CORRECTIONS TO SECTION 2.5 OF "DUALITY AND FLAT BASE CHANGE . . . " (CONTEMPORARY MATH. 244).

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Abstract. The section in question is about dualizing complexes on formal schemes. A weaker version of one oft-used—but flawed—Lemma is proved, and shown to be adequate for all the applications made of the original version.

Suresh Nayak pointed out that the proof of [DFS, p. Lemma 2.5.6] applies[1] only to $\mathcal{A}_{\vec{c}}(\mathfrak{X})$-complexes, not, as asserted, to arbitrary $\mathcal{F} \in \mathbf{D}_{\vec{c}}(\mathfrak{X})$. The Lemma is used four times in §2.5, so these four places need to be revisited. (There are no other references to Lemma 2.5.6 in the paper.)

First, in Remark (2) on p. 24, the reference to Lemma 2.5.6 is not necessary: the cited theorem 4.8 in [Y] (see also Proposition 5.3.1 in [DFS, p. 57]) shows that the t-dualizing complex $\mathcal{R}$ is $\mathbf{D}$-isomorphic to a bounded-below complex $\mathcal{X}'^{\bullet}$ of $\mathcal{A}_{\mathrm{qct}}$-injectives; and then one can proceed as indicated to show that for some n the (bounded) truncation $\sigma_{\le n}\mathcal{X}'^{\bullet}$ is $\mathcal{A}_{\mathrm{qct}}$-injective and $\mathbf{D}$-isomorphic to $\mathcal{X}'^{\bullet}$. (To follow the details, it helps to keep in mind 5.1.3 and 5.1.4 on p. 48.)

Since, by Remark (2), any t-dualizing complex is $\mathbf{D}$-isomorphic to a bounded complex of $\mathcal{A}_{\mathrm{qct}}$-injectives, in view of Propositions 3.3.1 and 5.1.2 one finds that the remaining three references to Lemma 2.5.6 can be replaced by references to the following boundedness result for certain functors. For the reference in the proof of 2.5.7(b) this is clear. The same is true for Remark (4) on p. 24, but $i > n_0$ at the end should be $i > n_0 + d$, where, by Remark (5), the Krull dimension d of $\mathfrak{X}$ is finite. Finally, for the reference in the proof of 2.5.12, one can note, via 5.1.4 and 5.1.2, that $\mathbf{D}^*_{\mathrm{c}} \subset \mathbf{D}_{\mathrm{qct}} \subset \mathbf{D}_{\vec{c}}$.

Let $\mathfrak{X}$ be a locally noetherian formal scheme, let $\mathcal{I}$ be a bounded complex of $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$-injectives, say $\mathcal{I}^i = 0$ for all $i > n$, and let $\mathcal{F} \in \mathbf{D}^+(\mathfrak{X})$, say $H^\ell(\mathcal{F}) = 0$ for all $\ell < -m$. Suppose there exists an open cover $(\mathfrak{X}_\alpha)$ of $\mathfrak{X}$ by completions of ordinary noetherian schemes X_α along closed subsets, with completion maps $\kappa_\alpha\colon \mathfrak{X}_\alpha \to X_\alpha$, such that for each α the restriction of $\mathcal{F}$ to $\mathfrak{X}_\alpha$ is $\mathbf{D}$-isomorphic to $\kappa_\alpha^* F_\alpha$ for some $F_\alpha \in \mathbf{D}(X_\alpha)$. Then

$$\mathcal{E}xt^i(\mathcal{F},\mathcal{I}) := H^i\mathbf{R}\mathcal{H}om^{\bullet}_{\mathfrak{X}}(\mathcal{F},\mathcal{I}) = 0 \quad \text{for all } i > m+n.$$

Moreover, if $\mathfrak{X}$ has finite Krull dimension d then

$$\mathrm{Ext}^i(\mathcal{F},\mathcal{I}) := H^i\mathbf{R}\mathrm{Hom}^{\bullet}_{\mathfrak{X}}(\mathcal{F},\mathcal{I}) = 0 \quad \text{for all } i > m+n+d.$$

Proof. The first assertion is local (see first sentence in the proof of Lemma 2.5.6), so one may assume that $\mathfrak{X}$ itself is a completion, with completion map $\kappa\colon \mathfrak{X} \to X$, and that in $\mathbf{D}(\mathfrak{X})$, $\mathcal{F} \cong \kappa^* F$ for some $F \in \mathbf{D}(X)$. As κ^*, being exact, commutes with

[1] when one interprets "$F \in \mathcal{A}_{\mathrm{qc}}(X)$" in that proof as meaning that F is a *complex* in $\mathcal{A}_{\mathrm{qc}}(X)$

the truncation functor $\sigma_{\ge -m}$, there are $\mathbf{D}$-isomorphisms (the first as in [H1, p. 70]):

$$\mathcal{F} \cong \sigma_{\ge -m}\mathcal{F} \cong \sigma_{\ge -m}\kappa^* F \cong \kappa^* \sigma_{\ge -m} F;$$

so one can replace F by $\sigma_{\ge -m}F$ and assume further that $F^\ell = 0$ for all $\ell < -m$.

As in the proof of Lemma 2.5.6, $\kappa_*\mathcal{I}$ is a bounded complex of $\mathcal{O}_X$-injectives, vanishing in degree $> n$. Since κ_* is exact, therefore for all $i > m+n$,

$$\begin{aligned}
\kappa_* H^i \mathbf{R}\mathcal{H}om^\bullet_{\mathfrak{X}}(\mathcal{F},\mathcal{I}) &\cong H^i \kappa_* \mathbf{R}\mathcal{H}om^\bullet_{\mathfrak{X}}(\kappa^* F,\mathcal{I}) \\
&\cong H^i \mathbf{R}\mathcal{H}om^\bullet_{X}(F,\kappa_*\mathcal{I}) \qquad\qquad [\text{Sp, p. 147, 6.7(2)}] \\
&\cong H^i \mathcal{H}om^\bullet_{X}(F,\kappa_*\mathcal{I}) = 0,
\end{aligned}$$

and hence $H^i \mathbf{R}\mathcal{H}om^\bullet_{\mathfrak{X}}(\mathcal{F},\mathcal{I}) = 0$.

If $\mathfrak{X}$ has Krull dimension d, and $\Gamma := \Gamma(\mathfrak{X}, -)$ is the global-section functor, then by a well-known theorem of Grothendieck the restriction of the derived functor $\mathbf{R}\Gamma$ to the category of abelian sheaves has cohomological dimension $\le d$; and so since $\mathbf{R}\mathrm{Hom}^\bullet_{\mathfrak{X}} \cong \mathbf{R}\Gamma\mathbf{R}\mathcal{H}om^\bullet_{\mathfrak{X}}$ [L4, Exercise 2.5.10(b)], the second assertion follows from [L4, Remark 1.11.2(iv)]. $\square$

References

[DFS] L. Alonso Tarrío, A. Jeremas López, and J. Lipman, Duality and flat base change on formal schemes, *Contemporary Math.* **244** (1999), 3–90.
Correction, *Proc. Amer. Math. Soc.* **131** (2003), 351–357.

[H1] R. Hartshorne, *Residues and Duality,* Lecture Notes in Math., no. **20**, Springer-Verlag, New York, 1966.

[L4] J. Lipman, *Notes on Derived Categories,* < `www.math.purdue.edu/~lipman/` >.

[Sp] N. Spaltenstein, Resolution of unbounded complexes, *Compositio Math.* **65** (1988), 121–154.

[Y] A. Yekutieli, Smooth formal embeddings and the residue complex, *Canadian J. Math.* **50** (1998), 863–896.

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