

# ON THE FUNDAMENTAL CLASS OF AN ESSENTIALLY SMOOTH SCHEME-MAP

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ABSTRACT. Let  $f: X \rightarrow Z$  be a separated essentially-finite-type flat map of noetherian schemes, and  $\delta: X \rightarrow X \times_Z X$  the diagonal map. The *fundamental class*  $C_f$  (globalizing residues) is a map from the relative Hochschild functor  $\mathbf{L}\delta^*\delta_*f^*$  to the relative dualizing functor  $f^!$ . A compatibility between this  $C_f$  and derived tensor product is shown. The main result is that, in a suitable sense,  $C_f$  generalizes Verdier's classical isomorphism for smooth  $f$  with fibers of dimension  $d$ , an isomorphism that binds  $f^!$  to relative  $d$ -forms.

## INTRODUCTION

**1.1.** (Underlying duality theory.) For a scheme  $X$ ,  $\mathbf{D}(X)$  is the derived category of the abelian category of  $\mathcal{O}_X$ -modules; and  $\mathbf{D}_{\text{qc}}(X) \subset \mathbf{D}(X)$  (resp.  $\mathbf{D}_{\text{qc}}^+(X) \subset \mathbf{D}(X)$ ) is the full subcategory spanned by the complexes  $C$  such that the cohomology modules  $H^i(C)$  are all quasi-coherent (resp. are all quasi-coherent, *and* vanish for all but finitely many  $i < 0$ ).

Grothendieck duality is concerned with a pseudofunctor  $(-)^!$  over the category  $\mathcal{E}$  of essentially-finite-type separated maps of noetherian schemes, taking values in  $\mathbf{D}_{\text{qc}}^+(-)$ . This pseudofunctor is uniquely determined up to isomorphism by the following three properties:

- (i) For formally étale  $\mathcal{E}$ -maps  $f$ ,  $f^!$  is the usual restriction functor  $f^*$ .
- (ii) (Duality) If  $f$  is a proper map of noetherian schemes then  $f^!$  is right-adjoint to  $\mathbf{R}f_*$ .
- (iii) Suppose there is given a fiber square in  $\mathcal{E}$

$$(1.1.1) \quad \begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \clubsuit & \downarrow f \\ Z' & \xrightarrow[u]{} & Z \end{array}$$

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2010 *Mathematics Subject Classification*. Primary: 14F05, 14F10. Secondary: 13D03.  
*Key words and phrases*. Grothendieck duality, relative dualizing complex, fundamental class.

This article is based on work supported by the National Science Foundation under Grant No. 0932078000, while the authors were in residence, during the Spring semester of 2013, at the Mathematical Sciences Research Institute in Berkeley, California.

with  $f$  (hence  $g$ ) proper and  $u$  (hence  $v$ ) formally étale. Then the functorial *base-change map*

$$(1.1.2) \quad \beta_{\clubsuit}(F): v^*f^!F \rightarrow g^!u^*F \quad (F \in \mathbf{D}_{\text{qc}}^+(Z)),$$

defined to be adjoint to the natural composition

$$\text{R}g_*v^*f^!F \xrightarrow{\sim} u^*\text{R}f_*f^!F \longrightarrow u^*F$$

(see (2.6.7) below), is identical with the natural composite isomorphism

$$(1.1.3) \quad v^*f^!F = v^!f^!F \xrightarrow{\sim} (fv)^!F = (ug)^!F \xrightarrow{\sim} g^!u^!F = g^!u^*F.$$

(N.B. The composite isomorphism (1.1.3) exists for *any*  $\mathcal{E}$ -maps  $f, g, u, v$  with  $u$  and  $v$  étale and  $fv = ug$ .)

The case of finite-type maps is treated in [L09, Theorem 4.8.1], from the category-theoretic viewpoint of Verdier and Deligne.<sup>1</sup> An extension to essentially-finite-type maps is given in [Nk09, §5.2]. The pseudofunctor  $(-)^!$  expands so as to take values in  $\mathbf{D}_{\text{qc}}(-)$  if one restricts to proper maps or to  $\mathcal{E}$ -maps of finite flat dimension [AJL11, §§5.7–5.9]—and even without such restrictions if one changes ‘pseudofunctor’ to ‘oplax functor,’ i.e., one allows that for an  $\mathcal{E}$ -diagram  $W \xrightarrow{g} X \xrightarrow{f} Z$  the associated map  $(fg)^! \rightarrow g^!f^!$  need not be an isomorphism, see [Nm14]. For flat  $\mathcal{E}$ -maps—the maps with which we shall be mainly concerned—the agreement of the oplax  $(-)^!$  with the preceding pseudofunctor results from [Nm14, Proposition 13.11].

**1.2.** (Verdier’s isomorphism, fundamental class and the “Ideal Theorem.”) In the original working out ([H66], amended in [C00], [C01]) of the duality theory conceived by Grothendieck and Verdier, the main result, Corollary 3.4 on p. 383, is roughly that in the presence of residual complexes,<sup>2</sup> when confined to finite-type scheme-maps and complexes with coherent homology the pseudofunctor  $(-)^!$  is obtained by pasting together concrete realizations of its restriction to smooth maps and to finite maps. This is a special case of the “Ideal Theorem” in [H66, p. 6].

For *finite*  $f: X \rightarrow Z$ , the canonical concrete realization is induced by the usual sheafified duality isomorphism (see (2.6.5)):

$$(1.2.1) \quad \text{R}f_*f^!F = \text{R}f_*\text{R}\mathcal{H}om(\mathcal{O}_X, f^!F) \xrightarrow{\sim} \text{R}\mathcal{H}om(f_*\mathcal{O}_X, F) \quad (F \in \mathbf{D}_{\text{qc}}^+(Z)).$$

For *formally smooth*  $f: X \rightarrow Z$  in  $\mathcal{E}$ , of relative dimension  $d$  (section 2.3), a canonical concrete realization is given by an isomorphism

$$(1.2.2) \quad f^!F \xrightarrow{\sim} \Omega_f^d[d] \otimes_X f^*F \quad (F \in \mathbf{D}_{\text{qc}}^+(Z)),$$

where  $\Omega_f^d$  is the sheaf of relative  $d$ -forms and “[ $d$ ]” denotes  $d$ -fold translation (shift) in  $\mathbf{D}(X)$ .

<sup>1</sup>The frequent references in this paper to [L09] are due much more to the approach and convenience of that source than to its originality.

<sup>2</sup>The theory of residual complexes presented in [H66, Chapters 6 and 7] is considerably generalized in [LNS05].

The initial avatar of such an isomorphism uses a trace map for residual complexes (that are assumed to exist). In particular, when  $f$  is proper there results an explicit (but somewhat complicated) description of Serre-Grothendieck duality. (See e.g., [C00, §3.4].)

A definition of an isomorphism with the same source and target as (1.2.2), but not requiring residual complexes, was given by Verdier in [V68, proof of Theorem 3]. We review and expand upon this classical isomorphism in §3. In particular, Proposition 3.4.4 explicates compatibility of the isomorphism with derived tensor product.

For any flat  $\mathcal{E}$ -map  $f: X \rightarrow Z$ , let  $\delta = \delta_f: X \rightarrow X \times_Z X$  be the diagonal of  $f$ . There is a  $\mathbf{D}_{\text{qc}}^+$ -map

$$C_f: \delta^* \delta_* f^* \rightarrow f^!$$

the *fundamental class of  $f$* , from the relative Hochschild functor to the relative dualizing functor, see Definition 4.2.

The fundamental class is compatible with derived tensor product, see Proposition 4.3.

When the flat map  $f$  is *equidimensional of relative dimension  $d$* , applying the homology functor  $H^{-d}$  to  $C_f$  leads to a map  $c_f$  from the target of the map (1.2.2) to the source (Section 4.4 below.) Proposition 2.4.2 in [AJL14] asserts that if, moreover,  $f$  is *formally smooth*, then  $c_f$  is an isomorphism. The proof uses Verdier’s isomorphism, and of course begs the question of whether that isomorphism is inverse to  $c_f$ .

*That this does hold is our main result, Theorem 4.5.*

Thus, the map  $c_f$  extends the inverse of Verdier’s isomorphism, from the class of formally smooth  $\mathcal{E}$ -maps to the class of arbitrary flat equidimensional  $\mathcal{E}$ -maps.

*Remarks 1.2.3.* (a) The above discussion has been limited, for simplicity, to  $\mathbf{D}_{\text{qc}}^+$ ; but the results will be established for  $\mathbf{D}_{\text{qc}}$ .

(b) The isomorphisms in play are not quite canonical: there are choices involved that affect the resulting homology maps up to sign. For example, for a scheme  $Y$  and an  $\mathcal{O}_Y$ -ideal  $\mathcal{I}$ , there are two natural identifications of  $\mathcal{I}/\mathcal{I}^2$  with  $\mathcal{T}or_1^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{I})$ , one the negative of the other; and we will have to choose one of them (see (3.2.1) *ff.*) We will also have to assign different roles to the two projections of  $X \times_Z X$  to  $X$ , necessitating another arbitrary choice. See also [S04, §7.1].

Our choices minimize sign considerations, under the constraint of respect for the usual triangulated structure on the derived sheaf-hom functor (see the remarks following equations (3.1.5) and (3.2.5)). Other choices might have done as well.

(c) In the present vein, we propose it as a nontrivial exercise (that, as far as we know, no one has carried out) to specify the relation between the above “initial avatar” of the fundamental class and Verdier’s isomorphism. (Both isomorphisms have been known for fifty years or so.)

We believe that the fundamental class is important enough, historically and technically, to merit the kind of scrutiny it gets in this paper. For instance, familiarity with various of its aspects could well prove useful in establishing that the pseudofunctor  $(-)^!$  together with the isomorphisms in (1.2.1) and (1.2.2) satisfy VAR 1–6 (mutatis mutandis) in [H66, p. 186]—a version over all of  $\mathcal{E}$  of the “Ideal Theorem.” It is hoped that a full treatment of this application will materialize in the not-too-distant future.

**1.3.** (Additional background: fundamental class and residues.) More history and motivation behind the fundamental class can be found in [AJL14, §0.6].

A preliminary version of the fundamental class, with roots in the dualizing properties of differentials on normal varieties, appears in [G60, p. 114], followed by some brief hints about connections with residues. In [L84], there is a concrete treatment of the case when  $S = \text{Spec}(k)$  with  $k$  a perfect field and  $f: X \rightarrow \text{Spec } k$  an integral algebraic  $k$ -scheme. The principal result (“Residue Theorem”) reifies  $c_f$  as a globalization of the local residue maps at the points of  $X$ , leading to explicit versions of local and global duality and the relation between them. This is generalized to certain maps of noetherian schemes in [HS93].

The close relation between the fundamental class and residues becomes clearer, and more general, over formal schemes, where local and global duality merge into a single theory with fundamental classes and residues conjoined. (See [L01, §5.5]; a complete exposition has yet to appear.)

## 2. PRELIMINARIES

In this section we describe those parts of the duality machinery, along with a few of their basic properties, that we will subsequently use. The reader is advised initially to skip to the next section, referring back to this one as needed.

**2.1.** Unless otherwise specified, we will be working exclusively with functors between full subcategories of categories of the form  $\mathbf{D}(X)$  (see beginning of §1.1); so to reduce clutter we will write:

- $f_*$  instead of  $\mathbf{R}f_*$  ( $f$  a scheme-map),
- $f^*$  instead of  $\mathbf{L}f^*$  ( $f$  a scheme-map),
- $\otimes$  instead of  $\otimes^{\mathbf{L}}$ .
- $\mathcal{H}om$  instead of  $\mathbf{R}\mathcal{H}om$ . (This is the derived *sheaf-hom* functor.)

**2.2.** For any scheme-map  $f: X \rightarrow Z$ , the functor  $f_*: \mathbf{D}(X) \rightarrow \mathbf{D}(Z)$  is *symmetric monoidal*; this entails, in particular, a functorial map

$$(2.2.1) \quad f_*E_1 \otimes_Z f_*E_2 \rightarrow f_*(E_1 \otimes_X E_2) \quad (E_i \in \mathbf{D}(X)),$$

see e.g., [L09, 3.4.4(b)].

The functor  $f^*: \mathbf{D}(Z) \rightarrow \mathbf{D}(X)$  is *pseudofunctorially* left-adjoint to  $f_*$  [L09, 3.6.7(d)]. This means that for scheme-maps  $X \xrightarrow{f} Z \xrightarrow{g} W$ , the canonical isomorphisms  $(gf)_* \xrightarrow{\sim} g_*f_*$  and  $f^*g^* \xrightarrow{\sim} (gf)^*$  are *conjugate*, see [L09, 3.3.7(a)].

For fixed  $B \in \mathbf{D}(X)$ , the functor  $- \otimes_X B$  is left-adjoint to  $\mathcal{H}om_X(B, -)$  (see e.g., [L09, (2.6.1)]). The resulting counit map

$$(2.2.2) \quad \text{ev}(X, B, C): \mathcal{H}om_X(B, C) \otimes_X B \rightarrow C \quad (C \in \mathbf{D}(X))$$

is referred to as “evaluation.”

**2.3.** A scheme-map  $f: X \rightarrow Z$  is *essentially of finite type* if every  $z \in Z$  has an affine open neighborhood  $\text{Spec } A$  whose inverse image is a union of finitely many affine open subschemes  $\text{Spec } B_i$  such that each  $B_i$  is a localization of a finitely generated  $A$ -algebra.

Throughout,  $\mathcal{E}$  will be the category of essentially-finite-type separated maps of noetherian schemes.

The category  $\mathcal{E}$  is closed under scheme-theoretic fiber product.

An  $\mathcal{E}$ -map  $f: X \rightarrow Z$  is *essentially smooth* if it is flat and it has geometrically regular fibers; or equivalently, if  $f$  is *formally smooth*, that is, for each  $x \in X$  and  $z := f(x)$ , the local ring  $\mathcal{O}_{X,x}$  is formally smooth over  $\mathcal{O}_{Z,z}$  for the discrete topologies, see [EGA4, 17.1.2, 17.1.6] and [EGA4, Chapter 0, 19.3.3, 19.3.5(iv) and 22.6.4 a)  $\Leftrightarrow$  c)]. (For this equivalence, which involves only local properties, it can be assumed that  $X = \text{Spec } A$  and  $Z = \text{Spec } B_S$  where  $B$  is a finite-type  $A$ -algebra and  $S \subset B$  is a multiplicatively closed subset; then the relevant local rings are the same for  $f$  or for the finite-type map  $g: \text{Spec } B \rightarrow \text{Spec } A$ , so that one only needs the equivalence for  $g$ , as given by [EGA4, 6.8.6].)

For essentially smooth  $f: X \rightarrow Z$ , the diagonal map  $X \rightarrow X \times_Z X$  is a regular immersion, see [EGA4, 16.9.2], [EGA4, 16.10.2] and [EGA4, 16.10.5]—whose proof is valid for  $\mathcal{E}$ -maps.

Arguing as in [EGA4, 17.10.2], one gets that for essentially smooth  $f$  the relative differential sheaf  $\Omega_f$  is locally free over  $\mathcal{O}_X$ ; moreover, when  $f$  is of finite type the rank of  $\Omega_f$  at  $x \in X$  is the dimension at  $x$  of the fiber  $f^{-1}f(x)$ . For any essentially smooth  $f$ , the (locally constant) rank of  $\Omega_f$  will be referred to as the *relative dimension of  $f$* .

An  $\mathcal{E}$ -map is *essentially étale* if it is essentially smooth and has relative dimension 0 (cf. [EGA4, 17.6.1]).

**2.4.** An  $\mathcal{O}_X$ -complex  $E$  ( $X$  a scheme) is *perfect* if each  $x \in X$  has an open neighborhood  $U = U_x$  such that the restriction  $E|_U$  is  $\mathbf{D}(U)$ -isomorphic to a bounded complex of finite-rank free  $\mathcal{O}_U$ -modules.

When applied to  $\mathcal{E}$ -maps, the term “perfect” means “having finite tor-dimension” (or “finite flat dimension”).

Perfection of maps is preserved under composition and under flat base change, see e.g., [I71, p. 243, 3.4 and p. 245, 3.5.1].

**2.5.** Let  $\mathbf{D}_{\text{qc}}^+(X) \subset \mathbf{D}_{\text{qc}}(X) \subset \mathbf{D}(X)$  be as at the beginning of §1.1.

For any  $\mathcal{E}$ -map  $f: X \rightarrow Z$ , there exists a functor  $f^\times: \mathbf{D}(Z) \rightarrow \mathbf{D}_{\text{qc}}(X)$  that is bounded below and right-adjoint to  $\mathbf{R}f_*$ . (See e.g., [L09, §4.1].) In particular, with  $\text{id}_X$  the identity map of  $X$  one has a functor  $\text{id}_X^\times$ , the *derived quasi-coherator*, right-adjoint to the inclusion  $\mathbf{D}_{\text{qc}}(X) \hookrightarrow \mathbf{D}(X)$ .

For any complexes  $A$  and  $B$  in  $\mathbf{D}_{\text{qc}}(X)$ , and with notation as in §2.1, set

$$(2.5.1) \quad \mathcal{H}om_X^{\text{qc}}(A, B) := \text{id}_X^\times \mathcal{H}om_X(A, B) \in \mathbf{D}_{\text{qc}}(X).$$

$\mathbf{D}_{\text{qc}}(X)$  is a (symmetric monoidal) closed category, with multiplication (derived)  $\otimes_X$  and internal hom  $\mathcal{H}om_X^{\text{qc}}$  (cf. e.g., [L09, 3.5.2(d)]).

For  $C \in \mathbf{D}_{\text{qc}}(X)$ , the counit map is a  $\mathbf{D}(X)$ -isomorphism  $\text{id}_X^\times C \xrightarrow{\sim} C$ . So for  $A, B \in \mathbf{D}(X)$ , the counit map  $\mathcal{H}om_X^{\text{qc}}(A, B) \rightarrow \mathcal{H}om_X(A, B)$  is an *isomorphism* when  $\mathcal{H}om_X(A, B) \in \mathbf{D}_{\text{qc}}(X)$ —for example, when  $B \in \mathbf{D}_{\text{qc}}^+(X)$  and the cohomology sheaves  $H^i A$  are coherent for all  $i$ , vanishing for  $i \gg 0$ , see [H66, p. 92, 3.3].

**2.6.** We will use some standard functorial maps, gathered together here, that are associated to an  $\mathcal{E}$ -map  $f: X \rightarrow Z$  and objects  $E_\bullet \in \mathbf{D}_{\text{qc}}(X)$ ,  $F_\bullet \in \mathbf{D}_{\text{qc}}(Z)$ ,  $G_\bullet \in \mathbf{D}_{\text{qc}}(Z)$ . For the most part, the definitions of these maps emerge category-theoretically from Section 2.2, cf. [L09, §3.5.4].

(a) The map

$$(2.6.1) \quad f^*(F \otimes_Z G) \longrightarrow f^*F \otimes_X f^*G$$

that is adjoint to the natural composite

$$F \otimes_Z G \longrightarrow f_* f^*F \otimes_Z f_* f^*G \xrightarrow{(2.2.1)} f_*(f^*F \otimes_X f^*G).$$

The map (2.6.1) is an *isomorphism* [L09, 3.2.4].

(b) The *sheafified adjunction isomorphism*:

$$(2.6.2) \quad f_* \mathcal{H}om_X(f^*G, E) \xrightarrow{\sim} \mathcal{H}om_Z(G, f_*E),$$

right-conjugate [ILN14, §1.6], for each fixed  $G$ , to the isomorphism (2.6.1).

(c) The *projection isomorphisms*, see, e.g., [L09, 3.9.4]:

$$(2.6.3) \quad f_*E \otimes_Z G \xrightarrow{\sim} f_*(E \otimes_X f^*G) \quad \text{and} \quad G \otimes_Z^{\mathbb{L}} f_*E \xrightarrow{\sim} f_*(f^*G \otimes_X^{\mathbb{L}} E).$$

These are, respectively, the natural composites

$$\begin{aligned} f_*E \otimes_Z G &\longrightarrow f_*E \otimes_Z f_*f^*G \xrightarrow{(2.2.1)} f_*(E \otimes_X f^*G), \\ G \otimes_Z f_*E &\longrightarrow f_*f^*G \otimes_Z f_*E \xrightarrow{(2.2.1)} f_*(f^*G \otimes_X E). \end{aligned}$$

(The definitions make sense for arbitrary scheme-maps, though in that generality the composites are not always isomorphisms.)

(d) The map

$$(2.6.4) \quad f^* \mathcal{H}om_Z(G, F) \longrightarrow \mathcal{H}om_X(f^*G, f^*F)$$

that is  $f^*$ - $f_*$  adjoint to the natural composite map

$$\mathcal{H}om_Z(G, F) \longrightarrow \mathcal{H}om_Z(G, f_*f^*F) \xrightarrow[(2.6.2)]{\sim} f_* \mathcal{H}om_X(f^*G, f^*F).$$

This is an isomorphism if the map  $f$  is perfect,  $G$  is homologically bounded-above, with coherent homology sheaves, and  $F \in \mathbf{D}_{\text{qc}}^+(Z)$ , see [L09, Proposition 4.6.6].

The map (2.6.4) is  $\mathcal{H}om$ - $\otimes$  adjoint to the natural composite map

$$f^* \mathcal{H}om_Z(G, F) \otimes_X f^* G \xrightarrow{(2.6.1)} f^*(\mathcal{H}om_Z(G, F) \otimes_Z G) \longrightarrow f^* F,$$

see [L09, Exercise 3.5.6(a)].

(e) The *duality isomorphism*:

$$(2.6.5) \quad \zeta(E, F): f_* \mathcal{H}om_X^{\text{qc}}(E, f^* F) \xrightarrow{\sim} \mathcal{H}om_Z^{\text{qc}}(f_* E, F),$$

right-conjugate, for each fixed  $E$  to the projection isomorphism

$$f_*(f^* G \otimes_X^{\mathbb{L}} E) \xleftarrow{\sim} G \otimes_Z^{\mathbb{L}} f_* E.$$

(f) The bifunctorial map

$$(2.6.6) \quad \chi_f(F, G): f^! F \otimes_X f^* G \rightarrow f^!(F \otimes_Z G),$$

defined in [Nm14, 13.1, 13.2 and 13.3] (with ‘ $\sigma$ ’ instead of ‘ $\chi$ ’), and shown in [Nm14, 13.11] to be an *isomorphism* whenever the map  $f$  is perfect.

When  $f$  is proper,  $f^!$  is right-adjoint to  $f_*$  and the map  $\chi_f$  is adjoint to the natural composite map

$$f_*(f^! F \otimes_X f^* G) \xrightarrow{(2.6.3)} f_* f^! F \otimes_Z G \longrightarrow F \otimes_Z G.$$

In particular,  $\chi_f(\mathcal{O}_Z, G)$  identifies with a map of triangulated functors

$$f^\# G := f^! \mathcal{O}_Z \otimes_X f^* G \rightarrow f^! G.$$

Note however that whereas the isomorphism  $f^!(G[1]) \xrightarrow{\sim} (f^! G)[1]$  associated with the triangulated structure on  $f^!$  is the identity map, the same is not true of  $f^\#$ . (See [L09, just before 1.5.5].)

(g) The *base change map*

$$(2.6.7) \quad \beta_{\clubsuit}(F): v^* f^! F \longrightarrow g^! u^* F$$

associated to a fiber square in  $\mathcal{E}$ :

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \clubsuit & \downarrow f \\ Z' & \xrightarrow{u} & Z \end{array}$$

with  $f$  any  $\mathcal{E}$ -map and  $u$  (hence  $v$ ) flat. This map is defined in [Nm14, 11.6 and 11.7], and denoted  $\theta: \mathfrak{R} \rightarrow \mathfrak{S}$ , where  $\mathfrak{R}, \mathfrak{S}$  are (repectively) the maps taking the cartesian square to  $v^* f^!$  and  $g^! u^*$ .

In general, this is not an isomorphism. But if  $F \in \mathbf{D}_{\text{qc}}^+(Z)$  then this is the isomorphism of [L09, Thm. 4.8.3] and [Nk09, Theorem 5.3]; and for any  $F \in \mathbf{D}_{\text{qc}}(Z)$ , if  $f$  is perfect then this is the isomorphism of [AJL11, §5.8.4]. (It actually suffices for  $g$  to be perfect, see [Nm14, 11.13(i)].)

(h) The bifunctorial map

$$(2.6.8) \quad \kappa_X(E_1, E_2, E_3): \mathcal{H}om_X(E_1, E_2) \otimes_X E_3 \rightarrow \mathcal{H}om_X(E_1, E_2 \otimes_X E_3)$$

adjoint to the natural composite—with  $\text{ev}$  the evaluation map (2.2.2)—

$$\mathcal{H}om_X(E_1, E_2) \otimes_X E_3 \otimes_X E_1 \xrightarrow{\sim} \mathcal{H}om_X(E_1, E_2) \otimes_X E_1 \otimes_X E_3 \xrightarrow{\text{via ev}} E_2 \otimes_X E_3.$$

The map (2.6.8) is an *isomorphism* if the complex  $E_1$  is *perfect*. Indeed, the question is local on  $X$ , so one can assume that  $E_1$  is a bounded complex of finite-rank free  $\mathcal{O}_X$ -modules, and conclude via a simple induction (like that in the second-last paragraph in the proof of [L09, 4.6.7]) on the number of degrees in which  $E_1$  doesn't vanish.

Similarly, (2.6.8) is an isomorphism if  $E_3$  is perfect.

The projection map (2.6.3) is compatible with derived tensor product, in the following sense:

**Lemma 2.7.** *Let  $f: X \rightarrow Z$  be a scheme map,  $E \in \mathbf{D}(X)$ ,  $F, G \in \mathbf{D}(Z)$ . The following natural diagram, where  $\otimes$  stands for  $\otimes_X$  or  $\otimes_Z$ , commutes.*

$$\begin{array}{ccccc} (f_*E \otimes F) \otimes G & \xrightarrow{(2.6.3)} & f_*(E \otimes f^*F) \otimes G & \xrightarrow{(2.6.3)} & f_*((E \otimes f^*F) \otimes f^*G) \\ \simeq \downarrow & & & & \downarrow \simeq \\ f_*E \otimes (F \otimes G) & \xrightarrow{(2.6.3)} & f_*(E \otimes f^*(F \otimes G)) & \xrightarrow{(2.6.1)} & f_*(E \otimes (f^*F \otimes f^*G)) \end{array}$$

*Proof.* After substituting for each instance of (2.6.3) its definition, and recalling (2.2.1), one comes down to proving commutativity of the border of the following diagram—whose maps are the obvious ones:

$$\begin{array}{ccccc} (f_*E \otimes F) \otimes G & \longrightarrow & (f_*E \otimes f_*f^*F) \otimes G & \longrightarrow & f_*(E \otimes f^*F) \otimes G \\ \downarrow & & \swarrow & \searrow & \downarrow \\ f_*E \otimes (F \otimes G) & \longrightarrow & f_*E \otimes (f_*f^*F \otimes G) & & f_*(E \otimes f^*F) \otimes f_*f^*G \\ \downarrow & & \swarrow & \searrow & \downarrow \\ f_*E \otimes f_*f^*(F \otimes G) & \longrightarrow & f_*E \otimes (f_*f^*F \otimes f_*f^*G) & & f_*((E \otimes f^*F) \otimes f^*G) \\ \downarrow & & \downarrow & & \downarrow \\ f_*(E \otimes f^*(F \otimes G)) & \longrightarrow & f_*(E \otimes f^*(f^*F \otimes f^*G)) & \xrightarrow{\text{⑦}} & f_*(E \otimes (f^*F \otimes f^*G)) \end{array}$$

The commutativity of subdiagram ⑥ follows directly from the definition of (2.6.1). Commutativity of ⑦ is given by symmetric monoidality of  $f_*$ , see §2.2. Commutativity of the other subdiagrams is pretty well obvious, whence the conclusion.  $\square$



For the map (2.6.8) we'll need a “transitivity” property—an instance of the Kelly-Mac Lane coherence theorem [KM71, p. 107, Theorem 2.4]:

**Lemma 2.8.** *With  $\kappa := \kappa_X$  as in (2.6.8),  $\kappa(E_1, E_2, E_3 \otimes E_4)$  factors as*

$$\begin{aligned} \mathcal{H}om_X(E_1, E_2) \otimes E_3 \otimes E_4 &\xrightarrow{\kappa(E_1, E_2, E_3) \otimes \text{id}} \mathcal{H}om_X(E_1, E_2 \otimes E_3) \otimes E_4 \\ &\xrightarrow{\kappa(E_1, E_2 \otimes E_3, E_4)} \mathcal{H}om_X(E_1, E_2 \otimes E_3 \otimes E_4). \end{aligned}$$

*Proof.* The assertion results from commutativity of the following natural diagram—where  $[-, -] := \mathcal{H}om_X(-, -)$ ,  $\otimes := \otimes_X$ , and various associativity isomorphisms are omitted:

$$\begin{array}{ccccc} [E_1, E_2] \otimes E_3 \otimes E_4 \otimes E_1 & \xrightarrow{\sim} & [E_1, E_2] \otimes E_3 \otimes E_1 \otimes E_4 & \xrightarrow{\sim} & [E_1, E_2] \otimes E_1 \otimes E_3 \otimes E_4 \\ \downarrow \text{via } \kappa & & \downarrow \text{via } \kappa & & \downarrow \text{via ev} \\ \text{①} & & \text{②} & & \\ [E_1, E_2 \otimes E_3] \otimes E_1 \otimes E_4 & & [E_1, E_2 \otimes E_3] \otimes E_1 \otimes E_4 & & \\ \uparrow \cong & & \downarrow \text{via ev} & & \\ \text{③} & & & & \\ [E_1, E_2 \otimes E_3] \otimes E_4 \otimes E_1 & \xrightarrow{\text{via } \kappa} & [E_1, E_2 \otimes E_3 \otimes E_4] \otimes E_1 & \xrightarrow{\text{ev}} & E_2 \otimes E_3 \otimes E_4 \end{array}$$

Commutativity of subdiagram ① is clear; and that of ② and ③ follow from the definitions of  $\kappa(E_1, E_2, E_3)$  and  $\kappa(E_1, E_2 \otimes E_3, E_4)$ , respectively.  $\square$

**2.9.** Let  $\delta: X \rightarrow Y$  be an  $\mathcal{E}$ -map, and  $p: Y \rightarrow X$  a scheme-map such that  $p\delta = \text{id}_X$ . We will be using the bifunctorial isomorphism

$$\psi = \psi(\delta, p, E, F): \delta^* \delta_* E \otimes_X F \xrightarrow{\sim} \delta^* \delta_*(E \otimes_X F) \quad (E, F \in \mathbf{D}_{\text{qc}}(X))$$

that is defined to be the natural composite

$$(2.9.1) \quad \begin{array}{ccc} \delta^* \delta_* E \otimes_X F & & \delta^* \delta_*(E \otimes_X F) \\ \cong \downarrow & & \uparrow \cong \\ \delta^* \delta_* E \otimes_X \delta^* p^* F & \xrightarrow[\text{(2.6.1)}]{\sim} & \delta^*(\delta_* E \otimes_Y p^* F) \xrightarrow[\text{(2.6.3)}]{\sim} \delta^* \delta_*(E \otimes_X \delta^* p^* F) \end{array}$$

(Cf. [AJL14, (2.2.6)].)

The rest of this subsection brings out properties of  $\psi$  needed later on.

**Lemma 2.9.2.** *With preceding notation, the isomorphism  $\psi(\delta, p, E, \delta^* p^* F)$  factors as*

$$\delta^* \delta_* E \otimes_X \delta^* p^* F \xrightarrow[\text{(2.6.1)}]{\sim} \delta^*(\delta_* E \otimes_Y p^* F) \xrightarrow[\text{(2.6.3)}]{\sim} \delta^* \delta_*(E \otimes_X \delta^* p^* F).$$

*Proof.* This results from the fact that the natural composites

$$\delta^* \delta_* E \otimes F \xrightarrow{\sim} \delta^* \delta_* E \otimes \delta^* p^* F \xrightarrow{\psi(\delta, p, E, \delta^* p^* F)} \delta^* \delta_*(E \otimes \delta^* p^* F) \xrightarrow{\sim} \delta^* \delta_*(E \otimes F)$$

and

$$\delta^* \delta_* E \otimes F \xrightarrow{\sim} \delta^* \delta_* E \otimes \delta^* p^* F \xrightarrow[\text{(2.6.3)} \circ \text{(2.6.1)}]{\sim} \delta^* \delta_*(E \otimes \delta^* p^* F) \xrightarrow{\sim} \delta^* \delta_*(E \otimes F)$$

are both equal to  $\psi(\delta, p, E, F)$  (the first by functoriality of  $\psi$ , and the second by definition).  $\square$

**Corollary 2.9.3.** *The following natural diagram commutes.*

$$\begin{array}{ccc} \delta^*\delta_*E \otimes F & \xrightarrow{\psi} & \delta^*\delta_*(E \otimes F) \\ & \searrow & \swarrow \\ & E \otimes F & \end{array}$$

*Proof.* One can replace  $F$  by the isomorphic complex  $\delta^*p^*F$ , whereupon the assertion follows from Lemma 2.9.2 via [L09, 3.4.6.2].  $\square$

**Lemma 2.9.4.** *For  $E, F_1, F_2 \in \mathbf{D}_{\text{qc}}(X)$ , with  $\psi(-, -) := \psi(\delta, p, -, -)$  and  $\otimes := \otimes_X$ , the isomorphism  $\psi(E, F_1 \otimes F_2)$  factors (modulo associativity isomorphisms) as*

$$\begin{aligned} \delta^*\delta_*E \otimes F_1 \otimes F_2 &\xrightarrow[\psi(E, F_1) \otimes \text{id}]{\sim} \delta^*\delta_*(E \otimes F_1) \otimes F_2 \\ &\xrightarrow[\psi(E \otimes F_1, F_2)]{\sim} \delta^*\delta_*(E \otimes F_1 \otimes F_2) \end{aligned}$$

*Proof.* The Lemma asserts commutativity the border of the next diagram, in which  $\otimes$  stands for  $\otimes_X$  or  $\otimes_Y$ , and the maps are the obvious ones:

$$\begin{array}{ccccc} \delta^*\delta_*E \otimes F_1 \otimes F_2 & \xrightarrow{\hspace{10em}} & \delta^*(\delta_*E \otimes p^*F_1) \otimes F_2 & & \\ \downarrow & & \swarrow & \searrow & \downarrow \\ & \textcircled{1} & \delta^*(\delta_*E \otimes p^*F_1) \otimes \delta^*p^*F_2 & & \delta^*\delta_*(E \otimes \delta^*p^*F_1) \otimes F_2 \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ \delta^*(\delta_*E \otimes p^*(F_1 \otimes F_2)) & \delta^*(\delta_*E \otimes p^*F_1 \otimes p^*F_2) & & \delta^*\delta_*(E \otimes \delta^*p^*F_1) \otimes \delta^*p^*F_2 & \delta^*\delta_*(E \otimes F_1) \otimes F_2 \\ \downarrow & \swarrow & \searrow & \downarrow & \downarrow \\ \delta^*\delta_*(E \otimes \delta^*(p^*F_1 \otimes p^*F_2)) & & & \delta^*\delta_*(E \otimes F_1) \otimes \delta^*p^*F_2 & \\ \downarrow & \swarrow & \searrow & & \downarrow \\ \delta^*\delta_*(E \otimes \delta^*p^*(F_1 \otimes F_2)) & \delta^*\delta_*(E \otimes \delta^*(p^*F_1 \otimes p^*F_2)) & & \delta^*(\delta_*(E \otimes \delta^*p^*F_1) \otimes p^*F_2) & \\ \downarrow & \swarrow & \searrow & \downarrow & \downarrow \\ & \textcircled{2} & \delta^*\delta_*(E \otimes \delta^*p^*F_1 \otimes \delta^*p^*F_2) & & \delta^*\delta_*(E \otimes F_1) \otimes \delta^*p^*F_2 \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ \delta^*\delta_*(E \otimes F_1 \otimes F_2) & \delta^*\delta_*(E \otimes \delta^*p^*F_1 \otimes \delta^*p^*F_2) & & \delta^*(\delta_*(E \otimes \delta^*p^*F_1) \otimes p^*F_2) & \\ \downarrow & \swarrow & \searrow & \downarrow & \downarrow \\ & \textcircled{3} & \delta^*\delta_*(E \otimes F_1 \otimes \delta^*p^*F_2) & & \delta^*(\delta_*(E \otimes F_1) \otimes p^*F_2) \\ \delta^*\delta_*(E \otimes F_1 \otimes F_2) & \longleftarrow & \delta^*\delta_*(E \otimes F_1 \otimes \delta^*p^*F_2) & \longleftarrow & \delta^*(\delta_*(E \otimes F_1) \otimes p^*F_2) \end{array}$$

Commutativity of the unlabeled subdiagrams is easy to verify.

Subdiagram ① expands naturally as

$$\begin{array}{ccccc}
 \delta^*\delta_*E \otimes F_1 \otimes F_2 & \longrightarrow & \delta^*\delta_*E \otimes \delta^*p^*F_1 \otimes F_2 & \longrightarrow & \delta^*(\delta_*E \otimes p^*F_1) \otimes F_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 \delta^*\delta_*E \otimes \delta^*p^*(F_1 \otimes F_2) & & \delta^*\delta_*E \otimes \delta^*p^*F_1 \otimes \delta^*p^*F_2 & & \\
 \downarrow & \searrow & \swarrow & \searrow & \downarrow \\
 & \delta^*\delta_*E \otimes \delta^*(p^*F_1 \otimes p^*F_2) & & & \\
 \downarrow & \swarrow & \searrow & & \downarrow \\
 \delta^*(\delta_*E \otimes p^*(F_1 \otimes F_2)) & \longrightarrow & \delta^*(\delta_*E \otimes p^*F_1 \otimes p^*F_2) & \longleftarrow & \delta^*(\delta_*E \otimes p^*F_1) \otimes \delta^*p^*F_2
 \end{array}$$

④ ⑤

For commutativity of ④, see the proof of [L09, 3.6.10]. For that of ⑤, see [L09, Example 3.4.4(b)]. Commutativity of the other two subdiagrams is clear, and so ① commutes.

Commutativity of ② is given by Lemma 2.7.

Finally, for commutativity of ③, see again the proof of [L09, 3.6.10].  $\square$

**Corollary 2.9.5.** *Upon identifying—as one may— $G \otimes \mathcal{O}_X$  with  $G$  for all  $G \in \mathbf{D}(X)$ , one has that  $\psi := \psi(E, \mathcal{O}_X)$  is the identity map of  $\delta^*\delta_*E$ .*

*Proof.* The case  $F_1 = F_2 = \mathcal{O}_X$  of Lemma 2.9.4 implies that  $\psi^2 = \psi$ , whence  $\psi = \psi^2\psi^{-1} = \psi\psi^{-1} = \text{id}$ .  $\square$

### 3. VERDIER'S ISOMORPHISM

Recall from Section 2.1 that, unless otherwise specified, all the functors that appear are functors between derived categories.

**3.1.** Let  $U \xrightarrow{i} X \xrightarrow{f} Z$  be  $\mathcal{E}$ -maps, with  $i$  essentially étale, and let

$$\begin{array}{ccc}
 Y := U \times_Z X & \xrightarrow{p_2} & X \\
 p_1 \downarrow & \spadesuit & \downarrow f \\
 U & \xrightarrow{fi} & Z
 \end{array}$$

be the resulting fiber square, where  $p_1$  and  $p_2$  are the canonical projections. With  $\delta_U$  the diagonal map and  $\text{id}_U$  the identity map of  $U$ , the composite

$$(3.1.1) \quad g: U \xrightarrow{\delta_U} U \times_Z U \xrightarrow{\text{id}_U \times i} U \times_Z X$$

is the graph of  $i$ , i.e.,  $p_1g = \text{id}_U$  and  $p_2g = i$ .

In Theorem 3 of [V68],  $fi$  is essentially smooth, of relative dimension, say,  $d$  (see Section 2.3 above); and that theorem says *there is a  $\mathbf{D}(U)$ -isomorphism*

$$(fi)^!\mathcal{O}_Z \xrightarrow{\sim} \Omega_{fi}^d[d] := (\wedge^d \Omega_{fi})[d].$$

A slight elaboration of Verdier’s approach produces, as follows, a functorial isomorphism

$$(3.1.2) \quad v_{f,i}(F): (fi)^!F \xrightarrow{\sim} \Omega_{fi}^d[d] \otimes_U (fi)^*F \quad (F \in \mathbf{D}_{\text{qc}}^+(Z)).$$

In particular, making the allowable identifications

$$\Omega_{fi}^d[d] \otimes_X (fi)^*\mathcal{O}_Z = \Omega_{fi}^d[d] \otimes_X \mathcal{O}_X = \Omega_{fi}^d[d],$$

we consider  $v_{f,i}(\mathcal{O}_Z)$  to be a map from  $(fi)^!\mathcal{O}_Z$  to  $\Omega_{fi}^d[d]$ .

(In §3.4, the definition of  $v_{f,i}(F)$  will be extended to all  $F \in \mathbf{D}_{\text{qc}}(Z)$ .)

The construction of  $v_{f,i}(F)$  uses two maps. The first is the following natural composite map  $\vartheta$ —whose definition needs only that  $fi$  be *flat*—with  $\chi_g$  as in (2.6.6), and  $\beta_{\spadesuit}$  as in (2.6.7) (an isomorphism since  $F \in \mathbf{D}_{\text{qc}}^+(Y)$ ):

$$(3.1.3) \quad \begin{aligned} \vartheta: g^!\mathcal{O}_Y \otimes_U (fi)^!F &\xrightarrow{\sim} g^!\mathcal{O}_Y \otimes_U i^*f^!F \\ &\xrightarrow{\sim} g^!\mathcal{O}_Y \otimes_U g^*p_2^*f^!F \\ &\xrightarrow{\chi_g} g^!p_2^*f^!F \\ &\xrightarrow{\beta_{\spadesuit}} g^!p_1^!(fi)^*F \\ &\xrightarrow{\sim} (fi)^*F. \end{aligned}$$

If  $fi$  is essentially smooth, then in (3.1.1), both  $\text{id}_U \times i$  and the regular immersion  $\delta_U$  are perfect, whence so is  $g$  (see §2.4), so that  $\chi_g$  is an isomorphism.<sup>3</sup> Thus, in this case,  $\vartheta$  is an isomorphism.

The second is an isomorphism that holds when  $fi$  is essentially smooth of relative dimension  $d$ ,

$$(3.1.4) \quad g^!\mathcal{O}_Y \xrightarrow{\sim} \mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d],$$

described in [H66, p. 180, Corollary 7.3] or [C00, §2.5] via the “fundamental local isomorphism” and Cartan-Eilenberg resolutions. An avatar (3.2.8) of (3.1.4) is reviewed in §3.2 below.

The  $\mathcal{O}_U$ -module  $\Omega_{fi}^d$  is invertible, so the complexes  $\mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d]$  and  $\mathcal{H}om_U(\Omega_{fi}^d[d], \mathcal{O}_U)$  are naturally isomorphic in  $\mathbf{D}(U)$  to the complex  $G$  which is  $(\Omega_{fi}^d)^{-1}$  in degree  $d$  and 0 elsewhere. Modulo these isomorphisms, the isomorphism

$$(3.1.5) \quad \mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d] \xrightarrow{\sim} \mathcal{H}om_U(\Omega_{fi}^d[d], \mathcal{O}_U),$$

resulting from the usual triangulated structure on the functor  $\mathcal{H}om(-, \mathcal{O}_U)$  is given by scalar multiplication in  $G$  by  $(-1)^{d^2+d(d-1)/2} = (-1)^{d(d+1)/2}$  (cf. [C00, p. 11, (1.3.8)].)

<sup>3</sup>As  $\delta_U$  is finite and  $\text{id}_U \times i$  is essentially étale, one can see this more concretely by showing that  $\delta_{U*}(\chi)$  is isomorphic to the natural map

$$\mathcal{H}om(\delta_{U*}\mathcal{O}_U, \mathcal{O}_{U \times_Z U}) \otimes G \longrightarrow \mathcal{H}om(\delta_{U*}\mathcal{O}_U, G) \quad (G := (\text{id}_U \times i)^*p_2^*f^!F).$$

Let

$$(3.1.6) \quad \gamma: g^! \mathcal{O}_Y \xrightarrow{\sim} \mathcal{H}om_U(\Omega_{f_i}^d[d], \mathcal{O}_U)$$

be the isomorphism obtained by composition from (3.1.4) and (3.1.5).

Thus when  $f_i$  is essentially smooth, so that  $\Omega_{f_i}^d$  is an *invertible*  $\mathcal{O}_U$ -module, one has a chain of natural functorial isomorphisms, the first being inverse to the evaluation map 2.2.2:

$$(3.1.7) \quad \begin{aligned} (f_i)^! F &\xrightarrow{\sim} \mathcal{H}om_U(\Omega_{f_i}^d[d], (f_i)^! F) \otimes_U \Omega_{f_i}^d[d] \\ &\xrightarrow{(2.6.8)} \mathcal{H}om_U(\Omega_{f_i}^d[d], \mathcal{O}_U) \otimes_U (f_i)^! F \otimes_U \Omega_{f_i}^d[d] \\ &\xrightarrow{(3.1.6)} g^! \mathcal{O}_Y \otimes_U (f_i)^! F \otimes_U \Omega_{f_i}^d[d] \\ &\xrightarrow{(3.1.3)} (f_i)^* F \otimes_U \Omega_{f_i}^d[d] \xrightarrow{\sim} \Omega_{f_i}^d[d] \otimes_U (f_i)^* F. \end{aligned}$$

The map  $v_{f_i}(F)$  is defined to be the composition of this chain.

**3.2.** Expanding a bit on [LS81, §2, II], we review the relation (3.2.7) between the normal bundle of a regular immersion  $\delta: U \rightarrow W$  and the relative dualizing complex  $\delta^! \mathcal{O}_W$ ; and from that deduce the isomorphism (3.1.4).

Let  $\delta: U \rightarrow W$  be any closed immersion of schemes, and  $I$  the kernel of the associated surjective map  $s: \mathcal{O}_W \rightarrow \delta_* \mathcal{O}_U$ . Then

$$\mathcal{O}_U \cong H^0 \delta^* \mathcal{O}_W \xrightarrow{H^0 \delta^* s} H^0 \delta^* \delta_* \mathcal{O}_U \cong \mathcal{O}_U,$$

is an *isomorphism*.

The natural triangle

$$\delta^* I \longrightarrow \delta^* \mathcal{O}_W \xrightarrow{\delta^* s} \delta^* \delta_* \mathcal{O}_U \xrightarrow{+}$$

gives rise to an exact sequence of  $\mathcal{O}_U$ -modules

$$\begin{array}{ccccccc} H^{-1} \delta^* \mathcal{O}_W & \longrightarrow & H^{-1} \delta^* \delta_* \mathcal{O}_U & \xrightarrow{t} & H^0 \delta^* I & \longrightarrow & \ker(H^0 \delta^* s) \\ \parallel & & & & \parallel & & \parallel \\ 0 & & & & I/I^2 & & 0 \end{array}$$

Clearly,  $t$  is an isomorphism, and so one has the natural  $\mathcal{O}_U$ -isomorphism

$$(3.2.1) \quad t^{-1}: I/I^2 \xrightarrow{\sim} H^{-1} \delta^* \delta_* \mathcal{O}_U.$$

(The isomorphism  $t$  is induced by the projection  $C[-1] \rightarrow P$ , where  $K \rightarrow I$  is a flat resolution of  $I$  and  $C$  is the mapping cone of the composite map  $K \rightarrow I \hookrightarrow \mathcal{O}_W$ . It is the *negative* of the connecting homomorphism

$$\mathcal{T}or_1^{\mathcal{O}_W}(\delta_* \mathcal{O}_U, \delta_* \mathcal{O}_U) \rightarrow \mathcal{T}or_0^{\mathcal{O}_W}(\delta_* \mathcal{O}_U, I)$$

usually attached to the natural exact sequence  $0 \rightarrow I \rightarrow \mathcal{O}_W \rightarrow \delta_* \mathcal{O}_U \rightarrow 0$ , see [L09, end of §1.4].)

There is an alternating graded  $\mathcal{O}_U$ -algebra structure on  $\bigoplus_{n \geq 0} H^{-n} \delta^* \delta_* \mathcal{O}_U$ , induced by the natural product map

$$\delta^* \delta_* \mathcal{O}_U \otimes_U \delta^* \delta_* \mathcal{O}_U \xrightarrow{\sim} \delta^* (\delta_* \mathcal{O}_U \otimes_W \delta_* \mathcal{O}_U) \longrightarrow \delta^* \delta_* (\mathcal{O}_U \otimes_U \mathcal{O}_U) \xrightarrow{\sim} \delta^* \delta_* \mathcal{O}_U.$$

(Cf. e.g., [B07, p. 201, Exercise 9(c)].) Hence (3.2.1) extends uniquely to a homomorphism of graded  $\mathcal{O}_U$ -algebras

$$(3.2.2) \quad \bigoplus_{n \geq 0} \Lambda^n(I/I^2) \longrightarrow \bigoplus_{n \geq 0} H^{-n} \delta^* \delta_* \mathcal{O}_U.$$

For example, over an affine open subset of  $W$ , if  $I$  is generated by a regular sequence of global sections  $\mathbf{t} := (t_1, t_2, \dots, t_d)$  then a finite free resolution of  $\delta_* \mathcal{O}_U$  is provided by the Koszul complex  $K(\mathbf{t}) := \bigotimes_{i=1}^d K_i$  where  $K_i$  is the complex which is  $\mathcal{O}_W \xrightarrow{t_i} \mathcal{O}_W$  in degrees -1 and 0 and vanishes elsewhere; and there results a  $\mathbf{D}(U)$ -map,

$$(3.2.3) \quad \bigoplus_{n=0}^d \Lambda^n(I/I^2)[n] \cong \delta^* K(\mathbf{t}) \xrightarrow{\sim} \delta^* \delta_* \mathcal{O}_U.$$

(Note that  $\delta^* K(\mathbf{t})$  is just the exterior algebra—with vanishing differentials—on  $\mathcal{O}_U^n$ , which is isomorphic to  $I/I^2$  via the natural map

$$\mathcal{O}_W^n = K^{-1}(\mathbf{t}) \hookrightarrow I \subset \mathcal{O}_W.)$$

One verifies that applying the functor  $H^{-n}$  to (3.2.3) produces the degree  $n$  component of (3.2.2) (a map that *does not depend on the choice of the generating family*  $\mathbf{t}$ ).

Next, for any  $\mathcal{O}_U$ -complex  $E$  with  $H^e E = 0$  for all  $e < 0$ , the natural map  $H^0 E \rightarrow E$  induces a map

$$(3.2.4) \quad H^0 \mathcal{H}om_U(E, \mathcal{O}_U) \longrightarrow H^0 \mathcal{H}om(H^0 E, \mathcal{O}_U) =: \mathcal{H}om^0(H^0 E, \mathcal{O}_U).$$

Hence for any integer  $n$  and any  $\mathcal{O}_U$ -complex  $F$  such that  $H^e F = 0$  for all  $e < -n$ , one has the natural composite

$$(3.2.5) \quad \begin{aligned} H^n \mathcal{H}om_U(F, \mathcal{O}_U) &\xrightarrow{\sim} H^0 \mathcal{H}om_U(F, \mathcal{O}_U)[n] \\ &\xrightarrow{\sim} H^0 \mathcal{H}om_U(F[-n], \mathcal{O}_U) \\ &\xrightarrow{(3.2.4)} \mathcal{H}om^0(H^0(F[-n]), \mathcal{O}_U) \xrightarrow{\sim} \mathcal{H}om^0(H^{-n} F, \mathcal{O}_U). \end{aligned}$$

(The second map—whose source and target are equal—is multiplication by  $(-1)^{n(n+1)/2}$ : replace  $\mathcal{O}_U$  by a quasi-isomorphic injective complex, and take  $p = 0$ ,  $m = -n$  in the expression  $(-1)^{pm+m(m-1)/2}$  after [C00, p. 11, (1.3.8)].)

Now if the closed immersion  $\delta$  is *regular, of codimension*  $d$ , i.e.,  $I$  is generated locally by regular sequences  $\mathbf{t}$  of length  $d$ , so that  $\delta_* \mathcal{O}_U$  is locally resolved by free complexes of the form  $K(\mathbf{t})$ , then there results the sequence

of isomorphisms

$$\begin{aligned}
 (3.2.6) \quad H^d \delta^* \delta_* \delta^! \mathcal{O}_W &\xrightarrow[(1.2.1)]{\simeq} H^d \delta^* \mathcal{H}om_W(\delta_* \mathcal{O}_U, \mathcal{O}_W) \\
 &\xrightarrow[(2.6.4)]{\simeq} H^d \mathcal{H}om_U(\delta^* \delta_* \mathcal{O}_U, \mathcal{O}_U) \\
 &\xrightarrow[(3.2.5)]{\simeq} \mathcal{H}om_U^0(H^{-d} \delta^* \delta_* \mathcal{O}_U, \mathcal{O}_U) \\
 &\xrightarrow[(3.2.2)]{\simeq} \mathcal{H}om_U^0(\wedge^d(I/I^2), \mathcal{O}_U).
 \end{aligned}$$

In particular, there is a canonical  $\mathcal{O}_U$ -isomorphism

$$H^d \delta^* \delta_* \delta^! \mathcal{O}_W \xrightarrow{\simeq} \mathcal{H}om_U^0(\wedge^d(I/I^2), \mathcal{O}_U) =: \nu_\delta.$$

Moreover, since  $\delta_* \delta^! \mathcal{O}_W \cong \mathcal{H}om_W(\delta_* \mathcal{O}_U, \mathcal{O}_W)$  has nonvanishing homology only in degree  $d$ , therefore the same holds for  $\delta^! \mathcal{O}_W$ , whence the natural maps are isomorphisms

$$\delta^! \mathcal{O}_W[d] \xrightarrow{\simeq} H^d \delta^! \mathcal{O}_W \xleftarrow{\simeq} H^d \delta^* \delta_* \delta^! \mathcal{O}_W.$$

Thus, when  $\delta$  is a regular immersion there is a canonical isomorphism

$$(3.2.7) \quad \boxed{\delta^! \mathcal{O}_W \xrightarrow{\simeq} \nu_\delta[-d].}$$

It is left to the interested reader to work out the precise relationship of (3.2.7) to the similar isomorphisms in [H66, p. 180, Corollary 7.3] and [C00, §2.5].

Now in (3.1.1), if  $fi$  is essentially smooth of relative dimension  $d$  then  $\delta_U: U \rightarrow W := U \times_Z U$  is a regular immersion and  $\Omega_{fi}^d$  is locally free of rank one (see §2.3), so the natural map  $\mathcal{H}om_U^0(\Omega_{fi}^d, \mathcal{O}_U) \xrightarrow{\simeq} \mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)$  is an isomorphism. Thus, there are natural isomorphisms, with  $Y := U \times_Z X$ ,

$$\begin{aligned}
 (3.2.8) \quad g^! \mathcal{O}_Y &\xrightarrow{\simeq} \delta_U^!(\text{id}_U \times i)^! \mathcal{O}_Y = \delta_U^!(\text{id}_U \times i)^* \mathcal{O}_Y = \delta_U^! \mathcal{O}_W \\
 &\xrightarrow[(3.2.7)]{\simeq} \mathcal{H}om_U^0(\wedge^d(I/I^2), \mathcal{O}_U)[-d] = \mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d].
 \end{aligned}$$

The isomorphism (3.1.4) is defined to be the resulting composition.

**3.3.** Though it is not *a priori* clear,  $v_{f,i}$  depends only on the map  $fi$  and not on its factorization into  $f$  and  $i$ . (In [V68],  $f$  is assumed proper.) Indeed:

**Proposition 3.3.1.** *For  $U \xrightarrow{i} X \xrightarrow{f} Z$  as in §3.1, if  $fi$  is essentially smooth then  $v_{f,i} = v_{fi, \text{id}_U}$ .*

*Proof.* There is a commutative  $\mathcal{E}$ -diagram, with  $i, f, p_1, p_2$  and  $g$  as in §3.1,  $\delta$  the diagonal map,  $q_1, q_2$  the canonical projections, and  $\spadesuit, \clubsuit$  and  $\heartsuit$  labeling the front, top and rear faces of the cube, respectively. (These three faces are fiber squares; and since  $fi$  is flat therefore so are  $p_2$  and  $q_2$ .)

$$\begin{array}{ccccc}
U & \xrightarrow{\delta} & W := U \times_Z U & \xrightarrow{q_2} & U \\
g \searrow & & \swarrow j := \text{id} \times i & & \swarrow i \\
Y := U \times_Z X & \xrightarrow{p_2} & X & & U \\
\downarrow p_1 & & \downarrow q_1 & & \downarrow f_i \\
U & \xrightarrow{f_i} & U & \xrightarrow{f} & Z \\
\downarrow p_1 & & \downarrow q_1 & & \downarrow f_i \\
U & \xrightarrow{f_i} & U & \xrightarrow{f} & Z \\
\downarrow p_1 & & \downarrow q_1 & & \downarrow f_i \\
U & \xrightarrow{f_i} & U & \xrightarrow{f} & Z
\end{array}$$

♣
♥
♠

Note that since  $i$  is essentially étale therefore  $i^! = i^*$  and  $j^! = j^*$ . There is, as in (3.2.8), a natural composite isomorphism

$$\xi: g^! \mathcal{O}_Y \xrightarrow{\sim} \delta^! j^! \mathcal{O}_Y = \delta^! j^* \mathcal{O}_Y \xrightarrow{\sim} \delta^! \mathcal{O}_W.$$

A detailed examination of the definition of  $v_{f,i}$  in (3.1.7), taking into account the definition (3.2.8) of (3.1.4), shows that it will suffice to prove commutativity of the border of the following natural functorial diagram.

$$\begin{array}{ccccccc}
g^! \mathcal{O}_Y \otimes (fi)^! & \xrightarrow{\text{via } \xi} & & & \delta^! \mathcal{O}_W \otimes (fi)^! & & \\
\cong \downarrow & & & & \downarrow \cong & & \\
g^! \mathcal{O}_Y \otimes i^* f^! & \longrightarrow & \delta^! j^! \mathcal{O}_Y \otimes i^* f^! & \xlongequal{\quad} & \delta^! j^* \mathcal{O}_Y \otimes i^* f^! & \longrightarrow & \delta^! \mathcal{O}_W \otimes \delta^* q_2^*(fi)^! \\
\cong \downarrow & & \textcircled{1} & & \downarrow \cong & & \downarrow \chi_\delta \\
g^! \mathcal{O}_Y \otimes g^* p_2^* f^! & \longrightarrow & \delta^! j^! \mathcal{O}_Y \otimes \delta^* j^* p_2^* f^! & \longrightarrow & \delta^! \mathcal{O}_W \otimes \delta^* q_2^* i^* f^! & \longrightarrow & \delta^! \mathcal{O}_W \otimes q_2^*(fi)^! \\
\downarrow \chi_g & & \textcircled{2} & & \downarrow \chi_\delta & & \downarrow \chi_\delta \\
g^! (\mathcal{O}_Y \otimes g^* p_2^* f^!) & \longrightarrow & \delta^! (j^! \mathcal{O}_Y \otimes j^* p_2^* f^!) & \longrightarrow & \delta^! (\mathcal{O}_W \otimes q_2^* i^* f^!) & \longrightarrow & \delta^! (\mathcal{O}_W \otimes q_2^*(fi)^!) \\
\parallel & & \downarrow \delta^! \chi_j & & \parallel & & \parallel \\
g^! p_2^* f^! & \longrightarrow & \delta^! j^! p_2^* f^! = \delta^! j^* p_2^* f^! & \longrightarrow & \delta^! q_2^* i^* f^! & \longrightarrow & \delta^! q_2^*(fi)^! \\
\downarrow g^! \beta_\clubsuit & & \downarrow \text{via } \beta_\clubsuit & & \textcircled{4} & & \downarrow \delta^! \beta_\heartsuit \\
g^! p_1^! (fi)^* & \longrightarrow & \delta^! j^! p_1^! (fi)^* & \longrightarrow & \delta^! (p_1 j)^! (fi)^* & \xlongequal{\quad} & \delta^! q_1^! (fi)^* \\
\cong \swarrow & & \textcircled{5} & & \swarrow \cong & & \\
& & (fi)^* & & & & 
\end{array}$$



Commutativity of subdiagram ① (resp. ⑤) results from the pseudofunctoriality of  $(-)^*$  (resp.  $(-)^!$ ).

Commutativity of ② is given by transitivity of  $\chi$  with respect to composition of maps, see [Nm14, 13.4] (or [L09, Exercises 4.7.3.4(d) and 4.9.3(d)]).

Commutativity of ③ follows from the fact that for any  $F \in \mathbf{D}_{\text{qc}}^+(Y)$ , the following natural diagram commutes:

$$\begin{array}{ccccc}
 j^! \mathcal{O}_Y \otimes j^* F & \xlongequal{\quad} & j^* \mathcal{O}_Y \otimes j^* F & \xrightarrow{\sim} & \mathcal{O}_W \otimes j^* F \\
 \chi_j \downarrow & \textcircled{3}_1 & \downarrow \simeq & \textcircled{3}_2 & \parallel \\
 j^! (\mathcal{O}_Y \otimes F) & \xlongequal{\quad} & j^* (\mathcal{O}_Y \otimes F) & \xlongequal{\quad} & j^* F
 \end{array}$$

For a sketch of the proof that ③<sub>1</sub> commutes see [L09, 4.9.2.3]. As for commutativity of ③<sub>2</sub>, replacement of  $F$  by a quasi-isomorphic flat complex reduces the problem to the context of ordinary (nonderived) functors, at which point the justification is left to the reader.

Commutativity of ④ is given by transitivity of  $\beta$  with respect to juxtaposition of fiber squares (see [Nm14, Theorem 11.9] or [L09, Theorem 4.8.3]), as applied to the following decomposition of the fiber square  $\heartsuit$ :

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{q_2} & \bullet & & \\
 j \downarrow & \clubsuit & \downarrow i & & \\
 \bullet & \xrightarrow{p_2} & \bullet & & \\
 p_1 \downarrow & \spadesuit & \downarrow f & & \\
 \bullet & \xrightarrow{fi} & \bullet & & 
 \end{array}$$

Here one needs to use that  $\beta_{\clubsuit}$  is the canonical isomorphism  $q_2^* i^* \xrightarrow{\sim} j^* p_2^*$  (see [Nm14, 11.4 and 11.5] with  $p = p' = \text{identity map}$  in the diagram of [Nm14, 11.4(i)], or [L09, 4.8.8(i)].)

Commutativity of the remaining subdiagrams is easy to check, whence the assertion.  $\square$

Accordingly, we restrict henceforth to the case  $U = X$  and  $i = \text{id}_X$ . The map  $g: X \rightarrow X \otimes_Z X$  is then the diagonal.

**3.4.** Again, let  $f: X \rightarrow Z$  be an essentially smooth  $\mathcal{E}$ -map of relative dimension  $d$ . With  $\chi_f$  as in (2.6.6), define  $v_f(F)$  for  $F \in \mathbf{D}_{\text{qc}}(Z)$  to be the composite isomorphism

$$(3.4.1) \quad f^! F \xrightarrow[\chi_f^{-1}]{\sim} f^! \mathcal{O}_Z \otimes_X f^* F \xrightarrow[\nu_{f, \text{id}_X}(\mathcal{O}_Z) \otimes \text{id}]{\sim} \Omega_f^d[d] \otimes_X f^* F.$$

As before (just after (3.1.2)), the identifications

$$\Omega_f^d[d] \otimes_X f^* \mathcal{O}_Z = \Omega_f^d[d] \otimes_X \mathcal{O}_X = \Omega_f^d[d],$$

allow us to consider  $v_f(\mathcal{O}_Z)$  to be a map from  $f^!\mathcal{O}_Z$  to  $\Omega_f^d[d]$ .

**Proposition 3.4.2.** *If  $F \in \mathbf{D}_{\text{qc}}^+(Z)$  then  $v_f(F) = v_{f, \text{id}_X}(F)$ .*

*Proof.* Let  $g: X \rightarrow Y := X \times_Z X$  be the diagonal map. Set  $\omega := \Omega_f^d[d]$  (a perfect complex), and set  $[A, B] := \mathcal{H}om_X(A, B)$  ( $A, B \in \mathbf{D}(X)$ ). Let  $\kappa = \kappa_X(\omega, -, -)$  be as in (2.6.8),  $\vartheta$  as in (3.1.3) and  $\gamma$  as in (3.1.6) (the last two with  $i := \text{id}_X$ ). Since  $f$  is essentially smooth, all of these maps are isomorphisms. By definition (see (3.1.7)), Proposition 3.4 says that the border of the following diagram (in which  $\otimes := \otimes_X$ ) commutes:

$$\begin{array}{ccc}
 f^!\mathcal{O}_Z \otimes f^*F & \xlongequal{\quad\quad\quad} & f^!\mathcal{O}_Z \otimes f^*F \\
 \swarrow \chi_f & \searrow \simeq & \downarrow \text{id} \\
 f^!F & & g^!\mathcal{O}_Y \otimes f^!\mathcal{O}_Z \otimes \omega \otimes f^*F \\
 \downarrow \simeq & \searrow \textcircled{2} & \downarrow v_{f, \text{id}_X}(\mathcal{O}_Z) \\
 [\omega, f^!F] \otimes \omega & & [\omega, f^!\mathcal{O}_Z] \otimes \omega \otimes f^*F \\
 \downarrow \kappa^{-1} & \swarrow \chi_f & \downarrow \vartheta \\
 [\omega, \mathcal{O}_X] \otimes f^!F \otimes \omega & & [\omega, f^!\mathcal{O}_Z] \otimes f^*F \otimes \omega \\
 \downarrow \kappa^{-1} & \swarrow \kappa & \downarrow \gamma^{-1} \\
 [\omega, \mathcal{O}_X] \otimes f^!\mathcal{O}_Z \otimes f^*F \otimes \omega & & f^*\mathcal{O}_Z \otimes \omega \otimes f^*F \\
 \downarrow \chi_f & \swarrow \kappa^{-1} & \downarrow \vartheta \\
 [\omega, \mathcal{O}_X] \otimes f^!F \otimes \omega & & f^*\mathcal{O}_Z \otimes f^*F \otimes \omega \\
 \downarrow \chi_f & \swarrow \simeq & \downarrow \vartheta \\
 [\omega, \mathcal{O}_X] \otimes f^!\mathcal{O}_Z \otimes f^*F \otimes \omega & \xrightarrow{\gamma^{-1}} & g^!\mathcal{O}_Y \otimes f^!\mathcal{O}_Z \otimes f^*F \otimes \omega \\
 \downarrow \chi_f & \swarrow \chi_f & \downarrow \chi_f \\
 [\omega, \mathcal{O}_X] \otimes f^!F \otimes \omega & \xrightarrow{\gamma^{-1}} & g^!\mathcal{O}_Y \otimes f^!F \otimes \omega
 \end{array}$$

Commutativity of the unlabeled subdiagrams is straightforward to verify.

Commutativity of ① follows from the definition (3.1.7) of  $v_{f, \text{id}_X}(\mathcal{O}_Z)$ .

Commutativity of ② is essentially the definition of the map  $\kappa$ .

Commutativity of ③ results from Lemma 2.8.

For ④, it's enough to have commutativity of the functorial diagram

$$\begin{array}{ccc}
 g^!\mathcal{O}_Y \otimes_X f^!\mathcal{O}_Z \otimes_X f^* & \xrightarrow{\text{via } \chi_f} & g^!\mathcal{O}_Y \otimes_X f^! \\
 \text{via } \vartheta \downarrow & & \downarrow \vartheta \\
 f^*\mathcal{O}_Z \otimes_X f^* & \xlongequal{\quad\quad} & f^*,
 \end{array}$$

that expands to the border of the next natural diagram, in which the omitted subscripts on  $\otimes$  symbols are the obvious ones, and, with reference to the

standard fiber square

$$\begin{array}{ccc}
 X \times_Z X & \xrightarrow{p_2} & X \\
 p_1 \downarrow & \diamond & \downarrow f \\
 X & \xrightarrow{f} & Z,
 \end{array}$$

$\beta_\diamond$  is as in (2.6.7):

$$\begin{array}{ccccc}
 g^! \mathcal{O}_Y \otimes f^! \mathcal{O}_Z \otimes f^* & \xrightarrow{\text{via } \chi_f} & g^! \mathcal{O}_Y \otimes f^! & & \\
 \downarrow \simeq & \searrow \simeq & \downarrow \simeq & & \\
 \textcircled{5} & g^! \mathcal{O}_Y \otimes g^* p_2^*(f^! \mathcal{O}_Z \otimes f^*) & \xrightarrow{\text{via } \chi_f} & g^! \mathcal{O}_Y \otimes g^* p_2^* f^! & \\
 \downarrow \simeq & \downarrow \simeq & \downarrow \simeq & \textcircled{6} & \downarrow \text{via } \beta_\diamond \\
 g^! \mathcal{O}_Y \otimes g^* p_2^* f^! \mathcal{O}_Z \otimes g^* p_2^* f^* & \xrightarrow{\simeq} & g^! \mathcal{O}_Y \otimes g^*(p_2^* f^! \mathcal{O}_Z \otimes p_2^* f^*) & & \\
 \downarrow \text{via } \beta_\diamond & & \downarrow \text{via } \beta_\diamond & \nearrow \text{via } \chi_{p_1} & \downarrow \chi_g \\
 g^! \mathcal{O}_Y \otimes g^* p_1^* f^* \mathcal{O}_Z \otimes g^* p_2^* f^* & \xrightarrow{\simeq} & g^! \mathcal{O}_Y \otimes g^*(p_1^* f^* \mathcal{O}_Z \otimes p_1^* f^*) & & g^! \mathcal{O}_Y \otimes g^* p_1^* f^* \\
 \downarrow \text{via } \chi_g & \textcircled{7} & \downarrow \text{via } \chi_g & & \downarrow \chi_g \\
 g^! p_1^* f^* \mathcal{O}_Z \otimes g^* p_1^* f^* & \xrightarrow{\chi_g} & g^!(p_1^* f^* \mathcal{O}_Z \otimes p_1^* f^*) & \xrightarrow{\text{via } \chi_{p_1}} & g^! p_1^* f^* \\
 \downarrow \simeq & & \downarrow \simeq & \textcircled{8} & \downarrow \simeq \\
 (p_1 g)^! f^* \mathcal{O}_Z \otimes (p_1 g)^* f^* & \xrightarrow{\chi_{p_1 g}} & (p_1 g)^!(f^* \mathcal{O}_Z \otimes f^*) & \xlongequal{\quad} & (p_1 g)^! f^* \\
 \downarrow \simeq & \textcircled{9} & \downarrow \simeq & & \downarrow \simeq \\
 f^* \mathcal{O}_Z \otimes f^* & \xrightarrow{\simeq} & f^* & \xlongequal{\quad} & f^*
 \end{array}$$

For commutativity of subdiagram  $\textcircled{5}$ , see the last two paragraphs of §3.6 in [L09].

Commutativity of  $\textcircled{6}$  results from [Nm14, 13.7] with applied to the diagram  $\diamond$  (cf. [L09, 4.9.3(c)].)

Commutativity of  $\textcircled{7}$ ,  $\textcircled{8}$  and  $\textcircled{9}$  are left mostly to the reader (cf. [L09, Exercises 4.7.3.4(a), (d) and (b)], as modified in [L09, 4.9.3(d)], describing the behavior of  $\chi$  vis-à-vis associativity of tensor product, composition of maps, and identity maps, respectively.) For more details on  $\textcircled{8}$ , see [Nm14, 13.4].

Commutativity of the remaining subdiagrams is easy to check, whence the assertion.  $\square$

*Remark 3.4.3* (Base change). Let there be given a fiber square in  $\mathcal{E}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{g} & Z \end{array}$$

in which  $f$  (and hence  $f'$ ) is essentially smooth of relative dimension  $d$ .

The isomorphism  $v_f(\mathcal{O}_Z)$  induces an  $\mathcal{O}_X$ -isomorphism

$$\omega_f := H^{-d} f^! \mathcal{O}_Z \xrightarrow{\sim} H^{-d}(\Omega_f^d[d]) = \Omega_f^d,$$

and similarly for  $f'$ . The resulting composite isomorphism

$$g'^* \omega_f \xrightarrow{\sim} g'^* \Omega_f^d \xrightarrow{\sim} \Omega_{f'}^d \xrightarrow{\sim} \omega_{f'},$$

is discussed in [S04, p. 740, Theorem 2.3.5].

The next Proposition expresses compatibility between  $v_f$  and (derived) tensor product.

**Proposition 3.4.4.** *For any  $F_1, F_2 \in \mathbf{D}_{\text{qc}}(Z)$ , the following diagram, where  $\otimes := \otimes_X$ , commutes.*

$$\begin{array}{ccc} f^! F_1 \otimes f^* F_2 & \xrightarrow{v_f(F_1) \otimes \text{id}} & \Omega_f^d[d] \otimes f^* F_1 \otimes f^* F_2 \\ \chi_f(F_1, F_2) \downarrow & & \downarrow \text{natural} \\ f^!(F_1 \otimes F_2) & \xrightarrow{v_f(F_1 \otimes F_2)} & \Omega_f^d[d] \otimes f^*(F_1 \otimes F_2). \end{array}$$

*Proof* (Sketch). The definition of  $v_f$  makes it enough to prove commutativity of the next diagram (expressing transitivity of  $\chi_f$  for any  $F_0 \in \mathbf{D}_{\text{qc}}(Z)$ ), and then to take  $F_0 := \mathcal{O}_Z$ .

$$\begin{array}{ccc} f^!(F_0 \otimes F_1) \otimes f^* F_2 & \xleftarrow{\chi_f(F_0, F_1) \otimes \text{id}} & f^! F_0 \otimes f^* F_1 \otimes f^* F_2 \\ \chi_f(F_0 \otimes F_1, F_2) \downarrow & & \downarrow \text{natural} \\ f^!(F_0 \otimes F_1 \otimes F_2) & \xleftarrow{\chi_f(F_0, F_1 \otimes F_2)} & f^! F_0 \otimes f^*(F_1 \otimes F_2). \end{array}$$

For this, one reduces easily, via a compactification of  $f$ , to the case where  $f$  is proper, a case dealt with (in outline) in [L09, Exercise 4.7.3.4(a)].  $\square$

## 4. THE FUNDAMENTAL CLASS

**4.1.** Let  $f: X \rightarrow Z$  be a flat  $\mathcal{E}$ -map. Set  $Y := X \times_Z X$ , let  $\delta: X \rightarrow Y$  be the diagonal map, and  $p_i: Y \rightarrow X$  ( $i = 1, 2$ ) the projections onto the first and second factors, respectively, so that we have the diagram, with fiber square  $\clubsuit$ ,

$$(4.1.1) \quad \begin{array}{ccccc} X & \xrightarrow{\delta} & Y & \xrightarrow{p_2} & X \\ & & p_1 \downarrow & \clubsuit & \downarrow f \\ & & X & \xrightarrow{f} & Z \end{array}$$

The maps  $p_i$  are flat.

There are maps of  $\mathbf{D}_{\text{qc}}$ -functors

$$(4.1.2) \quad \mu_i: \delta_* \rightarrow p_i^! \quad (i = 1, 2)$$

adjoint, respectively, to the natural maps  $\text{id} = (p_i\delta)^! \rightarrow \delta^!p_i^!$ . Thus  $\mu_i$  is the natural composite map

$$\delta_* = \delta_*(p_i\delta)^! \longrightarrow \delta_*\delta^!p_i^! \longrightarrow p_i^!.$$

Associated to  $\clubsuit$  is the functorial *base-change isomorphism* (see (2.6.7))

$$\beta = \beta_{\clubsuit}: p_2^*f^!F \xrightarrow{\sim} p_1^!f^*F \quad (F \in \mathbf{D}_{\text{qc}}(Z)).$$

**Definition 4.2.** With preceding notation, the *fundamental class of  $f$* ,

$$\mathbf{C}_f: \delta^*\delta_*f^* \rightarrow f^!,$$

a map between functors from  $\mathbf{D}_{\text{qc}}(Z)$  to  $\mathbf{D}_{\text{qc}}(X)$ , is given by the composite

$$\delta^*\delta_*f^* \xrightarrow{\text{via } \mu_1} \delta^*p_1^!f^* \xrightarrow[\delta^*\beta^{-1}]{\sim} \delta^*p_2^*f^! \xrightarrow[\text{natural}]{\sim} f^!.$$

*Remarks.* (Not used elsewhere). It results from [AJL14, 2.5 and 3.1] that *the fundamental class commutes with essentially étale localization on  $X$* . That is, if  $g: X' \rightarrow X$  is essentially étale then  $\mathbf{C}_{fg}$  is obtained from  $\mathbf{C}_f$  by applying the functor  $g^*$  and then making canonical identifications.

See also [AJL14, Theorem 5.1] for the behavior of  $\mathbf{C}_f$  under flat base change.

These results imply that if  $u: U \hookrightarrow X$  and  $v: V \hookrightarrow Z$  are open immersions such that  $f(U) \subset V$ , and  $f_0: U \rightarrow V$  is the restriction of  $f$ , then  $u^*(\mathbf{C}_f)$  can be identified with  $\mathbf{C}_{f_0}$ . Locally, then,  $\mathbf{C}_f$  reduces to the fundamental class of a flat  $\mathcal{E}$ -map of affine schemes, in which case a simple explicit description is given in [ILN14, Theorem 4.2.4].

The next Proposition expresses compatibility between  $\mathbf{C}_f$  and (derived) tensor product.

**Proposition 4.3.** *For any  $F_1, F_2 \in \mathbf{D}_{\text{qc}}(Z)$ , the following diagram, where  $\otimes := \otimes_X$ ,  $\chi_f$  is as in (2.6.6) and  $\psi := \psi(\delta, p_1, f^*F_1, f^*F_2)$ , commutes.*

$$\begin{array}{ccc}
\delta^* \delta_* f^* F_1 \otimes f^* F_2 & \xrightarrow{C_f(F_1) \otimes \text{id}} & f^! F_1 \otimes f^* F_2 \\
\downarrow \psi & & \downarrow \chi_f(F_1, F_2) \\
\delta^* \delta_* (f^* F_1 \otimes f^* F_2) & & \\
\downarrow (2.6.1) & & \downarrow \\
\delta^* \delta_* f^*(F_1 \otimes F_2) & \xrightarrow{C_f(F_1 \otimes F_2)} & f^!(F_1 \otimes F_2)
\end{array}$$

*Proof.* The diagram expands, naturally, as follows, where  $\alpha$  is induced by the composite  $p_2^* f^* \xrightarrow{\sim} (fp_2)^* = (fp_1)^* \xrightarrow{\sim} p_1^* f^*$  and by the base-change isomorphism  $\beta$ .

$$\begin{array}{ccccccc}
\delta^* \delta_* f^* F_1 \otimes f^* F_2 & \longrightarrow & \delta^* p_1^! f^* F_1 \otimes f^* F_2 & \longrightarrow & \delta^* p_2^* f^! F_1 \otimes f^* F_2 & \longrightarrow & f^! F_1 \otimes f^* F_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \delta^*(p_1^! f^* F_1 \otimes p_1^* f^* F_2) & \textcircled{1} & & \textcircled{2} & \\
& \nearrow & \downarrow & \swarrow \alpha & \downarrow & & \downarrow \\
\delta^*(\delta_* f^* F_1 \otimes p_1^* f^* F_2) & & \delta^* \chi_{p_1} & & \delta^*(p_2^* f^! F_1 \otimes p_2^* f^* F_2) & & \downarrow \\
& \downarrow \textcircled{3} & & & \downarrow & & \downarrow \\
\delta^* \delta_* (f^* F_1 \otimes \delta^* p_1^* f^* F_2) & & \delta^* p_1^! (f^* F_1 \otimes f^* F_2) & & \delta^* p_2^* (f^! F_1 \otimes f^* F_2) & \longrightarrow & f^! F_1 \otimes f^* F_2 \\
& \nearrow & \downarrow & & \downarrow & & \downarrow \\
\delta^* \delta_* (f^* F_1 \otimes f^* F_2) & & \delta^* p_1^! (f^* F_1 \otimes f^* F_2) & \textcircled{4} & \delta^* p_2^* \chi_f & & \downarrow \text{via } \chi_f \\
\delta^* \delta_* f^*(F_1 \otimes F_2) & \longrightarrow & \delta^* p_1^! f^*(F_1 \otimes F_2) & \longrightarrow & \delta^* p_2^* f^!(F_1 \otimes F_2) & \longrightarrow & f^!(F_1 \otimes F_2)
\end{array}$$

Commutativity of  $\textcircled{1}$  follows from that of the next diagram of natural isomorphisms, the commutativity of whose subdiagrams is either obvious or included in the pseudofunctoriality of  $(-)^*$ :

$$\begin{array}{ccc}
 f^* = (p_1\delta)^* f^* & \xlongequal{\quad\quad\quad} & (p_2\delta)^* f^* = f^* \\
 \downarrow & \searrow & \swarrow \\
 & (fp_1\delta)^* \xlongequal{\quad\quad\quad} (fp_2\delta)^* & \\
 & \downarrow & \downarrow \\
 & \delta^*(fp_1)^* \xlongequal{\quad\quad\quad} \delta^*(fp_2)^* & \\
 \swarrow & & \searrow \\
 \delta^* p_1^* f^* & \xlongequal{\quad\quad\quad} & \delta^* p_2^* f^*
 \end{array}$$

Subdiagram ② expands, naturally, as follows, with  $E_1 := f^!F_1$ ,  $E_2 := f^*F_2$  and  $p := p_2$ .

$$\begin{array}{ccccc}
 \delta^* p^* E_1 \otimes E_2 & \longrightarrow & (p\delta)^* E_1 \otimes E_2 & \xlongequal{\quad\quad\quad} & E_1 \otimes E_2 \\
 \parallel & & \parallel & \nearrow \textcircled{5} & \parallel \\
 \delta^* p^* E_1 \otimes (p\delta)^* E_2 & \longrightarrow & (p\delta)^* E_1 \otimes (p\delta)^* E_2 & \xrightarrow{\quad\gamma\quad} & (p\delta)^*(E_1 \otimes E_2) \\
 \downarrow & \nearrow & \textcircled{6} & & \downarrow \\
 \delta^* p^* E_1 \otimes \delta^* p^* E_2 & \longrightarrow & \delta^*(p^* E_1 \otimes p^* E_2) & \longrightarrow & \delta^* p^*(E_1 \otimes E_2)
 \end{array}$$

Commutativity of ⑥ is given by the dual of the commutative diagram [L09, 3.6.7.2] (see proof of [L09, 3.6.10]). As  $p\delta$  is the identity map of  $X$ , the same diagram, with  $g = f = \text{id}$ , yields that the isomorphism  $\gamma$  is idempotent, whence it is the identity map, so that ⑤ commutes.

Subdiagram ③ without  $\delta^*$  expands, naturally, to the following, with  $E_1 := f^*F_1$ ,  $E_2 := f^*F_2$ , and  $p := p_1$ :

$$\begin{array}{ccccc}
 \delta_* E_1 \otimes p^* E_2 & \longrightarrow & \delta_* \delta^! p^! E_1 \otimes p^* E_2 & \longrightarrow & p^! E_1 \otimes p^* E_2 \\
 \downarrow & & \downarrow & \textcircled{7} & \downarrow \\
 & & \delta_* (\delta^! p^! E_1 \otimes \delta^* p^* E_2) & \xrightarrow{\quad\delta_* \chi_\delta\quad} & \delta_* \delta^! (p^! E_1 \otimes p^* E_2) \\
 \downarrow & \nearrow & \downarrow & & \downarrow \chi_p \\
 \delta_* (E_1 \otimes \delta^* p^* E_2) & \longrightarrow & \delta_* ((p\delta)^! E_1 \otimes (p\delta)^* E_2) & \textcircled{8} & \delta_* \delta^! \chi_p \\
 \downarrow & \nearrow & \downarrow & & \downarrow \\
 \delta_* (E_1 \otimes E_2) & \longrightarrow & \delta_* \delta^! p^! (E_1 \otimes E_2) & \longrightarrow & p^! (E_1 \otimes E_2)
 \end{array}$$

Commutativity of subdiagram ⑦ is immediate from the definition of  $\chi_\delta$  ( $\delta$  being proper).

In [Nm14] there is a blanket convention that the functors and natural transformations strictly respect identities, hence  $(p\delta)^* = \text{id}^* = \text{id} = \text{id}^! = (p\delta)^!$ . The commutativity of ③ follows from [Nm14, Theorem 13.4].

Commutativity of subdiagram ④ is given by [Nm14, Proposition 13.7].

Commutativity of all the remaining (unlabeled) subdiagrams is clear.  $\square$

**4.4.** Let  $f: X \rightarrow Z$  be a flat  $\mathcal{E}$ -map. The map  $f$  is *equidimensional of relative dimension  $d$*  if for each  $x \in X$  that is a generic point of an irreducible component of the fiber  $f^{-1}f(x)$ , the transcendence degree of the residue field of the local ring  $\mathcal{O}_{X,x}$  over that of  $\mathcal{O}_{Z,f(x)}$  is  $d$ . When  $f$  is of finite type, this just means that every irreducible component of every fiber has dimension  $d$ .

An essentially smooth  $\mathcal{E}$ -map of constant relative dimension  $d$  is equidimensional of relative dimension  $d$ .

For any equidimensional such  $f$ , of relative dimension  $d$ , it holds that  $H^e f^! \mathcal{O}_Z = 0$  whenever  $e < -d$ . Indeed, this assertion is local on  $Z$  and  $X$  (see the remark right after (1.1.3)). One gets then from [EGA4, 13.3.1] that  $f$  may be assumed to be of the form  $\text{Spec } B \rightarrow \text{Spec } A$  where  $B$  is a localization of a module-finite quasi-finite algebra  $B_0$  over the polynomial ring  $A[T_1, \dots, T_d]$ . By Zariski's Main Theorem [EGA4, 8.12.6] the map  $\text{Spec } B_0 \rightarrow \text{Spec } A[T_1, \dots, T_d]$  factors as finite  $\circ$  (open immersion). The isomorphism (3.1.2) shows that the relative dualizing complex for the map  $\text{Spec } A[T_1, \dots, T_d] \rightarrow \text{Spec } A$  is concentrated in degree  $-d$ , so (1.2.1) implies that the assertion holds for any map  $\text{Spec } B_1 \rightarrow \text{Spec } A$  with  $B_1$  finite over  $A[T_1, \dots, T_d]$ , and then by (i) in 1.1, it holds for  $\text{Spec } B \rightarrow \text{Spec } A$ .

So there is a canonical composite map

$$(4.4.1) \quad \Omega_f^d[d] \xrightarrow{(3.2.2)} (H^{-d} \delta^* \delta_* \mathcal{O}_X)[d] \xrightarrow{H^{-d} c_f} (H^{-d} f^! \mathcal{O}_Z)[d] \longrightarrow f^! \mathcal{O}_Z,$$

whence a canonical map

$$c_f: \Omega_f^d[d] \otimes f^* \longrightarrow f^! \mathcal{O}_Z \otimes f^* \xrightarrow{(2.6.6)} f^!.$$

For essentially smooth  $f$  there is then an obvious question—the one that motivated the present paper, and to which the answer is affirmative:

**Theorem 4.5.** *For essentially smooth  $\mathcal{E}$ -maps  $f: X \rightarrow Z$  of relative dimension  $d$ , and  $F \in \mathbf{D}_{\text{qc}}(Z)$ , the fundamental class map  $c_f(F)$  is inverse to Verdier's isomorphism  $v_f(F)$ .*

*Proof.* As in Section 4.1. set  $Y := X \times_Z X$ , let  $\delta: X \rightarrow Y$  be the diagonal map, and  $p_i: Y \rightarrow X$  ( $i = 1, 2$ ) the projections onto the first and second factors, respectively, so that we have the diagram, with fiber square  $\clubsuit$ ,

$$(4.5.1) \quad \begin{array}{ccccc} X & \xrightarrow{\delta} & Y & \xrightarrow{p_2} & X \\ & & p_1 \downarrow & \clubsuit & \downarrow f \\ & & X & \xrightarrow{f} & Z \end{array}$$



Resolving  $\delta_*\mathcal{O}_X$  locally by a Koszul complex, one sees that  $H^e\delta^*\delta_*\mathcal{O}_X = 0$  if  $e < -d$ ; so one has natural maps

$$(4.5.2) \quad \omega := \Omega_f^d[d] \xrightarrow{(3.2.2)} (H^{-d}\delta^*\delta_*\mathcal{O}_X)[d] \longrightarrow \delta^*\delta_*\mathcal{O}_X,$$

and  $c_f$  is the composite

$$\omega \otimes f^* \xrightarrow{(4.5.2) \otimes \text{id}} \delta^*\delta_*\mathcal{O}_X \otimes f^* \xrightarrow{c_f(\mathcal{O}_Z) \otimes \text{id}} f^!\mathcal{O}_Z \otimes f^* \xrightarrow{(2.6.6)} f^!.$$

In view of Proposition 3.4.4, the theorem says then that the composite

$$\omega \otimes f^* \xrightarrow{(4.5.2) \otimes \text{id}} \delta^*\delta_*\mathcal{O}_X \otimes f^* \xrightarrow{c_f(\mathcal{O}_Z) \otimes \text{id}} f^!\mathcal{O}_Z \otimes f^* \xrightarrow{v_f(\mathcal{O}_Z) \otimes \text{id}} \omega \otimes f^*$$

is the identity map.

(Here we implicitly used commutativity of the diagram of natural isomorphisms

$$(4.5.3) \quad \begin{array}{ccc} f^*\mathcal{O}_Z \otimes f^* & \xrightarrow{\sim} & \mathcal{O}_X \otimes f^* \\ \simeq \downarrow & & \downarrow \simeq \\ f^*(\mathcal{O}_X \otimes \text{id}) & \xrightarrow{\sim} & f^* \end{array}$$

which commutativity holds since, *mutatis mutandis*, this diagram is *dual* [L09, 3.4.5] to the commutative subdiagram ② in the proof of [L09, 3.4.7(iii)].

It suffices therefore to prove Theorem 4.5 when  $F = \mathcal{O}_Z$ .

Let  $(-)'$  be the endofunctor  $\mathcal{H}om_X(-, \mathcal{O}_X)$  of  $\mathbf{D}(X)$ . The “mirror image” of the evaluation map  $\text{ev}(X, E, \mathcal{O}_X)$  (see (2.2.2)) is the natural composite

$$(4.5.4) \quad E \otimes E' \xrightarrow{\sim} E' \otimes E \xrightarrow{(2.2.2)} \mathcal{O}_X \quad (E \in \mathbf{D}(X)).$$

Now, after unwinding of the definitions involved, Theorem 4.5 for  $F = \mathcal{O}_Z$  states that the border of the next natural diagram commutes. (Going around clockwise from the upper left corner to the bottom right one gives  $c_f(\mathcal{O}_Z)$ , while going around counterclockwise gives  $v_f(\mathcal{O}_Z)^{-1}$ .)

In this diagram,  $\otimes := \otimes_X$ ,  $\psi_2^0 := \psi(\delta, p_2, \mathcal{O}_X, \mathcal{O}_X)$ —the identity map of  $\delta^*\delta_*\mathcal{O}_X$  (see 2.9.5), and  $\psi_2 := \psi(\delta, p_2, \mathcal{O}_X, \delta^!p_1^!\mathcal{O}_X)$ . Commutativity of the unlabeled subdiagrams is straightforward to check. The problem is to show commutativity of ①.

$$\begin{array}{ccccccc}
\mathcal{O}_X \otimes \omega & \xlongequal{\quad} & \omega \otimes \mathcal{O}_X & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \mathcal{O}_X & \xlongequal{\psi_2^0} & \delta^* \delta_* \mathcal{O}_X \\
\cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
\delta^! p_1^! \mathcal{O}_X \otimes \omega & \xleftarrow{\sim} & \omega \otimes \delta^! p_1^! \mathcal{O}_X & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \delta^! p_1^! \mathcal{O}_X & \xrightarrow{\psi_2} & \delta^* \delta_* \delta^! p_1^! \mathcal{O}_X \\
\cong \downarrow (2.6.7) & & \cong \downarrow (2.6.6) & & \textcircled{1} & & \downarrow \\
\delta^! p_2^* f^! \mathcal{O}_Z \otimes \omega & & \omega \otimes \delta^! \mathcal{O}_Y \otimes \delta^* p_1^! \mathcal{O}_X & \xrightarrow{(3.1.6)} & \omega \otimes \omega' \otimes \delta^* p_1^! \mathcal{O}_X & \xrightarrow{(4.5.4)} & \delta^* p_1^! \mathcal{O}_X \\
\cong \downarrow \chi_\delta^{-1} & & \downarrow & & \downarrow & & \downarrow (2.6.7) \cong \\
\delta^! \mathcal{O}_Y \otimes \delta^* p_2^* f^! \mathcal{O}_Z \otimes \omega & \xrightarrow{\cong} & \omega \otimes \delta^! \mathcal{O}_Y \otimes \delta^* p_2^* f^! \mathcal{O}_Z & \xrightarrow{(3.1.6)} & \omega \otimes \omega' \otimes \delta^* p_2^* f^! \mathcal{O}_Z & \xrightarrow{(4.5.4)} & \delta^* p_2^* f^! \mathcal{O}_Z \\
\cong \downarrow & & \downarrow & & \downarrow & & \downarrow \cong \\
\delta^! \mathcal{O}_Y \otimes f^! \mathcal{O}_Z \otimes \omega & \xrightarrow[\sim]{(3.1.6)} & \omega' \otimes f^! \mathcal{O}_Z \otimes \omega & \xrightarrow{\sim} & \omega \otimes \omega' \otimes f^! \mathcal{O}_Z & \xrightarrow[\sim]{(4.5.4)} & f^! \mathcal{O}_Z
\end{array}$$

We'll need another description of the map (3.1.6) (Lemma 4.5.9 below).

For this, begin by checking that the counit map  $\delta^* \delta_* G \rightarrow G$  ( $G \in \mathbf{D}(X)$ ) has right inverses  $\tau_p(G)$ , where  $p: Y \rightarrow X$  is any map such that  $p\delta = \text{id}_X$  (e.g.,  $p = p_1$  or  $p_2$ ), and  $\tau_p(G)$  is the natural composite

$$(4.5.5) \quad G \xrightarrow{\sim} \delta^* p^* G \xrightarrow{\sim} \delta^* p^* p_* \delta_* G \longrightarrow \delta^* \delta_* G,$$

that is, the composite  $G \xrightarrow{\sim} \delta^* p^* G \xrightarrow{\delta^* v(G)} \delta^* \delta_* G$ , where

$$(4.5.6) \quad v: p^* \rightarrow \delta_*$$

is the map adjoint to  $\text{id} \xrightarrow{\sim} p_* \delta_*$ .

Let  $\rho_0$  be the natural composite (of isomorphisms, since  $\delta_* \mathcal{O}_X$  is perfect)

$$\delta^* \delta_* \delta^! \mathcal{O}_Y \xrightarrow{(1.2.1)} \delta^* \mathcal{H}om_Y(\delta_* \mathcal{O}_X, \mathcal{O}_Y) \xrightarrow{(2.6.4)} (\delta^* \delta_* \mathcal{O}_X)',$$

and set

$$(4.5.7) \quad \rho = \rho_p := \rho_0 \tau_p(\delta^! \mathcal{O}_Y): \delta^! \mathcal{O}_Y \longrightarrow (\delta^* \delta_* \mathcal{O}_X)'.$$

As noted just before (3.2.7),  $H^e(\delta^! \mathcal{O}_Y) = 0$  for any  $e \neq d$ , whence the counit map  $\delta^* \delta_* \delta^! \mathcal{O}_Y \rightarrow \delta^! \mathcal{O}_Y$  is taken to an isomorphism by the functor  $H^d$ . The inverse of this isomorphism is  $H^d \tau_p(\delta^! \mathcal{O}_Y)$  because  $\tau_p(\delta^! \mathcal{O}_Y)$  is right-inverse to  $\delta^* \delta_* \delta^! \mathcal{O}_Y \rightarrow \delta^! \mathcal{O}_Y$ . It follows then from its description via (3.2.6) that (3.2.8), with  $U = X$ ,  $W = Y$  and  $n = d$ , is the natural composite

$$(4.5.8) \quad \begin{aligned} \delta^! \mathcal{O}_Y &\cong (H^d(\delta^! \mathcal{O}_Y))[-d] \\ &\xrightarrow{H^d(\rho)[-d]} (H^d((\delta^* \delta_* \mathcal{O}_X)'))[-d] \xrightarrow{\sim} (\Omega_f^d)'[-d], \end{aligned}$$

the last isomorphism arising via (3.2.5) (with  $n = d$ ) and (4.5.2).

**Lemma 4.5.9.** *For any  $\rho = \rho_p$  as in (4.5.7), the map (3.1.6) factors as*

$$\delta^! \mathcal{O}_Y \xrightarrow{\rho} (\delta^* \delta_* \mathcal{O}_X)' \xrightarrow{\text{via (4.5.2)}} \omega'.$$

*Proof.* For any complex  $E \in \mathbf{D}(X)$  set  $E^{\geq d} := t_{\geq d} E$ , with  $t_{\geq d}$  the truncation functor (denoted  $\tau_{\geq d}$  in [L09, § 10.1]). The natural map  $H^d(E)[-d] \rightarrow E^{\geq d}$  is an isomorphism if  $H^e E = 0$  for all  $e > d$ , a condition satisfied when  $E = (\delta^* \delta_* \mathcal{O}_X)'$  or  $\omega'$ .

Accordingly, and in view of the description of (3.1.6) via (3.2.8) = (4.5.8), the lemma asserts that the border of the following diagram, whose top row is (4.5.8), commutes.

$$\begin{array}{ccccccc} \delta^! \mathcal{O}_Y & \longrightarrow & (\delta^! \mathcal{O}_Y)^{\geq d} & \longrightarrow & ((\delta^* \delta_* \mathcal{O}_X)')^{\geq d} & \longrightarrow & (\Omega_f^d)'[-d] \\ \downarrow \rho & & \nearrow & & \text{via (4.5.2)} \downarrow & \textcircled{2} & \downarrow (3.1.5) \\ (\delta^* \delta_* \mathcal{O}_X)' & & & & (\omega')^{\geq d} & & \omega' \\ & & & & \text{via (4.5.2)} \longrightarrow & & \omega' \end{array}$$

The unlabeled subdiagrams clearly commute. Subdiagram ② expands naturally as follows, with  $\mathcal{H}om = \mathcal{H}om_X$ :

$$\begin{array}{ccc} H^d((\delta^* \delta_* \mathcal{O}_X)')[-d] & \xrightarrow{(3.2.5)} & ((H^{-d} \delta^* \delta_* \mathcal{O}_X)')[-d] \\ \simeq \uparrow & & \downarrow \text{cf. (4.5.2)} \\ ((\delta^* \delta_* \mathcal{O}_X)')^{\geq d} & & ((\Omega_f^d)')[-d] \\ \downarrow & & \downarrow (3.1.5) \\ ((H^{-d} \delta^* \delta_* \mathcal{O}_X[d])')^{\geq d} & & \omega' \\ \text{via (4.5.2)} \downarrow & & \downarrow \\ (\omega')^{\geq d} & \xlongequal{\quad\quad\quad} & \omega' \end{array}$$

The vertices of this diagram can all be identified with the complex  $G$  that is  $H^0 \mathcal{H}om(\Omega_f^d, \mathcal{O}_X)$  in degree  $d$  and 0 elsewhere. When this is done, all the maps in the diagram are identity maps except for the two labeled (3.2.5) and (3.1.5), which are both multiplication in  $G$  by  $(-1)^{d(d+1)/2}$ . (See the remarks following equations (3.2.5) and (3.1.5)). Hence subdiagram ② commutes, and Lemma 4.5.9 is proved.  $\square$

One has now that subdiagram ① without  $p_1^! \mathcal{O}_X$  expands naturally as follows, with  $\chi$  as in (2.6.6),  $\text{id}$  the identity functor, and

$$\begin{aligned}\psi_3 &:= \psi(\delta, p_2, \mathcal{O}_X, \delta^! \mathcal{O}_Y \otimes_Y \delta^*(-)), \\ \psi_4 &:= \psi(\delta, p_2, \delta^! \mathcal{O}_Y, \delta^*(-)), \\ \psi_7 &:= \psi(\delta, p_2, \mathcal{O}_X, \delta^! \mathcal{O}_Y).\end{aligned}$$

$$\begin{array}{ccccc} \omega \otimes \delta^! & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \delta^! & \xrightarrow{\psi_2} & \delta^* \delta_* \delta^! \\ \downarrow \chi_\delta^{-1} & & \downarrow \chi_\delta^{-1} & & \downarrow \chi_\delta^{-1} \\ \omega \otimes \delta^! \mathcal{O}_Y \otimes \delta^* & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \delta^! \mathcal{O}_Y \otimes \delta^* & \xrightarrow{\psi_3} & \delta^* \delta_* (\delta^! \mathcal{O}_Y \otimes_Y \delta^*) \\ \downarrow \text{via } \rho & & \downarrow \psi_7 \otimes \text{id} & \nearrow \psi_4 & \downarrow (2.6.3)^{-1} \\ \omega \otimes \delta^! \mathcal{O}_Y \otimes \delta^* & & \delta^* \delta_* \delta^! \mathcal{O}_Y \otimes \delta^* & \xrightarrow{(2.6.1)} & \delta^* (\delta_* \delta^! \mathcal{O}_Y \otimes_Y \text{id}) \\ \downarrow \text{via } \rho & & \downarrow \text{via } \rho & & \downarrow \text{via } \rho \\ \delta^* \delta_* \mathcal{O}_X \otimes (\delta^* \delta_* \mathcal{O}_X)' \otimes \delta^* & & \delta^* \mathcal{O}_Y \otimes \delta^* & \xrightarrow{(2.6.1)} & \delta^* (\mathcal{O}_Y \otimes_Y \text{id}) \\ \downarrow (4.5.2) & \nearrow (4.5.2) & \downarrow (4.5.4) & & \downarrow \sim \\ \omega \otimes (\delta^* \delta_* \mathcal{O}_X)' \otimes \delta^* & & \delta^* \mathcal{O}_Y \otimes \delta^* & \xrightarrow{(2.6.1)} & \delta^* (\mathcal{O}_Y \otimes_Y \text{id}) \\ \downarrow (4.5.2) & & \downarrow (4.5.4) & & \downarrow \sim \\ \omega \otimes \omega' \otimes \delta^* & \xrightarrow{(4.5.4)} & \mathcal{O}_X \otimes \delta^* & \xrightarrow{\sim} & \delta^* \end{array}$$

Commutativity of the unlabeled subdiagrams is clear;

Commutativity of ③ is given by Lemma 2.9.4.

Commutativity of ⑤ (without  $\delta^*$ ) is the definition of  $\chi_\delta = \chi_\delta(\mathcal{O}_Y, -)$ .

Commutativity of ⑥ is given by that of the first diagram in [L09, 3.5.3(h)], with  $(A, B, C) := (\delta^* \delta_* \mathcal{O}_X, \omega, \mathcal{O}_X)$ .

For commutativity of ⑧, cf. (4.5.3).

After restoring the term  $p_1^! \mathcal{O}_X \cong p_2^* f^! \mathcal{O}_Z$  omitted above, one gets commutativity of ④ from Lemma 2.9.2, with  $E := \delta^! \mathcal{O}_Y$ ,  $F := f^! \mathcal{O}_Z$ .

That leaves subdiagram  $\textcircled{7}$ , which without “ $\otimes \delta^*$ ” expands naturally as follows, where  $p := p_2$  and  $F'' := \mathcal{H}om_Y(F, \mathcal{O}_Y)$  ( $F \in \mathbf{D}(Y)$ ):

$$\begin{array}{ccccc}
 \delta^* \delta_* \mathcal{O}_X \otimes \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.1)} & \delta^*(\delta_* \mathcal{O}_X \otimes p^* \delta^! \mathcal{O}_Y) & \xrightarrow{(2.6.3)} & \delta^* \delta_*(\mathcal{O}_X \otimes \delta^* p^* \delta^! \mathcal{O}_Y) \\
 \text{via } \tau_p(\delta^! \mathcal{O}_Y) \downarrow & & \text{via (4.5.6)} \downarrow & & \downarrow \simeq \\
 \delta^* \delta_* \mathcal{O}_X \otimes \delta^* \delta_* \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.1)} & \delta^*(\delta_* \mathcal{O}_X \otimes \delta_* \delta^! \mathcal{O}_Y) & & \delta^* \delta_* \delta^* p^* \delta^! \mathcal{O}_Y \\
 \text{via (1.2.1)} \downarrow & & \text{via (1.2.1)} \downarrow & & \downarrow \simeq \\
 \delta^* \delta_* \mathcal{O}_X \otimes \delta^*(\delta_* \mathcal{O}_X)'' & \xrightarrow{(2.6.1)} & \delta^*(\delta_* \mathcal{O}_X \otimes (\delta_* \mathcal{O}_X)'') & & \delta^* \delta_* \delta^! \mathcal{O}_Y \\
 \text{via (2.6.4)} \downarrow & & \text{cf. (4.5.4)} \searrow & & \downarrow \\
 \delta^* \delta_* \mathcal{O}_X \otimes (\delta^* \delta_* \mathcal{O}_X)' & \xrightarrow{(4.5.4)} & \mathcal{O}_X & \xleftarrow{\sim} & \delta^* \mathcal{O}_Y
 \end{array}$$

Commutativity of subdiagram  $\textcircled{7}_1$  results from the definition of  $\tau_p$ .

Commutativity of subdiagram  $\textcircled{7}_2$  is clear.

Commutativity of  $\textcircled{7}_3$  follows from [L09, 3.5.6(a)] (with  $f := \delta$ ,  $A := \delta_* \mathcal{O}_X$  and  $B := \mathcal{O}_Y$ ).

Subdiagram  $\textcircled{7}_4$ , with the initial “ $\delta^*$ ” in each term dropped, expands naturally as follows:

$$\begin{array}{ccccccc}
 \delta_* \mathcal{O}_X \otimes p^* \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.3)} & & & & & \delta_*(\mathcal{O}_X \otimes \delta^* p^* \delta^! \mathcal{O}_Y) \\
 \downarrow \text{via (4.5.6)} & & & & & & \downarrow \simeq \\
 \delta_* \mathcal{O}_X \otimes \delta_* \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.3)} & \delta_*(\mathcal{O}_X \otimes \delta^* \delta_* \delta^! \mathcal{O}_Y) & & & & \delta_* \delta^* p^* \delta^! \mathcal{O}_Y \\
 \parallel & \searrow & \uparrow & & & & \downarrow \simeq \\
 & & \delta_* \mathcal{O}_X \otimes \delta_* \delta^* \delta_* \delta^! \mathcal{O}_Y & & & & \delta_* \delta^! \mathcal{O}_Y \\
 \parallel & \searrow & \uparrow & & & & \downarrow \simeq \\
 \delta_* \mathcal{O}_X \otimes \delta_* \delta^! \mathcal{O}_Y & \xrightarrow{(2.2.1)} & \delta_*(\mathcal{O}_X \otimes \delta^! \mathcal{O}_Y) & \xrightarrow{\sim} & \delta_* \delta^! \mathcal{O}_Y & & \\
 \downarrow \text{via (1.2.1)} & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & \delta_* \delta^! \mathcal{O}_Y \otimes \delta_* \mathcal{O}_X & \xrightarrow{(2.2.1)} & \delta_*(\delta^! \mathcal{O}_Y \otimes \mathcal{O}_X) & & \\
 & & \zeta(\mathcal{O}_X, \mathcal{O}_Y) \otimes \text{id} \searrow \text{see (2.6.5)} & & \downarrow & & \\
 \delta_* \mathcal{O}_X \otimes (\delta_* \mathcal{O}_X)'' & \xrightarrow{\sim} & (\delta_* \mathcal{O}_X)'' \otimes \delta_* \mathcal{O}_X & \xrightarrow{(2.2.2)} & \mathcal{O}_Y & & 
 \end{array}$$

Here, commutativity of the unlabeled subdiagrams is clear.

The commutativity of subdiagram  $\textcircled{7}_{41}$  is given by the definition of the projection isomorphism (2.6.3).

Commutativity of subdiagram  $\textcircled{7}_{42}$  (which says, incidentally, that the map  $v$  in (4.5.6) is adjoint to the natural isomorphism  $\delta^* p^* \xrightarrow{\sim} \text{id}$ ) follows easily from  $\tau_p$  in (4.5.5) being right inverse to the counit map  $\delta^* \delta_* G \rightarrow G$ .

Subdiagram  $\textcircled{7}_{43}$  commutes because  $\delta_*$  is a *symmetric* monoidal functor. (See §2.2.)

Finally, commutativity of subdiagram  $\textcircled{7}_{44}$  means that the map  $\zeta(\mathcal{O}_X, \mathcal{O}_Y)$  is adjoint to the natural composite

$$(4.5.10) \quad \delta_* \delta^! \mathcal{O}_Y \otimes_Y \delta_* \mathcal{O}_X \xrightarrow{(2.2.1)} \delta_*(\delta^! \mathcal{O}_Y \otimes_X \mathcal{O}_X) = \delta_* \delta^! \mathcal{O}_Y \longrightarrow \mathcal{O}_Y.$$

But for any  $F \in \mathbf{D}_{\text{qc}}(Y)$ ,  $\zeta(\mathcal{O}_X, F)$  is, by definition, right-conjugate to the projection isomorphism

$$(4.5.11) \quad \delta_* \delta^* G = \delta_*(\delta^* G \otimes_X \mathcal{O}_X) \xleftarrow[\text{(2.6.3)}]{\sim} G \otimes_Y \delta_* \mathcal{O}_X,$$

that is (see [L09, 3.3.5]),  $\zeta(\mathcal{O}_X, F)$  is adjoint to the natural composite

$$\delta_* \delta^! F \otimes_Y \delta_* \mathcal{O}_X \xrightarrow[\text{(4.5.11)}]{\sim} \delta_* \delta^* \delta_* \delta^! F \longrightarrow \delta_* \delta^! F \longrightarrow F,$$

which for  $F := \mathcal{O}_Y$  is the same as (4.5.10), since for any  $A$  and  $B \in \mathbf{D}(X)$  (e.g.,  $A = \delta^! \mathcal{O}_Y$  and  $B = \mathcal{O}_X$ ), the following natural diagram commutes (as follows easily from the definition of (2.6.3)):

$$\begin{array}{ccc} \delta_* A \otimes_Y \delta_* B & \xrightarrow{(2.2.1)} & \delta_*(A \otimes_X B) \\ \updownarrow & \searrow (2.6.3) & \uparrow \\ \delta_* \delta^* \delta_* A \otimes_Y \delta_* B & \xrightarrow{(2.2.1)} & \delta_*(\delta^* \delta_* A \otimes_X B) \end{array}$$

So subdiagram  $\textcircled{7}_{44}$  commutes.

Thus,  $\textcircled{7}$  commutes, whence so does  $\textcircled{1}$ . □

## 5. ACKNOWLEDGEMENT

We are grateful to Pramathanath Sastry for a number of comments from which this paper benefited.

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