DIFFERENTIAL INVARIANTS OF EMBEDDINGS OF MANIFOLDS IN COMPLEX SPACES

WEIMING HUANG AND JOSEPH LIPMAN

ABSTRACT. Let V be a reduced complex space, W a complex submanifold, and let (V', W') be another such pair. Let $f\colon V \to V'$ be a homeomorphism with $f(W) \subset W'$, such that f and f^{-1} are both continuously (real-)differentiable. Then f induces a component-(with multiplicity)-preserving homeomorphism $\mathbf{f}_0$ from the normal cone $C(V, W)$ to $C(V', W')$, respecting the natural $\mathbb{R}^*$ actions on these cones. Moreover, though $\mathbf{f}_0$ need not respect the $\mathbb{C}^*$ actions nevertheless the induced map on Borel-Moore homology $f_*\colon H_*(W) \to H_*(W')$ takes the Segre classes of the components of $C(V, W)$ to $\pm$those of the corresponding components of $C(V', W')$. In particular we recover the differential invariance of the multiplicity of W in V.

Introduction. In studying singularities one is interested in invariants, analytic (biholomorphic) or topological. And it can be an occasion for celebration when an analytic invariant turns out to be topological. For example, a famous open problem of Zariski is to determine whether the multiplicity of a hypersurface germ in $\mathbb{C}^n$ is invariant under ambient homeomorphisms.

In between the analytic and topological domains, there is a large and relatively unexplored territory populated by *differential* invariants, i.e, data which are associated to complex spaces and which are always the same for two C^s-homeomorphic spaces ($s > 0$). The multiplicity of a reduced complex space germ is such a differential invariant, for $s = 1$ [GL], but not a topological one, even for ambient homeomorphisms of curves in $\mathbb{C}^3$.

In this paper we consider a reduced complex space V with an r-dimensional connected submanifold $i\colon W \hookrightarrow V$. Assume for simplicity that all the irreducible components of V have the same dimension, say d, and that they all properly contain W. Let $\mathcal{I}$ be the kernel of the natural surjection $\mathcal{O}_V \to i_*\mathcal{O}_W$, let $\mathcal{G}$ be the graded $\mathcal{O}_W$-algebra $\oplus_{m\ge 0}\, i^*(\mathcal{I}^m/\mathcal{I}^{m+1})$, and let $C(V, W) := \operatorname{Specan}(\mathcal{G})$ be the normal cone of W in V (see §1), with (reduced, irreducible) components $(C_j)_{j\in J}$. The components P_j of the projectivized normal cone $P = P(V, W) := \operatorname{Projan}(\mathcal{G}) \xrightarrow{\wp} W$ correspond naturally to those of $C(V, W)$. For each j let $[P_j] \in H_{2d-2}(P)$ (Borel-Moore homology) be the natural image of the fundamental class of P_j. P carries a canonical invertible sheaf $\mathcal{O}(1)$, with Chern class, say, $c \in H^2(P, \mathbb{Z})$. The *Segre class* $s_i(C_j) \in H_{2r-2i}(W)$ is defined by $s_i(C_j) := \wp_*\big([P_j] \cap c^{d-1-r+i}\big)$.[1]

Our motivating result is that *these Segre classes are, up to sign,* C^1 *invariants of the pair* (V, W). (For a precise statement see Theorem (6.3).)

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[1]When V and W are algebraic varieties, this definition connects to the algebraic one in [Fn, Chap. 4] via the cycle map of *ibid.,* §19.1.

We first prove that the normal cone $C(V,W)$ is a differential invariant, even "as a cycle": given a second pair $V' \supset W'$, then any C^1 homeomorphism $f\colon V \to V'$ with f^{-1} also C^1 and $f(W) = W'$ induces a homeomorphism $\mathbf{f}_0$ from $C(V,W)$ onto $C(V',W')$ such that $\mathbf{f}_0$ maps each irreducible component of $C(V,W)$ onto one of $C(V',W')$ having the same multiplicity. (See Theorem (4.3.1); the case where W is a point was an important part of [GL].) This is shown via the standard deformation (see §2) of V to $C(V,W)$, *restricted however to real parameters* t. (So we have the trivial family $V_t \cong V$ for $0 \neq t \in \mathbb{R}$, together with $V_0 \cong C(V,W)$.) Of course the trivial part of this deformation, away from $t = 0$, behaves functorially; and one needs to show that the functoriality "extends continuously" to the entire deformation. This is done in Theorem (3.3), via the derivative of f. In §4 we prove the differential invariance of the multiplicities of the components by interpreting these numbers as intersection multiplicities along the components of V_0, and noting that such intersection numbers are known to be topological invariants.

Now in order to get at the Segre classes we must pass from $C(V,W)$ to $P(V,W)$, and so we have to quotient out the natural $\mathbb{C}^*$ action. The problem is that we used the derivative of f to establish functoriality of $C(V,W)$, and that derivative is only *real*-linear. Thus the $\mathbb{C}^*$ action may not be functorial.

To deal with this problem, we construct in §5 the *relative complexification* of $C := C(V,W)$ (in fact, of any cone over W), an analytic subset $\widetilde{C} \subset C \times_W C$ whose fibers are real-analytically isomorphic to the complexifications of the fibres of C, at least almost everywhere over W. This $\widetilde{C}$, together with a natural real-analytic $\mathbb{C}^*$ action, is indeed C^1-functorial (Theorem (5.3.1)). But we have not been able to extract any Segre classes directly from $\widetilde{C}$. Instead we use the $\mathbb{C}^*$-stable, analytic subset $\Lambda(C) \subset \widetilde{C}$ consisting of pairs (c_1, c_2) of points of C such that one of them lies in the orbit of the other with respect to the natural $\mathbb{C}^1$ action (reviewed in §1). Using the functoriality of $\widetilde{C}$, we find that $\Lambda(C)$ *is* C^1*-functorial.* Furthermore, off its vertex section, $\Lambda(C)$ together with its induced $\mathbb{C}^*$ action is topologically isomorphic to the rank two bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ (minus its 0-section) over $P(V,W)$. It follows that the Segre classes of the components of this rank two bundle become differential invariants, up to sign, when pushed down from $P(V,W)$ to W. These pushed-down classes are easily seen to be the Segre classes $s_i(C_j)$ such that $i - 1-$(codimension of W in V) is even. The remaining Segre classes can be obtained similarly, just by changing (V,W) to $(V \times \mathbb{C}^1, W \times \{0\})$ (which doesn't affect the total Segre class, but changes the codimension by one). For details see §§5–6.

Incidentally, with $e_j :=$ multiplicity of the component C_j of $C := C(V,W)$, the Segre classes $s_i(C)$ can be defined by $s_i(C) := \sum_j e_j s(C_j)$ (cf. [Fn, p. 74, Lemma 4.2]; the sums here are "locally finite" with respect to decomposition into irreducible components [BH, p. 465, 1.7].) As above, the e_j are differential invariants; but because of the sign ambiguity in Theorem (6.3), $s_i(C)$ may not be a differential invariant—though its image in $H_*(W,\mathbb{Z}_2)$ is.

In particular, $s_0(C) = m(V,W)[W]$, where $m(V,W)$ is the multiplicity of W in V [Fn, §4.3]. Hence Theorem (6.3) implies that $m(V,W)$ *is a differential invariant.* (That is the main result of [GL], where a more straightforward proof is given).

1. Normal cones. We begin with a brief review of some facts about normal cones, facts which are "well-known" but not, as a whole, easily accessible in the literature.

(1.1) Let $(V,\mathcal{O}_V)$ be a reduced complex analytic space, and let $(W,\mathcal{O}_W)$ be a (not necessarily reduced) complex subspace of V. Let $\mathcal{I}$ be the kernel of the surjection $\mathcal{O}_V \twoheadrightarrow i_*\mathcal{O}_W$ corresponding to the inclusion $i\colon W \hookrightarrow V$. The graded $\mathcal{O}_W$-algebra $\mathrm{gr}_W(V) := \oplus_{m\ge 0}\, i^*(\mathcal{I}^m/\mathcal{I}^{m+1})$ is finitely presentable, since $i^*(\mathcal{I}^m/\mathcal{I}^{m+1})$ is coherent for all m [MT, p. 2, Prop. 1.4]. So one can define the *normal cone* $C(V,W)$ *of* V *along* W to be

$$C(V,W) := \mathrm{Specan}(\mathrm{gr}_W(V))\,.$$

(For the definition of Specan, see [Ho, p. 19-02].) This cone is naturally equipped with a map

$$p = p(V,W)\colon C(V,W) \to W,$$

together with a "vertex" section

$$\sigma = \sigma(V,W)\colon W \to C(V,W)$$

($p\circ\sigma =$ identity), corresponding, via functoriality of Specan, to the obvious maps $\mathcal{O}_W \leftrightarrows \mathrm{gr}_W(V)$. Moreover, with $\mathbb{C}^1$ the affine line there is the map

$$\mu\colon \mathbb{C}^1 \times C(V,W) \to C(V,W)$$

corresponding to the map of $\mathcal{O}_W$-algebras $\mathrm{gr}_W(V) \to \mathrm{gr}_W(V)[T]$ (T an indeterminate) whose restriction to $\mathcal{I}^m/\mathcal{I}^{m+1}$ is multiplication by T^m ($m \ge 0$).

One checks via the corresponding $\mathcal{O}_W$-algebra maps that there are commutative diagrams (with "id" standing for "identity" and "mpn" for "multiplication"):

$$\begin{array}{ccc}
\mathbb{C}^1 \times C(V,W) & \xrightarrow{\;\mu\;} & C(V,W) \\
{\scriptstyle \mathrm{pr}_2}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
C(V,W) & \xrightarrow[\;p\;]{} & W
\end{array}$$

$$\begin{array}{ccccc}
\mathbb{C}^1 \times (\mathbb{C}^1 \times C(V,W)) & \xrightarrow{\;\mathrm{id}\times\mu\;} & \mathbb{C}^1 \times C(V,W) & \xrightarrow{\;\mu\;} & C(V,W) \\
\Big\| & & & & \Big\| \\
(\mathbb{C}^1 \times \mathbb{C}^1) \times C(V,W) & \xrightarrow[\;\mathrm{mpn}\times\mathrm{id}\;]{} & \mathbb{C}^1 \times C(V,W) & \xrightarrow[\;\mu\;]{} & C(V,W)
\end{array}$$

$$\begin{array}{ccc}
\mathbb{C}^0 \times C(V,W) & \xrightarrow{\;\sim\;} & C(V,W) \\
{\scriptstyle 1\times\mathrm{id}}\big\downarrow & & \Big\| \\
\mathbb{C}^1 \times C(V,W) & \xrightarrow[\;\mu\;]{} & C(V,W)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{C}^0 \times C(V,W) & \xrightarrow{\;\sim\;} & C(V,W) \\
{\scriptstyle 0\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma\circ p} \\
\mathbb{C}^1 \times C(V,W) & \xrightarrow[\;\mu\;]{} & C(V,W)
\end{array}$$

Restricting attention to underlying point sets, if for $a \in \mathbb{C}$ and $x \in C(V,W)$ we set $ax := \mu(a,x)$, then

$$\begin{aligned} p(ax) &= p(x) \\ a_1(a_2x) &= (a_1a_2)x \\ 1x &= x \\ 0x &= \sigma p(x). \end{aligned}$$

Remark **(1.1.1)**. The foregoing holds with $\mathrm{gr}_W(V)$ replaced by any finitely presented graded $\mathcal{O}_W$-algebra $\mathcal{G} = \oplus_{m\ge 0}\,\mathcal{G}_m$ ($\mathcal{G}_0 = \mathcal{O}_W$, and every $\mathcal{G}_m$ is a coherent $\mathcal{O}_W$-module).

(1.2) To get a picture of $p\colon C(V,W) \to W$ near a point $w \in W$, we embed the triple (V,W,w) locally into some $\mathbb{C}^n$, as follows. In the local ring $\mathcal{O}_{V,w}$ let $(\tau_1,\tau_2,\dots,\tau_s)$ generate the ideal corresponding to the germ of W. Denoting convergent power series rings by $\mathbb{C}\langle\cdots\rangle$, pick a surjective $\mathbb{C}$-algebra homomorphism

$$\alpha\colon \mathbb{C}\langle T_1,T_2,\dots,T_{r+s}\rangle \twoheadrightarrow \mathcal{O}_{V,w} \qquad (T_i \text{ indeterminates})$$

such that $\alpha(T_{r+i}) = \tau_i\ \ (1 \le i \le s)$. Correspondingly, with $n := r+s$, there is an open neighborhood V^* of w in V, an open neighborhood U of 0 in $\mathbb{C}^n$, a holomorphic map $\theta\colon V^* \to U$, and holomorphic functions $\varphi_i\colon U \to \mathbb{C}$ $(i = 1,2,\dots,e)$ such that

(i) θ induces an isomorphism of V^* onto the reduced analytic subspace V' of U consisting of the common zeros of the φ_i:

$$V' := \{\, z \in U \mid \varphi_1(z) = \varphi_2(z) = \cdots = \varphi_e(z) = 0 \,\}.$$

(ii) θ maps $W^* := W \cap V^*$ isomorphically onto the analytic space

$$W' := L \cap V' = L \times_{\mathbb{C}^n} V' \subset V'$$

where L is the reduced r-dimensional space

$$L := \{\, (z_1,\dots,z_n) \in U \mid z_{r+1} = z_{r+2} = \cdots = z_n = 0 \,\}.$$

(iii) $\theta(w) = 0$.

The embedding θ induces an isomorphism

$$C(V,W) \times_W W^* = C(V^*,W^*) \xrightarrow{\ \sim\ } C(V',W')$$

compatible with the canonical maps p, σ, and μ. So let us simply consider the case where $V = V'$ and $W = W'$. Then $\mathcal{I} = \mathcal{J}\mathcal{O}_V$, where $\mathcal{J}$ is the $\mathcal{O}_U$-ideal generated by the coordinate functions $\xi_{r+1},\dots,\xi_n$ (i.e., $\xi_h(z_1,\dots,z_n) = z_h$).

With $j\colon L \hookrightarrow U$ the inclusion, there is an isomorphism of graded $\mathcal{O}_L$-algebras

$$\mathrm{gr}_L(U) := \oplus_{m\ge 0}\, j^*(\mathcal{J}^m/\mathcal{J}^{m+1}) \xrightarrow{\ \sim\ } \mathcal{O}_L[T_1,\dots,T_s]$$

whose inverse takes T_h to the section of $j^*(\mathcal{J}/\mathcal{J}^2)$ given by ξ_{r+h} $(1 \le h \le s)$; and so we have an isomorphism

$$C(U,L) \xrightarrow{\ \sim\ } (L \times \mathbb{C}^s) \subset (\mathbb{C}^r \times \mathbb{C}^s) = \mathbb{C}^n.$$

This isomorphism identifies $p(U,L)$ with the projection $\mathrm{pr}_1\colon L\times\mathbb{C}^s\to L$, and $\sigma(U,L)$ with the map $\mathrm{id}\times 0\colon L \xrightarrow{\sim} L\times\{0\}\hookrightarrow L\times\mathbb{C}^s$. Furthermore, we have the closed immersion

$$C(V,W)\hookrightarrow C(U,L) \tag{1.2.1}$$

corresponding to the natural surjection $\mathrm{gr}_L(U)\twoheadrightarrow \mathrm{gr}_W(V)$. There results a commutative diagram, whose horizontal arrows represent embeddings:

$$\begin{array}{ccc} C(V,W) & \longrightarrow & L\times\mathbb{C}^s\subset\mathbb{C}^n \\ {\scriptstyle p}\big\downarrow\big\uparrow{\scriptstyle \sigma} & & {\scriptstyle \mathrm{pr}_1}\big\downarrow\big\uparrow{\scriptstyle \mathrm{id}\times 0} \\ W & \longrightarrow & L \end{array} \tag{1.2.2}$$

The action of $\mathbb{C}^1$ on $C(V,W)$ (via μ) is induced by the action on $C(U,L)\cong L\times\mathbb{C}^s$, easily checked to be given on underlying point sets by

$$a(x,z)=(x,az)\qquad (a\in\mathbb{C},\ x\in L,\ z\in\mathbb{C}^s). \tag{1.2.3}$$

In particular, the analytic group $\mathbb{C}^*=\mathbb{C}^1-\{0\}$ acts freely on $C(V,W)-\sigma(W)$.

The points of $C(V,W)$—identified via (1.2.2) with a subvariety of $W\times\mathbb{C}^s$—can be specified by equations as follows. Let $w\in W\subset L$. For any open neighborhood N of w in L, for $x\in N$, and for any polynomial

$$F(T_1,\dots,T_s)\in\Gamma(N,\mathcal{O}_L)[T_1,\dots,T_s],$$

let $F_x\in\mathbb{C}[T_1,\dots,T_s]$ be the polynomial obtained from F by evaluating coefficients at x, and define the function $\widetilde{F}\colon N\times\mathbb{C}^s\to\mathbb{C}$ by

$$\widetilde{F}(x,y)=F_x(y_1,\dots,y_s)\qquad (x\in N,\ y\in\mathbb{C}^s).$$

Set $V_N := V\cap(N\times\mathbb{C}^s)$. (Recall that $V\subset\mathbb{C}^n=\mathbb{C}^r\times\mathbb{C}^s$.) Then:

(1.2.4) The point $(w,z)\in W\times\mathbb{C}^s$ is in $C(V,W)$ $\Leftrightarrow$ for every $m\ge 0$ and for every N and F as above with F homogeneous of degree m, if the function $\widetilde{F}|V_N$ is in $\Gamma(V_N,\mathcal{I}^{m+1})$ then $\widetilde{F}(w,z)=0$.

The *proof,* an exercise on the definition of Specan, is left to the reader.

Remark. The following "initial form" characterization (1.2.5), suggested by [Hi2, p. 18, Remark 3.2], is readily seen to be equivalent to the one in (1.2.4). For s-tuples $\nu=(\nu_1,\dots,\nu_s)$ of non-negative integers, we set $|\nu|:=\nu_1+\cdots+\nu_s$; and for $z=(z_1,\dots,z_s)\in\mathbb{C}^s$, we set $z^\nu := z_1^{\nu_1}z_2^{\nu_2}\dots z_s^{\nu_s}$.

(1.2.5) The point $(w,z)\in W\times\mathbb{C}^s$ is in $C(V,W)$ $\Leftrightarrow$ for all open neighborhoods N_1 of w in $\mathbb{C}^r$ and N_2 of 0 in $\mathbb{C}^s$, and for all $m\ge 0$, if the holomorphic functions $f_\nu\colon N_1\times N_2\to\mathbb{C}$ are such that $\sum_{|\nu|=m} f_\nu(x,y)y^\nu=0$ for all $(x,y)\in V\cap(N_1\times N_2)$, then $\sum_{|\nu|=m} f_\nu(w,0)z^\nu=0$.

(Equivalently: for all holomorphic functions $f\colon N_1\times N_2\to\mathbb{C}$ vanishing on $V\cap(N_1\times N_2)$ and such that $\lim_{t\to 0} t^{-m}f(x,ty)<\infty$ for all $(x,y)\in N_1\times N_2$, we have $\lim_{t\to 0} t^{-m}f(w,tz)=0$.)

(1.3) Now here is a geometric description of $C(V,W)$. As in (1.2), we identify (V,W) with $(V',W')\subset(U,W')\subset(\mathbb{C}^r\times\mathbb{C}^s,\,\mathbb{C}^r)$. We denote by π_f the projection $\mathbb{C}^r\times\mathbb{C}^s\to\mathbb{C}^s$ ("f" stands for "fiber").

Proposition. *The point* $(w,z)\in W\times\mathbb{C}^s = C(U,W)$ *is in* $C(V,W)$ *iff there exist sequences* $v_i\in V$, $a_i\in\mathbb{C}$ $(0<i\in\mathbb{Z})$ *such that* $v_i\to w$ *and* $a_i\pi_f v_i\to z$. *Moreover, for any* $(w,z)\in C(V,W)$, *there exist such* a_i, v_i *with all the* a_i *real and positive.*

Proof. Suppose that there are sequences $v_i\in V$, $a_i\in\mathbb{C}$, such that $v_i\to w$ and $a_i\pi_f v_i\to z$. Set $v_i=(x_i,y_i)$, so that $x_i\to w$, $y_i\to 0$, and $a_iy_i=a_i\pi_f v_i\to z$. With notation as in (1.2.5), we have then (assuming, as we may, that $v_i\in N_1\times N_2$):

$$\begin{aligned}\sum_{|\nu|=m} f_\nu(w,0)z^\nu &= \lim_i \sum_{|\nu|=m} f_\nu(x_i,y_i)(a_iy_i)^\nu\\ &= \lim_i a_i^m \sum_{|\nu|=m} f_\nu(x_i,y_i)(y_i)^\nu = 0.\end{aligned}$$

Thus $(w,z)\in C(V,W)$.

For the converse, we have the following stronger statement, due to Hironaka [Hi, p. 131, Remark (2.3)].

Lemma (1.3.1). *If* $(w,z)\in C(V,W)$ *and* $z\neq 0$, *then there exists a real analytic map* $\varphi\colon(-1,1)\to V$ *with* $\varphi(0)=w$, $\varphi(t)\notin W$ *if* $t\neq 0$, *and such that*

$$z/|z| = \lim_{t\to 0^+} \pi_f\varphi(t)/|\pi_f\varphi(t)|\,.$$

A variant of Hironaka's proof will be given below, in (2.3).

2. Specialization to the normal cone. With $i\colon W\hookrightarrow V$ and $\mathcal{I}$ as in (1.1), consider the graded $\mathcal{O}_V$-algebra

$$\mathcal{R}=\mathcal{R}_{\mathcal{I}} := \oplus_{n\in\mathbb{Z}}\mathcal{I}^nT^{-n}\subset\mathcal{O}_V[T,T^{-1}]$$

where T is an indeterminate and $\mathcal{I}^n$ is defined to be $\mathcal{O}_V$ for all $n\le 0$. By [MT, p. 2, Prop. 1.4], $\mathcal{R}$ is finitely presentable, so we can set

$$\mathbf{V}=\mathbf{V}_W := \operatorname{Specan}(\mathcal{R}_{\mathcal{I}})\,.$$

$\mathbf{V}$ is called the *specialization of* (V,W) *to* $C(V,W)$, see [LT, pp. 556–557].

The terminology is explained as follows. We have natural maps

$$W\times\mathbb{C}^1\xrightarrow{\alpha}\mathbf{V}\xrightarrow{\beta}V\times\mathbb{C}^1$$

where α is the closed immersion corresponding to the obvious $\mathcal{O}_V$-algebra homomorphism

$$\mathcal{R}\twoheadrightarrow\mathcal{R}/\mathcal{I}T^{-1}\mathcal{R}\xrightarrow{\ \sim\ }\oplus_{n\ge 0}\,(\mathcal{O}_V/\mathcal{I})T^n = i_*\mathcal{O}_W[T]\,,$$

and β corresponds to the $\mathcal{O}_V$-algebra inclusion $\mathcal{O}_V[T]\hookrightarrow\mathcal{R}$. Note that $\beta\circ\alpha$ is the closed immersion $i\times\mathrm{id}\colon W\times\mathbb{C}^1\hookrightarrow V\times\mathbb{C}^1$. Let $\mathfrak{t}$ be the composition

$$\mathfrak{t}\colon\mathbf{V}\xrightarrow{\beta}V\times\mathbb{C}^1\xrightarrow{\mathrm{pr}}\mathbb{C}^1.$$

Denote the fiber $\mathfrak{t}^{-1}(0)$ by $\mathbf{V}_0$.

Proposition (2.1). (i) *The map* $\mathfrak{t}$ *is* flat.
(ii) β *induces an isomorphism of* $\mathbf{V} - \mathbf{V}_0$ *onto* $V \times (\mathbb{C}^1 - \{0\})$.
(iii) *There is a natural commutative diagram*

$$\begin{array}{ccccc} W & \xrightarrow{\ \sigma\ } & C(V,W) & \xrightarrow{\ p\ } & W \\ {\scriptstyle (\mathrm{id},0)}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle (i,0)} \\ W \times \{0\} & \xrightarrow[\alpha]{} & \mathbf{V}_0 & \xrightarrow[\beta]{} & V \times \mathbb{C}^1 \end{array}$$

with σ *and* p *as in* (1.1), *and* ρ *an isomorphism.*

Thus $\mathfrak{t}$ *gives us a flat family of closed immersions, isomorphic to* $i\colon W \hookrightarrow V$ *wherever* $\mathfrak{t} \neq 0$ *and to* $\sigma\colon W \hookrightarrow C(V,W)$ *where* $\mathfrak{t} = 0$.

Proof. We have $\mathrm{pr}^{-1}(0) = V \times \{0\} = \mathrm{Specan}(\mathcal{O}_V[T]/T\mathcal{O}_V[T])$, and it follows that $\mathbf{V}_0 = \mathrm{Specan}(\mathcal{R}/T\mathcal{R}) \subset \mathrm{Specan}(\mathcal{R})$. But there is an obvious isomorphism $\mathcal{R}/T\mathcal{R} \xrightarrow{\sim} \oplus_{n\geq 0}\, \mathcal{I}^n/\mathcal{I}^{n+1}$, whence an isomorphism $\rho\colon C(V,W) \xrightarrow{\sim} \mathbf{V}_0$.

The surjection $\mathcal{R}/T\mathcal{R} \twoheadrightarrow \mathcal{R}/(T\mathcal{R} + \mathcal{I}T^{-1}\mathcal{R})$ is naturally identifiable with the obvious surjection of $\oplus_{n\geq 0}\, \mathcal{I}^n/\mathcal{I}^{n+1}$ onto its degree 0 component $i_*\mathcal{O}_W$; thus the restriction of α to $\mathbf{V}_0$ gets identified with $\sigma\colon W \hookrightarrow C(V,W)$, and so the left square in (iii) commutes. The right square commutes because it is obtained by applying the functor Specan to a (clearly) commutative diagram of graded $\mathcal{O}_V$-algebras.

A morphism of analytic spaces $f\colon X \to \mathbf{V}$ factors through $\mathbf{V} - \mathfrak{t}^{-1}(0)$ iff the corresponding map $\Gamma(\mathbf{V},\mathcal{R}) \to \Gamma(X,\mathcal{O}_X)$ sends T to a unit, i.e., $\mathcal{R} \to f_*\mathcal{O}_X$ factors through $\mathcal{R}[T^{-1}]$. Consequently

$$\mathbf{V} - \mathbf{V}_0 = \mathrm{Specan}(\mathcal{R}[T^{-1}]) = \mathrm{Specan}(\mathcal{O}_V[T,T^{-1}])\,,$$

and (ii) follows.

In particular, off $\mathbf{V}_0$ the map $\mathfrak{t}$ coincides with the projection pr, which is flat. Since T is not a zero-divisor in $\mathcal{R}$, therefore the germ of $\mathfrak{t}$ in the local ring of any point on $\mathbf{V}_0$ is not a zero-divisor (see e.g., [Ho, p. 19-07, Corollaire]), and so $\mathfrak{t}$ is flat everywhere along $\mathbf{V}_0$ too. This proves (i). □

(2.2) Now let us see how the above specialization looks locally.

Assume as in (1.2) that $(V,W) \subset (\mathbb{C}^{r+s},\mathbb{C}^r)$. Let $\xi_1,\dots,\xi_{r+s}$ be the coordinate functions on $\mathbb{C}^{r+s}$, and for $i = 1,2,\dots,s$, set $\eta_i := \xi_{r+i}|V$. We embed $\mathbf{V}$ into $\mathbb{C}^{r+s+1}$ as follows. There is a surjective $\mathcal{O}_V$-algebra homomorphism

$$\psi\colon \mathcal{O}_V[T'_1,\dots,T'_s,T] \twoheadrightarrow \mathcal{R}$$

with

$$\psi(T'_i) = \eta_i T^{-1} \quad (1 \leq i \leq s), \qquad \psi(T) = T.$$

Correspondingly, there is an embedding $\mathbf{V} \hookrightarrow V \times \mathbb{C}^{s+1} \hookrightarrow \mathbb{C}^r \times \mathbb{C}^s \times \mathbb{C}^{s+1}$. But for each i, $\eta_i - T'_iT$ is a global section of the kernel of ψ; therefore the embedding factors through the subspace of $\mathbb{C}^r \times \mathbb{C}^s \times \mathbb{C}^{s+1}$ where these functions vanish, i.e., the reduced subspace whose points are of the form $(x_1,\dots,x_r,ay_1,\dots,ay_s,y_1,\dots,y_s,a)$, a subspace isomorphic to $\mathbb{C}^{r+s+1}$.

With $\mathbf{V}$ so regarded as a subspace of $\mathbb{C}^{r+s+1}$, the maps $\alpha\colon W\times\mathbb{C}^1\to\mathbf{V}$ and $\beta\colon\mathbf{V}\to V\times\mathbb{C}^1$ are given on underlying point sets by

$$\alpha(x_1,\dots,x_r,a)=(x_1,\dots,x_r,0,\dots,0,a)$$
$$\beta(x_1,\dots,x_r,y_1\dots,y_s,a)=(x_1,\dots,x_r,ay_1,\dots,ay_s,a)\,.$$

The map $\mathfrak{t}$ is induced by projection to the last coordinate. For $a\neq 0$, β maps the fiber $\mathbf{V}_a:=\mathfrak{t}^{-1}(a)$ isomorphically onto $V\times\{a\}$, i.e.,

(2.2.1) $$\mathbf{V}_a=\{\,(x_1,\dots,x_r,y_1\dots,y_s,a)\mid(x_1,\dots,x_r,ay_1,\dots,ay_s)\in V\,\}\,.$$

The embedding of $C(V,W)=\mathbf{V}_0$ in $\mathbf{V}\subset\mathbb{C}^r\times\mathbb{C}^s\times\mathbb{C}^{s+1}$ arises from the surjection $\overline{\psi}$ obtained from ψ by modding out T. This $\overline{\psi}$ factors as

$$\mathcal{O}_V[T_1',\dots,T_s']\twoheadrightarrow\mathcal{O}_W[T_1',\dots,T_s']\twoheadrightarrow\mathcal{R}/T\mathcal{R}\,.$$

Comparing this embedding to (1.2.1), we find that the underlying point set of $\mathbf{V}_0$ consists of all $(w,0,z,0)\in\mathbb{C}^r\times\mathbb{C}^s\times\mathbb{C}^s\times\mathbb{C}^1$ with $(w,z)\in\mathbb{C}(V,W)$, where $C(V,W)$ is regarded as being embedded into $\mathbb{C}^r\times\mathbb{C}^s$ as in (1.2); and then passing as above from $\mathbb{C}^r\times\mathbb{C}^s\times\mathbb{C}^s\times\mathbb{C}^1$ to $\mathbb{C}^{r+s+1}$, we can write

(2.2.2) $$\mathfrak{t}^{-1}(0)=\mathbf{V}_0=\{\,(w,z,0)\in\mathbb{C}^r\times\mathbb{C}^s\times\mathbb{C}^1\mid(w,z)\in C(V,W)\,\}\,.$$

(2.3). *To prove (1.3.1),* we first note that since $\mathfrak{t}$ is flat, therefore $\mathbf{V}_0$ is nowhere dense in $\mathbf{V}$, so that for any point $(w,z,0)\in\mathbf{V}_0$, there exists an analytic map

$$\phi\colon\mathbb{D}\to\mathbf{V}\qquad(\mathbb{D}:=\text{unit disc in }\mathbb{C}^1)$$

such that
$$\phi(\mathbb{D}-\{0\})\subset\mathbf{V}-\mathbf{V}_0\quad\text{and}\quad\phi(0)=(w,z,0)\,.$$

(This follows, e.g., from the Nullstellensatz and from the algebraic fact that in a noetherian local ring A—like the stalk at $(w,z,0)$ of $\mathcal{O}_{\mathbf{V}}$—any prime ideal is the intersection of all prime ideals $\wp$ containing it and such that $\dim(A/\wp)=1$.) Set

$$\phi(\xi)=(\lambda(\xi),\mu(\xi),\tau(\xi))\in\mathbb{C}^r\times\mathbb{C}^s\times\mathbb{C}^1\qquad(\xi\in\mathbb{D}).$$

For ξ sufficiently small, $\tau(\xi)$ is given by a convergent power series

$$\tau(\xi)=a\xi^q+a_1\xi^{q+1}+\dots\qquad(a\neq 0,\ q>0).$$

With $b\in\mathbb{C}$ such that ab^q is real and positive, we have then, for real $t>0$:

(2.3.1) $$\lim_{t\to 0^+}\tau(bt)/|\tau(bt)|=1.$$

Assuming, as we may, that $|b|=1$, consider the real analytic map $\varphi\colon(-1,1)\to V$ given by
$$\varphi(t)=\big(\lambda(bt),\ \mu(bt)\tau(bt)\big)\in V\qquad\big(t\in(-1,1)\big),$$
see (2.2.1). Then $\pi_f\varphi(t)=\mu(bt)\tau(bt)$; and since $\mu(0)=z$ and $\tau(0)=0$, (1.3.1) results from (2.3.1). $\square$

3. Differential functoriality of the specialization over $\mathbb{R}$. In this section we look at maps of analytic spaces primarily in terms of underlying topological spaces.

(3.1). Let $\mathfrak{t}\colon \mathbf{V} \to \mathbb{C}$ be the specialization of (V, W) to $C(V, W)$, see §2. The *specialization over* $\mathbb{R}$ (or $\mathbb{R}$-*specialization*) of (V, W) to $C(V, W)$ is the real analytic space

$$ {}_{\mathbb{R}}\mathbf{V} := \mathbb{R} \times_{\mathbb{C}} \mathbf{V} = \mathfrak{t}^{-1}(\mathbb{R}). $$

As in §2, we have natural maps

$$ W \times \mathbb{R} \xrightarrow{\alpha} {}_{\mathbb{R}}\mathbf{V} \xrightarrow{\beta} V \times \mathbb{R}\,. $$

The fibers $\mathbf{V}_a := \mathfrak{t}^{-1}(a)$ $(a \in \mathbb{R})$ of $\mathfrak{t}\colon {}_{\mathbb{R}}\mathbf{V} \to \mathbb{R}$ are all real-isomorphic, via β, to V, except for $\mathbf{V}_0 \cong C(V, W)$.

(3.2) Let (V, W) be as in (1.1), and let (V', W') be another such pair. Define $\mathfrak{t}'\colon {}_{\mathbb{R}}\mathbf{V}' \to \mathbb{R}$ as above (with respect to $W' \subset V'$).

Let $f\colon V \to V'$ be a C^1 (continuously differentiable) map such that $f(W) \subset W'$.

We recall the definition of C^1 map. A map $g\colon V \to \mathbb{R}^n$ is C^1 at $v \in V$ if for some analytic germ-embedding $(V, v) \hookrightarrow (\mathbb{C}^N, 0)$, there is an open neighborhood U of 0 in $\mathbb{C}^N$ and a C^1 map $U \to \mathbb{R}^n$ whose restriction to $V \cap U$ coincides with that of g. A germ-map $\gamma\colon (V, v) \to (V', v')$ is C^1 if its composition with some embedding $(V', v') \hookrightarrow (\mathbb{C}^M, 0)$ is C^1 at v. (If this property of γ holds for one choice of embeddings then it holds for any choice.) Finally, the above map f is C^1 if its germ at each $v \in V$ is C^1.

Define the C^1 map $\mathbf{f}\colon {}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0 \longrightarrow {}_{\mathbb{R}}\mathbf{V}' - \mathbf{V}'_0$ to be the composition

$$ {}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0 \xrightarrow[\beta]{\sim} V \times (\mathbb{R}^1 - \{0\}) \xrightarrow[f \times \mathrm{id}]{} V' \times (\mathbb{R}^1 - \{0\}) \xrightarrow[\beta'^{-1}]{\sim} {}_{\mathbb{R}}\mathbf{V}' - \mathbf{V}'_0. $$

Theorem (3.3). *With preceding notation, assume further that W is a complex submanifold of the analytic space V. Then the map $\mathbf{f}$ has a unique extension to a continuous map (still denoted $\mathbf{f}$)$\colon {}_{\mathbb{R}}\mathbf{V} \to {}_{\mathbb{R}}\mathbf{V}'$; and the following diagram commutes:*

$$ (3.3.1) \qquad \begin{array}{ccccc} W \times \mathbb{R} & \xrightarrow{\alpha} & {}_{\mathbb{R}}\mathbf{V} & \xrightarrow{\beta} & V \times \mathbb{R} \\ {\scriptstyle f \times \mathrm{id}}\downarrow & & {\scriptstyle \mathbf{f}}\downarrow & & \downarrow{\scriptstyle f \times \mathrm{id}} \\ W' \times \mathbb{R} & \xrightarrow[\alpha']{} & {}_{\mathbb{R}}\mathbf{V}' & \xrightarrow[\beta']{} & V' \times \mathbb{R} \end{array} $$

In particular, $\mathfrak{t}' \circ \mathbf{f} = \mathfrak{t}$. The restriction $\mathbf{f}_0$ of $\mathbf{f}$ to $\mathbf{V}_0 = C(V, W)$ is a continuous map from $C(V, W)$ to $C(V', W')$, fitting into a commutative diagram

$$ (3.3.2) \qquad \begin{array}{ccc} \mathbb{R} \times C(V, W) & \xrightarrow{\mathrm{id} \times \mathbf{f}_0} & \mathbb{R} \times C(V', W') \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ C(V, W) & \xrightarrow{\mathbf{f}_0} & C(V', W') \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle \sigma} & & {\scriptstyle p'}\downarrow\uparrow{\scriptstyle \sigma'} \\ W & \xrightarrow[f]{} & W' \end{array} $$

see (1.1), *and for each $w \in W$, the restriction of $\mathbf{f}_0$ to $p^{-1}(w)$ is real-analytic.*[2]

[2]See also Remark (5.4.3) below.

Proof. The assertions need only be verified near an arbitrary point $\nu \in \mathbf{V}_0$, so we can introduce coordinates as in (2.2). To be more precise, let $\pi\colon \mathbf{V} \to V$ be the canonical map, corresponding to the inclusion $\mathcal{O}_V \hookrightarrow \mathcal{R}$; and define $\pi'\colon \mathbf{V}' \to V'$ similarly. Let $w := \pi(\nu) = p\rho^{-1}(\nu) \in W$ (see (2.1), noting that π is β followed by the projection $V \times \mathbb{C}^1 \to V$), and let $w' := f(w) \in W'$. Choose neighborhoods V^* of w in V and V'^* of w' in V' such that $f(V^*) \subset V'^*$ and such that $(V^*, W \cap V^*, w)$ and $(V'^*, W' \cap V'^*, w')$ can be embedded into $(\mathbb{C}^r \times \mathbb{C}^s, \mathbb{C}^r, 0)$ and $(\mathbb{C}^{r'} \times \mathbb{C}^{s'}, \mathbb{C}^{r'}, 0)$ respectively, as in (1.2). Then $\mathbf{V}^* := \pi^{-1}(V^*)$ is the specialization of V^* to $C(V^*, W \cap V^*)$. From the definition of $\mathbf{f}$ and the relation between β and π, we see that $f\pi = \pi'\mathbf{f}$, so that $\mathbf{f}$ maps ${}_{\mathbb{R}}\mathbf{V}^* - \mathbf{V}_0$ into $\mathbf{V}'^* := \pi'^{-1}(V'^*)$. Hence we may—and do—assume that $(V, V', \mathbf{V}, \mathbf{V}') = (V^*, V'^*, \mathbf{V}^*, \mathbf{V}'^*)$, coordinatized as in (1.2) and (2.2). We may assume further, because W is a *submanifold* of V, that W is actually identical with the flat space L in (1.2).

Uniqueness of the extension holds because ${}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0$ is dense in ${}_{\mathbb{R}}\mathbf{V}$, as follows via (2.2.1) and (2.2.2) from Proposition (1.3): setting $v_i = (x_i, y_i)$ there, and with a_i real and positive, the sequence $(x_i, a_i y_i, a_i^{-1})$ in ${}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0$ has limit $(w, z, 0)$. (Since $y_i \to 0$, therefore $a_i \to \infty$ if $z \neq 0$; and if $z = 0$ then we can take $y_i = 0$ and $a_i = i$ for all i.)

Commutativity of the right half of (3.3.1) can be checked on the dense set ${}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0$, where it holds by the definition of $\mathbf{f}$. The left half can be also be checked outside of $\mathbf{V}_0$ (since $W \times (\mathbb{R} - \{0\})$ is dense in $W \times \mathbb{R}$), and there it is obvious because β' is bijective and $\beta \circ \alpha = i \times \mathrm{id}$, etc., see §2.

Now let us show that the asserted extension of $\mathbf{f}$ exists. The question comes down to the existence, for each $\nu \in \mathbf{V}_0$, of a point $\nu' \in \mathbf{V}'$ such that every sequence $(\nu_i)_{i>0}$ in ${}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0$ with $\nu_i \to \nu$ satisfies $\lim \mathbf{f}(\nu_i) = \nu'$. After embedding ${}_{\mathbb{R}}\mathbf{V}$ into $\mathbb{C}^r \times \mathbb{C}^s \times \mathbb{R}^1$ as above, we have $\nu = (0, z, 0)$ for some $z \in \mathbb{C}^s$, and $\nu_i = (x_i, y_i, a_i)$. The description of β preceding (2.2.1) gives an expression for $\mathbf{f}$ in coordinates:

$$\mathbf{f}(x, y, a) = (\xi, a^{-1}\eta, a), \quad \text{where} \;\; (\xi, \eta) := f(x, ay).$$

The question thus becomes whether the sequence $\mathbf{f}(x_i, y_i, a_i) = (\xi_i, a_i^{-1}\eta_i, a_i)$ has a limit depending only on z. Since $x_i \to 0$, $y_i \to z$, and $a_i \to 0$, and since f is continuous, therefore $(\xi_i, \eta_i) \to f(0,0) = (0,0)$, so that $\xi_i \to 0$. It remains to investigate $\lim a_i^{-1}\eta_i$.

By the definition of C^1 map, there exists a neighborhood U^* of $(0,0)$ in $\mathbb{C}^r \times \mathbb{C}^s$ and a C^1 map $F\colon U^* \to \mathbb{C}^{r'} \times \mathbb{C}^{s'}$ agreeing with f on $V \cap U^*$. To simplify, we multiply F by a C^∞ function $\psi\colon \mathbb{C}^r \times \mathbb{C}^s \to \mathbb{R}$ which takes the value 1 on a small neighborhood U_1 of $(0,0)$ and vanishes outside a compact subset $\bar{U}$ of U^*; then after replacing V by $V \cap U_1$, and F by the extension of ψF which takes the value $(0,0)$ outside $\bar{U}$, we may assume that $U^* = \mathbb{C}^r \times \mathbb{C}^s$. We may also assume that $F(\mathbb{C}^r \times \{0\}) \subset \mathbb{C}^{r'} \times \{0\}$ (take $U_1 \subset U$ where U is as in (1.2), recall that $L = W$, see above, and that $f(W) \subset W'$).

Denote the derivative of F at (x, y)—a real-linear map from $\mathbb{C}^r \times \mathbb{C}^s$ to $\mathbb{C}^{r'} \times \mathbb{C}^{s'}$—by $DF_{(x,y)}$. Set $F(x_i, 0) =: (x'_i, 0)$. Let $\mathrm{pr}_2\colon \mathbb{C}^{r'} \times \mathbb{C}^{s'} \to \mathbb{C}^{s'}$ be the projection, let q^j $(1 \le j \le 2s')$ be the real coordinate functions on $\mathbb{C}^{s'}$, and set $F^j := q^j \circ \mathrm{pr}_2 \circ F$. We are concerned with the limits (as $i \to \infty$):

$$\lim_i q^j(a_i^{-1}\eta_i) = \lim_i a_i^{-1} q^j \mathrm{pr}_2\big((\xi_i, \eta_i) - (x'_i, 0)\big) = \lim_i a_i^{-1}\big(F^j(x_i, a_i y_i) - F^j(x_i, 0)\big).$$

But a_i being *real,* the Mean Value Theorem gives

$$\lim_i a_i^{-1}\big(F^j(x_i,a_iy_i)-F^j(x_i,0)\big)=\lim_i DF^j_{(x_i,\,b_{ij}a_iy_i)}(0,y_i)\qquad (0<b_{ij}<1)$$
$$=DF^j_{(0,0)}(0,z),$$

the last equality by continuity of DF (needed only at points of W). Thus, the extended $\mathbf{f}$ exists.

It is clear that $\mathbf{f}$ maps $\mathbf{V}_0$ into $\mathbf{V}'_0$. Commutativity of (3.3.2) follows, via (2.2.2), (1.2.2), and (1.2.3), from the description of $\mathbf{f}_0$ entailed by the foregoing, viz.

$$\mathbf{f}_0(0,z,0)=\big(0,\mathrm{pr}_2 DF_{(0,0)}(z),0\big)\,. \tag{3.3.3}$$

This description also shows that the restriction of $\mathbf{f}_0$ to $p^{-1}(w)$ is real-analytic (even *real-linear* in these coordinates). □

For any subvariety (i.e., reduced analytic subspace) V_1 of V, set $W_1 := W\times_V V_1$, so that the deformation of V_1 to $C(V_1,W_1)$ is canonically embedded in $\mathbf{V}$. If in the preceding proof we have $(0,z,0)\in C(V_1,W_1)$, then by (1.3), we can choose $(x_i,y_i,a_i)\to(0,z,0)$ such that $(x_i,a_iy_i)\in V_1$, and consequently:

Corollary (3.4). *If V_1 and V'_1 are subvarieties of V and V' respectively, and if $f(V_1)\subset V'_1$, then $\mathbf{f}_0$ maps $C(V_1,W_1)$ continuously into $C(V'_1,W'_1)$.*

Remark **(3.5).** The same proof as in (3.3) shows that the C^1 map $\mathbf{F}$ defined by

$$\mathbf{F}(x,y,a):=(\xi,a^{-1}\eta,a)\qquad \big((\xi,\eta):=F(x,ay)\big)\qquad (x\in\mathbb{C}^r,\ y\in\mathbb{C}^s,\ 0\neq a\in\mathbb{R})$$

extends continuously to a map (still denoted $\mathbf{F}$) from $\mathbb{C}^r\times\mathbb{C}^s\times\mathbb{R}$ to $\mathbb{C}^{r'}\times\mathbb{C}^{s'}\times\mathbb{R}$ such that $\mathfrak{t}'\circ\mathbf{F}=\mathfrak{t}$, where now $\mathfrak{t}$ and $\mathfrak{t}'$ denote the respective projections to $\mathbb{R}$.

4. Multiplicities of components of $C(V,W)$. By *component* of a complex analytic space Z (not necessarily reduced) is meant an irreducible component of the reduced space Z_{red}. Let Y be such a component, with inclusion map $j\colon Y\hookrightarrow Z$, and let $\mathcal{P}$ be the defining $\mathcal{O}_Z$-ideal of Y, i.e., the kernel of the natural map $\mathcal{O}_Z\to j_*\mathcal{O}_Y$. Let y be any point of Y. Then the stalk $\mathcal{P}_y$ is an intersection of finitely many minimal prime ideals P_i in $\mathcal{O}_{Z,y}$, corresponding to the local components of Y at y.

Proposition-Definition (4.1). *The length e of the local artin ring $(\mathcal{O}_{Z,y})_{P_i}$ depends only on Y, and not on y or P_i. This integer is called the* multiplicity *of Y in Z, and denoted $e_{Y,Z}$.*

Proof. For each $n\ge 0$, set $\mathcal{G}_n := j^*(\mathcal{P}^n/\mathcal{P}^{n+1})$, a coherent $\mathcal{O}_Y$-module. Let k be the residue field of $(\mathcal{O}_{Z,y})_{P_i}$, and set

$$e_n := \dim_k(\mathcal{G}_{n,y})\otimes_{\mathcal{O}_{Y,y}} k,$$

so that $e_n=0$ for $n\gg 0$ and $e=\sum_{n=0}^{\infty}e_n$. Then by [Ho, p. 20-10, Prop. 6] the module $\mathcal{G}_n$ is locally free of rank e_n outside a nowhere dense analytic subspace of Y. Thus e_n (for given n), and hence e, depends only on Y. □

Remark **(4.1.1).** If U is an open subset of Z, and $Y_1,\dots,Y_r$ are the components of $Y\cap U$, then clearly $e_{Y_i,U}=e_{Y,Z}$ for all i.

(4.2). Let (V,W) be as in (1.1), and let $(V_\lambda)_{\lambda\in\Lambda}$ be the family of all components of V. For each λ, let $W_\lambda := (W\times_V V_\lambda)\subset V_\lambda$, and let $\mathcal{I}_\lambda$ be the defining $\mathcal{O}_{V_\lambda}$-ideal of W_λ. Set

$$\mathcal{R}_\lambda := \oplus_{n\in\mathbb{Z}}\mathcal{I}_\lambda^n T^{-n}\subset \mathcal{O}_{V_\lambda}[T,T^{-1}]$$

and

$$\mathbf{V}_\lambda := \operatorname{Specan}(\mathcal{R}_\lambda),$$

the specialization of (V_λ,W_λ) to $C(V_\lambda,W_\lambda)$. With notation as in §2, there is an obvious commutative diagram, whose vertical arrows are closed immersions:

$$\text{(4.2.1)}\qquad \begin{array}{ccccc} W_\lambda\times\mathbb{C}^1 & \xrightarrow{\alpha_\lambda} & \mathbf{V}_\lambda & \xrightarrow{\beta_\lambda} & V_\lambda\times\mathbb{C}^1\\ \downarrow & & \downarrow & & \downarrow\\ W\times\mathbb{C}^1 & \xrightarrow[\alpha]{} & \mathbf{V} & \xrightarrow[\beta]{} & V\times\mathbb{C}^1 \end{array}$$

The $\mathbf{V}_\lambda$ are all the components of $\mathbf{V}$: this need only be verified outside the nowhere dense analytic subset $\mathbf{V}_0$, where it follows from (2.1)(ii).

With $\mathfrak{t}_\lambda$ the restriction of $\mathfrak{t}$ to $\mathbf{V}_\lambda$ we have

$$C(V,W)=\mathfrak{t}^{-1}(0)=\bigcup_\lambda \mathfrak{t}_\lambda^{-1}(0)=\bigcup_\lambda C_\lambda(V_\lambda,W_\lambda).$$

Now W is covered by open subsets $U\subset V$ meeting only finitely many V_λ, and for such a U, $p^{-1}(U)\subset C(V,W)$ meets $C(V_\lambda,W_\lambda)$ only for those same λ; so the family $C(V_\lambda,W_\lambda)$ is locally finite in $C(V,W)$. Hence every component of $C(V,W)$ is a component of $C(V_\lambda,W_\lambda)$ for at least one and at most finitely many λ. Conversely, if $\dim V_\lambda=\dim V$ then every component of $C(V_\lambda,W_\lambda)$ is a component of $C(V,W)$ (since $\dim C(V,W)=\dim V$, by (2.1)).

Proposition (4.2.2). *Assume that V is equidimensional, i.e., all the components V_λ of V have the same dimension. Let C_* be a component of $C(V,W)$. Then*

$$e_{C_*,C(V,W)}=\sum_\lambda{}^{*} e_{C_*,C(V_\lambda,W_\lambda)}$$

the sum being over all λ such that C_* is a component of $C(V_\lambda,W_\lambda)$.

Proof. Note that after fixing $y\in C_*$ we can replace V by any open subset V^* containing $p(y)$ ($p\colon C(V,W)\to W$ the canonical map): first, by (4.1.1), the component of $C_*\cap p^{-1}(W\cap V^*)$ containing y has multiplicity $e_{C_*,C(V,W)}$ in $p^{-1}(W\cap V^*)=C(V^*,W\cap V^*)$, and similarly for $C(V_\lambda\cap V^*,W_\lambda\cap V^*)$; and second, though $V_\lambda\cap V^*$ may no longer be irreducible, that doesn't matter because (4.2.2) is clearly equivalent to a similar statement in which we assume only that $V=\cup V_\lambda$ where each V_λ is a union of components of V (all having the same dimension as V) and no two V_λ have a common component. So pick V^* as in (1.2), and embed $\mathbf{V}$ in $\mathbb{C}^{r+s+1}$ as in (2.2).

Now let B_* be a local component of C_* at y, and let P be the prime ideal in $\mathcal{O}_{\mathbf{V},y}$ consisting of germs of functions vanishing on B_*. Let $t \in \mathcal{O}_{\mathbf{V},y}$ be the germ of the function $\mathbf{t}\colon \mathbf{V} \to \mathbb{C}$, so that $\mathcal{O}_{\mathbf{V},y}/(t) = \mathcal{O}_{C(V,W),y}$, see (2.1). Then $e_{C_*,C(V,W)}$ is, by definition, the length of the artin local ring $\big(\mathcal{O}_{\mathbf{V},y}/(t)\big)_P$, i.e., (since $\mathbf{t}$ is flat and hence t is not a zero-divisor in $\mathcal{O}_{\mathbf{V},y}$) the multiplicity of the ideal $t(\mathcal{O}_{\mathbf{V},y})_P$. But by the equality of algebraic and topological intersection numbers (see e.g., [GL, p. 184, Fact]), that multiplicity is the intersection number $i\big((\mathbb{C}^{r+s} \times \{0\}) \cdot \mathbf{V}, C_*\big)$ defined in [BH, p. 482, 4.4]. (The intersection takes place in $\mathbb{C}^{r+s+1}$.) Similarly, with $\mathbf{V}_\lambda \subset \mathbf{V}$ as in (4.2.1), we have $e_{C_*,C(V_\lambda,W_\lambda)} = i\big((\mathbb{C}^{r+s} \times \{0\}) \cdot \mathbf{V}_\lambda, C_*\big)$. So the conclusion results from the equality

$$i\big((\mathbb{C}^{r+s} \times \{0\}) \cdot \mathbf{V}, C_*\big) = \sum_{\lambda}{}^{*}\, i\big((\mathbb{C}^{r+s} \times \{0\}) \cdot \mathbf{V}_\lambda, C_*\big)$$

given in [BH, p. 483]. □

(4.3) Suppose next that we have two equidimensional reduced analytic spaces V and V', along with complex submanifolds $W \subset V$ and $W' \subset V'$. We consider a situation as in §3, where there is a C^1 map $f\colon (V,W) \to (V',W')$; and we assume that f is *invertible,* i.e., that there is a C^1 map $g\colon (V',W') \to (V,W)$ such that $f \circ g$ and $g \circ f$ are both identity maps. Then by Theorem (3.3), f and g naturally induce inverse homeomorphisms $\mathbf{f}$ and $\mathbf{g}$ between $\mathbf{V}$ and $\mathbf{V}'$, restricting to homeomorphisms $\mathbf{f}_0$ and $\mathbf{g}_0$ between the respective subspaces $C(V,W)$ and $C(V',W')$.[3]

Theorem (4.3.1). *Under the preceding circumstances, the homeomorphism* $\mathbf{f}_0$ *gives a one-one multiplicity-preserving correspondence between the components of* $C(V,W)$ *and those of* $C(V',W')$.

Proof. The one-one correspondence obtains because any homeomorphism of analytic spaces maps each component of the source onto a component of the target, [GL, p. 172, (A8)]. We need to show that corresponding components C_* and C'_* have the same multiplicity (in $C(V,W)$, $C(V',W')$ respectively). The proof which follows is essentially the same as that in [GL, §D], to which we refer for more details.

Let $y \in C_* \subset \mathbf{V}_0$, and, $\pi\colon \mathbf{V} \to V$ being the canonical map, let $v := \pi(y)$. Using (4.1.1), and arguing as in the beginning of the proof of (3.3), we find that we may replace $\mathbf{V}$ by $\pi^{-1}(V^*)$ where V^* is an "embeddable" neighborhood of v (i.e., V^* is as in (1.2)) such that $V'^* := f(V^*)$ is also embeddable; and we may replace $\mathbf{V}'$ by $\pi^{-1}(V'^*)$. Thus we reduce to where V and V' are embedded in some $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ respectively, with $W = L$, see (1.2), and similarly for W'. Then as in the proof of (3.3) we can assume, after replacing V by a smaller neighborhood of v if necessary, that there is a C^1 map $F_n\colon \mathbb{C}^n \to \mathbb{C}^{n'}$ agreeing with f on V; and similarly assume that there is a C^1 map $G_{n'}\colon \mathbb{C}^{n'} \to \mathbb{C}^n$ agreeing with g on V'. We

[3]Continuity of the derivative of f (resp. g) need only hold at points of W (resp. W'), see proof of (3.3).

then define inverse C^1 maps

$$\mathbb{C}^n \times \mathbb{C}^{n'} \underset{G}{\overset{F}{\rightleftarrows}} C^{n'} \times \mathbb{C}^n$$

by

$$F(x,y) := \big(y + F_n(x),\, x - G_{n'}(y + F_n(x))\big),$$
$$G(x',y') := \big(y' + G_{n'}(x'),\, x' - F_n(y' + G_{n'}(x'))\big),$$

and verify that for $x \in V$ (resp. $x' \in V'$) we have

$$F(x,0) = \big(f(x),0\big) \quad \text{resp.} \quad G(x',0) = \big(g(x'),0\big).$$

Hence, if we embed V and V' in $\mathbb{C}^{n+n'}$ by

$$V \xrightarrow{\ \sim\ } V \times \{0\} \hookrightarrow \mathbb{C}^n \times \mathbb{C}^{n'} \quad \text{resp.} \quad V' \xrightarrow{\ \sim\ } V' \times \{0\} \hookrightarrow \mathbb{C}^{n'} \times \mathbb{C}^n,$$

and correspondingly embed ${}_{\mathbb{R}}\mathbf{V}$ and ${}_{\mathbb{R}}\mathbf{V}'$ in $\mathbb{C}^{n+n'} \times \mathbb{R}$, see (2.2), then (3.5) gives us *inverse homeomorphisms*

$$\mathbb{C}^{n+n'} \times \mathbb{R} \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \mathbb{C}^{n+n'} \times \mathbb{R}$$

with $\mathbf{F}({}_{\mathbb{R}}\mathbf{V}) \subset {}_{\mathbb{R}}\mathbf{V}'$ and $\mathbf{G}({}_{\mathbb{R}}\mathbf{V}') \subset {}_{\mathbb{R}}\mathbf{V}$. And finally, in view of (4.2.2) and (3.4) we can replace V by a component V_λ, i.e., we may assume V to be irreducible, of dimension, say, d.

Now the underlying idea is that, as we have just seen, the multiplicity of a component is an intersection multiplicity, and as such should be invariant under the homeomorphism $\mathbf{F}$. Technical complications arise from working with ${}_{\mathbb{R}}\mathbf{V}$ rather than with $\mathbf{V}$ (which has been necessitated by the real derivatives of f and g being not necessarily complex-linear).

Setting $N := n + n'$, we first deduce from [BH, p. 475, 2.15], applied to the inclusion of $\mathbb{C}^N \times \mathbb{R}^1$ (with fixed orientation) into $\mathbb{C}^N \times \mathbb{C}^1$, and to the smooth locus U of $\mathbf{V} - \mathbf{V}_0$ (which has real codimension ≥ 2 in $Y := \mathbf{V}$), that ${}_{\mathbb{R}}\mathbf{V}$ has a fundamental class ρ in the Borel-Moore homology $H_{2d+1}({}_{\mathbb{R}}\mathbf{V})$. Using the projection formula, we see further that $\pm\rho$ is the intersection of the fundamental classes of $\mathbb{C}^N \times \mathbb{R}$ and of $\mathbf{V}$. (Strictly speaking, the intersection class lies in $H^{\Phi}_{2d+1}(\mathbb{C}^N \times \mathbb{C}^1)$ where Φ is the family of closed subsets of ${}_{\mathbb{R}}\mathbf{V}$; but that group is canonically isomorphic to $H_{2d+1}({}_{\mathbb{R}}\mathbf{V})$.)[4] Then associativity of the intersection product and the relation

$$\mathbf{V}_0 = (\mathbb{C}^N \times \{0\}) \cap {}_{\mathbb{R}}\mathbf{V} = (\mathbb{C}^N \times i\mathbb{R}^1) \cap {}_{\mathbb{R}}\mathbf{V} = (\mathbb{C}^N \times i\mathbb{R}^1) \cap (\mathbb{C}^N \times \mathbb{R}^1) \cap \mathbf{V} \quad (i = \sqrt{-1})$$

show that $e_{C_*, C(V,W)}$ is the intersection number $i\big((\mathbb{C}^N \times \{0\}) \cdot {}_{\mathbb{R}}\mathbf{V}, C_*\big)$ (in $\mathbb{C}^N \times \mathbb{R}^1$), see [GL, p. 176, (B.5.2)]. Given the topological invariance (up to sign) of intersection numbers, the principal remaining problem is to show that the map

[4] Cf. [GL, p. 175, (B.3.5)], where the second $\overline{S}$ ($= {}_{\mathbb{R}}\mathbf{V}$) should be S ($= \mathbf{V}$).

$\mathbf{F}_*\colon H_{2d+1}({}_{\mathbb{R}}\mathbf{V}) \to H_{2d+1}({}_{\mathbb{R}}\mathbf{V}')$ induced by $\mathbf{F}$ takes ρ to $\pm$ the fundamental class ρ' of ${}_{\mathbb{R}}\mathbf{V}'$. (The corresponding statement for $\mathbb{C}^N \times \{0\}$ is straightforward.) One can proceed as in [GL, §(D.4)]. Another way, since $\mathbf{F}_*$ is an isomorphism, is to show that

$$H_{2d+1}({}_{\mathbb{R}}\mathbf{V}) \cong \mathbb{Z} \cong H_{2d+1}({}_{\mathbb{R}}\mathbf{V}'),$$

generated, necessarily, by ρ and ρ' respectively. This we now do.

Recall that for any locally compact space X, there are canonical isomorphisms

$$H_i(X \times \mathbb{R}^1) \xrightarrow{\ \sim\ } H_{i-1}(X) \qquad (i \in \mathbb{Z}).$$

These arise, upon identification of $\mathbb{R}^1$ with the open unit interval $(0,1)$, from the following exact sequence associated to the inclusion of the pair of points $\{0,1\}$ into the closed unit interval $I := [0,1]$, see [BH, p. 465, 1.6]:

$$\cdots \longrightarrow H_i(X) \oplus H_i(X) \xrightarrow{\alpha} H_i(X \times I) \longrightarrow H_i(X \times \mathbb{R}^1)$$
$$\xrightarrow{\beta} H_{i-1}(X) \oplus H_{i-1}(X) \xrightarrow{\gamma} H_{i-1}(X \times I) \longrightarrow \cdots$$

The point is that the (proper) projection $X \times I \to X$, being a homotopy equivalence, induces for every i an isomorphism $H_i(X \times I) \xrightarrow{\ \sim\ } H_i(X)$, whose inverse is given by $H_i(X) \xrightarrow{\ \sim\ } H_i(X \times \{a\}) \longrightarrow H_i(X \times I)$ for any $a \in I$ [BH, p. 465, 1.5]; hence α is surjective, and β maps $H_i(X \times \mathbb{R}^1)$ isomorphically onto the kernel of γ, which is isomorphic to $H_{i-1}(X)$ (diagonally embedded in $\oplus$).

As a corollary, we note that for any integers $i \neq j$, with $j \geq 0$, we have

$$H_i(\mathbb{R}^j) \cong H_{i-j}(\mathbb{R}^0) = 0, \tag{4.3.2}$$

the last equality by [BH, p. 464, 1.3]. (Similarly, $H_j(\mathbb{R}^j) = \mathbb{Z}$.)

Now consider the exact sequence

$$0 = H_{2d+1}(\mathbf{V}_0) \longrightarrow H_{2d+1}({}_{\mathbb{R}}\mathbf{V}) \longrightarrow H_{2d+1}({}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0) \xrightarrow{\delta} H_{2d}(\mathbf{V}_0) \xrightarrow{\epsilon} H_{2d}({}_{\mathbb{R}}\mathbf{V})$$

see [BH, p. 465, 1.6]. Note that $\mathbf{V}_0$ has complex dimension d, by (2.1)(i), hence cohomological dimension $2d$ [BH, p. 475, 3.1], whence the vanishing of $H_{2d+1}(\mathbf{V}_0)$ see [BH, p. 467, (1)]. By (2.1)(ii), ${}_{\mathbb{R}}\mathbf{V} - \mathbf{V}_0$ is homeomorphic to the disjoint union of two copies of $V \times \mathbb{R}^1$. Since V is, by assumption, irreducible, we have

$$H_{2d+1}(V \times \mathbb{R}^1) \cong H_{2d}(V) \cong \mathbb{Z},$$

the first isomorphism as above, the second by [BH, p. 476, 3.3]. Thus $H_{2d+1}({}_{\mathbb{R}}\mathbf{V})$ is free, of rank 1 or 2. (The rank is > 0 because $\rho \neq 0$, since as above, ρ gives rise via intersection to $e_{C_*, C(V,W)} > 0$.) Moreover, $H_{2d}(\mathbf{V}_0)$ is torsion-free [BH, p. 482, 4.3]. It will therefore suffice to show that δ is not the zero map. We do this by noting, with $[C_\mu]$ the fundamental class of the component C_μ of $C(V,W) = \mathbf{V}_0$, that

$$\epsilon\Bigl(\sum_\mu \pm e_{C_\mu, C(V,W)}[C_\mu]\Bigr) = 0. \tag{4.3.3}$$

Indeed, with the right choice of $\pm$, the left side is the image under ϵ of the intersection class $(\mathbb{C}^N \times \{0\}) \cdot {}_{\mathbb{R}}\mathbf{V}$ (see above). But by compatibility of intersections with "enlargement of families of supports" [BH, p. 468, 1.12], we have a commutative diagram, where $H^Z(-)$ stands for the Borel-Moore homology of $\mathbb{C}^N \times \mathbb{R}^1$ with supports in closed subsets of Z:

$$\begin{array}{ccc} H_{2N}^{\mathbb{C}^N\times\{0\}}(-) \times H_{2d+1}^{\mathbb{R}\mathbf{V}}(-) & \xrightarrow{\text{intersect}} & H_{2d}^{\mathbf{V}_0}(-) = H_{2d}(\mathbf{V}_0) \\ {\scriptstyle\text{natural}}\big\downarrow & & \big\downarrow{\scriptstyle\epsilon} \\ H_{2N}^{\mathbb{C}^N\times\mathbb{R}^1}(-) \times H_{2d+1}^{\mathbb{R}\mathbf{V}}(-) & \xrightarrow{\text{intersect}} & H_{2d}^{\mathbb{R}\mathbf{V}}(-) = H_{2d}({}_{\mathbb{R}}\mathbf{V}) \end{array}$$

in which the lower left corner vanishes, by (4.3.2); and (4.3.3) results. □

5. Relative complexification of the normal cone.

We now construct the *relative complexification* of a cone C, and for $C = C(V,W)$ establish C^1 functorial properties of this complexification (Theorem (5.3.1)).

Let C be a cone over a complex space W, i.e., $C = \mathrm{Specan}(\mathcal{G})$ for some finitely presented graded $\mathcal{O}_W$-algebra $\mathcal{G}$, see (1.1.1). Assume that all the irreducible components of C have the same dimension, and that all the fibers of the canonical map $C \to W$ have positive dimension. For example, if $V \supset W$ is as in (1.1), with V equidimensional and W nowhere dense in V, then (2.1) implies that $C(V,W)$ is equidimensional, of dimension $\dim C = \dim V > \dim W$, and hence the fibers of $p\colon C(V,W) \to W$ are all positive-dimensional.

Recall that a subset of a complex space X is *Zariski-open* if its complement is an analytic subset of X. (Analytic subsets of X are understood to be *closed*, defined locally by the vanishing of sections of $\mathcal{O}_X$.)

Lemma (5.1). *There exists a unique analytic subset $\widetilde{C}$ of $C \times_W C$ such that with $\tilde{p}\colon \widetilde{C} \to W$ the natural composition $\widetilde{C} \hookrightarrow C \times_W C \to W$,*

(i) *for any open dense $U \subset W$, $\tilde{p}^{-1}(U)$ is dense in $\widetilde{C}$; and*

(ii) *there is a dense Zariski-open subset W_0 of W such that for every $w \in W_0$, the reduced fiber $\widetilde{C}_w := \tilde{p}^{-1}(w)_{\mathrm{red}}$ is*

$$\widetilde{C}_w = \bigcup_{i \in I_w} C_w^i \times C_w^i \subset C \times_W C,$$

$(C_w^i)_{i\in I_w}$ being the family of irreducible components of the cone $C_w := p^{-1}(w)$.

In fact $\widetilde{C}$ is a union of irreducible components of $C \times_W C$, and so is stable under the natural $\mathbb{C}^1 \times \mathbb{C}^1$ action (given by μ in (1.1.1)).

We will call $\tilde{p}\colon \widetilde{C} \to W$ the *relative complexification* of $p\colon C \to W$. That's because for almost all $w \in W$ (e.g., $w \in W_0$), $\widetilde{C}_w$ is real-analytically isomorphic to a reduced complexification of $(C_w)_{\mathrm{red}}$, see (5.3.0). For example, if $\mathcal{G}$ is the symmetric algebra of a finite-rank locally free $\mathcal{O}_W$-module, i.e., C is a complex vector bundle over W, then $\widetilde{C} = C \times_W C$ together with the natural addition on the fibers and the $\mathbb{C}^1$ action specified immediately before Thm. (5.3.1) below, is just the usual complexification of the real vector bundle underlying C.

Proof. Uniqueness is immediate: if $(\widetilde{C}', W_0')$ and $(\widetilde{C}'', W_0'')$ are two pairs satisfying the conditions of (5.1), then $W_0' \cap W_0''$ is open and dense in W, and so $\widetilde{C}'$ and $\widetilde{C}''$ are both equal to the closure in $C \times_W C$ of

$$\bigcup_{w \in W_0' \cap W_0''} \Big(\bigcup_{i \in I_w} C_w^i \times C_w^i \Big).$$

As for existence, with $\sigma\colon W \to C$ as in (1.1.1) let C^* be the reduced space $C_{\mathrm{red}} - \sigma(W)$, on which $\mathbb{C}^*$ acts freely, preserving fibers of p; and set

$$P := C^*/\mathbb{C}^* = \mathrm{Projan}(\mathcal{G})_{\mathrm{red}}.$$

(Projan is constructed, in analogy with Specan, by pasting together subspaces of relative projective spaces $W_\alpha \times \mathbb{P}^{N_\alpha}$, with (W_α) a suitable open cover of W.) Let $\overline{P}$ be the normalization of P, and let

$$\phi\colon \overline{P} \to W, \qquad \Phi\colon \overline{P} \times_W \overline{P} \to W$$

be the natural maps, both of which are proper. Consider the commutative diagram

$$\begin{array}{ccc}
\overline{P} & \xrightarrow{\ \Delta\ } & \overline{P} \times_W \overline{P} \\
{\scriptstyle \phi'}\big\downarrow & & \big\downarrow{\scriptstyle \Phi'} \\
\mathrm{Specan}(\phi_*\mathcal{O}_{\overline{P}}) =: S & \xrightarrow{\ \delta\ } & T := \mathrm{Specan}(\Phi_*\mathcal{O}_{\overline{P} \times_W \overline{P}}) \\
\big\downarrow & & \big\downarrow \\
W & = & W
\end{array} \tag{5.1.1}$$

whose sides are the Stein factorizations of ϕ and Φ respectively [Fi, p. 71], where Δ is the diagonal map, and where δ corresponds to the natural map of $\mathcal{O}_W$-algebras

$$\Phi_*\mathcal{O}_{\overline{P} \times_W \overline{P}} \to \Phi_*\Delta_*\mathcal{O}_{\overline{P}} = \phi_*\mathcal{O}_{\overline{P}}.$$

Since S is proper over W, the map δ is proper (in fact, finite), and so $\delta(S)$ is an analytic subset of T. Let $\overline{Z} := \Phi'^{-1}\delta(S)$, an analytic subset of $\overline{P} \times_W \overline{P}$.[5] Let $\widetilde{Z}$ be the image of $\overline{Z}$ under the natural finite map $\overline{P} \times_W \overline{P} \to P \times_W P$, so that $\widetilde{Z}$ is an analytic subset of $P \times_W P$. Let $\widetilde{C}^* \subset C^* \times_W C^*$ be the inverse image of $\widetilde{Z}$ under the quotient map $C^* \times_W C^* \to P \times_W P$.

For any $w \in W$, the fiber P_w is non-empty (since C_w has positive dimension); the points in S_w correspond to the connected components of $\overline{P}_w$ (since S_w is finite and the fibers of $\phi'\colon \overline{P} \to S$ are non-empty and connected); the points of T_w correspond to the connected components of $\overline{P}_w \times \overline{P}_w$; and from commutativity of (5.1.1) it

[5] $\overline{Z}$ can be defined without reference to Stein factorization as being the support of the cokernel of the natural map $\Phi^*\Phi_*\mathcal{J} \to \mathcal{O}_{\overline{P} \times_W \overline{P}}$, where $\mathcal{J}$ is the kernel of $\mathcal{O}_{\overline{P} \times_W \overline{P}} \to \Delta_*\mathcal{O}_{\overline{P}}$. We do need Stein factorization to derive (5.1.2) below; but there might well be a more elementary argument.

follows for any $s \in S_w$ that if $\phi'^{-1}(s) = \overline{D}$ (a connected component of $\overline{P}_w$), then $\Phi'^{-1}(\delta s) = \overline{D} \times \overline{D}$ (the connected component of $\overline{P}_w \times \overline{P}_w$ containing $\Delta\overline{D}$). Thus

$$\overline{Z}_w = \bigcup_{i=1}^{m_w} \overline{D}_w^i \times \overline{D}_w^i \tag{5.1.2}$$

where m_w is the cardinality of S_w, and $\overline{D}_w^1, \dots, \overline{D}_w^{m_w}$ are the connected components of $\overline{P}_w$.

Now, there exists a dense Zariski-open subspace W_0 of W such that:

(a) W_0 is locally irreducible (as holds, e.g., at any smooth point of W_{red}).

(b) The natural (proper) map $\varphi\colon \widetilde{Z}_{\text{red}} \to W_{\text{red}}$ is flat everywhere on $\varphi^{-1}(W_0)$ (Frisch's generic flatness theorem [BF, (1.17)(2), (2.4), (2.5)(2), (2.7)(1)]).

(c) For each $w \in W_0$, the fiber C_w is equidimensional, of dimension equal to the codimension c_w of $\sigma(W)$ in C at $\sigma(w)$, see (1.1.1). (Apply generic flatness of the proper map $P \to W_{\text{red}}$, keeping in mind that C—and hence P—is equidimensional.)

(d) For each $w \in W_0$, the fiber P_w is reduced and the natural map $\pi_w\colon \overline{P}_w \to P_w$ is a normalization of P_w. (Generic simultaneous normalization for the map $P \to W_{\text{red}}$, see [BF, Theorem (2.13)].)

In view of (d), for $w \in W_0$, if $D_w^1, \dots, D_w^{n_w}$ are the irreducible components of P_w, then $n_w = m_w$, see above, and after relabeling we have $\pi_w^{-1}(D_w^i) = \overline{D}_w^i$ for all i. Hence the decomposition of $\widetilde{Z}_w$ into irreducible components is

$$\widetilde{Z}_w = \bigcup_{i=1}^{m_w} D_w^i \times D_w^i \qquad (w \in W_0). \tag{5.1.3}$$

Since the fibers of $C^* \times_W C^* \to P \times_W P$ (resp. $C^* \to P$) are all isomorphic to the manifold $\mathbb{C}^* \times \mathbb{C}^*$ (resp. $\mathbb{C}^*$), it follows, for $w \in W_0$, that the irreducible components of $\widetilde{C}_w^*$ are the reduced spaces $C_w^{*i} \times C_w^{*i}$, where the C_w^{*i} are the irreducible components of $C_w^* := C_w - \sigma(w)$. So we are approaching our goal.

Any irreducible component Γ^* of $\widetilde{C}^* \cap q^{-1}(W_0)$ ($q\colon C \times_W C \to W$ the natural map) is contained in a component Γ of $C \times_W C$. By (c) and (5.1.3), the fibers $\widetilde{Z}_w$ are equidimensional, each component having dimension $2 \dim P_w = 2(c_w - 1)$. It follows then from (a) and (b) that for any irreducible component Z^* of $\widetilde{Z} \cap \varphi^{-1}(W_0)$ and any $z \in Z^*$,

$$\dim_z Z^* = \dim_w W + 2(c_w - 1) \qquad (w = \varphi(z));$$

and therefore for any $x \in \Gamma^*$,

$$\dim \Gamma^* = \dim_w W + 2c_w \geq \dim \Gamma \qquad (w = q(x)). \tag{5.1.4}$$

Hence $\dim \Gamma^* = \dim \Gamma$ and

$$\Gamma^* = \Gamma \cap (C^* \times_W C^*) \cap q^{-1}(W_0), \tag{5.1.5}$$

so that Γ is Zariski open in Γ^*.

Finally, let $\widetilde{C}$ be the union of all those components Γ of $C \times_W C$ which contain a component, say Γ^*, of $\widetilde{C}^* \cap q^{-1}(W_0)$. Every such Γ—and hence $\widetilde{C}$—is mapped into itself under the $\mathbb{C}^1 \times \mathbb{C}^1$ action, since the image of the multiplication map $\mathbb{C}^1 \times \mathbb{C}^1 \times \Gamma \to C \times_W C$ is irreducible and contains Γ.

Let r be the restriction of $\tilde{p} := q|_{\widetilde{C}}$ to the Zariski open subset $\widetilde{C}^* \cap q^{-1}(W_0)$ of $\widetilde{C}$. In view of (5.14), in which

$$2c_w = 2(\dim P_w + 1) = 2\dim C_w = \dim_x q^{-1}q(x) \geq \dim_x r^{-1}r(x),$$

a theorem of Remmert [Fi, p. 142, 3.9], guarantees that r is an open map. So for any open dense $U \subset W$, $r^{-1}(U \cap W_0)$ is dense in $\widetilde{C}^* \cap q^{-1}(W_0)$, which is in turn dense in $\widetilde{C}$. Thus (5.1)(i) holds.

To finish, observe for $w \in W_0$ that the components C_w^i of C_w are given by $C_w^i = C_w^{*i} \cup \{\sigma(w)\}$ $(1 \leq i \leq m_w)$, and that by (5.1.5),

$$\begin{aligned}
\widetilde{C}_w^* \subset \widetilde{C}_w \subset \widetilde{C}_w^* &\cup \big(\{\sigma(w)\} \times C_w\big) \cup \big(C_w \times \{\sigma(w)\}\big) \\
&= \bigcup_{i=1}^{m_w} \big[\big(C_w^{*i} \times C_w^{*i}\big) \cup \big(\{\sigma(w)\} \times C_w^i\big) \cup \big(C_w^i \times \{\sigma(w)\}\big)\big] \\
&= \bigcup_{i=1}^{m_w} \big(C_w^i \times C_w^i\big).
\end{aligned}$$

Since $\widetilde{C}_w^* = \cup_i (C_w^{*i} \times C_w^{*i})$ is dense in $\cup_i (C_w^i \times C_w^i)$, and $\widetilde{C}_w$ is closed, therefore (5.1)(ii) results. □

Example (5.2). Again let $\mathcal{G} = \oplus_{m \geq 0} \mathcal{G}_m$ $(\mathcal{G}_0 = \mathcal{O}_W)$ be a finitely-presentable $\mathcal{O}_W$-algebra, set $C := \operatorname{Specan}(\mathcal{G})$, $P := \operatorname{Projan}(\mathcal{G})$, and let $p \colon C \to W$, $\wp \colon P \to W$ be the canonical maps. Points $x \in P$ correspond to $\mathbb{C}^1$-orbits of points in $C \setminus \sigma(W)$: the "line" L_x corresponding to x lies in the fiber $C_{\wp(x)}$.

Assume that $\mathcal{G}_m = \mathcal{G}_1^m$ for all $m \gg 0$. Let $\mathcal{L} \xrightarrow{\pi} C$ be the proper map obtained by blowing up $\sigma(W)$ where $\sigma \colon W \to C$ is the vertex section (see (1.1.1)). Then $\mathcal{L} \cong \operatorname{Specan}\big(\operatorname{Sym} \mathcal{O}_P(1)\big)$ is the canonical line bundle on P, and $\pi^{-1}\sigma(W) = \epsilon(P)$ where $\epsilon \colon P \to \mathcal{L}$ is the zero-section (cf. [GD, (8.7.8)]).

$$\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\ \pi\ } & C \\
{\scriptstyle\epsilon}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle\sigma} \\
P & \xrightarrow[\ \wp\ ]{} & W
\end{array}$$

For any $x \in P$, π maps the fiber $\mathcal{L}_x$ bijectively onto the line $L_x \subset C_{\wp(x)}$.

The map π is compatible with the multiplication maps $\mu_{\mathcal{L}}$, μ_C of (1.1), i.e., the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{C}^1 \times_P \mathcal{L} & \xrightarrow{\ \mu_{\mathcal{L}}\ } & \mathcal{L} \\
{\scriptstyle \mathrm{id}\times\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
\mathbb{C}^1 \times_W C & \xrightarrow[\ \mu_C\ ]{} & C
\end{array}$$

as can be checked, e.g., via commutativity of the diagrams

$$\begin{array}{ccc} \big(\oplus\,\mathcal{O}_P(n)\big)[T] & \xleftarrow{\text{mpn by } T^n} & \mathcal{O}_P(n) \\ {\scriptstyle\text{natural}}\big\uparrow & & \big\uparrow{\scriptstyle\text{natural}} \\ \big(\oplus\,\wp^*\mathcal{G}_n\big)[T] & \xleftarrow[\text{mpn by } T^n]{} & \wp^*\mathcal{G}_n \end{array} \qquad (n \ge 0).$$

Now consider the proper map

$$\pi\times\pi\colon \mathcal{L}\times_P\mathcal{L}\hookrightarrow \mathcal{L}\times_W\mathcal{L}\to C\times_W C.$$

The restriction of this map to the complement of $\epsilon(P)\times_P\epsilon(P)$ $(\cong P)$ is clearly injective. From (5.1)(ii) it follows that for any $x\in\wp^{-1}(W_0)$ the image of the induced map $\mathcal{L}_x\times\mathcal{L}_x\to C_{\wp(x)}\times C_{\wp(x)}$ lies in $\widetilde{C}$. Hence, $\widetilde{C}$ being closed in $C\times_W C$, if $\wp^{-1}(W_0)$ is dense in P (i.e., every component of P meets $\wp^{-1}(W_0)$) then the entire image of $\pi\times\pi$ lies in $\widetilde{C}$. Furthermore, if the fibers C_w all have dimension 1, then the maps $\wp$ and $\pi\times\pi$ are both finite, the image of $\pi\times\pi$ is $\widetilde{C}$ itself, and $\pi\times\pi$ induces a *homeomorphism*

$$(\mathcal{L}\times_P\mathcal{L})\setminus P \xrightarrow{\ \sim\ } \widetilde{C}\setminus W$$

where we have identified P (resp. W) with $\epsilon(P)\times_P\epsilon(P)$ (resp. $\sigma(W)\times_W\sigma(W)$).

All this happens, e.g., for $C := C(V,W)$ when W is a codimension-one submanifold of a reduced complex space V and V is equimultiple along W, because of Schickhoff's theorem [Li, p. 121, (2.6)].

(5.3). Every complex space $(V,\mathcal{O}_V)$ has a *conjugate space* $\overline{V}$, equal to $(V,\mathcal{O}_V)$ as a topological space with a sheaf of rings, but with $\mathcal{O}_{\overline{V}}=\mathcal{O}_V$ considered to be a $\mathbb{C}$-algebra via the composition

$$\mathbb{C}\xrightarrow{\text{conjugation}}\mathbb{C}\xrightarrow{\text{natural}}\mathcal{O}_V,$$

see [Hi, Definition (1.10)]. The identity map $V\to\overline{V}$ is a real-analytic isomorphism. Complex conjugation ρ_n in $\mathbb{C}^n$, along with the sheaf-isomorphism $\mathcal{O}_{\overline{\mathbb{C}^n}}\xrightarrow{\ \sim\ }\rho_{n*}\mathcal{O}_{\mathbb{C}^n}$ taking a holomorphic function $f(z)$ on an open set U to the holomorphic function $\rho_1 f(\rho_n z)$ on $\rho_n^{-1}(U)$, is a complex-analytic isomorphism of $\mathbb{C}^n$ onto $\overline{\mathbb{C}^n}$. Hence for any analytic subset V of $\mathbb{C}^n$, $\rho_n(V)$ can be regarded as an analytic subset of $\overline{\mathbb{C}^n}$ isomorphic to V, or as an analytic subset of $\mathbb{C}^n$ isomorphic to $\overline{V}$.

With $(V_i)_{i\in I}$ the family of (reduced) irreducible components of V, we set

$$V^{\mathrm{c}} := \bigcup_{i\in I}(V_i\times\overline{V_i})\subset(V\times\overline{V}). \tag{5.3.0}$$

We call V^{c} the *reduced complexification* of V, or simply the *complexification* of V when V itself is reduced. The reduced space V_{red} can be identified via the diagonal map with a real-analytic subvariety of V^{c}.

For example, with reference to (5.1), for each $w \in W$, there are natural inclusions

$$(C_w)^c \underset{j_w}{\hookrightarrow} C_w \times \overline{C_w} \underset{l_w}{\hookrightarrow} C \times_W C,$$

where l_w is a real-(but not necessarily complex-)analytic embedding. For $w \in W_0$, we have $l_w j_w((C_w)^c) = \widetilde{C}_w$.

From now on, when we regard a reduced fiber $\widetilde{C}_w$ $(w \in W_0)$ as a complex space, we mean it to be identical as such with $(C_w)^c$. (Thus we do not mean it to be a *complex* subspace of $C \times_W C$.) And when we refer to a $\mathbb{C}^1$ action on $C \times_W C$ or on one of its analytic subsets (for instance, $\widetilde{C}$) we mean the one given on point sets by

$$a(x, x') = (ax, \overline{a}x') \qquad \text{(see (1.1.1))}$$

where now $\overline{a}$ is the complex conjugate of $a \in \mathbb{C}$. This action is real-analytic, being obtained from the natural $\mathbb{C}^1 \times \mathbb{C}^1$ action on $C \times_W C$ via the (real-analytic) map $a \mapsto (a, \overline{a})$ from $\mathbb{C}^1$ to $\mathbb{C}^1 \times \mathbb{C}^1$.

The next result allows us to regard $\widetilde{C}$ as a "differential functor."

Consider a C^1 map $f\colon (V, W) \to (V', W')$ where now V and V' are reduced equidimensional complex analytic spaces and W (resp. W') is a nowhere-dense submanifold of V (resp. V'). Let

$$\mathbf{f}_0\colon C := C(V, W) \to C(V', W') =: C'$$

be the continuous map in Theorem (3.3).

Let C_g ("g" for "generic") be the union of those components of C whose image in W is not nowhere-dense.[6] Note that the (locally finite) union of the images of all the remaining components of C is nowhere dense in W, so that the fibre $(C_g)w$ is the same as C_w for all w in some dense open subset of W. Identify C with the diagonal in $C \times_W C$. Lemma (5.1) implies that the non-empty (hence dense) Zariski open subset $\tilde{p}^{-1}W_0 \cap C_g$ of C_g is contained in $\widetilde{C}$; and hence $C_g \subset \widetilde{C}$.

Assume further that the open subset of W on which the induced map $W \to W'$ is a submersion is dense in W, so that, submersions being open maps, the inverse image of any nowhere-dense subset of W' is nowhere dense in W. It follows that $\mathbf{f}_0(C_g) \subset C'_g$; and we set $\mathbf{f}_{0g} := \mathbf{f}_0|_{C_g}$.

Theorem (5.3.1). *In the preceding situation, let U be a dense open subset of W_0 such that $(C_g)_w = C_w$ for all $w \in U$. Then $\mathbf{f}_{0g}$ extends uniquely to a continuous map $\tilde{f}\colon \widetilde{C} \to \widetilde{C}'$, not depending on the choice of W_0 or of U, such that the following diagram commutes,*

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\ \tilde{f}\ } & \widetilde{C}' \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle \tilde{p}'} \\ W & \xrightarrow[\ f\ ]{} & W' \end{array}$$

and such that for each $w \in U$ the resulting map of (reduced) fibers $\widetilde{C}_w \to \widetilde{C}'_{f(w)}$ is complex-analytic. Moreover, $\tilde{f}$ commutes with the $\mathbb{C}^1$ actions on $\widetilde{C}$ and $\widetilde{C}'$.

[6]In fact the image is the same as that of the corresponding component of the projectivized normal cone $P(V, W)$, and so is an analytic subset of W. Thus any component of C_g maps onto a component of W.

Remark **(5.3.2).** Let $(V,W) \xrightarrow{f} (V',W') \xrightarrow{g} (V'',W'')$ be two maps satisfying the hypotheses of (5.3.1). Then gf also satisfies these hypotheses, because a composition of submersions is a submersion, and because submersions being open maps, the inverse image under f of a dense open subset of W' is dense and open in W.

Since, clearly, $(\mathbf{gf})_0 = \mathbf{g}_0\mathbf{f}_0$, we conclude from uniqueness in (5.3.1) and the denseness of $f^{-1}(W_0') \cap W_0$ in W_0 that $\widetilde{gf} = \tilde{g}\tilde{f}$.

To begin the *proof* of Theorem (5.3.1), we recall some simple facts.

Lemma (5.3.3). *Let $V^{\mathrm{c}} \subset V \times \overline{V}$ be the complexification of a reduced complex space V, so that V^{c} contains the diagonal $\Delta_V \subset V \times V = V \times \overline{V}$ as a real-analytic subspace. Then the only complex-analytic subset Z of V^{c} containing Δ_V is V^{c} itself.*

Proof. Let V_0 be the (open, dense) smooth locus of V. Then $\overline{V_0}$ is the smooth locus of $\overline{V}$, for example because smoothness at a point $v \in V$ means that the local ring $\mathcal{O}_{V,v} = \mathcal{O}_{\overline{V},v}$ is regular. Then $(V_0)^{\mathrm{c}}$ is a dense open subset of V^{c}, and the closed set $Z \cap (V_0)^{\mathrm{c}}$ contains Δ_{V_0}; hence we can replace V by V_0, i.e., we may assume that V is a manifold.

If $Z \neq V^{\mathrm{c}}$ then for some i, Z intersects the connected open and closed subspace $(V_i \times \overline{V_i})$ of V^{c} nowhere densely; so there exists for some $u \in \Delta_V$ a neighborhood U together with an isomorphism $\theta\colon (U,u) \xrightarrow{\sim} (B,0)$ where B is an open ball in some $\mathbb{C}^n$, and a non-zero holomorphic function $h\colon U \times \overline{U} \to \mathbb{C}$ vanishing on $Z \cap (U \times \overline{U})$, hence on $\Delta_U \cap (U \times \overline{U})$. There is a holomorphic open immersion $\Theta\colon U \times \overline{U} \to \mathbb{C}^n \times \mathbb{C}^n$ given, with $\rho =$ complex conjugation, by

$$\Theta(v,w) = \left(\frac{\theta(v) + \rho\theta(w)}{2}, \frac{\theta(v) - \rho\theta(w)}{2\sqrt{-1}}\right),$$

taking Δ_U onto an open subset of $\mathbb{R}^n \times \mathbb{R}^n \subset \mathbb{C}^n \times \mathbb{C}^n$. All the derivatives of the holomorphic function $h \circ \Theta^{-1}\colon \Theta(U \times \overline{U}) \to \mathbb{C}$ vanish everywhere on $\Theta(\Delta_U)$, and hence h vanishes everywhere in a neighborhood of Δ_U, contradicting the assumption that h is non-zero (since $U \times \overline{U}$ is connected). □

Corollary (5.3.3.1). *If a holomorphic map $\varphi\colon V^{\mathrm{c}} \to Y$ maps Δ_V into an analytic subset W of Y, then φ maps all of V^{c} into W.*

Corollary (5.3.3.2). *If two complex-analytic maps from V^{c} to a complex space X agree on Δ_V, then they must be identical.*

Proofs. For (5.3.3.1), let Z in (5.3.3) be $\varphi^{-1}(W)$.

For (5.3.3.2), let $\varphi\colon V^{\mathrm{c}} \to X \times X$ in (5.3.3.1) be the map whose coordinates are the two maps in question, and let W be the diagonal of $X \times X$. □

Uniqueness in Theorem (5.3.1) follows, in view of (5.1)(i), from (5.3.3.2) applied to each of the fibers $\widetilde{C}_w$ ($w \in U$). (For independence from W_0 and U, note that the intersection of two dense open subsets of W is again a dense open subset ...) This uniqueness guarantees that it is enough to prove existence with W replaced by an arbitrary member of an open covering (W_α) of W. (The global $\tilde{f}$ over all of W can then be obtained from the local maps $\tilde{f}_\alpha\colon \widetilde{C} \times_W W_\alpha \to \widetilde{C}'$ by pasting.) Thus, as in §(1.2), we can identify W with an open neighborhood of the origin in $\mathbb{C}^r$, embed C in $W \times \mathbb{C}^s$ ($p\colon C \to W$ being induced by projection to the first factor), and hence embed $C \times_W C$ in $\mathbb{C}^r \times \mathbb{C}^s \times \mathbb{C}^s$; and similarly for $p'\colon C' \to W'$, ...

Let $\lambda\colon \mathbb{C}^s \times \mathbb{C}^s \to \mathbb{C}^s \times \mathbb{C}^s$ be the real-linear automorphism taking (y, z) to (u, v), where with $i = \sqrt{-1}$ and $\bar{z}$ the complex conjugate of z,

$$u = \frac{y + \bar{z}}{2}, \qquad v = \frac{y - \bar{z}}{2i}.$$

The inverse automorphism is given by

$$y = u + iv, \qquad z = \overline{u} + i\overline{v}.$$

Then $y = z$ if and only if u and v are both real, i.e., λ maps the diagonal of $\mathbb{C}^s \times \mathbb{C}^s$ onto $\mathbb{R}^s \times \mathbb{R}^s \subset \mathbb{C}^s \times \mathbb{C}^s$.

Now recall from (3.3.3) that we can represent $\mathbf{f}_0$ locally by

$$\mathbf{f}_0(w, z) = \big(f(w), L_w(z)\big)$$

where

$$L_w\colon \mathbb{C}^s = \mathbb{R}^{2s} \to \mathbb{R}^{2s'} = \mathbb{C}^{s'} \tag{5.3.4}$$

is a real-linear map which depends continuously on w. In view of the relation $\lambda(y, y) = \big(\mathrm{re}(y), \mathrm{im}(y)\big)$, we see that the preceding identification of $\mathbb{C}^s$ (diagonally embedded in $\mathbb{C}^s \times \mathbb{C}^s$) with $\mathbb{R}^{2s}$ is given by λ. Hence, if L_w^{c} is the $\mathbb{C}$-linear map

$$L_w^{\mathrm{c}} := L_w \otimes_{\mathbb{R}} \mathbb{C}\colon \mathbb{C}^{2s} \to \mathbb{C}^{2s'},$$

then the continuous map $\tilde{f}\colon W \times \mathbb{C}^{2s} \to W' \times \mathbb{C}^{2s'}$ defined by

$$\tilde{f}(w, x) = \big(f(w),\, \lambda^{-1} L_w^{\mathrm{c}} \lambda(x)\big) \tag{5.3.5}$$

is an extension of $\mathbf{f}_0$ such that $q'\tilde{f} = fq$ (with q, q' the respective projections to W and W').

As before, complex conjugation $\rho_s\colon \mathbb{C}^s \to \mathbb{C}^s$ induces a complex-analytic isomorphism $\overline{B} \xrightarrow{\sim} \rho_s(B)$ for any analytic subset $B \subset \mathbb{C}^s$. The composition $\overline{\lambda}$ of λ with the real-linear map $\mathbb{C}^s \times \mathbb{C}^s \to \mathbb{C}^s \times \mathbb{C}^s$ taking $(y, \bar{z})$ to (y, z) is complex-linear. Thus if A and B are analytic subsets of $\mathbb{C}^s$, then λ induces a complex-analytic isomorphism of $A \times \overline{B}$ onto the analytic subset $\lambda(A \times B) = \overline{\lambda}(A \times \overline{B})$ of $\mathbb{C}^s$. In particular, λ maps $\mathbb{C}^s \times \overline{\mathbb{C}^s}$ isomorphically onto $\mathbb{C}^s \times \mathbb{C}^s$. Hence for every $w \in W$, $\tilde{f}$ induces a complex-analytic map

$$(C_w)^{\mathrm{c}} \subset C_w \times \overline{C_w} \subset \mathbb{C}^s \times \overline{\mathbb{C}^s} \to \mathbb{C}^{s'} \times \overline{\mathbb{C}^{s'}}.$$

Moreover, one checks that $\tilde{f}$ commutes with the $\mathbb{C}^1$ action on $W \times \mathbb{C}^s \times \overline{\mathbb{C}^s}$ (resp. $W \times \mathbb{C}^{s'} \times \overline{\mathbb{C}^{s'}}$) given by $c(w, x_1, x_2) = (w, cx_1, \bar{c}x_2)$.

We need only show now that $\tilde f(\widetilde C) \subset \widetilde C'$. Set $U := f^{-1}(W'_0) \cap W_0$. Because of (5.3.3.1), it suffices, since $\tilde p^{-1}(U)$ is dense in $\widetilde C$, see (5.1)(i) and (5.3.2), that $\tilde f(C_w) \subset \widetilde C'$ for each $w \in U$; and that's so since

$$\tilde f(C_w) = \mathbf{f}_0(C_w) \subset C'_{f(w)} \subset (C'_{f(w)})^{\mathrm c} = \widetilde C'_{f(w)}.$$

(5.4). A certain subvariety $\Lambda(C) \subset \widetilde C$ will play an important role in the subsequent discussion of Segre classes.

Recall from Example (5.2) the canonical line bundle $\mathcal L \to P := \mathrm{Projan}(\mathcal G)$. Assume that every component of P meets $\wp^{-1}(W_0)$, where $\wp\colon P \to W$ is the canonical map; or equivalently, that every component of C meets $p^{-1}(W_0)$, where $p\colon C \to W$ is the canonical map. (This assumption holds, e.g., for $C := C(V, W)$ when W is a submanifold of a reduced complex space V and V is equimultiple along W, by a theorem of Schickhoff [Li, p. 121, (2.6)].) Then (5.2) gives a natural map $\mathcal L \times_P \mathcal L \to \widetilde C$. Also, since $\widetilde C_w$ contains the diagonal of $C_w \times C_w$ for all $w \in W_0$ (see Lemma (5.1)(ii)), therefore $\widetilde C$ contains the dense subset $\{\,(x, x) \mid p(x) \in W_0\,\}$ of the diagonal of $C \times_W C$, and so $\widetilde C$ contains the entire diagonal of $C \times_W C$.

Let $\mathcal L^*$ be the real-analytic complex line bundle conjugate to $\mathcal L$, got by replacing every local trivialization $\varphi_U : U \times \mathbb C^1 \xrightarrow{\sim} \mathcal L_{|U}$ (U open in P) by its composition with $U \times \mathbb C^1 \xrightarrow{\mathrm{id}_U \times \rho} U \times \mathbb C^1$ ($\rho :=$ complex conjugation). Via the family $\{\mathrm{id}_U \times \rho\}$ we get an isomorphism of real-analytic spaces $\rho_{\mathcal L} : \mathcal L \xrightarrow{\sim} \mathcal L^*$. This preserves fibers over P, and addition on the fibers, but is not a line-bundle isomorphism since

$$\rho_{\mathcal L}(ax) = \bar a x \qquad (a \in \mathbb C,\ x \in \mathcal L).$$

Indeed, in the topological category $\mathcal L$ is isomorphic to a unitary bundle [Hz, p. 51, III], and so $\mathcal L^*$ is isomorphic to the dual bundle $\mathcal L^{-1} := \mathrm{Specan}\big(\mathrm{Sym}\,\mathcal O_P(-1)\big)$.

We identify the rank-two complex vector bundle $\mathcal L \oplus \mathcal L^*$ over P with $\mathcal L \times_P \mathcal L^*$. The composition

$$\gamma\colon \mathcal L \times_P \mathcal L^* \xrightarrow{\mathrm{id}\times\rho_{\mathcal L}} \mathcal L \times_P \mathcal L \xrightarrow{(5.2)} \widetilde C$$

commutes with the respective $\mathbb C^1$ actions. The image $\Lambda = \Lambda(C)$ of γ is an analytic subset of $\widetilde C$, being the image of the proper map $\pi \times \pi$ of §5.2. It consists of all points $(x, x') \in C \times_W C$ such that $x = ax'$ ($a \in \mathbb C$) or $x' = a'x$ ($a' \in \mathbb C$). In other words (verification left to reader):

$$\Lambda(C) = \{\,(b'cx, b\bar c x) \mid b', b \in \mathbb R,\ c \in \mathbb C,\ x \in C\,\}. \tag{5.4.1}$$

Recalling that λ identifies the diagonal of $\mathbb C^s \times \mathbb C^s$ with $\mathbb R^{2s}$, we see from (5.3.4) etc. that the map $\tilde f$ of Theorem (5.3.1) takes the diagonal of $C \times_W C$ into the diagonal of $C' \times_{W'} C'$. Moreover $\tilde f$ commutes with the $\mathbb C^1$ action on $\widetilde C$, as well as with the natural $\mathbb R \times \mathbb R$ action (since the maps L^c_w and λ used to construct $\tilde f$ both commute with the $\mathbb R \times \mathbb R$ action on $\mathbb C^s \times \mathbb C^s$). Hence:

Corollary (5.4.2). *Under the assumptions of Theorem* (5.3.1), $\tilde f(\Lambda(C)) \subset \Lambda(C')$.

6. Segre classes. As in §5, $C := \operatorname{Specan}(\mathcal{G})$ is a cone, with $\mathbb{C}^1$ action on $C \times_W C$ given on point sets by

$$a(x, x') = (ax, \bar{a}x').$$

Assume that all the irreducible components of the complex space W have the same dimension, say r. We identify W with its image under $\Delta \circ \sigma$ where $\sigma\colon W \to C$ is the vertex section and $\Delta\colon C \to C \times_W C$ is the diagonal map.

(6.1) For any closed analytic $\mathbb{C}^1$-stable subset Υ of $C \times_W C$,[7] all of whose irreducible components have the same complex dimension, we define the *topological Segre classes*

$$s_i(\Upsilon) \in H_{2(r-i)}(W) := H_{2(r-i)}(W, \mathbb{Z}) \qquad \text{(Borel-Moore homology)}$$

as follows.

Let Q be the topological quotient of $\Upsilon \setminus W$ under the induced (free) $\mathbb{C}^*$ action. The action preserves fibers over W, so the canonical map $(\Upsilon \setminus W) \to W$ induces a map $\nu\colon Q \to W$, which is *proper.* To see this, since Q is closed in the $\mathbb{C}^*$-quotient of $(C \times_W C) \setminus W$, we may assume $\Upsilon = C \times_W C$, and then, since the question is local over W, the definition of Specan allows us to assume that C is a closed subset of $W \times \mathbb{C}^s$ for some s (the zero-set of finitely many homogeneous polynomials in s variables, with coefficients which are analytic functions on W—see (1.2)). Then $C \times_W C$ is closed in $W \times \mathbb{C}^s \times \mathbb{C}^s$, so we may assume $\Upsilon = W \times \mathbb{C}^s \times \mathbb{C}^s$ (and $\Delta\sigma(W) = W \times \{0\} \times \{0\}$), with $\mathbb{C}^*$ action given by

$$a(w, z, z') = (w, az, \bar{a}z'). \tag{6.1.1}$$

For this action, every point in $(W \times \mathbb{C}^s \times \mathbb{C}^s) \setminus (W \times \{0\} \times \{0\})$ is equivalent to a point in $W \times S^{4s-1}$ where S^{4s-1} is the unit sphere in $\mathbb{C}^{2s}$; so there is a surjection $W \times S^{4s-1} \twoheadrightarrow Q$ whose composition with ν is the (proper) projection $W \times S^{4s-1} \to W$, whence ν itself is proper.

Next, *the quotient map* $q\colon \Upsilon \setminus W \to Q$ *is a principal real-analytic* $\mathbb{C}^*$*-bundle.* One can verify this via an open covering (U_ι) of Q together with commutative diagrams

$$\begin{array}{ccc} U_\iota \times \mathbb{C}^* & \xrightarrow{\ \phi_\iota\ } & q^{-1}U_\iota \\ {\scriptstyle \text{proj'n}}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ U_\iota & = & U_\iota \end{array}$$

where each ϕ_ι is a real-analytic homeomorphism commuting with the respective $\mathbb{C}^*$ actions (the action on $U_\iota \times \mathbb{C}^*$ being given by multiplication in $\mathbb{C}^*$). As before we reduce to consideration of the action (6.1.1) on $W \times \mathbb{C}^s \times \mathbb{C}^s$. The real-analytic homeomorphism $(w, z, z') \mapsto (w, z, \overline{z'})$ transforms the action into the relative diagonal one of $W \times (\mathbb{C}^{2m} \setminus \{0\})$ over W, the quotient of which is $W \times \mathbb{CP}^{2m-1}$, and here everything becomes straightforward.

[7] From (5.3.3.2) it follows that Υ is actually $\mathbb{C}^1 \times \mathbb{C}^1$-stable.

Lemma (6.1.2). *The quotient map q takes the non-singular locus V of $\Upsilon\setminus W$ onto an open subset $U \subset Q$ which is naturally a $2n$-dimensional real-analytic oriented manifold, and such that $Q \setminus U$ has topological dimension $\le 2n-2$.*

The proof is given below.

Lemma (6.1.2) guarantees that Q has a fundamental class $[Q] \in H_{2n}(Q)$ [BH, p. 469, Prop. 2.3]. Now let $c \in H^2(Q,\mathbb{Z})$ be the first Chern class of the principal C^*-bundle $q\colon \Upsilon \setminus W \to Q$, and, with $r := \dim W$, set

$$s_i(\Upsilon) := \nu_*([Q] \cap c^{i+n-r}) \in H_{2r-2i}(W),$$

where $\cap$ denotes "cap product" [BH, p. 505, Thm. 7.2], and

$$\nu_*\colon H_{2r-2i}(Q) \to H_{2r-2i}(W)$$

is defined because ν is proper [BH, p. 465, 1.5].

Example (6.1.3). If C is a vector bundle over W, with conjugate C^* (cf. (5.4)), then $C \times_W C$ with its $\mathbb{C}^*$ action (cf. (6.1.1)) can be identified with the bundle $C \oplus C^*$ with its standard (diagonal) $\mathbb{C}^*$ action; and the total Segre class

$$s(C \oplus C^*) := \sum_{i \ge 0} s_i(C \oplus C^*) \in \oplus_{i\ge 0} H_{2r-2i}(W)$$

is the cap product of the fundamental class $[W]$ with the multiplicative inverse (in the graded cohomology ring $\oplus_j H^j(W)$) of the total Chern class $\mathrm{ch}(C \oplus C^*)$ (cf. [Fn, p. 71, Prop. 4.1], where everything is algebraic, but corresponds to topological constructs as in *ibid.* Chap. 19). And since C^* is topologically isomorphic to the dual bundle of C [Hz, p. 51, III], we have, with $c_j \in H^{2j}(W)$ the j-th Chern class of C,

$$\mathrm{ch}(C \oplus C^*) = (1 + c_1 + c_2 + c_3 + \dots)(1 - c_1 + c_2 - c_3 + \dots),$$

the total Pontrjagin class of C [Hz, p. 65, Thm. 4.5.1].

In particular, this applies to $C(V,W)$ when V is a complex manifold and W is a submanifold (so that $C(V,W)$ is the normal bundle).

Proof of (6.1.2). The singular locus $S := \mathrm{Sing}(\Upsilon)$ is a closed analytic C^1-stable subset of Υ, of complex dimension $\le n$. As above, $S \setminus W$ is a principal $\mathbb{C}^*$-bundle over $q(S \setminus W)$, and so $q(S \setminus W) = Q \setminus q(V)$ has topological dimension $\le 2n-2$. So we can prove Lemma (6.1.2) by choosing for each $z \in V$ an open neighborhood $V_z \subset V$ in such a way that the sets $q(V_z)$ (which are open, since q is an open map) carry charts for a $2n$-dimensional canonically orientable real-analytic manifold structure on $q(V)$.

Let $N \subset \Upsilon$ be any neighborhood of the point $z_0 := \lim_{a\to 0} az$ ($a \in \mathbb{C}^*$). Replacing z by az for suitable a, we may assume that $z \in N$. Thus we may assume that $\Upsilon \subset W \times \mathbb{C}^s \times \mathbb{C}^s$, W being identified with $W_0 \times \{0\} \times \{0\}$ where W_0 is an open neighborhood of the origin in some $\mathbb{C}^r$, that $z_0 = (0,0,0)$, and that Υ is given in a polydisk neighborhood N_0 of z_0 by the vanishing of finitely many convergent power series

$$f_i(w,x,y) = \sum_{\alpha,\beta} c_{i\alpha\beta}(w) x^\alpha y^\beta$$

where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, etc. Moreover, the $\mathbb{C}^*$ action is as in (6.1.1). For any $(w_0,x_0,y_0) \in N_0$ and $t \in (0,1)$, then (since Υ is $\mathbb{C}^1$-stable),

$$\sum_{\alpha,\beta} c_{i\alpha\beta}(w_0) x_0^\alpha y_0^\beta = 0 \implies \sum_{\alpha,\beta} t^{|\alpha|+|\beta|} c_{i\alpha\beta}(w_0) x_0^\alpha y_0^\beta = 0 \qquad (|\alpha| := \alpha_1 + \dots + \alpha_s, \dots);$$

and it follows easily that for each $m \geq 0$,

$$\sum_{|\alpha|+|\beta|=m} c_{i\alpha\beta}(w_0)x_0^\alpha y_0^\beta = 0.$$

So we may assume that the f_i are homogeneous polynomials in x and y.

Furthermore (see (6.1.1)), for each $\theta \in \mathbb{R}$,

$$\sum_{|\alpha|+|\beta|=m} e^{\sqrt{-1}\theta(|\alpha|-|\beta|)} c_{i\alpha\beta}(w_0)x_0^\alpha y_0^\beta = 0,$$

and it follows, for fixed i, that $|\alpha| - |\beta|$ has the same value for all α, β such that $|\alpha| + |\beta| = m$ and $c_{i\alpha\beta} \neq 0$; in other words, f_i is a bihomogeneous polynomial in the two sets of variables x, y.

Now, on the open set O_1 where the coordinate x_1 does not vanish, q is induced by the map $\widetilde{q}\colon W \times \mathbb{C}^s \times \mathbb{C}^s \to W \times \mathbb{C}^{s-1} \times \mathbb{C}^s$ given by

$$\widetilde{q}(w, x_1, \dots, x_s, y_1, \dots, y_s) = (w, \frac{x_2}{x_1}, \dots, \frac{x_s}{x_1}, \frac{y_1}{\overline{x_1}}, \dots, \frac{y_s}{\overline{x_1}})$$

where "$^-$" denotes "complex conjugate." And since $f_i(w,x,y) = \sum_{\alpha,\beta} c_{i\alpha\beta}(w)x^\alpha y^\beta$ is bihomogeneous,

$$f_i(w,x,y) = x_1^{|\alpha|}\overline{x_1}^{|\beta|} f_i(w, 1, \frac{x_2}{x_1}, \dots, \frac{x_s}{x_1}, \frac{y_1}{\overline{x_1}}, \dots, \frac{y_s}{\overline{x_1}}).$$

Hence $q(\Upsilon \cap O_1)$ is homeomorphic to the complex-analytic variety U_1 defined by the vanishing of the power series $f_i(w, 1, \xi_2, \dots, \xi_s, \eta_1, \dots, \eta_s)$. Standard arguments show that for any $x \in V \cap O_1$, U_1 is a manifold in a neighborhood of $q(x)$.

Had we used a different embedding of Υ into $W \times \mathbb{C}^s \times \mathbb{C}^s$ (but with the same projection to W, and the same $\mathbb{C}^*$ action (6.1.1)), then the resulting chart would be complex-analytically equivalent to the one just described—that is a special case of the fact that if two graded $\mathcal{O}_W$-algebras have isomorphic Specans, then they are isomorphic and so have isomorphic Projans.

Similarly, working in O_i, where x_i doesn't vanish, we get another manifold chart; but on the overlap $O_i \cap O_1$ the two charts differ by a *real-analytic* coordinate transformation, of the form

$$(\dots, \xi_i, \dots, \xi_j, \dots, \eta_k, \dots) \mapsto (\dots, \frac{1}{\xi_i}, \dots, \frac{\xi_j}{\xi_i}, \dots, \frac{\eta_k}{\overline{\xi_i}}, \dots).$$

Similar remarks apply to the open sets O'_k where y_k doesn't vanish, and to the overlaps $O_i \cap O'_k$.

It should now be more or less apparent how $U := q(V)$ can be made into a real-analytic $2n$-dimensional manifold. The manifold U is *canonically orientable* because, q being a $\mathbb{C}^*$-bundle map, for each $u \in U$ there is an open set $O \subset \mathbb{C}^n$ together with a real-analytic homeomorphism ψ from O onto an open neighborhood U_u of u in U fitting into a commutative diagram

$$\begin{array}{ccc} O \times \mathbb{C}^* & \xrightarrow{\ \phi\ } & q^{-1}U_u \\ {\scriptstyle \text{proj'n}}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ O & \xrightarrow[\ \psi\ ]{} & U_u \end{array}$$

where ϕ is an *orientation-preserving* homeomorphism commuting with the respective $\mathbb{C}^*$ actions; and one checks that the charts ψ provide an orientation for U.

(6.2) We return to the situation in (5.4), assuming as we did there that every irreducible component of the cone C meets $p^{-1}(W_0)$. We assume further that both C (hence P) and W are equidimensional, with $\dim W < \dim C$. We are going to relate the Segre classes of components of $\Lambda(C)$ with the Segre classes of components of C, as described, algebraically, in [Fn, Chap. 4].

More specifically, the Segre classes $s_i(C_j) \in H_{2\dim W-2i}(W)$ of the irreducible components C_j of C can be defined topologically as above (and more easily, because we need only deal with the *complex*-analytic $\mathbb{C}^1$-action given by μ in (1.1.1)), cf. [Fn, Chap. 19]): viz., if P_j is the component of P corresponding to C_j (P_j is topologically the $\mathbb{C}^*$-quotient of $C_j \setminus \sigma(W)$), and $\iota_j\colon P_j \hookrightarrow P$ is the inclusion; if $\mathcal{L}_j := \iota_j^*\mathcal{L}$ and c_j is its first Chern class; and if $\wp\colon P \to W$ is, as before, the canonical map, then

$$s_i(C_j) := \wp_*\iota_{j*}\big([P_j] \cap c_j^{\dim C-1-\dim W+i}\big) = \wp_*\big(\iota_{j*}[P_j] \cap c^{\dim C-1-\dim W+i}\big).$$

where, with c the first Chern class of $\mathcal{L}$—so that $c_j = \iota_j^* c$—the equality is given by the projection formula [BH, p. 507, 7.5]. (Note that C_j is a cone, P_j is its projectivization, and $\mathcal{L}_j$ is the canonical line bundle on P_j.)

Now the construction of Segre classes in §6.1 applies in particular when $W = P$ and $C = \mathcal{L}$, in which case $C \times_W C$ with its $\mathbb{C}^*$ action can be identified with the rank two bundle $\mathcal{L} \oplus \mathcal{L}^*$ with its standard (diagonal) $\mathbb{C}^*$ action. As noted in (5.4), $\mathcal{L}^*$ is topologically isomorphic to $\mathcal{L}^{-1}$, so the total Chern class $\mathrm{ch}(\mathcal{L}_j \oplus \mathcal{L}_j^*)$ is $1 - c_j^2$ where c_j is the first Chern class of $\mathcal{L}_j$. Hence (see Example (6.1.3))

$$s(\mathcal{L}_j \oplus \mathcal{L}_j^*) = [P_j] \cap (1 + c_j^2 + c_j^4 + \dots).$$

As noted in (5.4), the proper map $\gamma\colon (\mathcal{L} \oplus \mathcal{L}^*) \setminus P \to \Lambda(C) \setminus W$ is bijective, hence is a homeomorphism, and it commutes with the respective $\mathbb{C}^*$ actions, but since it involves one complex conjugation it reverses the natural orientations. Since homeomorphisms of analytic spaces take components to components [GL, p. 172, (A8)], it follows that any irreducible component of $\Lambda(C)$ is $\mathbb{C}^1$-stable, so that its total Segre class is defined, and indeed can be obtained by applying $-\wp_*$ to the Segre class of the corresponding component of $\mathcal{L} \oplus \mathcal{L}^*$. The components in question correspond to those of P, and so to those of C. Hence, for the component of $\Lambda(C)$ corresponding to the component C_j the total Segre class is

$$-\wp_*\iota_{j*}s(\mathcal{L}_j \oplus \mathcal{L}_j^*) = -\sum_{i\ge 0} s_{2i-\dim C+1+\dim W}(C_j) \in \oplus_{i\ge 0} H_{2(\dim C-2i-1)}(W).$$

We can thus recover from $\Lambda(C)$ about half of the total Segre class $s(C_j)$. To recover the rest, proceed likewise with $\Lambda(C \times \mathbb{C}^1)$, noting that

$$s(C \times \mathbb{C}^1) = s(C)$$

(cf. [Fn, p. 71, 4.1.1]), and of course

$$\dim(C \times \mathbb{C}^1) = \dim C + 1.$$

Here $C \times \mathbb{C}^1$ is viewed as the cone corresponding to the grading of $\mathcal{G}[T]$ (T an indeterminate) with degree n piece $\oplus_{i=0}^n \mathcal{G}_i T^{n-i}$, a cone whose components are naturally in one-one correspondence with those of C.

Theorem (6.3). *Let V and V' be reduced equidimensional complex spaces, and let $W \subset V$ and $W' \subset V'$ be nowhere dense equidimensional complex submanifolds. Let $f\colon V \to V'$ be a C^1 homeomorphism such that f^{-1} is C^1 and $f(W) = W'$. Let C_j be an irreducible component of $C := C(V,W)$ and let C'_j be the corresponding component of $C' := C(V',W')$ (see Theorem (4.3.1)). Then*

$$f_* s(C_j) = \pm s(C'_j).$$

Proof. As in Corollary (5.4.2), f induces a homeomorphism $\tilde{f}\colon \Lambda(C) \to \Lambda(C')$ which takes W to W' (see (5.3.5)), and which commutes with the $\mathbb{C}^*$ actions. Similarly, the map $f \times 1\colon V \times \mathbb{C}^1 \to V' \times \mathbb{C}^1$ induces a homeomorphism

$$\Lambda\big(C(V \times \mathbb{C}^1, W \times \{0\})\big) = \Lambda(C \times \mathbb{C}^1) \to \Lambda(C' \times \mathbb{C}^1) = \Lambda\big(C(V' \times \mathbb{C}^1, W' \times \{0\})\big).$$

These homeomorphisms respect irreducible components [GL, p. 172, (A8)], and so the induced homology maps take fundamental classes of components to fundamental classes of components, up to multiplication by ± 1.

The theorem results easily now from the foregoing procedure to recover the Segre classes of components of C from those of the corresponding components of $\Lambda(C)$ and $\Lambda(C \times \mathbb{C}^1)$. □

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