{O}_Z$ and $F := v^*G$ ($G \in \mathbf{D}_Z$), to the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\Gamma'"'] & Y \arrow[d, "\Gamma"] \\
X \times Z \arrow[r, "r := u\times 1"'] \arrow[d, "p'"'] & Y \times Z \arrow[d, "p"] \\
X \arrow[r, "u"'] & Y
\end{tikzcd}$$

where p and p' are the natural projections.

Lemma 6.4.2. *Let*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\Gamma'"'] \arrow[dr, phantom, "\mathbf{d}"] & Y \arrow[d, "\Gamma"] \\
X' \arrow[r, "r"'] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\
X \arrow[r, "u"'] & Y
\end{tikzcd}$$

be a commutative diagram of qcqs maps such that $p' \circ \Gamma' = 1_X$ and $p \circ \Gamma = \mathrm{id}_Y$. Let $\mu = \mu_{\Gamma,p}$ and $\mu' = \mu_{\Gamma',p'}$ be defined as in (2.2.6), ν as in the paragraph before (6.4) and $\phi = \phi_{\mathbf{d}}$ as in (1.5.1). Then the following diagram commutes for all E and F in $\mathbf{D}_Y$.

$$\begin{tikzcd}
\Gamma'^*\Gamma'_*u^*(E\otimes F) \arrow[d, "\text{via }\nu"'] & u^*\Gamma^*\Gamma_*(E\otimes F) \arrow[l, "\phi"'] \arrow[d, "\text{via }\mu"] \\
\Gamma'^*\Gamma'_*(u^*E\otimes u^*F) \arrow[d, "\mu'"'] & u^*(E\otimes \Gamma^*\Gamma_*F) \arrow[d, "\nu"] \\
u^*E\otimes \Gamma'^*\Gamma'_*u^*F & u^*E\otimes u^*\Gamma^*\Gamma_*F \arrow[l, "\mathrm{id}\otimes\phi"]
\end{tikzcd}$$

Proof. The definitions of μ and ϕ lead to the following expansion of the preceding diagram, where "$=\!=$" indicates the isomorphism ps^*, and the other maps are the obvious ones:

$$
\begin{tikzcd}
\Gamma'^*\Gamma'_*u^*(E\otimes F) \arrow[dd] \arrow[dr, equal] & & \Gamma'^*r^*\Gamma_*(E\otimes F) \arrow[ll] \arrow[r, equal] \arrow[dd, equal] & u^*\Gamma^*\Gamma_*(E\otimes F) \arrow[d, equal] \\
 & \Gamma'^*\Gamma'_*u^*(\Gamma^*p^*E\otimes F) \arrow[dd] & & u^*\Gamma^*\Gamma_*(\Gamma^*p^*E\otimes F) \arrow[dl, equal] \arrow[dd] \\
\Gamma'^*\Gamma'_*(u^*E\otimes u^*F) \arrow[dd, equal] \arrow[dr, equal] & & \Gamma'^*r^*\Gamma_*(\Gamma^*p^*E\otimes F) \arrow[ul] \arrow[dd] & \\
 & \Gamma'^*\Gamma'_*(u^*\Gamma^*p^*E\otimes u^*F) \arrow[dd, equal] & & u^*\Gamma^*(p^*E\otimes \Gamma_*F) \arrow[dl, equal] \arrow[dd] \\
\Gamma'^*\Gamma'_*(\Gamma'^*p'^*u^*E\otimes u^*F) \arrow[dd] \arrow[dr, equal] & & \Gamma'^*r^*(p^*E\otimes \Gamma_*F) \arrow[dd] \arrow[dl, phantom, "\mathbf{B_{31}}"] & \\
 & \Gamma'^*\Gamma'_*(\Gamma'^*r^*p^*E\otimes u^*F) \arrow[dd] & & u^*(\Gamma^*p^*E\otimes \Gamma^*\Gamma_*F) \arrow[dd] \arrow[dl, phantom, "\mathbf{B_{32}}"] \\
\Gamma'^*(p'^*u^*E\otimes \Gamma'_*u^*F) \arrow[dd] \arrow[dr, equal] & & \Gamma'^*(r^*p^*E\otimes r^*\Gamma_*F) \arrow[dl] \arrow[dd] & \\
 & \Gamma'^*(r^*p^*E\otimes \Gamma'_*u^*F) \arrow[dd] & & u^*\Gamma^*p^*E\otimes u^*\Gamma^*\Gamma_*F \arrow[dl, equal] \arrow[ddd, equal] \\
\Gamma'^*p'^*u^*E\otimes \Gamma'^*\Gamma'_*u^*F \arrow[dd, equal] \arrow[dr, equal] & & \Gamma'^*r^*p^*E\otimes \Gamma'^*r^*\Gamma_*F \arrow[dl] \arrow[dd, equal] & \\
 & \Gamma'^*r^*p^*E\otimes \Gamma'^*\Gamma'_*u^*F \arrow[dl, equal] & & \\
u^*E\otimes \Gamma'^*\Gamma'_*u^*F & & u^*E\otimes \Gamma'^*r^*\Gamma_*F \arrow[ll] \arrow[r, equal] & u^*E\otimes u^*\Gamma^*\Gamma_*F
\end{tikzcd}
$$

Here commutativity of the unlabeled subdiagrams is clear, by naturality of the transformations involved and by the transitivity property of pseudofunctoriality isomorphisms; that of $\mathbf{B_{31}}$ follows from [L3, Proposition 3.7.3] with $(f, f', g, g', P, Q) := (\Gamma, \Gamma', r, u, p^*E, F)$; and that of $\mathbf{B_{32}}$ results from the fact that the contravariant pseudofunctor $(-)^*$ is *monoidal* (see [L3, (3.6.7)(b) and (3.6.10)]). $\square$

This completes the proof that subdiagram $\mathbf{B}$ in (6.3.1) commutes.

6.5 (*Step IIA*)**.** We show next that diagram **A** commutes.

Recall the diagram formed by the last two rows of (6.1.2):

$$
\begin{array}{ccccccc}
X & \xrightarrow{\;g\;} & X\times_Z Y & \xrightarrow{\;k\;} & X\times Y & \xrightarrow{\;p_Y\;} & Y\\
 & & {\scriptstyle r}\big\downarrow & & {\scriptstyle \mathrm{id}\times v}\big\downarrow & & \big\downarrow{\scriptstyle v}\\
 & & X & \xrightarrow[\Gamma_{vu}]{} & X\times Z & \xrightarrow[p_Z]{} & Z
\end{array}
$$

where k is the natural closed immersion; g is the graph of u, i.e., the unique closed immersion such that $kg = \Gamma_u$; r is projection onto the first factor; and p_Y, p_Z are the projections onto the second factor, so that $p_Y kg = u$ and $p_Z\Gamma_{vu} = vu$. Recall also the isomorphisms B (§2.1.6) and θ (§1.4.4). Further, let $p\colon X\times Y\to X$ be the canonical projection, so that $k^*p^* = r^*$ and $\Gamma_u^*p^* = 1 := \mathrm{id}_X$.

Referring to the definitions of its constituent maps, expand **A** as follows, where the maps labeled ?? are induced by a map $\xi\colon g_*u^*v^* \longrightarrow r^*u^*v^!$ to be defined below (6.5.2); the ones labeled ??? are induced by the composition (with $\mathrm{id} := \mathrm{id}_{X\times_Z Y}$)

$$
\begin{aligned}
&\Gamma_u^* k_*(r^*u^!\mathcal{O}_Y\otimes \mathrm{id}) \overset{\text{via ps}^*}{=\!=} \Gamma_u^* k_*(k^*p^*u^!\mathcal{O}_Y\otimes \mathrm{id})\\
&\text{(via (1.3.5))}\qquad \xrightarrow{\sim} \Gamma_u^*(p^*u^!\mathcal{O}_Y\otimes k_*)\\
&\text{(see (1.3.3))}\qquad \underset{\nu}{\xrightarrow{\sim}} \Gamma_u^*p^*u^!\mathcal{O}_Y\otimes \Gamma_u^*k_*;
\end{aligned}
$$

and with $q\colon X\times Z\longrightarrow X$ the canonical projection, so that $p^* = (1\times v)^*q^*$, the map ???? is the composite isomorphism

$$
\begin{aligned}
&\Gamma_u^*(1\times v)^*\Gamma_{vu*}(u^!\mathcal{O}_Y\otimes u^*v^!) \overset{\text{via ps}^*}{=\!=} \Gamma_u^*(1\times v)^*\Gamma_{vu*}(\Gamma_{vu}^*q^*u^!\mathcal{O}_Y\otimes u^*v^!)\\
&\text{(see (1.3.5))}\qquad \xrightarrow{\sim} \Gamma_u^*(1\times v)^*(q^*u^!\mathcal{O}_Y\otimes \Gamma_{vu*}u^*v^!)\\
&\text{(see (1.3.3))}\qquad \underset{\nu}{\xrightarrow{\sim}} \Gamma_u^*(1\times v)^*q^*u^!\mathcal{O}_Y\otimes \Gamma_u^*(1\times v)^*\Gamma_{vu*}u^*v^!\\
&\qquad\qquad \underset{\text{via ps}^*}{=\!=} \Gamma_u^*p^*u^!\mathcal{O}_Y\otimes \Gamma_u^*(1\times v)^*\Gamma_{vu*}u^*v^!.
\end{aligned}
$$

$$
\begin{tikzcd}
\Gamma_u^*\Gamma_{u*}u^!v^* \arrow[dd, equal, "\text{via } \mathsf{ps}_*"'] \arrow[dr, equal] & & u^!\mathcal{O}_Y\otimes\Gamma_u^*\Gamma_{u*}u^*v^* \arrow[dr, "\text{via } \phi^{-1}"] \arrow[ddd, equal, "\text{via } \mathsf{ps}^*"'] & \\
& \Gamma_u^*\Gamma_{u*}(u^!\mathcal{O}_Y\otimes u^*v^*) \arrow[ur, "\mu_u"] \arrow[dd, equal, "\text{via } \mathsf{ps}_*"] & & u^!\mathcal{O}_Y\otimes u^*\delta_Y^*\delta_{Y*}v^* \arrow[dddddddddddddd, "\text{via } \mathbf{a}_v"] \\
\Gamma_u^*k_*g_*u^!v^* \arrow[dr, equal] \arrow[dd, equal] & & & \\
& \Gamma_u^*k_*g_*(u^!\mathcal{O}_Y\otimes u^*v^*) \arrow[dd, equal, "\text{via } \mathsf{ps}^*"] & \Gamma_u^*p^*u^!\mathcal{O}_Y\otimes\Gamma_u^*\Gamma_{u*}u^*v^* \arrow[ddd, equal, "\text{via } \mathsf{ps}_*"'] & \\
\Gamma_u^*k_*g_*(p_Ykg)^!v^* \arrow[dd, "\text{via } \int_g^{p_Yk}"'] & & \mathbf{A_1} & \\
& \Gamma_u^*k_*g_*(g^*r^*u^!\mathcal{O}_Y\otimes u^*v^*) \arrow[dd, "(1.3.5)"] & & \\
\Gamma_u^*k_*(p_Yk)^!v^* \arrow[dd, "\text{via } \mathsf{B}^{-1}"'] & & \Gamma_u^*p^*u^!\mathcal{O}_Y\otimes\Gamma_u^*k_*g_*u^*v^* \arrow[dd, "??"'] & \\
\mathbf{A_2} & \Gamma_u^*k_*(r^*u^!\mathcal{O}_Y\otimes g_*u^*v^*) \arrow[ur, "???"'] \arrow[dd, "??"] & & \mathbf{A_3} \\
\Gamma_u^*k_*r^*(vu)^! \arrow[dd, equal, "\text{via } \mathsf{ps}^!"'] & & \Gamma_u^*p^*u^!\mathcal{O}_Y\otimes\Gamma_u^*k_*r^*u^*v^! \arrow[dddd, "\text{via } \theta^{-1}"'] & \\
& \Gamma_u^*k_*(r^*u^!\mathcal{O}_Y\otimes r^*u^*v^!) \arrow[ur, "???"'] \arrow[dd, "(1.3.3)"] & & \\
\Gamma_u^*k_*r^*u^!v^! \arrow[dr, equal] \arrow[dd, "\text{via } \theta^{-1}"'] & & & \\
& \Gamma_u^*k_*r^*(u^!\mathcal{O}_Y\otimes u^*v^!) \arrow[dd, "\text{via } \theta^{-1}"] & \mathbf{A_4} & \\
\Gamma_u^*(1\times v)^*\Gamma_{vu*}u^!v^! \arrow[dr, equal] \arrow[dd, equal, "\text{via } \mathsf{ps}^*"'] & & \Gamma_u^*p^*u^!\mathcal{O}_Y\otimes\Gamma_u^*(1\times v)^*\Gamma_{vu*}u^*v^! \arrow[dd, equal, "\mathsf{ps}^*"'] & \\
& \Gamma_u^*(1\times v)^*\Gamma_{vu*}(u^!\mathcal{O}_Y\otimes u^*v^!) \arrow[ur, "????"'] \arrow[dd, equal, "\mathsf{ps}^*"] & \mathbf{A_5} & \\
\Gamma_{vu}^*\Gamma_{vu*}u^!v^! \arrow[dr, equal] & & u^!\mathcal{O}_Y\otimes\Gamma_{vu}^*\Gamma_{vu*}u^*v^! \arrow[dr, "\text{via } \phi^{-1}"] & \\
& \Gamma_{vu}^*\Gamma_{vu*}(u^!\mathcal{O}_Y\otimes u^*v^!) \arrow[ur, "\mu_{vu}"'] & & u^!\mathcal{O}_Y\otimes u^*\Gamma_v^*\Gamma_{v*}v^!
\end{tikzcd}
$$

Commutativity of the unlabeled squares is transparent.

Commutativity of $\mathbf{A_5}$ becomes clear upon expansion of ???? and μ_{vu} according to their definitions, and identification via the pseudofunctor $(-)^*$ of $\Gamma_u^*(1\times v)^*$ with Γ_{vu}^*. Details are left to the reader.

Commutativity of $\mathbf{A_1}$ can be seen by expanding it as follows, according to the definitions of the maps involved, with $E := u^!\mathcal{O}_Y$, $F := u^*v^*G$ ($G \in \mathbf{D}_Z$), and $\pi_\bullet$ denoting a projection isomorphism, see (1.3.5):

$$
\begin{tikzcd}
\Gamma_u^*\Gamma_{u*}(E\otimes F) \arrow[rrr, "\mu_u"] \arrow[d, equal, "\text{via } \mathsf{ps}_*"'] \arrow[dr, equal, "\text{via } \mathsf{ps}^*"'] & & & E\otimes\Gamma_u^*\Gamma_{u*}F \arrow[dd, equal, "\text{via } \mathsf{ps}^*"] \\
\Gamma_u^*k_*g_*(E\otimes F) \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \Gamma_u^*\Gamma_{u*}(\Gamma_u^*p^*E\otimes F) \arrow[r, "\pi_\Gamma"] \arrow[d, equal, "\text{via } \mathsf{ps}_*"'] & \Gamma_u^*(p^*E\otimes\Gamma_{u*}F) \arrow[dd, equal, "\text{via } \mathsf{ps}_*"'] \arrow[dr, "\nu"] & \\
\Gamma_u^*k_*g_*(g^*r^*E\otimes F) \arrow[ddd, "\pi_g"'] \arrow[dr, equal, "\text{via } \mathsf{ps}^*"'] & \Gamma_u^*k_*g_*(g^*r^*E\otimes F) \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \mathbf{A_{11}} & \Gamma_u^*p^*E\otimes\Gamma_u^*\Gamma_{u*}F \arrow[dd, equal, "\text{via } \mathsf{ps}_*"] \\
& \Gamma_u^*k_*g_*(g^*k^*p^*E\otimes F) \arrow[d, "\pi_g"'] & \Gamma_u^*(p^*E\otimes k_*g_*F) \arrow[dr, "\nu"'] & \\
\Gamma_u^*k_*(r^*E\otimes g_*F) \arrow[r, equal, "\text{via } \mathsf{ps}^*"'] & \Gamma_u^*k_*(k^*p^*E\otimes g_*F) \arrow[ur, "\pi_k"'] & & \Gamma_u^*p^*E\otimes\Gamma_u^*k_*g_*F
\end{tikzcd}
$$

Here, commutativity of the unlabeled subdiagrams is easily checked; and that of $\mathbf{A_{11}}$ results from transitivity of the projection isomorphism with respect to the composition $\Gamma_u = kg$, cf. [L3, Proposition 3.7.1].

As for $\mathbf{A_4}$, apply [L3, Proposition 3.7.3] to

$$
\begin{tikzcd}
X\times_Z Y \arrow[r, "r"] \arrow[d, "k"'] & X \arrow[d, "\Gamma_{vu}=\Gamma"] \\
X\times Y \arrow[r, "1\times v"'] & X\times Z
\end{tikzcd}
$$

to obtain the commutativity of the following diagram, with $P := q^*u^!\mathcal{O}_Y$, $Q := u^*v^!G$ ($G \in \mathbf{D}_Z$), $\nu_\bullet$ coming from (1.3.3), and $\pi_\bullet$ denoting a projection isomorphism, see (1.3.5)—commutativity from which, with a bit of patience, one readily deduces commutativity of $\mathbf{A_4}$ (details left to the reader):

$$
\begin{tikzcd}
k_*(k^*(1\times v)^*P\otimes r^*Q) \arrow[r, equal, "\text{via } \mathsf{ps}^*"] & k_*(r^*\Gamma^*P\otimes r^*Q) & k_*r^*(\Gamma^*P\otimes Q) \arrow[l, "k_*\nu_r"'] \\
 & & (1\times v)^*\Gamma_*(\Gamma^*P\otimes Q) \arrow[u, "\theta"'] \\
(1\times v)^*P\otimes k_*r^*Q \arrow[uu, "\pi_k"] \arrow[d, equal] & (1\times v)^*P\otimes(1\times v)^*\Gamma_*Q \arrow[l, "\text{via }\theta"] \arrow[d, equal] & (1\times v)^*(P\otimes\Gamma_*Q) \arrow[l, "\nu_{1\times v}"] \arrow[u, "\text{via }\pi_\Gamma"'] \\
(1\times v)^*q^*u^!\mathcal{O}_Y\otimes k_*r^*Q \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & (1\times v)^*q^*u^!\mathcal{O}_Y\otimes(1\times v)^*\Gamma_*Q \arrow[l, "\text{via }\theta"] \arrow[d, equal, "\text{via } \mathsf{ps}^*"] & \\
p^*u^!\mathcal{O}_Y\otimes k_*r^*Q & p^*u^!\mathcal{O}_Y\otimes(1\times v)^*\Gamma_*Q \arrow[l, "\text{via }\theta"] &
\end{tikzcd}
$$

This leaves us with $\mathbf{A_2}$ and $\mathbf{A_3}$, for which we first need to define the above map ξ. Consider the fiber square diagram, with $1:=\mathrm{id}_Y$,

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathbf{e}"] & Y \arrow[d, "\delta_v"] & \\
X\times_Z Y \arrow[r, "w:=u\times_Z 1"] \arrow[d, "k"'] \arrow[dr, phantom, "\mathbf{f}"] & Y\times_Z Y \arrow[r, "t_1"] \arrow[d, "i'"] \arrow[dr, phantom, "\mathbf{g}"] & Y \arrow[d, "\Gamma_v"] \\
X\times Y \arrow[r, "u\times 1"'] & Y\times Y \arrow[r, "1\times v"'] \arrow[d, "p'_Y"] & Y\times Z \arrow[d, "p'_Z"] \\
 & Y \arrow[r, "v"'] & Z
\end{tikzcd}
$$

Here t_1 is the projection onto the first factor, k and i' are the natural maps, g and Γ_v are graph maps (of u and v respectively), δ_v is the diagonal map, and p'_Y, p'_Z are the projections onto the second factor, so that $p'_Z\circ\Gamma_v=v$. Setting $t_2:=p'_Y i'$, one has then the composite functorial map

$$
\bar\lambda\colon \delta_{v*}v^*=\delta_{v*}(t_2\delta_v)^!v^*\xrightarrow{\int^{t_2}_{\delta_v}} t_2^!v^*\xrightarrow{\mathsf{B}^{-1}} t_1^*v^! \tag{6.5.1}
$$

that shows up in a factorization of the map $\delta_{Y*}v^*\to(1\times v)^*\Gamma_{v*}v^!$ occurring in the definition of the map $\mathbf{a}_v$ in $\mathbf{A_3}$ (see (2.2.4)), namely the map

$$
\delta_{Y*}v^*\overset{\mathsf{ps}_*}{=\!=}i'_*\delta_{v*}v^*\xrightarrow{i'_*\bar\lambda}i'_*t_1^*v^!\xrightarrow{\theta^{-1}}(1\times v)^*\Gamma_{v*}v^!.
$$

We define $\xi\colon g_*u^*v^* \to r^*u^*v^!$ to be the natural composition

$$\xi\colon g_*u^*v^* \xrightarrow{\text{via }\theta^{-1}} w^*\delta_{v*}v^* \xrightarrow{\text{via }\bar\lambda} w^*t_1^*v^! \overset{\mathsf{ps}^*}{=\!=} r^*u^*v^!. \tag{6.5.2}$$

To dispose of $\mathbf{A_3}$ one sees, after expanding according to the definitions of the maps in play, that it's enough to show the next diagram commutes. In that diagram, unlabeled arrows represent maps induced by isomorphisms of the type θ^{-1}.

$$\begin{tikzcd}
\Gamma_u^*\Gamma_{u*}u^*v^* \arrow[r] \arrow[d, equal, "\text{via } \mathsf{ps}_*"'] & \Gamma_u^*(u\times 1)^*\delta_{Y*}v^* \arrow[r, equal, "\text{via } \mathsf{ps}^*"] \arrow[dd, equal, "\text{via } \mathsf{ps}_*"'] & u^*\delta_Y^*\delta_{Y*}v^* \arrow[dd, equal, "\text{via } \mathsf{ps}_*"] \\
\Gamma_u^*k_*g_*u^*v^* \arrow[d, "\text{via } \theta^{-1}"'] \arrow[r, phantom, "\mathbf{A_{31}}"] & & \\
\Gamma_u^*k_*w^*\delta_{v*}v^* \arrow[r] \arrow[d, "\text{via } \bar\lambda"'] & \Gamma_u^*(u\times 1)^*i'_*\delta_{v*}v^* \arrow[r, equal, "\text{via } \mathsf{ps}^*"] \arrow[d, "\text{via } \bar\lambda"'] & u^*\delta_Y^*i'_*\delta_{v*}v^* \arrow[d, "\text{via } \bar\lambda"] \\
\Gamma_u^*k_*w^*t_1^*v^! \arrow[r] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \Gamma_u^*(u\times 1)^*i'_*t_1^*v^! \arrow[r, equal, "\text{via } \mathsf{ps}^*"] \arrow[d, "\text{via } \theta^{-1}"'] & u^*\delta_Y^*i'_*t_1^*v^! \arrow[d, "\text{via } \theta^{-1}"] \\
\Gamma_u^*k_*r^*u^*v^! \arrow[d, "\text{via } \theta^{-1}"'] \arrow[r, phantom, "\mathbf{A_{32}}"] & \Gamma_u^*(u\times 1)^*(1\times v)^*\Gamma_{v*}v^! \arrow[r, equal, "\text{via } \mathsf{ps}^*"] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & u^*\delta_Y^*(1\times v)^*\Gamma_{v*}v^! \arrow[dd, equal, "\text{via } \mathsf{ps}^*"] \\
\Gamma_u^*(1\times v)^*\Gamma_{vu*}u^*v^! \arrow[r] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \Gamma_u^*(1\times v)^*(u\times 1)^*\Gamma_{v*}v^! \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \\
\Gamma_{vu}^*\,\Gamma_{vu*}u^*v^! \arrow[r] & \Gamma_{vu}^*\,(u\times 1)^*\Gamma_{v*}v^! \arrow[r, equal, "\text{via } \mathsf{ps}^*"'] & u^*\Gamma_v^*\Gamma_{v*}v^!
\end{tikzcd}$$

It is straightforward to see that the unlabeled subdiagrams commute.

Application of [L3, 3.7.2(ii)] to the composite diagram $\mathbf{f}\circ\mathbf{e}$ above—for which $i'\delta_v = \delta_Y$ and $kg = \Gamma_u$—yields commutativity of $\mathbf{A_{31}}$.

As for $\mathbf{A_{32}}$, we can ignore Γ_u^* and $v^!$, and expand the rest as follows, where $\tilde p = t_1w = ur$ is the projection from $X\times_Z Y$ onto Y, and unlabeled arrows represent maps induced by isomorphisms of type θ^{-1}:

$$\begin{tikzcd}
k_*w^*t_1^* \arrow[r] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & (u\times 1)^*i'_*t_1^* \arrow[r] & (u\times 1)^*(1\times v)^*\Gamma_{v*} \arrow[d, equal, "\mathsf{ps}^*"] \\
k_*\tilde p^* \arrow[rr] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & & (u\times v)^*\Gamma_{v*} \arrow[d, equal, "\mathsf{ps}^*"] \\
k_*r^*u^* \arrow[r] & (1\times v)^*\Gamma_{vu*}u^* \arrow[r] & (1\times v)^*(u\times 1)^*\Gamma_{v*}
\end{tikzcd} \tag{6.5.3}$$

Application of [L3, 3.7.2(iii)] to the composite diagram $\mathbf{g}\circ\mathbf{f}$ above and to

$$\begin{tikzcd}
X\times_Z Y \arrow[r, "r"] \arrow[d, "k"'] & X \arrow[r, "u"] \arrow[d, "\Gamma_{vu}"] & Y \arrow[d, "\Gamma_v"] \\
X\times Y \arrow[r, "1\times v"'] & X\times Z \arrow[r, "u\times 1"'] & Y\times Z
\end{tikzcd}$$

gives commutativity of the top (respectively bottom) half of (6.5.3), whence the commutativity of $\mathbf{A_{32}}$.

Thus $\mathbf{A_3}$ commutes.

It remains to consider $\mathbf{A_2}$. Work with the fiber-square diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "g"'] & Y \arrow[d, "\delta_v"] & \\
X\times_Z Y \arrow[r, "w:=u\times_Z \mathrm{id}_Y"] \arrow[d, "r"'] & Y\times_Z Y \arrow[r, "t_2"] \arrow[d, "t_1"] & Y \arrow[d, "v"] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z
\end{tikzcd} \tag{6.5.4}$$

where r and t_1 are the canonical projections onto the first factor, and t_2 is the canonical projection onto the second factor. Set $\tau := t_2 w$.

Using the definition of (2.1.3.2), and the equalities $u=\tau g$, $t_2\delta_v=\mathrm{id}$, one sees that for commutativity of $\mathbf{A_2}$ it suffices to prove commutativity of the following expanded diagram (6.5.5), in which $\mathcal{O}:=\mathcal{O}_{Y\times_Z Y}$, the map $\bar\lambda$ is as in (6.5.1), the unlabeled maps are isomorphisms coming out of (1.3.5) or (2.1.6.1) or (1.4.1) (see 1.4.4), and the isomorphisms denoted by "$=\!=$" are induced by ps^* or $\mathsf{ps}^!$, or have other obvious interpretations.

The commutativity of the unlabeled subdiagrams is clear; that of $\mathbf{A_{25}}$ results from the definition of the isomorphism $w^!t_2^! \overset{\mathsf{ps}^!}{=\!=} (t_2w)^!$, see [AJL, (5.7.5)]; of $\mathbf{A_{26}}$ from the horizontal transitivity of B (see §2.1.6); and of $\mathbf{A_{27}}$ from the definition of B [AJL, 5.8.4].

Commutativity of $\mathbf{A_{21}}$ is the same as commutativity of the following diagram of isomorphisms coming from (1.3.5), in which $E=r^*u^!\mathcal{O}_Y$ and $F=\tau^*v^*G$ $(G\in\mathbf{D}_Z)$.

(6.5.5)

$$
\begin{tikzcd}[column sep=small]
g_*u^!\mathcal{O}_Y\otimes\tau^*v^* \arrow[ddddddd, equal] \arrow[dr, equal, "\text{via } \mathsf{ps}^*"] & & & g_*(u^!\mathcal{O}_Y\otimes g^*\tau^*v^*) \arrow[lll] \arrow[rr, equal] \arrow[d, equal] & & g_*(u^!\mathcal{O}_Y\otimes u^*v^*) \arrow[dd, equal] \\
& g_*g^*r^*u^!\mathcal{O}_Y\otimes\tau^*v^* \arrow[rr] \arrow[d, equal] & & g_*(g^*r^*u^!\mathcal{O}_Y\otimes g^*\tau^*v^*) \arrow[d] \arrow[drr, equal] & & \\
& g_*(g^*r^*u^!\mathcal{O}_Y\otimes\mathcal{O}_X)\otimes\tau^*v^* \arrow[d] & \mathbf{A_{21}} & r^*u^!\mathcal{O}_Y\otimes g_*g^*\tau^*v^* \arrow[d, equal] \arrow[drr, equal] & & g_*(g^*r^*u^!\mathcal{O}_Y\otimes u^*v^*) \arrow[d] \\
& r^*u^!\mathcal{O}_Y\otimes g_*\mathcal{O}_X\otimes\tau^*v^* \arrow[rr] \arrow[d, equal] & & r^*u^!\mathcal{O}_Y\otimes g_*(\mathcal{O}_X\otimes g^*\tau^*v^*) & & r^*u^!\mathcal{O}_Y\otimes g_*u^*v^* \arrow[dddd] \\
\mathbf{A_{23}} & r^*u^!\mathcal{O}_Y\otimes g_*u^*\mathcal{O}_Y\otimes\tau^*v^* \arrow[d] & & \mathbf{A_{22}} & & \\
& r^*u^!\mathcal{O}_Y\otimes w^*\delta_{v*}\mathcal{O}_Y\otimes\tau^*v^* \arrow[rr] \arrow[d] & & r^*u^!\mathcal{O}_Y\otimes w^*(\delta_{v*}\mathcal{O}_Y\otimes t_2^*v^*) \arrow[dr] & & \\
& w^!\mathcal{O}\otimes w^*\delta_{v*}\mathcal{O}_Y\otimes\tau^*v^* \arrow[dr] \arrow[dd, equal] & & & r^*u^!\mathcal{O}_Y\otimes w^*\delta_{v*}(\mathcal{O}_Y\otimes\delta_v^*t_2^*v^*) \arrow[dd] \arrow[dr, equal] & \\
g_*(\tau g)^!\mathcal{O}_Y\otimes\tau^*v^* \arrow[dd, equal] & & w^!\mathcal{O}\otimes w^*(\delta_{v*}\mathcal{O}_Y\otimes t_2^*v^*) \arrow[drr] & & & r^*u^!\mathcal{O}_Y\otimes w^*\delta_{v*}v^* \arrow[ddddddd, "\text{via } \bar\lambda"] \\
& w^!\delta_{v*}\mathcal{O}_Y\otimes\tau^*v^* \arrow[dr, "\text{via } \bar\lambda"'] & & & w^!\mathcal{O}\otimes w^*\delta_{v*}(\mathcal{O}_Y\otimes\delta_v^*t_2^*v^*) \arrow[dd, equal] & \\
g_*g_+^!\tau^!\mathcal{O}_Y\otimes\tau^*v^* \arrow[dd, "(2.1.2)(\mathrm{i})"'] & & w^!t_1^*v^!\mathcal{O}_Z\otimes\tau^*v^* \arrow[dl] & \mathbf{A_{24}} & & \\
& w^!t_2^!\mathcal{O}_Y\otimes\tau^*v^* \arrow[r, equal] \arrow[dl, equal] & w^!t_2^!\mathcal{O}_Y\otimes w^*t_2^*v^* \arrow[d] & & w^!\mathcal{O}\otimes w^*\delta_{v*}v^* \arrow[dl, equal] \arrow[dddd, "\text{via } \bar\lambda"] & \\
\tau^!\mathcal{O}_Y\otimes\tau^*v^* \arrow[dd, equal] & & w^!\mathcal{O}\otimes w^*(t_2^!\mathcal{O}_Y\otimes t_2^*v^*) \arrow[ddl, equal] & & & \\
& \mathbf{A_{25}} & & w^!\delta_{v*}v^* \arrow[ddl, "\text{via } \bar\lambda"'] & & \\
\tau^!v^* \arrow[d, equal] & w^!(t_2^!\mathcal{O}_Y\otimes t_2^*v^*) \arrow[d, equal] & & & & \\
(t_2w)^!v^* \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\mathbf{A_{26}}"] & w^!t_2^!v^* & w^!t_1^*v^! \arrow[l] \arrow[rr, equal] \arrow[d] \arrow[drr, phantom, "\mathbf{A_{27}}"] & & w^!\mathcal{O}\otimes w^*t_1^*v^! \arrow[r] & r^*u^!\mathcal{O}_Y\otimes w^*t_1^*v^! \arrow[d, equal] \\
r^*(vu)^! \arrow[rr, equal] & & r^*u^!v^! \arrow[rr, equal] & & r^*(u^!\mathcal{O}_Y\otimes u^*v^!) \arrow[r] & r^*u^!\mathcal{O}_Y\otimes r^*u^*v^!
\end{tikzcd}
$$

$$
\begin{tikzcd}
g_*g^*E\otimes F \arrow[rr] \arrow[d, equal] & & g_*(g^*E\otimes g^*F) \arrow[dd] \\
g_*(g^*E\otimes\mathcal{O}_X)\otimes F \arrow[r] \arrow[d] & g_*(g^*E\otimes\mathcal{O}_X\otimes g^*F) \arrow[ur, equal] \arrow[d] & \\
E\otimes g_*\mathcal{O}_X\otimes F \arrow[r] \arrow[ur, phantom, "\mathbf{A_{211}}"] & E\otimes g_*(\mathcal{O}_X\otimes g^*F) \arrow[r, equal] & E\otimes g_*g^*F
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams results from functoriality of the isomorphisms in (1.3.5).

Commutativity of $\mathbf{A_{211}}$ is a special case of that of the natural diagram, for any scheme-map $f\colon X\to Y$ and $A\in\mathbf{D}(Y)$, $B\in\mathbf{D}(X)$, $C\in\mathbf{D}(Y)$:

$$
\begin{tikzcd}
f_*(f^*A\otimes_X B)\otimes_Y C \arrow[rrrr] & & & & f_*\big(f^*A\otimes_X(B\otimes_X f^*C)\big) \\
C\otimes_Y f_*(f^*A\otimes_X B) \arrow[u] \arrow[rr] & & f_*(f^*C\otimes_X f^*A\otimes_X B) \arrow[rr] \arrow[urr] \arrow[u, phantom, "1"] & \arrow[u, phantom, "2"] & f_*\big(f^*A\otimes_X(f^*C\otimes_X B)\big) \arrow[u] \\
& f_*(f^*(C\otimes_Y A)\otimes_X B) \arrow[rr] \arrow[ur] \arrow[u, phantom, "3"] & & f_*(f^*(A\otimes_Y C)\otimes_X B) \arrow[ur] \arrow[u, phantom, "4"] & \\
C\otimes_Y A\otimes_Y f_*B \arrow[uu] \arrow[rr] \arrow[ur] & & A\otimes_Y C\otimes_Y f_*B \arrow[rr] \arrow[ur] \arrow[u, phantom, "5"] & \arrow[u, phantom, "6"] & A\otimes_Y f_*(f^*C\otimes_X B) \arrow[uu] \\
A\otimes_Y f_*B\otimes_Y C \arrow[u] \arrow[rrrr] \arrow[urr] & \arrow[u, phantom, "7"] & & \arrow[u, phantom, "8"] & A\otimes_Y f_*(B\otimes_X f^*C) \arrow[u]
\end{tikzcd}
$$

Here commutativity of subdiagram 5 results from functoriality of the projection isomorphisms (1.3.5), that of 4 results from the *dual* [L3, 3.4.5] of the second diagram in [L3, (3.4.2.2)], that of 3 and 6 from [L3, 3.4.7 (iv)], that of 1 and 8 from [L3, 3.4.6.1], and that of 2 and 7 from the third diagram in [L3, (3.4.1.1)].

Thus, $\mathbf{A_{21}}$ commutes.

For subdiagram $\mathbf{A_{22}}$, it is enough to check commutativity of the following natural diagram, where $\delta := \delta_v$ and $t := t_2$ (so that $\tau = tw$ and $t\delta = \mathrm{id}_Y$):

$$
\begin{tikzcd}
g_*\mathcal{O}_X \otimes \tau^* \arrow[r] \arrow[d, equal] & g_*(\mathcal{O}_X \otimes g^*\tau^*) \arrow[r, equal] \arrow[d, equal] & g_*g^*\tau^* \arrow[r, equal] & g_*u^* \arrow[d, equal] \\
g_*u^*\mathcal{O}_Y \otimes \tau^* \arrow[d] \arrow[r, phantom, "\mathbf{A_{221}}"] & g_*(u^*\mathcal{O}_Y \otimes u^*\delta^*t^*) \arrow[r, equal] \arrow[d] \arrow[r, phantom, shift right=4ex, "\mathbf{A_{222}}"] & g_*(\mathcal{O}_X \otimes u^*\delta^*t^*) \arrow[r, equal] \arrow[d, equal] & g_*(\mathcal{O}_X \otimes u^*) \arrow[d, equal] \\
w^*\delta_*\mathcal{O}_Y \otimes \tau^* \arrow[d] & g_*u^*(\mathcal{O}_Y \otimes \delta^*t^*) \arrow[r, equal] \arrow[d] & g_*u^*\delta^*t^* \arrow[r, equal] & g_*u^* \arrow[d] \\
w^*(\delta_*\mathcal{O}_Y \otimes t^*) \arrow[r] & w^*\delta_*(\mathcal{O}_Y \otimes \delta^*t^*) \arrow[r, equal] & w^*\delta_*\delta^*t^* \arrow[r, equal] & w^*\delta_*
\end{tikzcd}
$$

Commutativity of $\mathbf{A_{221}}$ follows from [L3, 3.7.3], with $(f, g, f', g', P, Q) := (\delta, w, g, u, t^*G, \mathcal{O}_Y)$ $(G \in \mathbf{D}(Y))$, except that there the factors in the tensor products need to be switched, as do the two projection maps defined above in (1.3.5)—all of which is made permissible by [L3, 3.4.6.1] and the *dual* ([L3, 3.4.5]) of the second diagram in [L3, (3.4.2.2)]. Commutativity of $\mathbf{A_{222}}$ results from the dual of the first diagram in [L3, (3.4.2.2)]. Commutativity of the unlabeled diagrams is easy to check.

Thus $\mathbf{A_{22}}$ commutes.

Next we expand $\mathbf{A_{24}}$—again dropping v^*, setting $\delta := \delta_v$ and $t := t_2$, and substituting w^*t^* for τ^*. The map

$$
\bar\lambda_0 \colon \delta_* = \delta_*(t\delta)^! \xrightarrow{\int^t_{\delta_v}} t^! \tag{6.5.6}
$$

is as in the definition (6.5.1) of $\bar\lambda$.

$$
\begin{tikzcd}
w^!\mathcal{O} \otimes w^*\delta_*\mathcal{O}_Y \otimes w^*t^* \arrow[r] \arrow[d, equal] & w^!\mathcal{O} \otimes w^*(\delta_*\mathcal{O}_Y \otimes t^*) \arrow[rr] \arrow[d, equal] \arrow[dddl, "\text{via } \bar\lambda_0"] & & w^!\mathcal{O} \otimes w^*\delta_*(\mathcal{O}_Y \otimes \delta^*t^*) \arrow[dd, equal] \arrow[dddl, equal] \\
w^!\delta_*\mathcal{O}_Y \otimes w^*t^* \arrow[dd, "\text{via } \bar\lambda_0"'] & w^!(\delta_*\mathcal{O}_Y \otimes t^*) \arrow[dddd, "\text{via } \bar\lambda_0"'] \arrow[ddr] & & \\
 & & & w^!\mathcal{O} \otimes w^*\delta_* \arrow[dd, equal] \\
w^!t^!\mathcal{O}_Y \otimes w^*t^* \arrow[d] & & w^!\delta_*(\mathcal{O}_Y \otimes \delta^*t^*) \arrow[dr, equal] \arrow[d, phantom, "\mathbf{A_{241}}"] & \\
w^!\mathcal{O} \otimes w^*(t^!\mathcal{O}_Y \otimes t^*) \arrow[r] & w^!(t^!\mathcal{O}_Y \otimes t^*) \arrow[r, equal] & w^!t^! & w^!\delta_* \arrow[l, "\text{via } \bar\lambda_0"]
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams is easy to verify.

For commutativity $\mathbf{A_{241}}$, it is enough, by definition of the maps involved, to verify commutativity of the natural diagram (in which the unlabeled maps are the obvious ones):

$$
\begin{tikzcd}
\delta_*\mathcal{O}_Y\otimes t^* \arrow[r, "(1.3.5)"] \arrow[d, equal, "\text{via } \mathsf{ps}^!"'] & \delta_*(\mathcal{O}_Y\otimes\delta^*t^*) \arrow[r, equal] \arrow[d, equal, "\text{via } \mathsf{ps}^!"'] & \delta_* \arrow[r, equal] \arrow[d, phantom, "\mathbf{A_{2411}}"] & \delta_*(t\delta)^! \arrow[d, equal, "\text{via } \mathsf{ps}^!"', "\text{and } \mathsf{ps}^*"] \\
\delta_*(t\delta)^!\mathcal{O}_Y\otimes t^* \arrow[r, "(1.3.5)"] \arrow[d, equal, "\text{via } \mathsf{ps}^!"', "\text{and } \mathsf{ps}^*"] \arrow[drrr, equal] & \delta_*((t\delta)^!\mathcal{O}_Y\otimes\delta^*t^*) \arrow[rr, equal] \arrow[d, phantom, "\mathbf{A_{2412}}"] & & \delta_*(\delta^!t^!\mathcal{O}_Y\otimes\delta^*t^*) \arrow[d, "(1.3.5)"] \\
\delta_*(\delta^!t^!\mathcal{O}_Y\otimes\delta^*t^*\mathcal{O}_Y)\otimes t^* \arrow[rrr, equal] \arrow[d, "\text{via } (1.3.5)"'] & \mathbf{A_{2413}} & & \delta_*\delta^!t^!\mathcal{O}_Y\otimes t^* \arrow[d] \\
\delta_*\delta^!t^!\mathcal{O}_Y\otimes t^*\mathcal{O}_Y\otimes t^* \arrow[r] \arrow[urrr, equal] & t^!\mathcal{O}_Y\otimes t^*\mathcal{O}_Y\otimes t^* \arrow[rr, equal] & & t^!\mathcal{O}_Y\otimes t^*
\end{tikzcd}
$$

It is evident that the unlabeled diagrams commute.

Subdiagram $\mathbf{A_{2412}}$ (without $\otimes t^*$ and without δ_*) expands as

$$
\begin{tikzcd}
(t\delta)^!\mathcal{O}_Y \arrow[r, equal, "\mathsf{ps}^!"] \arrow[d, equal] & \delta^!t^!\mathcal{O}_Y \arrow[dd, equal] \\
(t\delta)^!\mathcal{O}_Y\otimes(t\delta)^*\mathcal{O}_Y \arrow[d, equal, "\text{via } \mathsf{ps}^!"', "\text{and } \mathsf{ps}^*"] \arrow[dr, equal, "\text{via } \mathsf{ps}^!"] & \\
\delta^!t^!\mathcal{O}_Y\otimes\delta^*t^*\mathcal{O}_Y \arrow[r, equal, "\text{via } \mathsf{ps}^*"'] & \delta^!t^!\mathcal{O}_Y\otimes(t\delta)^*\mathcal{O}_Y
\end{tikzcd}
$$

This expanded diagram is easily seen to commute.

Commutativity of $\mathbf{A_{2413}}$ results from [L3, 3.4.7(iii)].

Subdiagram $\mathbf{A_{2411}}$ (without δ_*) expands as follows (with id the identity functor on $\mathbf{D}_Y$):

$$
\begin{tikzcd}
\mathcal{O}_Y\otimes\delta^*t^* \arrow[r, equal, "\text{via } \mathsf{ps}^*"] \arrow[dd, equal] & \mathcal{O}_Y\otimes\mathrm{id} \arrow[r, equal] \arrow[d, equal] & \mathcal{O}_Y\otimes(t\delta)^! \arrow[r, equal] \arrow[d, phantom, "\mathbf{A_{2414}}"] & (t\delta)^! \arrow[d, equal] \\
& (t\delta)^!\mathcal{O}_Y\otimes\mathrm{id} \arrow[rr, equal] & & (t\delta)^!\mathcal{O}_Y\otimes(t\delta)^* \arrow[d, equal, "\text{via } \mathsf{ps}^!"', "\text{and } \mathsf{ps}^*"] \\
(t\delta)^!\mathcal{O}_Y\otimes\delta^*t^* \arrow[rrr, equal, "\text{via } \mathsf{ps}^!"'] \arrow[ur, equal, "\text{via } \mathsf{ps}^*"] & & & \delta^!t^!\mathcal{O}_Y\otimes\delta^*t^*
\end{tikzcd}
$$

Subdiagram $\mathbf{A_{2414}}$ commutes because all its maps are identity maps—see paragraph following (2.1.2.1). The rest is clear.

It remains to show commutativity of $\mathbf{A_{23}}$, for which one can omit "$\otimes\tau^*v^*$."

Before proceeding, recall from §2.1.2 that for perfect maps the restriction of $(-)^!$ to $\mathbf{D}^+_{\mathsf{qc}}$ is *pseudofunctorially* isomorphic to $(-)^!_+$. Moreover, this isomorphism "respects flat base change." More specifically, referring to (2.1.6) and [L3, Exercise 4.9.3(c)] one finds that the following diagram commutes:

$$
\begin{tikzcd}
r^*u^!\mathcal{O}_Y \arrow[r, "\mathsf{B}"] \arrow[d, "\simeq"', "(2.1.2.2)"] & w^!t_1^*\mathcal{O}_Y = w^!\mathcal{O} \arrow[d, "(2.1.2.2)"', "\simeq"] \\
r^*u^!_+\mathcal{O}_Y \arrow[r, "\bar{\mathsf{B}}"'] & w^!_+t_1^*\mathcal{O}_Y = w^!_+\mathcal{O}
\end{tikzcd}
$$

Next, the composed map

$$g_*u^!\mathcal{O}_Y \overset{\text{via}\,\mathsf{ps}^*}{=\!=} g_*g^*r^*u^!\mathcal{O}_Y = g_*(g^*r^*u^!\mathcal{O}_Y\otimes\mathcal{O}_X)$$

at the top of $\mathbf{A_{23}}$ is the same as

$$g_*u^!_+\mathcal{O}_Y = g_*(u^!_+\mathcal{O}_Y\otimes u^*\mathcal{O}_Y) \overset{\text{via}\,\mathsf{ps}^*}{=\!=} g_*(g^*r^*u^!_+\mathcal{O}_Y\otimes\mathcal{O}_X).$$

Also, using that $\delta_*(\delta^!_+t^!\mathcal{O}_Y\otimes\delta^*t^*\mathcal{O}_Y)\to\delta_*\delta^!_+t^!\mathcal{O}_Y\otimes t^*\mathcal{O}_Y$ from (1.3.5) is the identity map (see [L3, 3.4.7(iii)], and the remarks following (2.1.2.1)), one finds that the map $\bar\lambda_0(\mathcal{O}_Y)$ (see (6.5.6)), that forms part of the definition of the map $\bar\lambda(\mathcal{O}_Z)$ near the bottom of $\mathbf{A_{23}}$, factors as

$$\delta_*\mathcal{O}_Y = \delta_*(t\delta)^!_+\mathcal{O}_Y \xrightarrow{\delta_*\mathsf{ps}^!_+} \delta_*\delta^!_+t^!_+\mathcal{O}_Y \xrightarrow{\int_+} t^!_+\mathcal{O}_Y,$$

with $\int_+$ the unit map for the adjunction $(-)_*\dashv(-)^!_+$ in 2.1.2(i).

Hence, with ϖ the associated unit map, $\delta := \delta_v$, $t := t_2$, θ as in (1.4.1), and recalling that $f^!\mathcal{O}_Y = f^!_+\mathcal{O}_Y$ for any flat S-map f, one can expand $\mathbf{A_{23}}$ as

$$
\begin{tikzcd}
g_*(\tau g)^!_+\mathcal{O}_Y \arrow[rr, equal] \arrow[dd, "g_*\mathsf{ps}^!_+"'] & & g_*u^!_+\mathcal{O}_Y \arrow[rr, equal] \arrow[dd, "\text{via}\,\varpi"] \arrow[dddl, equal, "\text{via}\,\mathsf{ps}^!_+"] & & g_*(u^!_+\mathcal{O}_Y\otimes u^*\mathcal{O}_Y) \arrow[dd, equal, "\text{via}\,\mathsf{ps}^*"] \\
& g_*g^!_+w^!_+t^!_+\mathcal{O}_Y \arrow[dd, equal, "\text{via}\,\mathsf{ps}^!_+"'] & & & \\
g_*g^!_+\tau^!_+\mathcal{O}_Y \arrow[ur, equal, "\text{via}\,\mathsf{ps}^!_+"] \arrow[ddd, "\int_+"'] & & g_*u^!_+\delta^!_+\delta_*\mathcal{O}_Y \arrow[dd, equal, "\text{via}\,\mathsf{ps}^!_+"] \arrow[dddl, equal, "\text{via}\,\mathsf{ps}^!_+"] & & g_*(g^*r^*u^!_+\mathcal{O}_Y\otimes u^*\mathcal{O}_Y) \arrow[dd, "(1.3.5)"] \\
& g_*u^!_+\delta^!_+t^!_+\mathcal{O}_Y \arrow[dd, shift left, "\text{via}\,\varpi"] & & & \\
& & g_*g^!_+w^!_+\delta_*\mathcal{O}_Y \arrow[dddl, equal, "\text{via}\,\mathsf{ps}^!_+"] \arrow[dddd, "\int_+"] & & r^*u^!_+\mathcal{O}_Y\otimes g_*u^*\mathcal{O}_Y \arrow[dd, "\text{via}\,\theta^{-1}"] \\
\tau^!_+\mathcal{O}_Y \arrow[ddd, equal, "\mathsf{ps}^!_+"'] & g_*u^!_+\delta^!_+\delta_*\delta^!_+t^!_+\mathcal{O}_Y \arrow[uu, shift left, "\text{via}\,\int_+"] \arrow[dd, equal, "\text{via}\,\mathsf{ps}^!_+"'] & & \mathbf{A_{231}} & \\
& & & & r^*u^!_+\mathcal{O}_Y\otimes w^*\delta_*\mathcal{O}_Y \arrow[dd, "\text{via}\,(2.1.6.1)"] \\
& g_*g^!_+w^!_+\delta_*\delta^!_+t^!_+\mathcal{O}_Y \arrow[d, "\int_+"'] \arrow[uuuuuu, bend left=60, "\text{via}\,\int_+"] & & & \\
w^!_+t^!_+\mathcal{O}_Y & w^!_+\delta_*\delta^!_+t^!_+\mathcal{O}_Y \arrow[l, "w^!\int_+"] \arrow[r, equal, "\text{via}\,\mathsf{ps}^!_+"'] & w^!_+\delta_*\mathcal{O}_Y \arrow[rr, equal] & & w^!_+\mathcal{O}\otimes w^*\delta_*\mathcal{O}_Y
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams is readily checked. Also, the composition

$$\delta_+^! \xrightarrow{\text{via } \varpi} \delta_+^!\delta_*\delta_+^! \xrightarrow{\int_+} \delta_+^!$$

is the identity map. Diagram chasing shows then that we need "only" check that $\mathbf{A_{231}}$ commutes.[3] For this purpose we can even drop the final $\mathcal{O}_Y$ at each vertex, regarding what's left as a diagram of functors defined on $\mathbf{D}^+_{\mathsf{qc}}$.

Henceforth we will use the symbol "π" to refer to either of the projection isomorphisms in (1.3.5), or their inverses.

One last preparatory remark: the isomorphism (2.1.2.2) is a special case of a canonical functorial map, defined in [Nk, 5.8][4] *for any* S*-map* $f\colon X \to Y$,

$$\kappa(F)\colon f_+^!\mathcal{O}_Y \otimes f^*F \to f_+^!F \qquad \big(F \in \mathbf{D}^+_{\mathsf{qc}}(Y)\big).$$

If f is proper, $\kappa(F)$ is adjoint to the composition

$$f_*(f_+^!\mathcal{O}_Y \otimes f^*F) \xrightarrow{\pi} f_*f_+^!\mathcal{O}_Y \otimes F \xrightarrow{\text{via } \int_+} \mathcal{O}_Y \otimes F = F.$$

If f is essentially étale, so that $f_+^! = f^*$, then $\kappa(F)$ is the identity map of f^*F.

One checks that $\kappa(\mathcal{O}_Y)$ is the identity map.

It should now be clear that the next Proposition will complete the proof.

Proposition 6.5.7. *Not assuming u or w flat, consider any commutative* S*-diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{e}"] & Y \arrow[d, "\delta"] \\ X' \arrow[r, "w"] \arrow[d, "r"'] \arrow[dr, phantom, "\mathsf{e}'"] & Y' \arrow[d, "t"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

with $rg = \mathrm{id}_X$, $t\delta = \mathrm{id}_Y$, e' *(hence* e*) a fiber square, and t (hence r) flat (cf.* (6.5.4)*).*

---

[3]For the authors, this was the most elusive point in the present proof of Theorem 3.1.

[4]where qct should be qc

The following diagram of $\mathbf{D}_{\mathsf{qc}}^{+}$*-valued functors commutes.*

$$
\begin{tikzcd}
g_*u^!_+ \arrow[d, "\text{via }\varpi"'] & g_*(u^!_+\mathcal{O}_Y\otimes u^*) \arrow[l, "g_*\kappa"'] \arrow[d, equal, "\text{via }\mathsf{ps}^*"] \\
g_*u^!_+\delta^!_+\delta_* \arrow[d, equal, "\text{via }\mathsf{ps}^!_+"'] & g_*(g^*r^*u^!_+\mathcal{O}_Y\otimes u^*) \\
g_*g^!_+w^!_+\delta_* \arrow[dd, "\int_+"'] & r^*u^!_+\mathcal{O}_Y\otimes g_*u^* \arrow[u, "\pi"'] \\
 & r^*u^!_+\mathcal{O}_Y\otimes w^*\delta_* \arrow[u, "\text{via }\theta"'] \arrow[d, "\simeq"', "\text{via (2.1.6.1)}"] \\
w^!_+\delta_* & w^!_+\mathcal{O}_{Y'}\otimes w^*\delta_* \arrow[l, "\kappa(\delta_*)"]
\end{tikzcd}
\tag{6.5.7.1}
$$

Proof. We deal only with the pseudofunctor $(-)^!_+$, and not with $(-)^!$, so to reduce notational clutter we will denote $f^!_+$ by $f^!$, for any S-map f. Likewise, we will denote $\int_+$ simply by $\int$.

We will prove 6.5.7 when u (hence w) is essentially étale (see §1.1), and then when u (hence w) is proper. Then finally we will use the fact that any S-map is of the form (proper)$\circ$(essentially étale) [Nk, 4.1 and 2.7] to establish the general case.

Let us assume then, to begin, that u and w are essentially étale, so that $u^! = u^*$ and $w^! = w^*$. Note that since g and δ have left inverses, they are closed immersions [Gr1, p. 278, (5.2.4)].

In the next diagram, subrectangle 4 is as in (6.5.7.1); the "base change" map

$$\bar{\mathsf{B}} := \bar{\mathsf{B}}_{\mathbf{e}}\colon u^*\delta^! \to g^!w^*$$

is defined to be adjoint to the natural composition

$$g_*u^*\delta^! \xrightarrow{\theta^{-1}} w^*\delta_*\delta^! \xrightarrow{\int} w^*;$$

and the unlabeled maps are the obvious ones. (In particular, three maps in the rightmost column are *identity maps,* see the last paragraph in §1.3.)

$$
\begin{tikzcd}
g_*u^* \arrow[rrrr, equal] \arrow[dr, equal] \arrow[dd] & & & & g_*(u^*\mathcal{O}_Y \otimes u^*) \arrow[dd, equal] \\
& g_*u^! \arrow[d] & \textcircled{2} & g_*(u^!\mathcal{O}_Y \otimes u^*) \arrow[ll, "g_*\kappa"'] \arrow[ur, equal] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \\
g_*u^*\delta^!\delta_* \arrow[r, equal] \arrow[d, "\text{via } \bar{\mathsf{B}}"'] & g_*u^!\delta^!\delta_* \arrow[d, equal, "\text{via } \mathsf{ps}^!_+"] & & g_*(g^*r^*u^!\mathcal{O}_Y \otimes u^*) \arrow[r, equal] & g_*(g^*r^*u^*\mathcal{O}_Y \otimes u^*) \arrow[d, equal] \\
g_*g^!w^*\delta_* \arrow[r, equal] \arrow[ddd] & g_*g^!w^!\delta_* \arrow[dd] & \textcircled{4} & r^*u^!\mathcal{O}_Y \otimes g_*u^* \arrow[u] \arrow[r, equal] & r^*u^*\mathcal{O}_Y \otimes g_*u^* \\
& & & r^*u^!\mathcal{O}_Y \otimes w^*\delta_* \arrow[u] \arrow[r, equal] \arrow[d, "\text{via } (2.1.6.1)"'] & r^*u^*\mathcal{O}_Y \otimes w^*\delta_* \arrow[u, "\theta"'] \arrow[dd, equal] \\
& w^!\delta_* \arrow[dl, equal] & \textcircled{2}' & w^!\mathcal{O}_{Y'} \otimes w^*\delta_* \arrow[ll, "\kappa(\delta_*)"'] \arrow[dr, equal] & \textcircled{3} \\
w^*\delta_* \arrow[rrrr, equal] & & & & w^*\mathcal{O}_{Y'} \otimes w^*\delta_*
\end{tikzcd}
$$

It is clear that the unlabeled subdiagrams commute. Subdiagram ① commutes by 2.1.2(iii), ② and ②′ commute by [L3, definition of 4.9.1.1], and ③ commutes, in view of 2.1.2(iii), by the description of $r^*u^!\mathcal{O}_Y \to w^!\mathcal{O}_{Y'}$ in 2.1.6 with $(f, u, g, v) := (u, t, w, r)$—all valid for *essentially* étale maps. So to achieve our goal of proving that ④ commutes, we need only do the same for the outer one.

Proving commutativity of the outer rectangle means showing that the left column composes to θ^{-1}, that is, the next diagram commutes:

$$
\begin{tikzcd}
g_*u^*\delta^!\delta_* \arrow[d, "\text{via } \bar{\mathsf{B}}"'] \arrow[dr, "\theta^{-1}"] & & g_*u^* \arrow[ll, "\text{via } \varpi"'] \arrow[dd, "\theta^{-1}"] \\
g_*g^!w^*\delta_* \arrow[d] \arrow[r, phantom, "\textcircled{5}"] & w^*\delta_*\delta^!\delta_* \arrow[dl, "\text{via } \int"] & \\
w^*\delta_* \arrow[rr, equal] & & w^*\delta_* \arrow[ul, "\text{via } \varpi"']
\end{tikzcd}
$$

Here, subdiagram ⑤ commutes by the definition of $\bar{\mathsf{B}}$, and commutativity of the other two subdiagrams is obvious.

Thus Proposition 6.5.7 holds when u and w are essentially étale.

Suppose next that u and w are *proper.*

It will suffice to show commutativity of the adjoint of diagram (6.5.7.1), namely subdiagram 6 in

$$
\begin{tikzcd}
\delta_*u_*u^! \arrow[r, equals, "\mathsf{ps}_*"] \arrow[dddddr, "\text{via}\int"'] & w_*g_*u^! \arrow[d, "\text{via}\,\varpi"] & & & w_*g_*(u^!\mathcal{O}_Y\otimes u^*) \arrow[lll, "w_*g_*\kappa"'] \arrow[d, equals, "\text{via}\,\mathsf{ps}^*"] \\
& w_*g_*u^!\delta^!\delta_* \arrow[d, equals, "\text{via}\,\mathsf{ps}^!_+"] & & & w_*g_*(g^*r^*u^!\mathcal{O}_Y\otimes u^*) \\
& w_*g_*g^!w^!\delta_* \arrow[d, "\text{via}\int"] & 6 & & w_*(r^*u^!\mathcal{O}_Y\otimes g_*u^*) \arrow[u, "\text{via}\,\pi"'] \\
& w_*w^!\delta_* \arrow[dd, "\int"] & & & w_*(r^*u^!\mathcal{O}_Y\otimes w^*\delta_*) \arrow[u, "\text{via}\,\theta"'] \arrow[dd, "\text{via (2.1.6.1)}"] \\
& & \mathcal{O}_{Y'}\otimes\delta_* \arrow[dl, equals] & w_*w^!\mathcal{O}_{Y'}\otimes\delta_* \arrow[l, "\text{via}\int"] & \\
& \delta_* & & & w_*(w^!\mathcal{O}_{Y'}\otimes w^*\delta_*) \arrow[ul, "\pi"]
\end{tikzcd}
$$

The subtriangle commutes by [L3, 3.10.4(c)], applied to the map denoted there by $\phi\colon g_*u^! \to u^!f_*$. (Recall that over proper maps the pseudofunctor $(-)^! := (-)^!_+$ is right-adjoint to $(-)_*$, and so may be identified with the pseudofunctor $(-)^\times$ in [L3].) So it's enough to show commutativity of the outer border.

Fill in that border as follows (with id the identity functor on $\mathbf{D}^+_{\mathsf{qc}}(Y)$). In this diagram, the maps α, β and γ are the respective composites

$$\alpha\colon g^*w^*\delta_* \overset{\mathsf{ps}^*}{=\!=} u^*\delta^*\delta_* \xrightarrow{u^*\epsilon_\delta} u^*.$$

$$\beta\colon (\delta_*-\otimes\delta_*-) \xrightarrow{\pi} \delta_*(-\otimes\delta^*\delta_*-) \xrightarrow{\text{via}\,\epsilon_\delta} \delta_*(-\otimes-).$$

$$\gamma\colon r^* \xrightarrow{\eta_g} g_*g^*r^* \overset{g_*\mathsf{ps}^*}{=\!=\!=} g_*.$$

$$
\begin{tikzcd}
\delta_*u_*u^! \arrow[r, equal, "\mathsf{ps}_*"] \arrow[dd, "\text{via}\int"'] & w_*g_*u^! & w_*g_*(u^!\mathcal{O}_Y\otimes u^*) \arrow[l, "w_*g_*\kappa"'] \arrow[r, equal] & w_*g_*(u^!\mathcal{O}_Y\otimes u^*) \arrow[d, equal, "\text{via }\mathsf{ps}^*"] \\
 & \delta_*u_*(u^!\mathcal{O}_Y\otimes u^*) \arrow[ul, "\delta_*u_*\kappa"'] \arrow[ur, equal, "\mathsf{ps}_*"] & w_*g_*(u^!\mathcal{O}_Y\otimes g^*w^*\delta_*) \arrow[u, "\text{via }\alpha"'] & w_*g_*(g^*r^*u^!\mathcal{O}_Y\otimes u^*) \\
\delta_* \arrow[dd, equal] & \delta_*(u_*u^!\mathcal{O}_Y\otimes \mathrm{id}) \arrow[u, "\text{via }\pi"'] \arrow[ddl, "\text{via}\int"'] \arrow[r, phantom, "7"] & w_*(g_*u^!\mathcal{O}_Y\otimes w^*\delta_*) \arrow[u, "\text{via }\pi"] \arrow[r, phantom, "9"] & w_*(r^*u^!\mathcal{O}_Y\otimes g_*u^*) \arrow[u, "\text{via }\pi"'] \\
 & \delta_*u_*u^!\mathcal{O}_Y\otimes\delta_* \arrow[u, "\text{via }\beta"'] \arrow[r, equal, "\text{via }\mathsf{ps}_*"] \arrow[d, "\text{via}\int"'] & w_*g_*u^!\mathcal{O}_Y\otimes\delta_* \arrow[u, "\pi"] & \\
\delta_*(\mathcal{O}_Y\otimes\mathrm{id}) \arrow[d, equal] & \delta_*\mathcal{O}_Y\otimes\delta_* \arrow[l, "\text{via }\beta"] \arrow[r, phantom, "8"] & w_*r^*u^!\mathcal{O}_Y\otimes\delta_* \arrow[u, "\text{via }\gamma"] \arrow[d, "\text{via (2.1.6.1)}"'] & w_*(r^*u^!\mathcal{O}_Y\otimes w^*\delta_*) \arrow[l, "\pi"] \arrow[uul, "\text{via }\gamma"'] \arrow[uu, "\text{via }\theta"'] \arrow[d, "\text{via (2.1.6.1)}"] \\
\mathcal{O}_{Y'}\otimes\delta_* & & w_*w^!\mathcal{O}_{Y'}\otimes\delta_* \arrow[ll, "\text{via}\int"] & w_*(w^!\mathcal{O}_{Y'}\otimes w^*\delta_*) \arrow[l, "\pi"]
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams is easily checked. (For the leftmost, see the preparatory remarks just before Proposition 6.5.7).

For showing commutativity of 7, expand it as follows, with $A := u^!\mathcal{O}_Y$:

$$
\begin{tikzcd}
\delta_*u_*(A\otimes u^*) \arrow[rrr, equal, "\mathsf{ps}_*"] & & & w_*g_*(A\otimes u^*) \\
\delta_*(u_*A\otimes\mathrm{id}) \arrow[u, "\text{via }\pi"] & & & \\
 & \delta_*u_*(A\otimes u^*\delta^*\delta_*) \arrow[uul, "\text{via }\epsilon_\delta"'] \arrow[r, equal, "\mathsf{ps}_*"] \arrow[dd, equal, "\mathsf{ps}_*"', "\mathsf{ps}^*"] & w_*g_*(A\otimes u^*\delta^*\delta_*) \arrow[uur, "\text{via }\epsilon_\delta"] \arrow[r, equal, "\mathsf{ps}^*"'] \arrow[dd, equal, "\mathsf{ps}_*"', "\mathsf{ps}^*"] & w_*g_*(A\otimes g^*w^*\delta_*) \arrow[uu, "\text{via }\alpha"'] \\
\delta_*(u_*A\otimes\delta^*\delta_*) \arrow[uu, "\text{via }\epsilon_\delta"] \arrow[ur, "\text{via }\pi"'] & & & \\
 & (\delta u)_*(A\otimes(\delta u)^*\delta_*) \arrow[r, equal] & (wg)_*(A\otimes(wg)^*\delta_*) & \\
 \arrow[r, phantom, "7_1"] & (\delta u)_*A\otimes\delta_* \arrow[u, "\pi"] \arrow[r, equal] & (wg)_*A\otimes\delta_* \arrow[u, "\pi"'] \arrow[r, phantom, "7_2"] & w_*(g_*A\otimes w^*\delta_*) \arrow[uuu, "\text{via }\pi"'] \\
\delta_*u_*A\otimes\delta_* \arrow[uuu, "\pi"] \arrow[ur, equal, "\text{via }\mathsf{ps}_*"'] \arrow[rrr, equal, "\text{via }\mathsf{ps}_*"'] & & & w_*g_*A\otimes\delta_* \arrow[ul, equal, "\text{via }\mathsf{ps}_*"] \arrow[u, "\pi"']
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagrams is clear. Subdiagrams ⑦$_1$ and ⑦$_2$ commute by [L3, 3.7.1], *mutatis mutandis.*

Thus ⑦ commutes.

Expand ⑧ as follows, where $\bar\gamma$ is the composition $t^* \xrightarrow{\eta_\delta} \delta_*\delta^*t^* \xrightarrow{\delta_*\mathsf{ps}^*} \delta_*$:

$$
\begin{tikzcd}
\delta_*(\mathcal{O}_Y\otimes \mathrm{id}) \arrow[d, equal] & \delta_*\mathcal{O}_Y\otimes\delta_* \arrow[l, "\text{via}\,\beta"'] & \delta_*u_*u^!\mathcal{O}_Y\otimes\delta_* \arrow[l, "\text{via}\int"'] \arrow[r, equal, "\text{via}\,\mathsf{ps}_*"] & w_*g_*u^!\mathcal{O}_Y\otimes\delta_* \\
\delta_* \arrow[d, equal] \arrow[r, phantom, "⑧_1"] & t^*\mathcal{O}_Y\otimes\delta_* \arrow[u, "\text{via}\,\bar\gamma"] \arrow[dl, equal] & t^*u_*u^!\mathcal{O}_Y\otimes\delta_* \arrow[l, "\text{via}\int"'] \arrow[u, "\text{via}\,\bar\gamma"] \arrow[ur, phantom, "⑧_2"] \arrow[d, phantom, "⑧_3"] & w_*r^*u^!\mathcal{O}_Y\otimes\delta_* \arrow[l, "\text{via}\,\theta^{-1}"'] \arrow[u, "\text{via}\,\gamma"'] \arrow[d, "\text{via}\,(2.1.6.1)"] \\
\mathcal{O}_{Y'}\otimes\delta_* & & & w_*w^!\mathcal{O}_{Y'}\otimes\delta_* \arrow[lll, "\text{via}\int"]
\end{tikzcd}
$$

Commutativity of the unlabeled subdiagram is obvious.

Next, commutativity of subdiagram ⑧$_1$ is equivalent to that of its adjoint, which, since $\pi\colon \delta_*\mathcal{O}_Y\otimes\delta_* \to \delta_*(\mathcal{O}_Y\otimes\delta^*\delta_*)$ is adjoint to the composition

$$
\delta^*(\delta_*\mathcal{O}_Y\otimes\delta_*) \xrightarrow{(1.3.3)} \delta^*\delta_*\mathcal{O}_Y\otimes\delta^*\delta_* \xrightarrow{\text{via}\,\epsilon_\delta} \mathcal{O}_Y\otimes\delta^*\delta_*
$$

(cf. [L3, 3.4.6.2]), is the outer border of

$$
\begin{tikzcd}
\mathcal{O}_Y\otimes\delta^*\delta_* \arrow[dd, equal] \arrow[dr, equal, "\text{via}\,\mathsf{ps}^*"] & & & \delta^*\delta_*\mathcal{O}_Y\otimes\delta^*\delta_* \arrow[lll, "\text{via}\,\epsilon_\delta"'] \arrow[d, equal, "\text{via}\,\mathsf{ps}^*"] & \delta^*(\delta_*\mathcal{O}_Y\otimes\delta_*) \arrow[l, "(1.3.3)"'] \arrow[d, equal, "\text{via}\,\mathsf{ps}^*"] \\
& \delta^*t^*\mathcal{O}_Y\otimes\delta^*\delta_* \arrow[d, equal] \arrow[drr, equal] & & \delta^*\delta_*\delta^*t^*\mathcal{O}_Y\otimes\delta^*\delta_* \arrow[ll, "\text{via}\,\epsilon_\delta"'] & \delta^*(\delta_*\delta^*t^*\mathcal{O}_Y\otimes\delta_*) \arrow[l, "(1.3.3)"'] \\
\mathcal{O}_Y\otimes\delta^*\delta_* \arrow[r, equal] \arrow[d, "\text{via}\,\epsilon_\delta"'] \arrow[ur, phantom, "⑧_{11}"] \arrow[dr, phantom, "⑧_{12}"] & \delta^*\mathcal{O}_{Y'}\otimes\delta^*\delta_* \arrow[rr, equal] & & \delta^*t^*\mathcal{O}_Y\otimes\delta^*\delta_* \arrow[u, "\text{via}\,\eta_\delta"] & \\
\mathcal{O}_Y\otimes \mathrm{id} \arrow[r, equal] & \mathrm{id} & \delta^*\delta_* \arrow[l, "\text{via}\,\epsilon_\delta"] \arrow[r, equal] & \delta^*(\mathcal{O}_{Y'}\otimes\delta_*) \arrow[ull, "(1.3.3)"] \arrow[r, equal] & \delta^*(t^*\mathcal{O}_Y\otimes\delta_*) \arrow[ul, "(1.3.3)"] \arrow[uu, "\text{via}\,\eta_\delta"']
\end{tikzcd}
$$

For commutativity of ⑧$_{11}$, see [L3, 3.6.7(b)]. For commutativity of ⑧$_{12}$, see [L3, 3.4.4(b)]. Commutativity of the remaining subdiagrams is easily verified. Thus ⑧$_1$ commutes.

Next, commutativity of ⑧$_2$ (without $u^!\mathcal{O}_Y \otimes \delta_*$) is implied by that of the following expanded diagram:

$$
\begin{tikzcd}
\delta_*u_* \arrow[rr, equal] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \arrow[d, phantom, "⑧_{21}"] & \delta_*u_* \arrow[r, equal, "\mathsf{ps}_*"] \arrow[d, equal, "\text{via } \mathsf{ps}^*"] & w_*g_* \arrow[d, equal, "\text{via } \mathsf{ps}^*"] \\
\delta_*\delta^*t^*u_* \arrow[r, "\text{via } \theta"] & \delta_*\delta^*w_*r^* \arrow[r, "\text{via } \theta"] & \delta_*u_*g^*r^* \arrow[r, equal, "\mathsf{ps}_*"] \arrow[d, phantom, "⑧_{22}"] & w_*g_*g^*r^* \\
t^*u_* \arrow[u, "\eta_\delta"] \arrow[r, "\text{via } \theta"'] & w_*r^* \arrow[u, "\text{via } \eta_\delta"'] \arrow[rr, equal] & & w_*r^* \arrow[u, "\text{via } \eta_g"']
\end{tikzcd}
$$

Commutativity of ⑧$_{21}$ is [L3, 3.7.2(ii)], applied to the composite diagram $\mathbf{e}'\circ\mathbf{e}$ in Proposition 6.5.7.

Commutativity of ⑧$_{22}$ results from the adjointness of $\theta\colon \delta^*w_* \to u_*g^*$ and the composite map $w_* \xrightarrow{w_*\eta_g} w_*g_*g^* \overset{\mathsf{ps}_*}{=\!=} \delta_*u_*g^*$, which holds by [L3, 3.7.2(i)(b)], with $(f,g,f',g') := (w,\delta,u,g)$.

Commutativity of the other two subdiagrams is clear. Thus ⑧$_2$ commutes.

Next, since u and w are proper, commutativity of ⑧$_3$ is an immediate consequence of the definition of the base-change map B in (2.1.6.1) (with $(f,g,u,v) := (u,w,t,r)$), see [AJL, 5.8.2, 5.8.5].

Thus ⑧ commutes.

Subdiagram ⑨ is the outer border of

$$
\begin{tikzcd}
w_*g_*(u^!\mathcal{O}_Y \otimes u^*\delta^*\delta_*) \arrow[rrr, "\text{via } \epsilon_\delta"] \arrow[d, equal, "\text{via } \mathsf{ps}^*"'] & \arrow[d, phantom, "⑨_1"] & & w_*g_*(u^!\mathcal{O}_Y \otimes u^*) \arrow[dddd, equal, "\text{via } \mathsf{ps}^*"] \\
w_*g_*(u^!\mathcal{O}_Y \otimes g^*w^*\delta_*) \arrow[r, "\text{via } \theta"] & w_*g_*(u^!\mathcal{O}_Y \otimes g^*g_*u^*) \arrow[urr, "\text{via } \epsilon_g"'] & ⑨_2 & \\
w_*(g_*u^!\mathcal{O}_Y \otimes w^*\delta_*) \arrow[u, "\text{via } \pi"] \arrow[r, "\text{via } \theta"] & w_*(g_*u^!\mathcal{O}_Y \otimes g_*u^*) \arrow[u, "\text{via } \pi"] \arrow[dr, "\text{via } \pi"] & & ⑨_3 \\
& & w_*g_*(g^*g_*u^!\mathcal{O}_Y \otimes u^*) \arrow[uuur, "\text{via } \epsilon_g"] & \\
w_*(r^*u^!\mathcal{O}_Y \otimes w^*\delta_*) \arrow[uu, "\text{via } \gamma"] \arrow[r, "\text{via } \theta"'] & w_*(r^*u^!\mathcal{O}_Y \otimes g_*u^*) \arrow[uu, "\text{via } \gamma"] \arrow[rr, "\text{via } \pi"'] & & w_*g_*(g^*r^*u^!\mathcal{O}_Y \otimes u^*) \arrow[ul, "\text{via } \gamma"']
\end{tikzcd}
$$

Subdiagram ⑨$_1$ commutes because θ is, by definition, $g^*\dashv g_*$-adjoint to α.

Commutativity of ⑨$_3$ results from the obvious commutativity of

$$\begin{tikzcd}
g^*g_* \arrow[rr, "\text{via}\,\epsilon_g"] \arrow[dr, equal, "g^*g_*\mathsf{ps}^*"'] & & \mathrm{id} \arrow[dd, equal, "\mathsf{ps}^*"] \\
& g^*g_*g^*r^* \arrow[dr, "\epsilon_g"] & \\
g^*r^* \arrow[uu, "g^*\gamma"] \arrow[ur, "g^*\eta_g"] \arrow[rr, equal] & & g^*r^*
\end{tikzcd}$$

As for commutativity of ⑨$_2$, after dropping w_* and setting $A := u^!\mathcal{O}_Y$, $B := u^*$, one need only show commutativity of

$$\begin{tikzcd}
& g_*A \otimes g_*B \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\
g_*(A \otimes g^*g_*B) \arrow[dr, "\text{via}\,\epsilon_g"'] & & g_*(g^*g_*A \otimes B) \arrow[dl, "\text{via}\,\epsilon_g"] \\
& g_*(A \otimes B) &
\end{tikzcd} \tag{6.5.8}$$

In the following diagram

$$\begin{tikzcd}
& & g^*g_*A \otimes g^*g_*B \arrow[dll, "\text{via}\,\epsilon_g"'] \arrow[d, "(1.3.3)"', "\simeq"] \arrow[drr, "\text{via}\,\epsilon_g"] & & \\
& \textcircled{9}_{21} & g^*(g_*A \otimes g_*B) \arrow[dl, "g^*\pi"'] \arrow[dr, "g^*\pi"] & \textcircled{9}_{22} & \\
A \otimes g^*g_*B \arrow[ddrr, "\text{via}\,\epsilon_g"'] & g^*g_*(A \otimes g^*g_*B) \arrow[l, "\epsilon_g"'] \arrow[dr, "\text{via}\,\epsilon_g"'] & & g^*g_*(g^*g_*A \otimes B) \arrow[r, "\epsilon_g"] \arrow[dl, "\text{via}\,\epsilon_g"] & g^*g_*A \otimes B \arrow[ddll, "\text{via}\,\epsilon_g"] \\
& \textcircled{9}_{23} & g^*g_*(A \otimes B) \arrow[d, "\epsilon_g"'] & \textcircled{9}_{24} & \\
& & A \otimes B & &
\end{tikzcd}$$

commutativity of the outer border is clear, as is that of subdiagrams ⑨$_{23}$ and ⑨$_{24}$; and commutativity of ⑨$_{21}$ and ⑨$_{22}$ results from [L3, 3.4.6.2]. Looking inside the diagram one sees then that the $g^* \dashv g_*$-adjoint of (6.5.8)—hence (6.5.8) itself—commutes.

Thus ⑨$_2$—and finally ⑨ itself—commutes.

This completes the proof of Proposition 6.5.7 in case u and w are proper.

In the general case, u factors as $X \xrightarrow{u_2} Z \xrightarrow{u_1} Y$ where u_1 is proper and u_2 is essentially étale [Nk, 4.1 and 2.7]. It follows that the diagram $\mathbf{e}'\circ\mathbf{e}$ in Proposition 6.5.7 expands as

$$\begin{tikzcd}
X \arrow[r, "u_2"] \arrow[d, "g"'] & Z \arrow[r, "u_1"] \arrow[d, "h"'] & Y \arrow[d, "\delta"] \\
X' \arrow[r, "w_2"] \arrow[d, "r"'] & Z\times_Y Y' \arrow[r, "w_1"] \arrow[d, "s"'] & Y' \arrow[d, "t"] \\
X \arrow[r, "u_2"'] & Z \arrow[r, "u_1"'] & Y
\end{tikzcd}$$

where w_1 and s are the natural projections; h is the unique map such that $w_1h = \delta u_1$ and $sh = \mathrm{id}_Z$; and w_2 is the unique map such that $sw_2 = u_2r$ and $w_1w_2 = w$. One checks that all the subsquares are fiber squares; so w_1 is proper and w_2 is essentially étale (see second-last paragraph in §1.1). Since u_2 is flat, the map $\kappa\colon u_2^!\mathcal{O}_Z \otimes u_2^* \to u_2^!$ is an isomorphism on $\mathbf{D}^+_{\mathsf{qc}}(Z)$; and likewise for w_2.

Straightforward use of the isomorphisms

$$u^! \overset{\mathsf{ps}^!}{=\!=} u_2^!u_1^!, \quad u^* \overset{\mathsf{ps}^*}{=\!=} u_2^*u_1^*, \quad w^! \overset{\mathsf{ps}^!}{=\!=} w_2^!w_1^!, \quad w^* \overset{\mathsf{ps}^*}{=\!=} w_2^*w_1^*$$

transforms the assertion in Proposition 6.5.7 to that of commutativity of the border of the next diagram (6.5.9), in which $\mathcal{O} := \mathcal{O}_Y$, $\mathcal{O}' := \mathcal{O}_{Y'}$, $\mathcal{O}'' := \mathcal{O}_{Z\times_Y Y'}$, and the unlabeled maps are the obvious ones:

Diagram chasing shows it suffices now to prove commutativity of all the subdiagrams.

Commutativity of the unlabeled subdiagrams is clear.

Commutativity of ⓒ follows easily from the essentially étale case of 6.5.7, applied to the diagram

$$\begin{tikzcd}
X \arrow[r, "u_2"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\
X' \arrow[r, "w_2"] \arrow[d, "r"'] & Z\times_Y Y' \arrow[d, "s"] \\
X \arrow[r, "u_2"'] & Z
\end{tikzcd}$$

(6.5.9)

$$
\begin{tikzcd}[column sep=small]
g_*u_2^!u_1^!\delta^!\delta_* \arrow[ddddddddddddd, equal] & & & g_*u_2^!u_1^! = g_*u^! \arrow[lll] \arrow[ddll] & & g_*(u^!\mathcal{O}\otimes u^*) \arrow[ll] \arrow[dd, equal] \\
 & & & g_*u_2^!(u_1^!\mathcal{O}\otimes u_1^*) \arrow[u] & \text{a} & \\
 & g_*u_2^!h^!h_*u_1^! \arrow[dd, equal] & & g_*(u_2^!\mathcal{O}_Z\otimes u_2^*u_1^!\mathcal{O}\otimes u_2^*u_1^*) \arrow[u, "\mathrm{via}\,\kappa"'] & & g_*(u_2^!u_1^!\mathcal{O}\otimes u_2^*u_1^*) \arrow[ll, "\mathrm{via}", "\kappa^{-1}"'] \arrow[d, equal] \\
 & & & g_*(g^*r^*u_2^!\mathcal{O}_Z\otimes u_2^*(u_1^!\mathcal{O}\otimes u_1^*)) \arrow[u] & & g_*(g^*r^*u_2^!u_1^!\mathcal{O}\otimes u_2^*u_1^*) \\
 & g_*g^!w_2^!h_*u_1^! \arrow[dddd] & \text{c} & r^*u_2^!\mathcal{O}_Z\otimes g_*u_2^*(u_1^!\mathcal{O}\otimes u_1^*) \arrow[u] & \text{d} & r^*u_2^!u_1^!\mathcal{O}\otimes g_*u_2^*u_1^* \arrow[u] \\
\text{b} & & & w_2^!\mathcal{O}''\otimes w_2^*h_*(u_1^!\mathcal{O}\otimes u_1^*) \arrow[u] \arrow[dl, "\kappa"] \arrow[ddr, equal] & & r^*u_2^!u_1^!\mathcal{O}\otimes w_2^*w_1^*\delta_* \arrow[u] \\
 & & w_2^!h_*(u_1^!\mathcal{O}\otimes u_1^*) \arrow[ddl] \arrow[dd, equal] & & & \\
 & & & & w_2^!\mathcal{O}''\otimes w_2^*h_*(h^*s^*u_1^!\mathcal{O}\otimes u_1^*) \arrow[dll, "\kappa"] & \\
 & w_2^!h_*u_1^! \arrow[dd] & w_2^!h_*(h^*s^*u_1^!\mathcal{O}\otimes u_1^*) \arrow[dd, "\simeq"] & & & \\
 & & \text{e} & & w_2^!\mathcal{O}''\otimes w_2^*(s^*u_1^!\mathcal{O}\otimes h_*u_1^*) \arrow[uu, "\simeq"'] \arrow[dll, "\kappa"] & \\
 & w_2^!h_*u_1^!\delta^!\delta_* \arrow[dd, equal] & w_2^!(s^*u_1^!\mathcal{O}\otimes h_*u_1^*) \arrow[dd, "\simeq"] & & & \\
 & & & & w_2^!\mathcal{O}''\otimes w_2^*w_1^!t^*\mathcal{O}\otimes w_2^*w_1^*\delta_* \arrow[uu, "\simeq"'] \arrow[dll, "\kappa"] & \\
 & w_2^!h_*h^!w_1^!\delta_* \arrow[drr] & w_2^!(w_1^!t^*\mathcal{O}\otimes w_1^*\delta_*) \arrow[dr] & & \text{f} & w_2^!w_1^!t^*\mathcal{O}\otimes w_2^*w_1^*\delta_* \arrow[ul, "\mathrm{via}\,\kappa^{-1}"] \arrow[uuuuuuu] \arrow[d, equal] \\
g_*g^!w_2^!w_1^!\delta_* \arrow[rrr] & & & w_2^!w_1^!\delta_* = w^!\delta_* & & w^!\mathcal{O}'\otimes w^*\delta_* \arrow[ll]
\end{tikzcd}
$$

Commutativity of ⓔ results from the proper case of 6.5.7, applied to

$$
\begin{tikzcd}
Z \arrow[r, "u_1"] \arrow[d, "h"'] & Y \arrow[d, "\delta"] \\
Z \times_Y Y' \arrow[r, "w_1"] \arrow[d, "s"'] & Y' \arrow[d, "t"] \\
Z \arrow[r, "u_1"'] & Y
\end{tikzcd}
$$

Subdiagram ⓑ has the following, clearly commutative, expansion (where the maps are the obvious ones):

$$
\begin{tikzcd}
g_*u_2^!u_1^!\delta^!\delta_* \arrow[rr, equal] \arrow[ddd, equal] & & g_*u_2^!u_1^!\delta^!\delta_* \arrow[dl, equal] \arrow[d] & g_*u_2^!u_1^! \arrow[l] \arrow[d] \\
& g_*u_2^!h^!w_1^!\delta_* \arrow[d] \arrow[ddl, equal] & g_*u_2^!h^!h_*u_1^!\delta^!\delta_* \arrow[dl, equal] \arrow[dd, equal] & g_*u_2^!h^!h_*u_1^! \arrow[l] \arrow[dd, equal] \\
& g_*u_2^!h^!h_*h^!w_1^!\delta_* \arrow[d, equal] & & \\
g_*g^!w_2^!w_1^!\delta_* \arrow[d] & g_*g^!w_2^!h_*h^!w_1^!\delta_* \arrow[l] \arrow[r, equal] \arrow[d] & g_*g^!w_2^!h_*u_1^!\delta^!\delta_* \arrow[d] & g_*g^!w_2^!h_*u_1^! \arrow[l] \arrow[d] \\
w_2^!w_1^!\delta_* & w_2^!h_*h^!w_1^!\delta_* \arrow[l] \arrow[r, equal] & w_2^!h_*u_1^!\delta^!\delta_* & w_2^!h_*u_1^! \arrow[l]
\end{tikzcd}
$$

Commutativity of subdiagram ⓐ follows from [L3, 4.9.3(d)] as regards [L3, 4.7.3.4(d)] with $(f, g, E) := (u_2, u_1, \mathcal{O})$—in view of [L3, 4.9.3(d)] as regards [L3, 4.7.3.4(a)] with $(f, E, F, G) := (u_2, \mathcal{O}_Z, u_1^!\mathcal{O}, u_1^*)$, which gives that (via κ)$\circ$(via κ^{-1}) in ⓐ is the map $g_*\chi^{u_2}_{u_1^!\mathcal{O}, u_1^*-}$ coming from [L3, (4.9.1.1)], as extended to S-maps in the manner of [Nk, 5.8]. A similar argument shows that ⓕ commutes. Details are left to the reader.

Subdiagram d expands as follows, with the map π coming from (1.3.5). (Recall: $u_2^! = u_2^*$, $w_2^! = w_2^*$.)

$$
\begin{tikzcd}
g_*(g^*r^*u_2^!\mathcal{O}_Z \otimes u_2^*(u_1^!\mathcal{O} \otimes u_1^*)) & & & g_*(u_2^!\mathcal{O}_Z \otimes u_2^*u_1^!\mathcal{O} \otimes u_2^*u_1^*) \arrow[lll] \\
r^*u_2^!\mathcal{O}_Z \otimes g_*u_2^*(u_1^!\mathcal{O} \otimes u_1^*) \arrow[u] & & g_*(g^*r^*u_2^!\mathcal{O}_Z \otimes u_2^*u_1^!\mathcal{O} \otimes u_2^*u_1^*) \arrow[ull] \arrow[ur, equal] & \\
& r^*u_2^!\mathcal{O}_Z \otimes g_*(u_2^*u_1^!\mathcal{O} \otimes u_2^*u_1^*) \arrow[ul] & & \\
w_2^!\mathcal{O}'' \otimes w_2^*h_*(u_1^!\mathcal{O} \otimes u_1^*) \arrow[uu] & & & g_*(u_2^!u_1^!\mathcal{O} \otimes u_2^*u_1^*) \arrow[uuu] \\
d_1 & r^*u_2^!\mathcal{O}_Z \otimes g_*(g^*r^*u_2^*u_1^!\mathcal{O} \otimes u_2^*u_1^*) \arrow[uu, equal] & & \\
w_2^!\mathcal{O}'' \otimes w_2^*h_*(h^*s^*u_1^!\mathcal{O} \otimes u_1^*) \arrow[uu, equal] & & & \\
& & g_*(g^*r^*u_2^!\mathcal{O}_Z \otimes g^*r^*u_2^*u_1^!\mathcal{O} \otimes u_2^*u_1^*) \arrow[ul, "\pi"'] \arrow[uuuuu, equal] & \\
w_2^!\mathcal{O}'' \otimes w_2^*(s^*u_1^!\mathcal{O} \otimes h_*u_1^*) \arrow[uu] & d_2 & & g_*(g^*r^*u_2^!u_1^!\mathcal{O} \otimes u_2^*u_1^*) \arrow[ul] \arrow[uuuu, equal] \\
& r^*u_2^!\mathcal{O}_Z \otimes r^*u_2^*u_1^!\mathcal{O} \otimes g_*u_2^*u_1^* \arrow[uuuu] \arrow[uur] & & \\
w_2^!\mathcal{O}'' \otimes w_2^*s^*u_1^!\mathcal{O} \otimes w_2^*h_*u_1^* \arrow[uu] \arrow[ur] \arrow[r, equal] & w_2^!s^*u_1^!\mathcal{O} \otimes w_2^*h_*u_1^* \arrow[rr, "\sim"] & & r^*u_2^!u_1^!\mathcal{O} \otimes g_*u_2^*u_1^* \arrow[uu] \arrow[ull, equal] \\
& & d_3 & \\
w_2^!\mathcal{O}'' \otimes w_2^*w_1^!\mathcal{O}' \otimes w_2^*w_1^*\delta_* \arrow[uu] & w_2^!w_1^!t^*\mathcal{O} \otimes w_2^*w_1^*\delta_* \arrow[l] \arrow[uu] \arrow[rr, "\sim"] & & r^*u_2^!u_1^!\mathcal{O} \otimes w_2^*w_1^*\delta_* \arrow[uu]
\end{tikzcd}
$$

Diagram chasing shows that to prove commutativity of the border it will suffice to prove commutativity of all the subdiagrams.

Commutativity of the unlabeled subdiagrams is easily verified.

Commutativity of d_3 results from transitivity of (2.1.6.1) and of (1.4.1). (See [L3, 3.7.2(iii)], having in mind that u_2 and w_2, as well as r, s and t, are flat.)

Commutativity of d_2 results from [L3, 3.4.7(iii)], with $(f, A, B, C) := (g, r^*u_2^!\mathcal{O}_Z, r^*u_2^*u_1^!\mathcal{O}, u_2^*u_1^*-)$.

Last, in the next diagram of *isomorphisms,* with $A, B \in \mathbf{D}^+_{\mathsf{qc}}(Z)$, the border commutes by [L3, 3.7.3] with $(f, f', g, g', P, Q) := (h, g, w_2, u_2, s^*A, B)$, and commutativity of the unlabeled subdiagrams is easy to check (the one at the bottom by pseudofunctoriality of $(-)^*$), and hence d_1' commutes. Setting $A := u_1^!\mathcal{O}$, $B := u_1^*-$, one obtains commutativity of d_1 from that of d_1'.

$$
\begin{tikzcd}
 & w_2^*(s^*A \otimes h_*B) \arrow[r] & w_2^*h_*(h^*s^*A \otimes B) \arrow[dr] \arrow[dd, equal] & \\
 & & & g_*u_2^*(h^*s^*A \otimes B) \arrow[dddd] \\
 & w_2^*s^*A \otimes w_2^*h_*B \arrow[uu] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\textcircled{d}'_1"] & w_2^*h_*(A \otimes B) \arrow[d] & \\
w_2^*s^*A \otimes g_*u_2^*B \arrow[r] \arrow[dd] & r^*u_2^*A \otimes g_*u_2^*B \arrow[d] & g_*u_2^*(A \otimes B) & \\
 & g_*(g^*r^*u_2^*A \otimes u_2^*B) \arrow[r, equal] \arrow[dl] & g_*(u_2^*A \otimes u_2^*B) \arrow[u] \arrow[dr, equal] & \\
g_*(g^*w_2^*s^*A \otimes u_2^*B) \arrow[rrr, equal, "\text{via } \mathsf{ps}^*"'] & & & g_*(u_2^*h^*s^*A \otimes u_2^*B)
\end{tikzcd}
$$

With this, Proposition 6.5.7, Step IIA and Theorem 3.1, are proved. $\square$

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