LOCAL HOMOLOGY AND COHOMOLOGY ON SCHEMES

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Abstract. We present a sheafified derived-category generalization of Greenlees-May duality (a far-reaching generalization of Grothendieck's local duality theorem): for a quasi-compact separated scheme X and a "proregular" subscheme Z—for example, any separated noetherian scheme and any closed subscheme—there is a sort of adjointness between local cohomology supported in Z and left-derived completion along Z. In particular, left-derived completion can be identified with *local homology,* i.e., the homology of $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, -)$.

Generalizations of a number of duality theorems scattered about the literature result: the *Peskine-Szpiro duality sequence* (generalizing local duality), the *Warwick Duality* theorem of Greenlees, the *Affine Duality* theorem of Hartshorne. Using Grothendieck Duality, we also get a generalization of a *Formal Duality* theorem of Hartshorne, and of a related local-global duality theorem.

In a sequel we will develop the latter results further, to study Grothendieck duality and residues on formal schemes.

Introduction. We redevelop here some basic facts about local homology and cohomology on quasi-compact separated schemes, in the context of derived categories. While our results are not fundamentally new, they do, we believe, add value and meaning to what is already known, through a more general and in some ways more transparent approach—leading for example to a unification of several duality theorems scattered about the literature. Furthermore, the derived category formulation provides an essential link between Grothendieck Duality on ordinary and on formal schemes, the latter to be treated in a subsequent paper.

The main result is the Duality Theorem (0.3) on a quasi-compact separated scheme X around a *proregularly embedded* closed subscheme Z. This asserts a sort of sheafified adjointness between local cohomology supported in Z and left-derived completion functors along Z. (For complexes with quasi-coherent homology, the precise derived-category adjoint of local cohomology is described in (0.4)(a).) A special case—and also a basic point in the proof—is that:

(∗) these left-derived completion functors can be identified with *local homology,* i.e., the homology of $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, -)$.

The technical condition "Z proregularly embedded," treated at length in §3, is just what is needed to make cohomology supported in Z enjoy some good properties which are standard when X is noetherian. Indeed, it might be said that these properties hold in the noetherian context because (as follows immediately from the definition) *every closed subscheme of a noetherian scheme is proregularly embedded.*

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The assertion $(*)$ is a sheafified derived-category version of Theorem 2.5 in [GM]. (The particular case where Z is regularly embedded in X had been studied, over commutative rings, by Strebel [St, pp. 94–95, 5.9] and, in great detail, by Matlis [M2, p. 89, Thm. 20]. Also, a special case of Theorem (0.3) appeared in [Me, p. 96] at the beginning of the proof of 2.2.1.3.) More specifically, our Proposition (4.1) provides another approach to the Greenlees-May duality isomorphism—call it Ψ—from local homology to left-derived completion functors. Though this Ψ is a priori local and depends on choices, it is in fact canonical: Corollary (4.2) states that a certain natural global map from left-derived completion functors to local homology restricts locally to an inverse of Ψ.

We exhibit in §5 how Theorem (0.3) provides a unifying principle for a substantial collection of other duality results from the literature. For example, as noted by Greenlees and May [GM, p. 450, Prop. 3.8], their theorem contains the standard Local Duality theorem of Grothendieck. (See Remark (0.4)(c) below for more in this vein).

To describe things more precisely, we need some notation. Let X be a quasi-compact separated scheme, let $\mathcal{A}(X)$ be the category of all $\mathcal{O}_X$-modules, and let $\mathcal{A}_{\mathrm{qc}}(X) \subset \mathcal{A}(X)$ be the full (abelian) subcategory of quasi-coherent $\mathcal{O}_X$-modules. The derived category $\mathbf{D}(X)$ of $\mathcal{A}(X)$ contains full subcategories $\mathbf{D}_{\mathrm{qc}}(X) \supset \mathbf{D}_{\mathrm{c}}(X)$ whose objects are the $\mathcal{O}_X$-complexes with quasi-coherent, respectively coherent, homology sheaves.

Let $Z \subset X$ be a closed subset. If $X \setminus Z$ is quasi-compact then by induction on $\min\{\, n \mid X \text{ can be covered by } n \text{ affine open subsets}\,\}$, and [GrD, p. 318, (6.9.7)], one shows that Z is the support $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$ for some *finite-type* quasi-coherent $\mathcal{O}_X$-ideal $\mathcal{I}$ (and conversely). We assume throughout that Z satisfies this condition.

The left-exact functor $\varGamma_Z\colon \mathcal{A}(X) \to \mathcal{A}(X)$ associates to each $\mathcal{O}_X$-module $\mathcal{F}$ its subsheaf of sections with support in Z. We define the subfunctor $\varGamma'_Z \subset \varGamma_Z$ by

$$\varGamma'_Z\mathcal{F} := \varinjlim_{n>0} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{F}) \qquad \big(\mathcal{F} \in \mathcal{A}(X)\big), \tag{0.1}$$

which depends only on Z (not $\mathcal{I}$). If $\mathcal{F}$ is quasi-coherent, then $\varGamma'_Z\mathcal{F} = \varGamma_Z\mathcal{F}$. The functor $\varGamma_Z$ (resp. $\varGamma'_Z$) has a right-derived functor $\mathbf{R}\varGamma_Z : \mathbf{D}(X) \to \mathbf{D}(X)$ (resp. $\mathbf{R}\varGamma'_Z\colon \mathbf{D}(X) \to \mathbf{D}(X)$), as does any functor from $\mathcal{A}(X)$ to an abelian category, via K-injective resolutions [Sp, p. 138, Thm. 4.5].[1]

By the universal property of derived functors, there is a unique functorial map

$$\gamma\colon \mathbf{R}\varGamma'_Z\mathcal{E} \to \mathcal{E}$$

whose composition with $\varGamma'_Z\mathcal{E} \to \mathbf{R}\varGamma'_Z\mathcal{E}$ is the inclusion map $\varGamma'_Z\mathcal{E} \hookrightarrow \mathcal{E}$.

[1] See also [*ibid.,* p. 133, Prop. 3.11] or [BN, §2] for the existence of such resolutions in module categories. (Actually, as recently observed by Weibel, Cartan-Eilenberg resolutions, totalized via products, will do in this case.) Moreover, Neeman has a strikingly simple proof that hence such resolutions exist in any abelian quotient category of a module category, i.e., by a theorem of Gabriel-Popescu, in any abelian category—for instance $\mathcal{A}(X)$—with a generator and with exact filtered $\varinjlim$. (Private communication.)

For proregularly embedded $Z \subset X$, *the derived-category map* $\mathbf{R}\varGamma'_Z\mathcal{E} \to \mathbf{R}\varGamma_Z\mathcal{E}$ *induced by the inclusion* $\varGamma'_Z \hookrightarrow \varGamma_Z$ *is an isomorphism for any complex* $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$ (Corollary (3.2.4)). This isomorphism underlies the well-known homology isomorphisms (of sheaves)[2]

$$\varinjlim_{n>0} \mathcal{E}\mathrm{xt}^i(\mathcal{O}_X/\mathcal{I}^n, \mathcal{F}) \xrightarrow{\ \sim\ } H^i_Z(\mathcal{F}) \qquad (i \geq 0,\ \mathcal{F} \in \mathcal{A}_{\mathrm{qc}}(X)). \tag{0.1.1}$$

We also consider the *completion functor* $\Lambda_Z : \mathcal{A}_{\mathrm{qc}}(X) \to \mathcal{A}(X)$ given by

$$\Lambda_Z\mathcal{F} := \varprojlim_{n>0} \big((\mathcal{O}_X/\mathcal{I}^n) \otimes \mathcal{F}\big) \qquad \big(\mathcal{F} \in \mathcal{A}_{\mathrm{qc}}(X)\big). \tag{0.2}$$

This depends only on Z. We will show in §1 that Λ_Z has a left-derived functor $\mathbf{L}\Lambda_Z \colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}(X)$, describable via flat quasi-coherent resolutions. By the universal property of derived functors, there is a unique functorial map

$$\lambda\colon \mathcal{F} \to \mathbf{L}\Lambda_Z\mathcal{F}$$

whose composition with $\mathbf{L}\Lambda_Z\mathcal{F} \to \Lambda_Z\mathcal{F}$ is the completion map $\mathcal{F} \to \Lambda_Z\mathcal{F}$.

Theorem (0.3). *For any quasi-compact separated scheme X and any proregularly embedded closed subscheme Z (Definition (3.0.1)), there is a functorial isomorphism*

$$\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathbf{R}\varGamma'_Z\mathcal{E}, \mathcal{F}) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}, \mathbf{L}\Lambda_Z\mathcal{F})$$
$$\big(\mathcal{E} \in \mathbf{D}(X),\ \mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)\big)$$

whose composition with the map $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet\big(\mathcal{E}, \mathcal{F}\big) \to \mathbf{R}\mathcal{H}\mathrm{om}^\bullet\big(\mathbf{R}\varGamma'_Z\mathcal{E}, \mathcal{F}\big)$ *induced by γ is the map* $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet\big(\mathcal{E}, \mathcal{F}\big) \to \mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}, \mathbf{L}\Lambda_Z\mathcal{F})$ *induced by λ.*

The *proof* occupies §§1–4; an outline is given in §2. Miscellaneous corollaries and applications appear in §5.

From Theorem (0.3) we get a commutative diagram

$$\begin{array}{ccc}
\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathbf{R}\varGamma'_Z\mathcal{E}, \mathcal{F}) & \xrightarrow[\alpha]{\ \sim\ } & \mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}, \mathbf{L}\Lambda_Z\mathcal{F}) \\
\downarrow & \searrow^{\lambda'} & \downarrow{\scriptstyle \gamma'} \\
\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathbf{R}\varGamma'_Z\mathbf{R}\varGamma'_Z\mathcal{E}, \mathcal{F}) & \xrightarrow{\ \sim\ } & \mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathbf{R}\varGamma'_Z\mathcal{E}, \mathbf{L}\Lambda_Z\mathcal{F})
\end{array}$$

with horizontal isomorphisms as in (0.3), λ' induced by λ, and γ' induced by γ. It follows readily from Lemma (3.1.1)(2) that the natural map $\mathbf{R}\varGamma'_Z\mathbf{R}\varGamma'_Z\mathcal{E} \to \mathbf{R}\varGamma'_Z\mathcal{E}$ is an isomorphism; hence both λ' and γ' are isomorphisms, and α has the *explicit description* $\alpha = {\gamma'}^{-1} \circ \lambda'$. Conversely, if we knew beforehand that λ' and γ' are isomorphisms, then we could *define* $\alpha := {\gamma'}^{-1} \circ \lambda'$ and recover Theorem (0.3). Thus we can restate the Theorem as:

[2]See [H, p. 273], where, however, the proof seems incomplete—"way-out" needs to begin with [Gr, p. 22, Thm. 6]. Alternatively, one could use quasi-coherent injective resolutions . . .

Theorem (0.3)(bis). *For any quasi-compact separated scheme X and proregularly embedded closed subscheme Z, the maps λ and γ induce functorial isomorphisms*

$$\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma'_Z\mathcal{E},\mathcal{F}) \underset{\lambda'}{\xrightarrow{\ \sim\ }} \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma'_Z\mathcal{E},\mathbf{L}\Lambda_Z\mathcal{F}) \underset{\gamma'}{\xleftarrow{\ \sim\ }} \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathbf{L}\Lambda_Z\mathcal{F})$$
$$\big(\mathcal{E}\in\mathbf{D}(X),\ \mathcal{F}\in\mathbf{D}_{\mathrm{qc}}(X)\big).$$

As explained in Remark (5.1.2), that λ' is an isomorphism amounts to the following Corollary. Recall that *proregularity* of a finite sequence $\mathbf{t}:=(t_1,t_2,\dots,t_\mu)$ in a commutative ring A is defined in (3.0.1) (where X can be taken to be $\mathrm{Spec}(A)$); and that every sequence in a noetherian ring is proregular.

Corollary (0.3.1). *Let $\mathbf{t}$ be a proregular sequence in a commutative ring A, and let F be a flat A-module, with $\mathbf{t}$-adic completion $\widehat{F}$. Then the natural local homology maps $H^n_{\mathbf{t}A}(F)\to H^n_{\mathbf{t}A}(\widehat{F})$ $(n\ge 0)$ are all isomorphisms.*

In other words, the natural Koszul-complex map $\mathcal{K}^\bullet_\infty(\mathbf{t})\otimes F\to\mathcal{K}^\bullet_\infty(\mathbf{t})\otimes\widehat{F}$ is a quasi-isomorphism (*see* (3.1.1)(2)).

Suppose now that X is affine, say $X=\mathrm{Spec}(A)$, let $\mathbf{t}:=(t_1,t_2,\dots,t_\mu)$ be a proregular sequence in A, and set $Z:=\mathrm{Spec}(A/\mathbf{t}A)$. With $\mathbf{t}^n:=(t_1^n,\dots,t_\mu^n)$, consider the A-module functors

$$\Gamma_{\mathbf{t}}(G):=\varinjlim_{n>0}\mathrm{Hom}_A(A/\mathbf{t}^nA,G),$$
$$\Lambda_{\mathbf{t}}(G):=\varprojlim_{n>0}\big((A/\mathbf{t}^nA)\otimes G\big)\qquad(\mathbf{t}\text{-adic completion}).$$

This is the situation in [GM], and when the sequence $\mathbf{t}$ is A-*regular,* in [M2]. The arguments used here to prove Theorem (0.3) apply as well in the simpler ring-theoretic context, yielding an isomorphism in the derived A-module category $\mathbf{D}(A)$:

$$(0.3)_{\mathrm{aff}}\qquad \mathbf{R}\mathrm{Hom}^\bullet_A(\mathbf{R}\Gamma_{\mathbf{t}}E,F)\xrightarrow{\ \sim\ }\mathbf{R}\mathrm{Hom}^\bullet_A(E,\mathbf{L}\Lambda_{\mathbf{t}}F)\qquad\big(E,F\in\mathbf{D}(A)\big).$$

In fact $(0.3)_{\mathrm{aff}}$ (with the isomorphism explicated as in (0.3) or (0.3)bis) is essentially equivalent to (0.3) for $\mathcal{E}\in\mathbf{D}_{\mathrm{qc}}$, see Remark (0.4)(d).

Suppose, for example, that $\mathbf{t}$ is A-regular, so that there is an isomorphism

$$\mathbf{R}\Gamma_{\mathbf{t}}(A)[\mu]\xrightarrow{\ \sim\ }H^\mu_{\mathbf{t}A}(A)=:K.$$

Then for any A-complex F, there is a natural isomorphism $K[-\mu]\underline{\underline{\otimes}}F\xrightarrow{\ \sim\ }\mathbf{R}\Gamma_{\mathbf{t}}(F)$ (cf. Corollary (3.2.5)), and so we have a composed isomorphism

$$\begin{aligned}H^0\mathbf{L}\Lambda_{\mathbf{t}}(F)&\xrightarrow{\ \sim\ }H^0\mathbf{R}\mathrm{Hom}^\bullet_A(\mathbf{R}\Gamma_{\mathbf{t}}A,F)\\&\xrightarrow{\ \sim\ }H^0\mathbf{R}\mathrm{Hom}^\bullet_A(\mathbf{R}\Gamma_{\mathbf{t}}A,\mathbf{R}\Gamma_{\mathbf{t}}F)\xrightarrow{\ \sim\ }\mathrm{Hom}_{\mathbf{D}(A)}(K,K\underline{\underline{\otimes}}F)\end{aligned}$$

corresponding to the First Representation Theorem of [M2, p. 91].[3]

[3]Matlis states the theorem for A-modules F which are "K-torsion-free" i.e. ([*ibid,* p. 86]), the canonical map $K\underline{\underline{\otimes}}F\to K\otimes F$ is an isomorphism; and he shows for such F that the natural map $H^0\mathbf{L}\Lambda_{\mathbf{t}}(F)\to\Lambda_{\mathbf{t}}(F)$ is an isomorphism [*ibid,* p. 89, Thm. 21, (2)].

Remarks (0.4). (a) Fix a quasi-compact separated scheme X, and write $\mathcal{A}$, $\mathcal{A}_{\mathrm{qc}}$, $\mathbf{D}$, $\mathbf{D}_{\mathrm{qc}}$, for $\mathcal{A}(X)$, $\mathcal{A}_{\mathrm{qc}}(X)$, $\mathbf{D}(X)$, $\mathbf{D}_{\mathrm{qc}}(X)$, respectively. Let $Z \subset X$ be a proregularly embedded closed subscheme. Corollary (3.2.5)(iii) gives us the functor $\mathbf{R}\varGamma_Z \colon \mathbf{D}_{\mathrm{qc}} \to \mathbf{D}_{\mathrm{qc}}$. Theorem (0.3) yields a *right adjoint* for this functor, as follows.

The inclusion functor $\mathcal{A}_{\mathrm{qc}} \hookrightarrow \mathcal{A}$ has a right adjoint Q, the "quasi-coherator" [I, p. 187, Lemme 3.2]. The functor Q, having an exact left adjoint, preserves K-injectivity, and it follows then that $\mathbf{R}Q$ is right-adjoint to the natural functor $\boldsymbol{j} \colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}) \to \mathbf{D}$, see [Sp, p. 129, Prop. 1.5(b)]. Since $\boldsymbol{j}$ induces an *equivalence of categories* $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}) \approx \mathbf{D}_{\mathrm{qc}}$ (see §1), therefore the inclusion functor $\mathbf{D}_{\mathrm{qc}} \hookrightarrow \mathbf{D}$ has a right adjoint, which—mildly abusing notation—we continue to denote by $\mathbf{R}Q$. Thus there is a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(\mathcal{E}, \mathbf{L}\Lambda_Z\mathcal{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}}(\mathcal{E}, \mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) \qquad (\mathcal{E}, \mathcal{F} \in \mathbf{D}_{\mathrm{qc}}).$$

Recalling that $\mathbf{R}\varGamma'_Z$ coincides with $\mathbf{R}\varGamma_Z$ on $\mathbf{D}_{\mathrm{qc}}$, and applying the functor $H^0\mathbf{R}\Gamma$ to the isomorphism in (0.3),[4] we deduce an *adjunction isomorphism*

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}}(\mathbf{R}\varGamma_Z\mathcal{E}, \mathcal{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}}(\mathcal{E}, \mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) \qquad (\mathcal{E}, \mathcal{F} \in \mathbf{D}_{\mathrm{qc}}).$$

(In this form the isomorphism doesn't sheafify, since for open immersions $i \colon U \to X$ the canonical functorial map $i^*Q_X \to Q_U i^*$ is usually *not* an isomorphism.)

For example, if X is *affine,* say $X = \mathrm{Spec}(A)$, then for any $\mathcal{L} \in \mathcal{A}(X)$, $Q(\mathcal{L})$ is the quasi-coherent $\mathcal{O}_X$-module $\Gamma(X, \mathcal{L})^\sim$ associated to the A-module $\Gamma(X, \mathcal{L})$; and hence

$$\mathbf{R}Q(\mathcal{G}) \cong \big(\mathbf{R}\Gamma(X, \mathcal{G})\big)^\sim \qquad (\mathcal{G} \in \mathbf{D}).$$

Any complex in $\mathbf{D}_{\mathrm{qc}}$ is isomorphic to a K-flat quasi-coherent $\mathcal{F}$ (Prop. (1.1)). For such an $\mathcal{F}$, with $\mathcal{F}_{/Z}$ the *completion* of $\mathcal{F}$ along Z [GrD, p. 418, (10.8.2)], Remark (d) below, with $E = A$, implies

$$\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F} \cong Q\Lambda_Z\mathcal{F} = \big(\Gamma(Z, \mathcal{F}_{/Z})\big)^\sim.$$

If furthermore A is noetherian, $Z = \mathrm{Spec}(A/I)$, and $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$, then one finds, as in (0.4.1) below, that with $\hat{A}$ the I-adic completion of A,

$$\Gamma(Z, \mathcal{F}_{/Z}) \cong \Gamma(X, \mathcal{F}) \otimes_A \hat{A}.$$

In more detail, Theorem (0.3)—at least for $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$—can be expressed via category-theoretic properties of the endofunctors $S := \mathbf{R}\varGamma_Z$ and $T := \mathbf{R}Q\mathbf{L}\Lambda_Z$ of $\mathbf{D}_{\mathrm{qc}}(X)$. (In the commutative-ring context, use $S := \mathbf{R}\Gamma_{\mathbf{t}}$ and $T := \mathbf{L}\Lambda_{\mathbf{t}}$ instead.)

Theorem (0.3)*. *The canonical maps $S \xrightarrow{\gamma} \mathbf{1} \xrightarrow{\nu} T$ (where $\mathbf{1}$ is the identity functor of $\mathbf{D}_{\mathrm{qc}}(X)$) induce functorial isomorphisms*

$$\mathrm{Hom}(S\mathcal{E}, S\mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, T\mathcal{F}) \cong \mathrm{Hom}(\mathcal{E}, T\mathcal{F}) \cong \mathrm{Hom}(T\mathcal{E}, T\mathcal{F}).$$

Proof. (See also (5.1.1.)) The first isomorphism is given by Lemma (0.4.2) below. The next two follow from Theorem (0.3)(bis), giving the adjointness of S and T, as well as the isomorphism $S \xrightarrow{\ \sim\ } ST$ in the following Corollary. Hence:

$$\mathrm{Hom}(\mathcal{E}, T\mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}(ST\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}(T\mathcal{E}, T\mathcal{F}). \qquad \square$$

Conversely, Theorem (0.3)*, applied to arbitrary affine open subsets of X, yields Theorem (0.3)(bis).

[4] Note that $H^0\mathbf{R}\Gamma\mathbf{R}\mathcal{H}om^\bullet = H^0\mathbf{R}\mathrm{Hom}^\bullet = \mathrm{Hom}_{\mathbf{D}}$, see e.g., [Sp, 5.14, 5.12, 5.17]. (In order to combine left- and right-derived functors, we must deal with *unbounded* complexes.)

Corollary. *The maps γ and ν induce functorial isomorphisms*

(i) $S^2 \xrightarrow{\sim} S$.
(ii) $T \xrightarrow{\sim} T^2$.
(iii) $TS \xrightarrow{\sim} T$.
(iv) $S \xrightarrow{\sim} ST$.

Proof. (i) can be deduced, for example, from the functorial isomorphism (see above) $\mathrm{Hom}(S\mathcal{E}, S^2\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(S\mathcal{E}, S\mathcal{F})$, applied when $\mathcal{E} = \mathcal{F}$ and when $\mathcal{E} = S\mathcal{F}$.

(ii): equivalent to (i) by adjunction.

(iii): use $\mathrm{Hom}(\mathcal{E}, TS\mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, S\mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{E}, T\mathcal{F})$.

(iv): use $\mathrm{Hom}(S\mathcal{E}, S\mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, T\mathcal{F}) \cong \mathrm{Hom}(S\mathcal{E}, ST\mathcal{F})$. □

The fact that ν induces isomorphisms $T \xrightarrow{\sim} T^2$ and $\mathrm{Hom}(T\mathcal{E}, T\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{E}, T\mathcal{F})$ implies that the derived completion functor T, together with $\nu\colon \mathbf{1} \to T$, is a *Bousfield localization* of $\mathbf{D}_{\mathrm{qc}}(X)$ with respect to the triangulated subcategory whose objects are the complexes $\mathcal{E}$ such that $T\mathcal{E} = 0$, or equivalently, by (iii) and (iv), such that $S\mathcal{E} = 0$, i.e., $\mathcal{E} = \mathbf{R}i_*i^*\mathcal{E}$ where $i\colon X \setminus Z \hookrightarrow X$ is the inclusion (see (0.4.2.1).

(b) With notation as (a), suppose that the separated scheme X is noetherian, so that any closed subscheme Z is proregularly embedded. On coherent $\mathcal{O}_X$-modules the functor Λ_Z is *exact.* This suggests (but doesn't prove) the following concrete interpretation for the restriction of the derived functor $\mathbf{L}\Lambda_Z$ to $\mathbf{D}_{\mathrm{c}} \subset \mathbf{D}_{\mathrm{qc}}$ (i.e., to $\mathcal{O}_X$-complexes whose homology sheaves are coherent). Let $\kappa = \kappa_Z$ be the canonical ringed-space map from the formal completion $X_{/Z}$ to X, so that κ_* and κ^* are exact functors [GrD, p. 422, (10.8.9)]. For $\mathcal{F} \in \mathcal{A}_{\mathrm{qc}}$, following [GrD, p. 418, (10.8.2)] we denote by $\mathcal{F}_{/Z}$ the restriction of $\Lambda_Z\mathcal{F}$ to Z. From the map $\kappa^*\mathcal{F} \to \mathcal{F}_{/Z}$ which is adjoint to the natural map $\mathcal{F} \to \Lambda_Z\mathcal{F} = \kappa_*\mathcal{F}_{/Z}$ we get a functorial map $\kappa_*\kappa^*\mathcal{F} \to \kappa_*\mathcal{F}_{/Z} = \Lambda_Z\mathcal{F}$; and since $\kappa_*\kappa^*$ is exact, there results a functorial map

$$\lambda_*^*\colon \kappa_*\kappa^*\mathcal{F} \to \mathbf{L}\Lambda_Z\mathcal{F} \qquad (\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}).$$

Proposition (0.4.1). *The map λ_*^* is an isomorphism for all $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}$.*

Proof. The question being local, we may assume X affine. As indicated at the end of §2, the functor $\mathbf{L}\Lambda_Z$ is bounded above (i.e., "way-out left") and also bounded below (i.e., "way-out right"); and the same is clearly true of $\kappa_*\kappa^*$. So by [H, p. 68, Prop. 7.1] (dualized) we reduce to where $\mathcal{F}$ is a single finitely-generated free $\mathcal{O}_X$-module, in which case the assertion is obvious since by §1, $\mathbf{L}\Lambda_Z\mathcal{P} = \Lambda_Z\mathcal{P}$ for any quasi-coherent flat complex $\mathcal{P}$. □

Via the natural isomorphism $\kappa_*\mathbf{R}\mathcal{H}om^\bullet_{X_{/Z}}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E}, \kappa_*\kappa^*\mathcal{F})$ [Sp, p. 147, Prop. 6.7], *the isomorphism in* (0.3) *now becomes, for $\mathcal{E} \in \mathbf{D}$, $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}$:*

$$(0.3)_{\mathrm{c}} \qquad \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma'_Z\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \kappa_*\mathbf{R}\mathcal{H}om^\bullet_{X_{/Z}}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F}),$$

or—by Lemma (0.4.2) below, and since as before $\mathbf{R}\varGamma'_Z\mathcal{F} \cong \mathbf{R}\varGamma_Z\mathcal{F}$:

$$(0.3)'_{\mathrm{c}} \qquad \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma'_Z\mathcal{E}, \mathbf{R}\varGamma'_Z\mathcal{F}) \xrightarrow{\sim} \kappa_*\mathbf{R}\mathcal{H}om^\bullet_{X_{/Z}}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F}).$$

Explicitly, all these isomorphisms fit into a natural commutative diagram:

$$\begin{array}{ccccc}
 & & \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma'_Z\mathcal{E},\mathcal{F}) & \xleftarrow[\text{(0.4.2)}]{\sim} & \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma'_Z\mathcal{E},\mathbf{R}\varGamma'_Z\mathcal{F}) \\
 & \nearrow & \simeq\big\downarrow\text{(0.3)} & & \\
\mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E},\mathbf{L}\Lambda_Z\mathcal{F}) & & \simeq\big\downarrow\text{(0.3)}'_{\mathrm{c}} \\
 & \searrow & \simeq\big\uparrow\text{via }\lambda^*_* & & \\
 & & \mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E},\kappa_*\kappa^*\mathcal{F}) & \xleftarrow{\sim} & \kappa_*\mathbf{R}\mathcal{H}om^\bullet_{X_{/Z}}(\kappa^*\mathcal{E},\kappa^*\mathcal{F})
\end{array}$$

Lemma (0.4.2). *Let X be a scheme, let $Z \subset X$ be a closed subset, and let $i\colon (X \setminus Z) \hookrightarrow X$ be the inclusion. Let $\mathcal{G} \in \mathbf{D}(X)$ be exact off Z, i.e., $i^*\mathcal{G} = 0$. Then for any $\mathcal{F} \in \mathbf{D}(X)$ the natural map $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{G},\mathbf{R}\varGamma_Z\mathcal{F}) \to \mathbf{R}\mathcal{H}om^\bullet(\mathcal{G},\mathcal{F})$ is an isomorphism. In particular, for any $\mathcal{E} \in \mathbf{D}(X)$ there are natural isomorphisms*

$$\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{E},\mathbf{R}\varGamma_Z\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{E},\mathcal{F}),$$
$$\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma'_Z\mathcal{E},\mathbf{R}\varGamma_Z\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma'_Z\mathcal{E},\mathcal{F}).$$

Proof. If $\mathcal{J}$ is an injective K-injective resolution of $\mathcal{F}$ [Sp, p. 138, Thm. 4.5] then $i^*\mathcal{J}$ is K-injective and the natural sequence $0 \to \varGamma_Z\mathcal{J} \to \mathcal{J} \to i_*i^*\mathcal{J} \to 0$ is exact; hence there is a natural triangle

$$\mathbf{R}\varGamma_Z\mathcal{F} \to \mathcal{F} \to \mathbf{R}i_*i^*\mathcal{F} \to \mathbf{R}\varGamma_Z\mathcal{F}[1]. \tag{0.4.2.1}$$

Apply the functor $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{G},-)$ to this triangle, and conclude via the isomorphism $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{G},\mathbf{R}i_*i^*\mathcal{F}) \cong \mathbf{R}i_*\mathbf{R}\mathcal{H}om^\bullet(i^*\mathcal{G},i^*\mathcal{F}) = 0$ [Sp, p. 147, Prop. 6.7]. □

(c) (Local Duality). Let A be a noetherian commutative ring (so that any finite sequence in A is proregular), let J be an A-ideal, let $\hat{A}$ be the J-adic completion, and let Γ_J be the functor of A-modules described by

$$\Gamma_J(M) := \{\, x \in M \mid J^n x = 0 \quad \text{for some } n > 0 \,\}.$$

The derived A-module category $\mathbf{D}(A)$ has the full subcategory $\mathbf{D}_{\mathrm{c}}(A)$ consisting of those complexes whose homology modules are finitely generated. Arguing as in Remark (b), one deduces from $(0.3)_{\mathrm{aff}}$ the *duality isomorphism*

$$\mathbf{R}\mathrm{Hom}^\bullet_A(\mathbf{R}\Gamma_J E,\mathbf{R}\Gamma_J F) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_A(E, F \otimes_A \hat{A}) \qquad \big(E \in \mathbf{D}(A),\ F \in \mathbf{D}_{\mathrm{c}}(A)\big). \tag{$(0.3)'_{\mathrm{aff,\,c}}$}$$

(This is of course closely related to $(0.3)'_{\mathrm{c}}$, see Remark (d). For example, when J is a maximal ideal and $Z := \{J\} \subset X := \mathrm{Spec}(A)$, just check out the germ of $(0.3)'_{\mathrm{c}}$ at the closed point $J \in X$.)

Now suppose that E and F are both in $\mathbf{D}_{\mathrm{c}}(A)$, and one of the following holds:

(1) $E \in \mathbf{D}_{\mathrm{c}}^{-}(A)$ and $F \in \mathbf{D}_{\mathrm{c}}^{+}(A)$;[5] or

(2) F has finite injective dimension (i.e., F is $\mathbf{D}$-isomorphic to a bounded injective complex); or

(3) E has finite projective dimension.

Then the natural map

$$\mathbf{R}\mathrm{Hom}^{\bullet}_A(E,F)\otimes_A \hat{A} \to \mathbf{R}\mathrm{Hom}^{\bullet}_A(E,F\otimes_A \hat{A})$$

is an isomorphism. To see this, reduce via "way-out" reasoning [H, p. 68] to where E is a bounded-above complex of finitely generated projectives and F is a single finitely generated A-module. Similarly, $\mathrm{Ext}^n_A(E,F) := H^n\big(\mathbf{R}\mathrm{Hom}^{\bullet}_A(E,F)\big)$ is finitely generated. Hence $(0.3)'_{\mathrm{aff,\,c}}$ yields homology isomorphisms

$$\mathrm{Ext}^n_A(\mathbf{R}\Gamma_J E, \mathbf{R}\Gamma_J F) \xrightarrow{\ \sim\ } \mathrm{Ext}^n_A(E,F)\hat{}\qquad (n\in\mathbb{Z}).$$

In particular, if m is a maximal ideal and $D \in \mathbf{D}_{\mathrm{c}}(A)$ is a *dualizing complex* (which has, by definition finite injective dimension), normalized so that $\mathbf{R}\Gamma_m D$ is an injective hull I_m of the A-module A/m [H, p. 284, Prop. 7.3], then there are hyperhomology duality isomorphisms, generalizing [H, p. 280, Cor. 6.5]:

$$\mathrm{Hom}_{\hat{A}}(\mathbb{H}^{-n}_m E, I_m) \xrightarrow{\ \sim\ } \mathrm{Ext}^n_A(E,D)\hat{}\qquad \big(n\in\mathbb{Z},\ E\in\mathbf{D}_{\mathrm{c}}(A)\big).$$

And since $\mathrm{Ext}^n_A(E,D)\hat{}$ is a noetherian $\hat{A}$-module therefore $\mathbb{H}^{-n}_m E$ is artinian, and Matlis dualization yields the Local Duality theorem of [H, p. 278]. (One checks that the isomorphisms derived here agree with those in [H].)

More generally, if J is any A-ideal and $\hat{}$ denotes J-adic completion then with $\kappa\colon \mathrm{Spf}(\hat{A}) = \widehat{X} \to X := \mathrm{Spec}(A)$ the canonical map, $U := X\setminus\{m\}$, and $\mathcal{E} := \widetilde{E}$, $\mathcal{D} := \widetilde{D}$ the quasi-coherent $\mathcal{O}_X$-complexes generated by E and D, there is a triangle

$$\mathrm{Hom}^{\bullet}_A(\mathbf{R}\Gamma_J E, I_m) \to \mathbf{R}\mathrm{Hom}^{\bullet}_A(E,D)\otimes_A\hat{A} \to \mathbf{R}\mathrm{Hom}^{\bullet}_{\widehat{U}}(\kappa^*\mathcal{E},\kappa^*\mathcal{D}) \xrightarrow{+}$$

whose exact homology sequence looks like

$$(0.4.3)\qquad \cdots \to \mathrm{Hom}_{\hat{A}}(\mathbb{H}^{-n}_J E, I_m) \to \mathrm{Ext}^n_A(E,D)\hat{} \to \mathrm{Ext}^n_{\widehat{U}}(\kappa^*\mathcal{E},\kappa^*\mathcal{D}) \to \dots$$

The particular case when A is Gorenstein of dimension d—so that $D\cong A[d]$—and $E = A$, is [PS, p. 107, Prop. (2.2)]. See §5.4 for details.

Incidentally, we have here a characterization of $D\otimes_A\hat{A}$ ($\hat{A} :=$ m-adic completion):

$$D\otimes_A\hat{A} \underset{(0.3)'_{\mathrm{aff,\,c}}}{\cong} \mathbf{R}\mathrm{Hom}^{\bullet}_A(\mathbf{R}\Gamma_m A, \mathbf{R}\Gamma_m D) = \mathbf{R}\mathrm{Hom}^{\bullet}_A(\mathbf{R}\Gamma_m A, I_m) \underset{(0.3)_{\mathrm{aff}}}{\cong} \mathbf{L}\Lambda_m I_m.$$

Thus if $E^{\bullet}$ is an injective resolution of A, so that $\mathrm{Hom}_A(E^{\bullet}, I_m)$ is a flat resolution of I_m [M, p. 95, Lemma 1.4], then $D\otimes_A\hat{A}\cong \mathrm{Hom}_A(E^{\bullet}, I_m)\hat{}$.

[5]For any derived category $\mathbf{D}_*$, $\mathbf{D}^+_*$ (resp. $\mathbf{D}^-_*$) is the full subcategory whose objects are the complexes $C\in\mathbf{D}_*$ having bounded-below (resp. bounded-above) homology, i.e., $H^n(C)=0$ for $n\ll 0$ (resp. $n\gg 0$). $\mathbf{D}^+_*$ (resp. $\mathbf{D}^-_*$) is isomorphic to the derived category of the homotopy category of such C. This notation differs from that in [H], where C itself is assumed bounded.

(d) Not surprisingly, but also not trivially, $(0.3)_{\mathrm{aff}}$ can be derived from (0.3)—and vice-versa when $\mathcal{E}\in\mathbf{D}_{\mathrm{qc}}(X)$—in brief as follows. The general case of (0.3) reduces to the case where $\mathcal{E}=\mathcal{O}_X$, see §2; thus once one knows the existence of $\mathbf{L}\Lambda_Z$ (§1), (0.3) is essentially equivalent to $(0.3)_{\mathrm{aff}}$.

The functor $\Gamma_X := \Gamma(X,-)$ $(X := \mathrm{Spec}(A))$ has an exact left adjoint, taking an A-module M to its associated quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$. Hence Γ_X preserves K-injectivity, and there is a functorial isomorphism

$$\mathbf{R}\mathrm{Hom}^{\bullet}_A(E,\mathbf{R}\Gamma_X\mathcal{G}) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_X(\widetilde{E},\mathcal{G}) \qquad \big(E\in\mathbf{D}(A),\ \mathcal{G}\in\mathbf{D}(X)\big).$$

Next, if $\mathcal{G}$ is any $\mathcal{O}_X$-complex of Γ_X-acyclics (i.e., $\Gamma_X\mathcal{G}^n\to\mathbf{R}\Gamma_X\mathcal{G}^n$ is an isomorphism for all n), then $\Gamma_X\mathcal{G}\to\mathbf{R}\Gamma_X\mathcal{G}$ is an isomorphism. (This is well-known when $\mathcal{G}$ is bounded below; and in the general case can be deduced from [BN, §5] or found explicitly in [L, (3.9.3.5)].) So for any A-complex F there are natural isomorphisms $F\xrightarrow{\ \sim\ }\Gamma_X\widetilde{F}\xrightarrow{\ \sim\ }\mathbf{R}\Gamma_X\widetilde{F}$, and hence

$$(0.4.4)\qquad \mathbf{R}\mathrm{Hom}^{\bullet}_A(E,F) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_X(\widetilde{E},\widetilde{F}) \qquad \big(E,F\in\mathbf{D}(A)\big).$$

There are also natural isomorphisms

$$(0.4.5)\qquad \mathbf{R}\varGamma_Z\widetilde{E} \xrightarrow{\ \sim\ } \widetilde{\mathbf{R}\Gamma_{\mathbf{t}}E}, \qquad \mathbf{L}\Lambda_{\mathbf{t}}F \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_X\mathbf{L}\Lambda_Z\widetilde{F}.$$

The first obtains via Koszul complexes, see (3.2.3). For the second, we may assume F flat and K-flat, in which case we are saying that $\Lambda_{\mathbf{t}}F=\Gamma_X\Lambda_Z\widetilde{F}\to\mathbf{R}\Gamma_X\Lambda_Z\widetilde{F}$ is an isomorphism, which as above reduces to where F is a single flat A-module, and then follows from [EGA, p. 68, (13.3.1)].

Thus there are natural isomorphisms

$$(\#)\qquad \begin{aligned} \mathbf{R}\mathrm{Hom}^{\bullet}_A(E,\mathbf{L}\Lambda_{\mathbf{t}}F) &\xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_A(E,\mathbf{R}\Gamma_X\mathbf{L}\Lambda_Z\widetilde{F}) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_X(\widetilde{E},\mathbf{L}\Lambda_Z\widetilde{F}),\\ \mathbf{R}\mathrm{Hom}^{\bullet}_A\big(\mathbf{R}\Gamma_{\mathbf{t}}E,F\big) &\xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_A\big(\widetilde{\mathbf{R}\Gamma_{\mathbf{t}}E},\widetilde{F}\big) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_X\big(\mathbf{R}\varGamma_Z\widetilde{E},\widetilde{F}\big). \end{aligned}$$

Hence (0.3) implies $(0.3)_{\mathrm{aff}}$.

Conversely, (0.3)(bis) (with $\mathcal{E}\in\mathbf{D}_{\mathrm{qc}}(X)$) follows from $(0.3)_{\mathrm{aff}}$. Indeed, it suffices to see that the maps λ' and γ' are made into isomorphisms by the functor $\mathbf{R}\Gamma_U$ for any affine open $U\subset X$. Moreover, we may assume that the complexes $\mathcal{E}$ and $\mathcal{F}$ are quasi-coherent (see §1). Then (#) provides what we need.

1. Left-derivability of the completion functor. Let X be a quasi-compact separated scheme and let $Z\subset X$ be a closed subscheme. We show in this section that *the completion functor* $\Lambda_Z\colon\mathcal{A}_{\mathrm{qc}}(X)\to\mathcal{A}(X)$ *of* (0.2) *has a left-derived functor* $\mathbf{L}\Lambda_Z\colon\mathbf{D}_{\mathrm{qc}}(X)\to\mathbf{D}(X)$.

Proposition (1.1). *On a quasi-compact separated scheme X, every $\mathcal{E}\in\mathbf{D}_{\mathrm{qc}}(X)$ is isomorphic to a quasi-coherent K-flat complex $\mathcal{P}_{\mathcal{E}}$.*

The *proof* will be given below, in (1.2).

If $\mathcal{P} \in \mathbf{D}(X)$ is a K-flat exact *quasi-coherent* complex, then $\Lambda_Z\mathcal{P}$ is exact. Indeed, all the complexes $\mathcal{P}_n := (\mathcal{O}_X/\mathcal{I}^n) \otimes \mathcal{P} \quad (n > 0)$ are exact [Sp, p. 140, Prop. 5.7], and hence the same is true after taking global sections over any affine open subset U of X. Also, the natural map of complexes $\Gamma(U, \mathcal{P}_{n+1}) \to \Gamma(U, \mathcal{P}_n)$ is surjective for every n. So by [EGA, p. 66, (13.2.3)], the complex

$$\Gamma(U, \Lambda_Z\mathcal{P}) = \varprojlim \Gamma(U, \mathcal{P}_n)$$

is exact, whence the assertion.

Consequently (see [H, p. 53], where condition 1 for the triangulated subcategory L whose objects are all the quasi-coherent K-flat complexes can be replaced by the weaker condition in our Proposition (1.1)), after choosing one $\mathcal{P}_\mathcal{E}$ for each $\mathcal{E}$ we have a left-derived functor $\mathbf{L}\Lambda_Z$ with $\mathbf{L}\Lambda_Z\mathcal{E} := \Lambda_Z(\mathcal{P}_\mathcal{E})$. For simplicity we take $\mathcal{P}_\mathcal{E} = \mathcal{E}$ whenever $\mathcal{E}$ itself is quasi-coherent and K-flat, so then $\mathbf{L}\Lambda_Z\mathcal{E} = \Lambda_Z\mathcal{E}$.

(1.2). Here is the *proof of Proposition* (1.1). It uses a simple-minded version of some simplicial techniques found e.g., in [Ki, §2]. We will recall as much as is needed.

Let $\mathcal{U} = (U_\alpha)_{1\le\alpha\le n}$ be an affine open cover of the quasi-compact separated scheme $(X, \mathcal{O}_X)$. Denote the set of subsets of $\{1, 2, \dots, n\}$ by $\mathfrak{P}_n$. For $i \in \mathfrak{P}_n$, set

$$U_i := \bigcap_{\alpha\in i} U_\alpha, \qquad \mathcal{O}_i := \mathcal{O}_{U_i} = \mathcal{O}_X|_{U_i}.$$

(In particular, $U_\phi = X$.) For $i \supset j$ in $\mathfrak{P}_n$, let $\lambda_{ij} \colon U_i \hookrightarrow U_j$ be the inclusion map. A $\mathcal{U}$-module is, by definition, a family $\mathcal{F} = (\mathcal{F}_i)_{i\in\mathfrak{P}_n}$ where $\mathcal{F}_i$ is an $\mathcal{O}_i$-module, together with a family of $\mathcal{O}_j$-homomorphisms

$$\varphi_{jk} \colon \lambda^*_{jk}\mathcal{F}_k \to \mathcal{F}_j \qquad (j \supset k)$$

such that φ_{jj} is the identity map of $\mathcal{F}_j$, and whenever $i \supset j \supset k$ we have $\varphi_{ik} = \varphi_{ij} \circ \big(\varphi_{jk}|_{U_i}\big)$, i.e., φ_{ik} factors as

$$\lambda^*_{ik}\mathcal{F}_k = \lambda^*_{ij}\lambda^*_{jk}\mathcal{F}_k \xrightarrow{\lambda^*_{ij}(\varphi_{jk})} \lambda^*_{ij}\mathcal{F}_j \xrightarrow{\varphi_{ij}} \mathcal{F}_i\,.$$

We say the $\mathcal{U}$-module $\mathcal{F}$ is *quasi-coherent* (resp. *flat*, resp. ...) if each one of the $\mathcal{O}_i$-modules $\mathcal{F}_i$ is such.

The $\mathcal{U}$-modules together with their morphisms (defined in the obvious manner) form an abelian category with $\varinjlim$ and $\varprojlim$. For example, given a direct system $(\mathcal{F}^\rho)_{\rho\in R}$ of $\mathcal{U}$-modules, set $\mathcal{F}_i := \varinjlim \mathcal{F}^\rho_i$ $(i \in \mathfrak{P}_n)$, define φ_{ij} $(i \supset j)$ to be the adjoint of the natural composed map

$$\mathcal{F}_j = \varinjlim \mathcal{F}^\rho_j \xrightarrow{\text{via } \psi^\rho_{ij}} \varinjlim \lambda_{ij*}\mathcal{F}^\rho_i \longrightarrow \lambda_{ij*}\mathcal{F}_i$$

where $\psi^\rho_{ij} \colon \mathcal{F}^\rho_j \to \lambda_{ij*}\mathcal{F}^\rho_i$ is adjoint to $\varphi^\rho_{ij} \colon \lambda^*_{ij}\mathcal{F}^\rho_j \to \mathcal{F}^\rho_i$; and check that $\mathcal{F} := (\mathcal{F}_i, \varphi_{ij}) = \varinjlim \mathcal{F}^\rho$ in the category of $\mathcal{U}$-modules.

Lemma(1.2.1). *Any quasi-coherent $\mathcal{U}$-module $\mathcal{F}$ is a homomorphic image of a flat quasi-coherent $\mathcal{U}$-module.*

Proof. For each i we can find an epimorphism of quasi-coherent $\mathcal{O}_i$-modules $\mathcal{Q}_i \twoheadrightarrow \mathcal{F}_i$ with $\mathcal{Q}_i$ flat. Set $\mathcal{P}_i := \oplus_{i\supset j}\,\lambda^*_{ij}\mathcal{Q}_j$. Map $\mathcal{P}_i$ surjectively to $\mathcal{F}_i$ via the family of composed maps

$$\lambda^*_{ij}\mathcal{Q}_j \longrightarrow \lambda^*_{ij}\mathcal{F}_j \xrightarrow{\varphi_{ij}} \mathcal{F}_i.$$

Let

$$\varphi'_{ki} \colon \lambda^*_{ki}\mathcal{P}_i = \oplus_{i\supset j}\,\lambda^*_{kj}\mathcal{Q}_j \longrightarrow \oplus_{k\supset j}\,\lambda^*_{kj}\mathcal{Q}_j = \mathcal{P}_k$$

be the natural map. Then $\mathcal{P} := (\mathcal{P}_i, \varphi'_{ij})$ is a flat $\mathcal{U}$-module, and the maps $\mathcal{P}_i \to \mathcal{F}_i$ constitute an epimorphism of $\mathcal{U}$-modules. □

The tensor product of two $\mathcal{U}$-modules is defined in the obvious way. A complex of $\mathcal{U}$-modules is *K-flat* if its tensor product with any exact complex is again exact.

Corollary(1.2.2). (Cf. [Ki, p. 303, Satz 2.2.]) *Any complex of quasi-coherent $\mathcal{U}$-modules is the target of a quasi-isomorphism from a K-flat complex of quasi-coherent $\mathcal{U}$-modules.*

Proof. (Sketch.) Any bounded-above complex of flat $\mathcal{U}$-modules is K-flat, so the assertion for bounded-above complexes follows from Lemma (1.2.1) (see [H, p. 42, 4.6, 1) (dualized)]). In the general case, express an arbitrary complex as the $\varinjlim$ of its truncations, and then use the $\varinjlim$ of a suitable direct system of K-flat resolutions of these truncations. (Clearly, $\varinjlim$ preserves K-flatness. For more details, see [Sp, p. 132, Lemma 3.3] or [L, (2.5.5)].) □

The *Čech functor* $\check{C}^\bullet$ from $\mathcal{U}$-complexes (i.e., complexes of $\mathcal{U}$-modules) to $\mathcal{O}_X$-complexes is defined as follows:

Let $|i|$ be the cardinality of $i \in \mathfrak{P}_n$, and let $\lambda_i := \lambda_{i\phi}$ be the inclusion map $U_i \hookrightarrow X$. For any $\mathcal{U}$-module $\mathcal{F}$, set

$$\begin{aligned} \check{C}^m(\mathcal{F}) &:= \bigoplus_{|i|=m+1} \lambda_{i*}\mathcal{F}_i \qquad && 0 \le m < n \\ &:= 0 && \text{otherwise.} \end{aligned}$$

Whenever j is obtained from $k = \{k_0 < k_1 < \cdots < k_m\} \in \mathfrak{P}_n$ by removing a single element, say k_a, we set $\epsilon_{kj} := (-1)^a$. The boundary map $\delta^m \colon \check{C}^m(\mathcal{F}) \to \check{C}^{m+1}(\mathcal{F})$ is specified by the family of maps

$$\delta^m_{kj} \colon \lambda_{j*}\mathcal{F}_j \to \lambda_{k*}\mathcal{F}_k$$

with δ^m_{kj} the natural composition

$$\lambda_{j*}\mathcal{F}_j \longrightarrow \lambda_{j*}\lambda_{kj*}\lambda^*_{kj}\mathcal{F}_j = \lambda_{k*}\lambda^*_{kj}\mathcal{F}_j \xrightarrow{\lambda_{k*}(\epsilon_{kj}\varphi_{kj})} \lambda_{k*}\mathcal{F}_k$$

if $j \subset k$, and $\delta^m_{kj} = 0$ otherwise. Then $\delta^{m+1} \circ \delta^m = 0$ for all m, and so we have a functor $\check{C}^\bullet$ from $\mathcal{U}$-modules to $\mathcal{O}_X$-complexes. For any $\mathcal{U}$-complex $\mathcal{F}^\bullet$, $\check{C}^\bullet(\mathcal{F}^\bullet)$ is defined to be the total complex associated to the double complex $\check{C}^p(\mathcal{F}^q)$.

Remarks. (a) If $\mathcal{G}$ is an $\mathcal{O}_X$-module and $\mathcal{G}'$ is the $\mathcal{U}$-module such that $\mathcal{G}'_i := \lambda^*_i\mathcal{G}$ and φ_{ij} is the identity map of $\mathcal{G}'_i = \lambda^*_{ij}\mathcal{G}_j$ for all $i \supset j$, then $\check{C}^\bullet(\mathcal{G}')$ is the usual Čech resolution of $\mathcal{G}$ [Go, p. 206, Thm. 5.2.1].

(b) Since all the maps λ_i are *affine* (X being separated) and *flat*, therefore $\check{C}^\bullet$ takes flat quasi-coherent $\mathcal{U}$-complexes to flat quasi-coherent $\mathcal{O}_X$-complexes. Moreover, $\check{C}^\bullet$ commutes with $\varinjlim$. (We need this only for quasi-coherent complexes, for which the proof is straightforward; but it also holds for arbitrary complexes, [Ke, §2].)

Lemma (1.2.3). *The functor $\check{C}^\bullet$ takes quasi-isomorphisms between* quasi-coherent *complexes to quasi-isomorphisms.*

Proof. One checks that $\check{C}^\bullet$ commutes with degree-shifting: $\check{C}^\bullet(\mathcal{F}^\bullet[1]) = \check{C}^\bullet(\mathcal{F}^\bullet)[1]$; and that $\check{C}^\bullet$ preserves mapping cones. Since quasi-isomorphisms are just those maps whose cones are exact, it suffices to show that $\check{C}^\bullet$ takes exact quasi-coherent $\mathcal{U}$-complexes $\mathcal{F}^\bullet$ to exact $\mathcal{O}_X$-complexes. But since the maps λ_i are affine, each row $\check{C}^p(\mathcal{F}^\bullet)$ of the double complex $\check{C}^p(\mathcal{F}^q)$ is exact, and all but finitely many rows vanish, whence the conclusion. □

Now by [BN, p. 230, Corollary 5.5], any $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$ is isomorphic to a quasi-coherent complex; so to prove (1.1) we may as well assume that $\mathcal{E}$ itself is quasi-coherent. Define the $\mathcal{U}$-complex $\mathcal{E}'$ as in remark (a) and let $\mathcal{P} \to \mathcal{E}'$ be a quasi-isomorphism of quasi-coherent $\mathcal{U}$-complexes with $\mathcal{P}$ a $\varinjlim$ of bounded-above flat complexes, see proof of Corollary (1.2.2). Lemma (1.2.3) provides a quasi-isomorphism $\mathcal{P}_{\mathcal{E}} := \check{C}^\bullet(\mathcal{P}) \to \check{C}^\bullet(\mathcal{E}')$; and there is a natural quasi-isomorphism $\mathcal{E} \to \check{C}^\bullet(\mathcal{E}')$ (remark (a)), so that $\mathcal{E}$ is isomorphic in $\mathbf{D}(X)$ to $\mathcal{P}_{\mathcal{E}}$. Moreover, $\mathcal{P}_{\mathcal{E}}$ is a $\varinjlim$ of bounded-above quasi-coherent flat $\mathcal{O}_X$-complexes (remark (b)), and hence is quasi-coherent and K-flat. This proves Proposition (1.1). □

For completeness, and for later use, we present a slightly more elaborate version of the just-quoted Corollary 5.5 in [BN, p. 230]. Recall from Remark (0.4)(a) the definition of *quasi-coherator.*

Proposition (1.3). *Let X be a quasi-compact separated scheme. Then the natural functor*

$$\boldsymbol{j}_X\colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X)) \to \mathbf{D}_{\mathrm{qc}}(X)$$

is an equivalence of categories, having as quasi-inverse the derived quasi-coherator $\mathbf{R}Q_X$.

Corollary (1.3.1). *In the category $\mathsf{C}_{\mathrm{qc}}(X)$ of quasi-coherent $\mathcal{O}_X$-complexes, every object has a K-injective resolution.*

Proof. The Proposition asserts that the natural maps $\mathcal{E} \to \mathbf{R}Q_X\boldsymbol{j}_X\mathcal{E}$ $\big(\mathcal{E} \in \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))\big)$ and $\boldsymbol{j}_X\mathbf{R}Q_X\mathcal{F} \to \mathcal{F}$ $\big(\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)\big)$ are isomorphisms. The Corollary results: since Q_X has an exact left adjoint therefore Q_X takes K-injective $\mathcal{O}_X$-complexes to complexes which are K-injective in $\mathsf{C}_{\mathrm{qc}}(X)$, so if $\mathcal{E} \xrightarrow{\sim} \mathbf{R}Q_X\boldsymbol{j}_X\mathcal{E}$ and if $\mathcal{E} \to I_{\mathcal{E}}$ is a quasi-isomorphism with $I_{\mathcal{E}}$ a K-injective $\mathcal{O}_X$-complex [Sp, p. 134, 3.13], then the resulting map $\mathcal{E} \to Q_X I_{\mathcal{E}}$ is still a quasi-isomorphism, and thus $\mathcal{E}$ has a K-injective resolution in $\mathsf{C}_{\mathrm{qc}}(X)$.

We will show that the functor $\mathbf{R}Q_X|_{\mathbf{D}_{\mathrm{qc}}(X)}$ is *bounded-above,* i.e., there is an integer $d \ge 0$ such that for any $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$ and $q \in \mathbb{Z}$, if $H^p(\mathcal{F}) = 0$ for all $p \ge q$ then $H^p(\mathbf{R}Q_X\mathcal{F}) = 0$ for all $p \ge q + d$. Then by the way-out Lemma [H, p. 68] it suffices to prove the above isomorphism assertions when $\mathcal{E}$ and $\mathcal{F}$ are single quasi-coherent sheaves, and this case is dealt with in [I, p. 189, Prop. 3.5]. (It follows then from $\boldsymbol{j}_X\mathbf{R}Q_X\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ that we can take $d = 0$.)

We proceed by induction on $n(X)$, the least among all natural numbers n such that X can be covered by n affine open subschemes. If $n(X) = 1$, i.e., X is affine, then for any $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$, $\mathbf{R}Q_X(\mathcal{F})$ is the sheafification of the complex $\mathbf{R}\Gamma_X(\mathcal{F}) := \mathbf{R}\Gamma(X, \mathcal{F})$; so to show boundedness we can replace $\mathbf{R}Q_X$ by $\mathbf{R}\Gamma_X$. For a K-injective resolution I of $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$, use a "special" inverse limit of injective resolutions I_q of the truncations $\tau_{\ge -q}(F)$, as in [Sp, p. 134, 3.13]. If C_q is the kernel of the split surjection $I_q \to I_{q-1}$, then $C_q[-q]$ is an injective resolution of the quasi-coherent $\mathcal{O}_X$-module $H^{-q}(\mathcal{F})$, and hence $H^p\Gamma_X(C_q) = 0$ for $p > -q$. Applying [Sp, p.126, Lemma], one finds then that for $p \ge -q$ the natural map $H^p\Gamma_X(I) \to H^p\Gamma_X(I_q)$ is an isomorphism; and so if $\tau_{\ge -q}(\mathcal{F}) = 0$, then $H^p\Gamma_X(I) = 0$. Thus $\mathbf{R}\Gamma_X|_{\mathbf{D}_{\mathrm{qc}}(X)}$ is indeed bounded above (with $d = 0$).

Now suppose that $n := n(X) > 1$, and let $X = X_1 \cup \dots \cup X_n$ be an affine open cover. Set $U := X_1$, $V := X_2 \cup \dots \cup X_n$, $W := U \cap V$, and let $u\colon U \hookrightarrow X$, $v\colon V \hookrightarrow X$, $w\colon W \hookrightarrow X$ be the inclusions. Note that $n(U) = 1$, $n(V) = n-1$, and $n(W) \le n-1$ (X separated $\Rightarrow X_1 \cap X_i$ affine).

By the inductive hypothesis, $\mathcal{E} \xrightarrow{\sim} \mathbf{R}Q_V\boldsymbol{j}_V\mathcal{E}$ for any $\mathcal{E} \in \mathsf{C}_{\mathrm{qc}}(V)$. Hence, as above, $\mathcal{E}$ has a K-injective resolution in $\mathsf{C}_{\mathrm{qc}}(V)$, so the functor $v_*^{\mathrm{qc}}\colon \mathcal{A}_{\mathrm{qc}}(V) \to \mathcal{A}_{\mathrm{qc}}(X)$ ($:=$ restriction of v_*) has a right-derived functor $\mathbf{R}v_*^{\mathrm{qc}}$, and there is a functorial isomorphism $\mathbf{R}(v_*^{\mathrm{qc}}Q_V) \xrightarrow{\sim} \mathbf{R}v_*^{\mathrm{qc}}\mathbf{R}Q_V$. Since the left adjoint v^* of v_* is exact, therefore v_* preserves K-injectivity of complexes, and so there is a functorial isomorphism $\mathbf{R}(Q_Xv_*) \xrightarrow{\sim} \mathbf{R}Q_X\mathbf{R}v_*$; and furthermore it is easily seen, via adjointness of v^* and v_*, that $Q_Xv_* = v_*^{\mathrm{qc}}Q_V$. Thus we have a functorial isomorphism

$$\mathbf{R}Q_X\mathbf{R}v_* \xrightarrow{\sim} \mathbf{R}(Q_Xv_*) = \mathbf{R}(v_*^{\mathrm{qc}}Q_V) \xrightarrow{\sim} \mathbf{R}v_*^{\mathrm{qc}}\mathbf{R}Q_V.$$

Similar remarks apply to u and w.

Now we can apply $\mathbf{R}Q_X$ to the Mayer-Vietoris triangle

$$\mathcal{F} \to \mathbf{R}u_*u^*\mathcal{F} \oplus \mathbf{R}v_*v^*\mathcal{F} \to \mathbf{R}w_*w^*\mathcal{F} \to \mathcal{F}[1]$$

to get the triangle

$$\mathbf{R}Q_X\mathcal{F} \to \mathbf{R}u_*^{\mathrm{qc}}\mathbf{R}Q_Uu^*\mathcal{F} \oplus \mathbf{R}v_*^{\mathrm{qc}}\mathbf{R}Q_Vv^*\mathcal{F} \to \mathbf{R}w_*^{\mathrm{qc}}\mathbf{R}Q_Ww^*\mathcal{F} \to \mathbf{R}Q_X\mathcal{F}[1].$$

So it's enough to show: *if V is any quasi-compact open subset of X with $n(V) < n(X)$, and $v\colon V \hookrightarrow X$ is the inclusion, then the functor $\mathbf{R}v_*^{\mathrm{qc}}$ is bounded above.* (This derived functor exists, as before, by the induction hypothesis.)

We induct on $n(V)$, the case $n(V)=1$ being trivial, since then the map v is affine and the functor $v_*^{\mathrm{qc}}\colon \mathcal{A}_{\mathrm{qc}}(V)\to\mathcal{A}_{\mathrm{qc}}(X)$ is exact. Suppose then that $n:=n(V)>1$. V has an open cover $V=V_1\cup V_2$ with $n(V_1)=1$, $n(V_2)=n-1$, and $n(V_1\cap V_2)\le n-1$. Let $i_1\colon V_1\hookrightarrow V$, $i_2\colon V_2\hookrightarrow V$, and $i_{12}\colon V_{12}=V_1\cap V_2\hookrightarrow V$ be the inclusions. Since $n(V_s)<n(X)$ ($s=1,2$, or 12), we may assume that $\boldsymbol{j}_{V_s}\colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(V_s))\to\mathbf{D}_{\mathrm{qc}}(V_s)$ is an equivalence of categories with quasi-inverse $\mathbf{R}Q_{V_s}$, so that we have isomorphisms

$$\mathbf{R}Q_V\mathbf{R}i_{s*}i_s^*\boldsymbol{j}_V\mathcal{E}\cong\mathbf{R}i_{s*}^{\mathrm{qc}}\mathbf{R}Q_{V_s}\boldsymbol{j}_{V_s}i_s^*\mathcal{E}\cong\mathbf{R}i_{s*}^{\mathrm{qc}}i_s^*\mathcal{E}\qquad\big(\mathcal{E}\in\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(V)\big).$$

Similarly, $\mathbf{R}Q_V\boldsymbol{j}_V\mathcal{E}\cong\mathcal{E}$. Hence application of $\mathbf{R}Q_V$ to the Mayer-Vietoris triangle

$$\boldsymbol{j}_V\mathcal{E}\to\mathbf{R}i_{1*}i_1^*\boldsymbol{j}_V\mathcal{E}\oplus\mathbf{R}i_{2*}i_2^*\boldsymbol{j}_V\mathcal{E}\to\mathbf{R}i_{12*}i_{12}^*\boldsymbol{j}_V\mathcal{E}\to\boldsymbol{j}_V\mathcal{E}[1]$$

gives rise to a triangle

$$\mathcal{E}\to\mathbf{R}i_{1*}^{\mathrm{qc}}i_1^*\mathcal{E}\oplus\mathbf{R}i_{2*}^{\mathrm{qc}}i_2^*\mathcal{E}\to\mathbf{R}i_{12*}^{\mathrm{qc}}i_{12}^*\mathcal{E}\to\mathcal{E}[1].$$

Since i_{s*}^{qc} has an exact left adjoint i_s^*, therefore i_{s*}^{qc} preserves K-injectivity, and consequently $\mathbf{R}v_*^{\mathrm{qc}}\mathbf{R}i_{s*}^{\mathrm{qc}}=\mathbf{R}(vi_s)_*^{\mathrm{qc}}$. So we can apply $\mathbf{R}v_*^{\mathrm{qc}}$ to the preceding triangle and use the induction hypothesis to see that $\mathbf{R}v_*^{\mathrm{qc}}\mathcal{E}$ is one vertex of a triangle whose other two vertices are obtained by applying bounded-above functors to $\mathcal{E}$, whence the conclusion. □

2. Proof of Theorem (0.3)—outline. We first define bifunctorial maps

$$\text{(2.1)}\qquad \begin{aligned}\psi\colon \mathcal{E}\underline{\underline{\otimes}}\mathbf{R}\varGamma_Z\mathcal{F}&\to\mathbf{R}\varGamma_Z(\mathcal{E}\underline{\underline{\otimes}}\mathcal{F})\\ \psi'\colon \mathcal{E}\underline{\underline{\otimes}}\mathbf{R}\varGamma'_Z\mathcal{F}&\to\mathbf{R}\varGamma'_Z(\mathcal{E}\underline{\underline{\otimes}}\mathcal{F})\end{aligned}\qquad\big(\mathcal{E},\mathcal{F}\in\mathbf{D}(X)\big)$$

(where $\underline{\underline{\otimes}}$ denotes derived tensor product.) To do so, we may assume that $\mathcal{E}$ is K-flat and $\mathcal{F}$ is K-injective, and choose a quasi-isomorphism $\mathcal{E}\otimes\mathcal{F}\to\mathcal{J}$ with $\mathcal{J}$ K-injective. The obvious composed map of complexes $\mathcal{E}\otimes\varGamma_Z\mathcal{F}\to\mathcal{E}\otimes\mathcal{F}\to\mathcal{J}$ has image in $\varGamma_Z\mathcal{J}$, and so we can define ψ to be the resulting composition in $\mathbf{D}(X)$:

$$\mathcal{E}\underline{\underline{\otimes}}\mathbf{R}\varGamma_Z\mathcal{F}\cong\mathcal{E}\otimes\varGamma_Z\mathcal{F}\to\varGamma_Z\mathcal{J}\cong\mathbf{R}\varGamma_Z(\mathcal{E}\underline{\underline{\otimes}}\mathcal{F}).$$

The map ψ' is defined similarly, *mutatis mutandis.*

Under the hypotheses of Theorem (0.3), assertion (i) in Cor. (3.2.5) (resp. (3.1.4)) gives that *ψ is an isomorphism if $\mathcal{E}$ and $\mathcal{F}$ are both in $\mathbf{D}_{\mathrm{qc}}(X)$* (resp. *$\psi'$ is an isomorphism for all $\mathcal{E},\mathcal{F}$*).[6]

In view of the canonical isomorphism $\mathbf{R}\varGamma'_Z\mathcal{O}_X\xrightarrow{\sim}\mathbf{R}\varGamma_Z\mathcal{O}_X$ (Cor. (3.2.4)) and of [Sp, p. 147, Prop. 6.6], we have then natural isomorphisms

$$\begin{aligned}\mathbf{R}\mathcal{H}om^\bullet\big(\mathbf{R}\varGamma'_Z\mathcal{E},\mathcal{F}\big)&\xrightarrow{\sim}\mathbf{R}\mathcal{H}om^\bullet\big(\mathcal{E}\underline{\underline{\otimes}}\mathbf{R}\varGamma_Z\mathcal{O}_X,\mathcal{F}\big)\\&\xrightarrow{\sim}\mathbf{R}\mathcal{H}om^\bullet\big(\mathcal{E},\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X,\mathcal{F})\big).\end{aligned}$$

It remains to find a natural isomorphism

$$\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X,\mathcal{F})\xrightarrow{\sim}\mathbf{L}\Lambda_Z\mathcal{F}\qquad\big(\mathcal{F}\in\mathbf{D}_{\mathrm{qc}}(X)\big).$$

[6] The ring-theoretic avatar of this result is closely related to results of Matlis [M, p. 114, Thm. 3.7], [M2, p. 83, Thm. 10], and Strebel [St, p. 94, 5.8].

To get this we define below a natural map $\Phi\colon \mathbf{L}\Lambda_Z\mathcal{F} \to \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathcal{F})$, and after reducing to where X is affine and $\mathcal{F}$ is a single flat quasi-coherent $\mathcal{O}_X$-module, prove in §4 that Φ is an isomorphism by constructing Φ^{-1} via the representability of $\mathbf{R}\varGamma_Z\mathcal{O}_X$ as a limit of Koszul complexes.[7]

Assuming X to be quasi-compact and separated, so that $\mathbf{L}\Lambda_Z$ exists, let us then define Φ. Let $\mathcal{I}$ be a finite-type quasi-coherent $\mathcal{O}_X$-ideal such that $Z = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ (see Introduction). For any $\mathcal{O}_X$-complexes $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, the natural map

$$\big(\mathcal{P}\otimes\mathcal{Q}\big)\otimes\big(\mathcal{H}om^\bullet(\mathcal{Q},\mathcal{R})\big) \cong \mathcal{P}\otimes\big(\mathcal{Q}\otimes\mathcal{H}om^\bullet(\mathcal{Q},\mathcal{R})\big) \to \mathcal{P}\otimes\mathcal{R}$$

induces (via $\otimes$–$\mathcal{H}om$ adjunction) a functorial map

$$\mathcal{P}\otimes\mathcal{Q} \to \mathcal{H}om^\bullet\big(\mathcal{H}om^\bullet(\mathcal{Q},\mathcal{R}), \mathcal{P}\otimes\mathcal{R}\big).$$

Letting $\mathcal{Q}$ run through the inverse system $\mathcal{O}_X/\mathcal{I}^n$ $(n>0)$ one gets a natural map

$$\begin{aligned}\Lambda_Z\mathcal{P} = \varprojlim\,(\mathcal{P}\otimes\mathcal{O}_X/\mathcal{I}^n) &\to \varprojlim\, \mathcal{H}om^\bullet\big(\mathcal{H}om^\bullet(\mathcal{O}_X/\mathcal{I}^n,\mathcal{R}), \mathcal{P}\otimes\mathcal{R}\big)\\ &\cong \mathcal{H}om^\bullet\big(\varinjlim\, \mathcal{H}om^\bullet(\mathcal{O}_X/\mathcal{I}^n,\mathcal{R}), \mathcal{P}\otimes\mathcal{R}\big)\\ &\cong \mathcal{H}om^\bullet\big(\varGamma'_Z\mathcal{R}, \mathcal{P}\otimes\mathcal{R}\big).\end{aligned}$$

For $\mathcal{F}\in\mathbf{D}_{\mathrm{qc}}(X)$, $\mathcal{G}\in\mathbf{D}(X)$, taking $\mathcal{P}$ to be $\mathcal{P}_\mathcal{F}$ (Proposition (1.1)) and $\mathcal{R}$ to be a K-injective resolution of $\mathcal{G}$ one gets a composed derived-category map

$$\begin{aligned}\Phi(\mathcal{F},\mathcal{G})\colon \mathbf{L}\Lambda_Z\mathcal{F} \cong \Lambda_Z\mathcal{P} &\to \mathcal{H}om^\bullet\big(\varGamma'_Z\mathcal{R}, \mathcal{P}\otimes\mathcal{R}\big)\\ &\to \mathbf{R}\mathcal{H}om^\bullet\big(\varGamma'_Z\mathcal{R}, \mathcal{P}\otimes\mathcal{R}\big)\\ &\cong\ \mathbf{R}\mathcal{H}om^\bullet\big(\mathbf{R}\varGamma'_Z\mathcal{G}, \mathcal{F}\underset{=}{\otimes}\mathcal{G}\big),\end{aligned} \tag{2.2}$$

which one checks to be independent of the choice of $\mathcal{P}$ and $\mathcal{R}$.

As indicated above we want to show that $\Phi(\mathcal{F},\mathcal{O}_X)$ is an isomorphism. The question is readily seen to be local on X,[8] so we may assume X to be affine. The idea is then to apply way-out reasoning [H, p. 69, (iii)] to reduce to where $\mathcal{F}$ is a single *flat* quasi-coherent $\mathcal{O}_X$-module, which case is disposed of in Corollary (4.2).

But to use *loc. cit.,* we need the functors $\mathcal{H}_Z := \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, -)$ and $\mathbf{L}\Lambda_Z$ from $\mathbf{D}_{\mathrm{qc}}(X)$ to $\mathbf{D}(X)$ to be bounded above (= "way-out left") and also bounded below (= "way-out right"). Boundedness of $\mathcal{H}_Z$ is shown in Lemma (4.3). That $\mathbf{L}\Lambda_Z(-)$ is bounded above is clear, since X is now affine and so if $\mathcal{E}\in\mathbf{D}_{\mathrm{qc}}(X)$ is such that $H^i(\mathcal{E})=0$ for all $i>i_0$ then there is a flat $\mathcal{P}_\mathcal{E}$ as in (1.1) vanishing in all degrees $>i_0$. Now by [H, p. 69, (ii), (iv)] (dualized), the case where $\mathcal{F}$ is a flat quasi-coherent $\mathcal{O}_X$-module (Cor. (4.2)) implies that $\Phi(\mathcal{F},\mathcal{O}_X)\colon \mathcal{H}_Z\mathcal{F}\to\mathbf{L}\Lambda_Z\mathcal{F}$ is an isomorphism for *all* $\mathcal{F}\in\mathbf{D}^-_{\mathrm{qc}}(X)$. That, and $\mathcal{H}_Z$ being bounded below, lets us conclude, via [H, p. 68, Example 1] (dualized, with P the class of quasi-coherent flat $\mathcal{O}_X$-modules), that $\mathbf{L}\Lambda_Z$ is bounded below. (See also [GM, p. 445, Thm. 1.9, (iv)].)

[7](A short proof for the case $\mathcal{F}=\mathcal{O}_X$ over smooth algebraic $\mathbb{C}$-varieties is given in [Me, p. 97]. Cf. also [H2, §4].) The ring-theoretic avatar of the isomorphism Φ underlies the duality theorem of Strebel [St, p. 94, 5.9] and Matlis [M2, p. 89, Thm. 20], and the more general results of Greenlees and May [GM, p. 449, Prop. 3.1 and p. 447, Thm. 2.5].

[8]Using the exact functor "extension by zero," one shows that restriction to an open $U\subset X$ takes any K-injective (resp. K-flat) $\mathcal{O}_X$-complex to a K-injective (resp. K-flat) $\mathcal{O}_U$-complex.

For the last assertion of Theorem (0.3), it suffices to verify the commutativity of the following diagram, where $\mathcal{E}$ may be taken to be K-flat, and as above, $\mathcal{P} = \mathcal{P}_{\mathcal{F}}$. This verification is straightforward (though not entirely effortless) and so will be left to the reader.

$$\begin{array}{ccc}
\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E},\mathcal{P}) & \xrightarrow{\text{via }\lambda} & \mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E},\Lambda_Z\mathcal{P}) \\
\Vert & & \big\downarrow \text{via } \Phi(\mathcal{P},\mathcal{O}_X) \\
\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E},\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{O}_X,\mathcal{P})) & \longrightarrow & \mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E},\mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_Z'\mathcal{O}_X,\mathcal{P})) \\
\Vert & & \big\downarrow \simeq \\
\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}\otimes\mathcal{O}_X,\mathcal{P}) & \longrightarrow & \mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}\otimes\mathbf{R}\varGamma_Z'\mathcal{O}_X,\mathcal{P}) \\
\simeq\big\downarrow & & (2.1)\big\downarrow \simeq \\
\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E},\mathcal{F}) & \xrightarrow[\text{via }\gamma]{} & \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_Z'\mathcal{E},\mathcal{F})
\end{array}$$

This completes the outline of the proof of Theorem (0.3).

3. Proregular embeddings. In this section we explore the basic condition of *proregularity,* as defined in (3.0.1). This definition, taken from [GM, p. 445], seems unmotivated at first sight; but as mentioned in the Introduction, it is precisely what is needed to make local cohomology on quite general schemes behave as it does on noetherian schemes (where *every* closed subscheme is proregularly embedded), for example with respect to Koszul complexes. What this amounts to basically is an elaboration of [Gr, Exposé II] in the language of derived categories of sheaves.[9] We work throughout with *unbounded* complexes, which sometimes introduces technical complications, but which will ultimately be quite beneficial in situations involving combinations of right- and left-derived functors.

Rather than explain further, we simply suggest a perusal of the salient results—Lemma (3.1.1) (especially (1) ⇔ (2)), (3.1.3)–(3.1.7), (3.2.3)–(3.2.7). For completeness we have included several results which are not used elsewhere in this paper. Some readers may prefer going directly to §4, referring back to §3 as needed.

Definition (3.0.1). Let X be a topological space and $\mathcal{O}$ a sheaf of commutative rings on X. A sequence $\mathbf{t} := (t_1, t_2, \dots, t_\mu)$ in $\Gamma(X,\mathcal{O})$ is *proregular* if for each $i = 1, 2, \dots, \mu$ and each $r > 0$ there exists an $s > r$ such that in $\mathcal{O}$,

$$(t_1^s,\dots,t_{i-1}^s)\mathcal{O} : t_i^s \subset (t_1^r,\dots,t_{i-1}^r)\mathcal{O} : t_i^{s-r}.$$

A closed subspace $Z \subset X$ is *proregularly embedded in* X if there exists an open covering $(X_\alpha)_{\alpha\in A}$ of X and for each α a proregular sequence $\mathbf{t}_\alpha$ in $\Gamma(X_\alpha,\mathcal{O}_\alpha)$ (where $\mathcal{O}_\alpha := \mathcal{O}|_{X_\alpha}$) such that $Z \cap X_\alpha$ is the support of $\mathcal{O}_\alpha/\mathbf{t}_\alpha\mathcal{O}_\alpha$.

[9] More generally, to do the same for [*ibid.,* Exposé VI], replace $\mathcal{O}$ in what follows by an $\mathcal{O}$-module $\mathcal{M}$, $\mathcal{P}$ by $\mathcal{M}\otimes\mathcal{P}$ ($\mathcal{P}$ flat), $\mathcal{J}$ by $\mathcal{H}om(\mathcal{M},\mathcal{J})$ ($\mathcal{J}$ injective), and the functor $\varGamma_{\mathbf{t}}'(-)$ by $\mathcal{H}om_{\mathbf{t}}(\mathcal{M},-) := \varinjlim \mathcal{H}om(\mathcal{M}/\mathbf{t}^n\mathcal{M},-)$ ...

Examples. (a) Suppose that X is *quasi-compact* (not necessarily Hausdorff, but every open cover has a finite subcover), and that the $\mathcal{O}$-module $\mathcal{O}$ is *coherent.* Then $\mathbf{t}$ is proregular if (and clearly only if) for each i, r as above and each $x \in X$, there exists an $s = s(x) > r$ such that in the stalk $\mathcal{O}_x$,

$$(t_1^s, \dots, t_{i-1}^s)\mathcal{O}_x : t_i^s \subset (t_1^r, \dots, t_{i-1}^r)\mathcal{O}_x : t_i^{s-r}. \tag{3.0.2}$$

Indeed, the ideal sheaves appearing in (3.0.1) are all coherent, and so we can take $s(y) = s(x)$ for all points y in some neighborhood W_x of x. If (3.0.2) holds for s then it holds for all $s' > s$; and since X can be covered by finitely many of the W_x, the condition in (3.0.1) is satisfied.

Note that (3.0.2) holds whenever the ring $\mathcal{O}_x$ is *noetherian,* since then

$$(t_1^r, \dots, t_{i-1}^r)\mathcal{O}_x : t_i^{s-r} = (t_1^r, \dots, t_{i-1}^r)\mathcal{O}_x : t_i^s \quad \text{for } s \gg r.$$

Thus if X is quasi-compact, $\mathcal{O}$ is coherent, and all the stalks $\mathcal{O}_x$ are noetherian, then every sequence $\mathbf{t}$ is proregular.

(b) If (3.0.2) holds, then it also holds when $\mathcal{O}_x$ is replaced by any *flat* $\mathcal{O}_x$-algebra. It follows, for example, that if R is a ring of fractions of a polynomial ring (with any number of indeterminates) over a noetherian ring, then every sequence $\mathbf{t}$ in $R = \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is proregular; and every closed subscheme $Z \subset \mathrm{Spec}(R)$ such that $\mathrm{Spec}(R) \setminus Z$ is quasi-compact is proregularly embedded.

(c) For an example by Verdier of a non-proregular sequence, and the resulting homological pathologies, see [I, pp. 195–198].

(3.1). Let $(X, \mathcal{O})$ be as in Definition (3.0.1). Denote the category of $\mathcal{O}$-modules by $\mathcal{A}$, and let $\mathbf{D}$ be the derived category of $\mathcal{A}$. Fix a sequence $\mathbf{t} = (t_1, \dots, t_\mu)$ in $\Gamma(X, \mathcal{O})$, and set

$$\mathbf{t}^n := (t_1^n, \dots, t_\mu^n) \qquad (n > 0).$$

Define the functor $\varGamma'_{\mathbf{t}} : \mathcal{A} \to \mathcal{A}$ by

$$\varGamma'_{\mathbf{t}}\mathcal{G} := \varinjlim_{n>0} \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathbf{t}^n\mathcal{O}, \mathcal{G}) \qquad (\mathcal{G} \in \mathcal{A}).$$

The stalk of $\varGamma'_{\mathbf{t}}\mathcal{G}$ at any point $x \in X$ is

$$(\varGamma'_{\mathbf{t}}\mathcal{G})_x = \varinjlim_{n>0} \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{O}_x/\mathbf{t}^n\mathcal{O}_x, \mathcal{G}_x) \qquad (x \in X).$$

The (homological) derived functors of $\varGamma'_{\mathbf{t}}$ are

$$H^i\mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{G} = \varinjlim_{n>0} \mathcal{E}xt^i_{\mathcal{O}}(\mathcal{O}/\mathbf{t}^n\mathcal{O}, \mathcal{G}) \qquad (i \ge 0,\ \mathcal{G} \in \mathcal{A}).$$

If $\mathbf{s}$ is another finite sequence in $\Gamma(X, \mathcal{O})$ such that $\sqrt{\mathbf{s}\mathcal{O}} = \sqrt{\mathbf{t}\mathcal{O}}$ then $\varGamma'_{\mathbf{s}} = \varGamma'_{\mathbf{t}}$. If $(X, \mathcal{O})$ is a scheme and $Z := \mathrm{Supp}(\mathcal{O}/\mathbf{t}\mathcal{O})$ then $\varGamma'_{\mathbf{t}} = \varGamma'_Z$, see (0.1).

For $t \in \Gamma(X, \mathcal{O})$, let $\mathcal{K}^\bullet(t)$ be the complex $\cdots \to 0 \to \mathcal{O} \xrightarrow{t} \mathcal{O} \to 0 \to \cdots$ which in degrees 0 and 1 is multiplication by t from $\mathcal{O} =: \mathcal{K}^0(t)$ to $\mathcal{O} =: \mathcal{K}^1(t)$, and which vanishes elsewhere. For $0 \le r \le s$, there is a map of complexes $\mathcal{K}^\bullet(t^r) \to \mathcal{K}^\bullet(t^s)$ which is the identity in degree 0 and multiplication by t^{s-r} in degree 1; and so we get a direct system of complexes, whose $\varinjlim$ we denote by $\mathcal{K}^\bullet_\infty(t)$. The stalk of $\mathcal{K}^\bullet_\infty(t)$ at $x \in X$ looks in degrees 0 and 1 like the localization map $\mathcal{O}_x \to (\mathcal{O}_x)_t = \mathcal{O}_x[1/t]$.

With $\otimes = \otimes_{\mathcal{O}}$, set

$$\mathcal{K}^\bullet(\mathbf{t}) := \mathcal{K}^\bullet(t_1) \otimes \cdots \otimes \mathcal{K}^\bullet(t_\mu),$$
$$\mathcal{K}^\bullet_\infty(\mathbf{t}) := \varinjlim_{n>0} \mathcal{K}^\bullet(\mathbf{t}^n) = \mathcal{K}^\bullet_\infty(t_1) \otimes \cdots \otimes \mathcal{K}^\bullet_\infty(t_\mu);$$

and for any complex $\mathcal{F}$ of $\mathcal{O}$-modules set

$$\mathcal{K}^\bullet(\mathbf{t}, \mathcal{F}) := \mathcal{K}^\bullet(\mathbf{t}) \otimes \mathcal{F}, \qquad \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}) := \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{F}.$$

Since the complex $\mathcal{K}^\bullet_\infty(\mathbf{t})$ is flat and bounded, the functor of complexes $\mathcal{K}^\bullet_\infty(\mathbf{t}, -)$ takes quasi-isomorphisms to quasi-isomorphisms [H, p. 93, Lemma 4.1, b2], and so may be regarded as a functor from $\mathbf{D}$ to $\mathbf{D}$.

After choosing a quasi-isomorphism φ from $\mathcal{F}$ to a K-injective $\mathcal{O}$-complex $\mathcal{L}^\bullet$ [Sp, p. 138, Thm. 4.5], we can use the natural identifications

$$\varGamma'_{\mathbf{t}}\mathcal{L}^j = \varinjlim \ker\big(\mathcal{K}^0(\mathbf{t}^n, \mathcal{L}^j) \to \mathcal{K}^1(\mathbf{t}^n, \mathcal{L}^j)\big) = \ker\big(\mathcal{K}^0_\infty(\mathbf{t}, \mathcal{L}^j) \to \mathcal{K}^1_\infty(\mathbf{t}, \mathcal{L}^j)\big) \quad (j \in \mathbb{Z})$$

to get a $\mathbf{D}$-morphism

$$\delta' = \delta'(\mathcal{F}) \colon \mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{F} \cong \varGamma'_{\mathbf{t}}\mathcal{L}^\bullet \hookrightarrow \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{L}^\bullet) \cong \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}),$$

easily checked to be functorial in $\mathcal{F}$ (and in particular, independent of φ).

In proving the next Lemma, we will see that proregularity of $\mathbf{t}$ implies that $\delta'(\mathcal{F})$ is always an isomorphism. And the converse holds if cohomology on X commutes with filtered direct limits, for example if X is *compact* (i.e., quasi-compact and Hausdorff) [Go, p. 194, Thm. 4.12.1], or if X is *quasi-noetherian* [Ke, p. 641, Thm. 8]. Kempf defines X to be quasi-noetherian if its topology has a base of quasi-compact open sets, if the intersection of any two quasi-compact open subsets of X is again quasi-compact, and if X itself is quasi-compact. We prefer to use the term *concentrated*. For example, if X is noetherian (i.e., every open subset is quasi-compact) then X is concentrated. A scheme is concentrated iff it is quasi-compact and quasi-separated [GrD, p. 296, Prop. (6.1.12)].[10]

Lemma (3.1.1). *Let $\mathbf{t} = (t_1, \dots, t_\mu)$ $\big(t_i \in \Gamma(X, \mathcal{O})\big)$ and δ' be as above, and suppose that X is compact or concentrated. Then the following are equivalent:*

(1) *The sequence $\mathbf{t}$ is proregular (Definition (3.0.1)).*
(2) *For any $\mathcal{F} \in \mathbf{D}$, the map $\delta'(\mathcal{F}) \colon \mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{F} \to \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F})$ is an isomorphism.*
(2)′ *For any injective $\mathcal{O}$-module $\mathcal{J}$ and every $i \neq 0$, $H^i\mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{J}) = 0$.*
(3) *For any flat $\mathcal{O}$-module $\mathcal{P}$ and every $i \neq 0$, the inverse system*

$$\big(H_i(\mathbf{t}^r, \mathcal{P})\big)_{r>0} := \big(H^{-i}\mathcal{H}om_{\mathcal{O}}(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P})\big)_{r>0}$$

is essentially null, i.e., for each r there is an $s > r$ such that the natural map $H_i(\mathbf{t}^s, \mathcal{P}) \to H_i(\mathbf{t}^r, \mathcal{P})$ is the zero map.
(3)′ *For every $i \neq 0$ the inverse system $\big(H_i(\mathbf{t}^r, \mathcal{O})\big)_{r>0}$ is essentially null.*
(3)″ *The inverse system $\big(H_1(\mathbf{t}^r, \mathcal{O})\big)_{r>0}$ is essentially null.*

[10] where, for the implication d) ⇒ a), the family (U_α) should be a base of the topology.

Proof. We proceed as follows.

(A): $(1) \Rightarrow (3)' \Leftrightarrow (3) \Rightarrow (3)'' \Rightarrow (1)$.

(B): $(3)' \Rightarrow (2)' \Leftrightarrow (2) \Rightarrow (3)$.

The hypothesis "X compact or concentrated" will be needed only for $(2) \Rightarrow (3)$.

(A). Assuming (1), we prove $(3)'$ by induction on μ. For $\mu = 1$, the assertion amounts to the vanishing (in $\mathcal{O}$) of $t_1^{s-r}(0 : t_1^s)$ when $s \gg r$, which we get by taking $i = 1$ in Definition (3.0.1). For $\mu > 1$, there is an obvious direct system of split exact sequences of complexes

$$(3.1.1.1) \qquad 0 \to \mathcal{O}'_r[-1] \to \mathcal{K}^\bullet(t_\mu^r) \to \mathcal{O}_r \to 0 \qquad (r > 0)$$

where $\mathcal{O}'_r := \mathcal{O} =: \mathcal{O}_r$ for all r, where the map $\mathcal{O}'_s \to \mathcal{O}'_r$ $(s > r)$ is multiplication by t_μ^{s-r}, and $\mathcal{O}_s \to \mathcal{O}_r$ is the identity. From this system we derive an inverse system of exact sequences

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{K}^\bullet((t_1^r, \dots, t_{\mu-1}^r)) \otimes \mathcal{O}_r, \mathcal{O}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{K}^\bullet((t_1^r, \dots, t_{\mu-1}^r)) \otimes \mathcal{K}^\bullet(t_\mu^r), \mathcal{O})$$
$$\to \mathcal{H}om_{\mathcal{O}}(\mathcal{K}^\bullet((t_1^r, \dots, t_{\mu-1}^r)) \otimes \mathcal{O}'_r, \mathcal{O})[1] \to 0$$

whence an inverse system of exact homology sequences, with $\mathcal{I}^{[r]} := (t_1^r, \dots, t_{\mu-1}^r)\mathcal{O}$,

$$\cdots \longrightarrow H_i\big((t_1^r, \dots, t_{\mu-1}^r), \mathcal{O}\big) \xrightarrow{t_\mu^r} H_i\big((t_1^r, \dots, t_{\mu-1}^r), \mathcal{O}\big) \longrightarrow H_i\big(\mathbf{t}^r, \mathcal{O}\big)$$
$$\longrightarrow H_{i-1}\big((t_1^r, \dots, t_{\mu-1}^r), \mathcal{O}\big) \xrightarrow{t_\mu^r} H_{i-1}\big((t_1^r, \dots, t_{\mu-1}^r), \mathcal{O}\big) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_1\big((t_1^r, \dots, t_{\mu-1}^r), \mathcal{O}\big) \longrightarrow H_1\big(\mathbf{t}^r, \mathcal{O}\big) \longrightarrow (\mathcal{I}^{[r]} : t_\mu^r)/\mathcal{I}^{[r]} \longrightarrow 0.$$

Now the inductive hypothesis quickly reduces the problem to showing that the inverse system $T_r := (\mathcal{I}^{[r]} : t_\mu^r)/\mathcal{I}^{[r]}$, with maps $T_s \to T_r$ $(s > r)$ given by multiplication by t_μ^{s-r}, is essentially null; and that results from Definition (3.0.1) with $i = \mu$.

Thus (1) implies $(3)'$. Since $\mathcal{P}$ is flat, $H_i(\mathbf{t}^r, \mathcal{P}) = H_i(\mathbf{t}^r, \mathcal{O}) \otimes \mathcal{P}$, so $(3)' \Rightarrow (3)$; and obviously $(3) \Rightarrow (3)' \Rightarrow (3)''$.

Conversely, assuming $(3)''$ we get (1) from the surjections (as above):

$$H_1\big((t_1^r, \dots, t_i^r), \mathcal{O}\big) \twoheadrightarrow \big((t_1^r, \dots, t_{i-1}^r)\mathcal{O} : t_i^r\big)/(t_1^r, \dots, t_{i-1}^r)\mathcal{O} \qquad (1 \le i \le \mu).$$

(B). If $\mathcal{J}$ is an injective $\mathcal{O}$-module, then

$$H^i\mathcal{K}_\infty^\bullet(\mathbf{t}, \mathcal{J}) = H^i \varinjlim_{r>0} \mathcal{K}^\bullet(\mathbf{t}^r, \mathcal{J}) \cong \varinjlim_{r>0} H^i \mathcal{H}om\big(\mathcal{H}om(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{O}), \mathcal{J}\big)$$
$$= \varinjlim_{r>0} \mathcal{H}om\big((H_{-i}(\mathbf{t}^r, \mathcal{O})), \mathcal{J}\big)$$

and consequently $(3)' \Rightarrow (2)'$.

$(2)'$ implies, for any $\mathcal{O}$-complex $\mathcal{F}$, that if $\mathcal{F} \to \mathcal{L}^\bullet$ is a quasi-isomorphism with $\mathcal{L}^\bullet$ both K-injective and injective [Sp, p. 138, 4.5], then the j-th column $\mathcal{K}_\infty^\bullet(\mathbf{t}, \mathcal{L}^j)$ of the double complex $(\mathcal{K}_\infty^i(\mathbf{t}) \otimes \mathcal{L}^j)_{0 \le i \le \mu,\, j \in \mathbb{Z}}$ is a finite resolution of $\varGamma'_{\mathbf{t}}\mathcal{L}^j$, so that the inclusion $\varGamma'_{\mathbf{t}}\mathcal{L}^\bullet \hookrightarrow \mathcal{K}_\infty^\bullet(\mathbf{t}, \mathcal{L}^\bullet)$ is a quasi-isomorphism; and (2) follows. Conversely, since $\mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{J} \cong \varGamma'_{\mathbf{t}}\mathcal{J}$, $(2)'$ follows from (2).

To deduce (3) from (2)′ we imitate [Gr, p. 24]. There exists a monomorphism of the $\mathcal{O}$-module $H^i\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r),\mathcal{P})$ into an injective $\mathcal{O}$-module $\mathcal{J}'$, giving rise naturally to an element of

$$\varinjlim_{s>r}\operatorname{Hom}\bigl(H^i\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^s),\mathcal{P}),\,\mathcal{J}'\bigr); \tag{3.1.1.2}$$

and it will suffice to show that this element is zero. Noting that homology commutes with the exact functor $\operatorname{Hom}(-,\mathcal{J}')$ and with $\varinjlim$, noting that $\mathcal{K}^\bullet(\mathbf{t}^s)$ is a finite-rank free $\mathcal{O}$-complex, setting $\Gamma(-):=\Gamma(X,-)$, and setting $\mathcal{J}:=\mathcal{H}om^\bullet(\mathcal{P},\mathcal{J}')$ (which is an injective $\mathcal{O}$-module since $\mathcal{P}$ is flat), we can rewrite (3.1.1.2) as

$$\begin{aligned}
&H^i\varinjlim\Gamma\mathcal{H}om^\bullet\bigl(\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^s),\mathcal{P}),\,\mathcal{J}'\bigr)\\
&=H^i\varinjlim\Gamma\mathcal{H}om^\bullet\bigl(\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^s),\mathcal{O})\otimes\mathcal{P},\,\mathcal{J}'\bigr)\\
&=H^i\varinjlim\Gamma\mathcal{H}om^\bullet\bigl(\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^s),\mathcal{O}),\,\mathcal{J}\bigr)\\
&=H^i\varinjlim\Gamma\bigl(\mathcal{K}^\bullet(\mathbf{t}^s)\otimes\mathcal{J}\bigr),
\end{aligned}$$

or again, since Γ commutes with $\varinjlim$ (X being compact or concentrated), as $H^i\Gamma\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{J})$. But by (2)′, $\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{J})$ is a resolution of $\varGamma'_Z\mathcal{J}$, and as a $\varinjlim$ of injective complexes, is a complex of Γ-acyclic sheaves (since $H^i(X,-)$ commutes with $\varinjlim$); also $\varGamma'_Z\mathcal{J}$, the $\varinjlim$ of the flabby sheaves $\mathcal{H}om(\mathcal{O}/\mathbf{t}^n\mathcal{O},\mathcal{J})$, is Γ-acyclic; and so

$$H^i\Gamma\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{J})=H^i(X,\varGamma'_Z\mathcal{J})=0\qquad(i\neq 0).$$

This completes the proof of Lemma (3.1.1). □

With no assumption on the topological space X we define as in (2.1) *mutatis mutandis* a functorial map

$$\psi'_{\mathbf{t}}(\mathcal{E},\mathcal{F})\colon\mathcal{E}\underline{\underline{\otimes}}\,\mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{F}\xrightarrow{\ \sim\ }\mathbf{R}\varGamma'_{\mathbf{t}}(\mathcal{E}\underline{\underline{\otimes}}\,\mathcal{F})\qquad(\mathcal{E},\mathcal{F}\in\mathbf{D}).$$

Corollary (3.1.2). *If* $\mathbf{t}$ *is proregular then* $\psi'_{\mathbf{t}}(\mathcal{E},\mathcal{F})$ *is an isomorphism for all* $\mathcal{E},\mathcal{F}$.

Proof. Assume, as one may, that $\mathcal{E}$ is K-flat, and check that the following diagram—whose bottom row is the natural isomorphism—commutes:

$$\begin{array}{ccc}
\mathcal{E}\otimes\mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{F} & \xrightarrow{\psi'_{\mathbf{t}}(\mathcal{E},\mathcal{F})} & \mathbf{R}\varGamma'_{\mathbf{t}}(\mathcal{E}\otimes\mathcal{F})\\
{\scriptstyle\text{via }\delta'(\mathcal{F})}\downarrow & & \downarrow{\scriptstyle\delta'(\mathcal{E}\otimes\mathcal{F})}\\
\mathcal{E}\otimes\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{F}) & \xrightarrow{\ \sim\ } & \mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{E}\otimes\mathcal{F})
\end{array}$$

By the implication (1) ⇒ (2) in Lemma (3.1.1) (whose proof did not need X to be compact or concentrated), the maps $\delta'(\mathcal{F})$ and $\delta'(\mathcal{E}\otimes\mathcal{F})$ are also isomorphisms, and the assertion follows. □

Corollary (3.1.3). *If* $\mathbf{t}$ *and* $\mathbf{t}^*$ *in* $\Gamma(X,\mathcal{O})$ *are such that* $\mathbf{t}^*$ *and* $(\mathbf{t},\mathbf{t}^*)$ *are both proregular, then the natural map* $\mathbf{R}\mathit{\Gamma}'_{(\mathbf{t},\mathbf{t}^*)} \to \mathbf{R}\mathit{\Gamma}'_{\mathbf{t}} \circ \mathbf{R}\mathit{\Gamma}'_{\mathbf{t}^*}$ *is an isomorphism.*

Proof. Proregularity of $(\mathbf{t},\mathbf{t}^*)$ trivially implies that of $\mathbf{t}$ (and also, when X is compact or concentrated, of $\mathbf{t}^*$, see remark preceding (3.1.5) below). By (3.1.1)(2), the assertion results from the equality $\mathcal{K}^\bullet_\infty((\mathbf{t},\mathbf{t}^*),-) = \mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{K}^\bullet_\infty(\mathbf{t}^*,-))$. □

Corollary (3.1.4). *Let* $(X,\mathcal{O})$ *be a scheme and* $Z \subset X$ *a proregularly embedded subscheme.*

(i) *The map* $\psi'\colon \mathcal{E} \underline{\underline{\otimes}} \mathbf{R}\mathit{\Gamma}'_Z\mathcal{F} \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_Z(\mathcal{E} \underline{\underline{\otimes}} \mathcal{F})$ *of* (2.1) *is an isomorphism for all* $\mathcal{E},\mathcal{F} \in \mathbf{D}(X)$.

(ii) *If* $Z^* \subset X$ *is a closed subscheme such that* Z^* *and* $Z \cap Z^*$ *are both proregularly embedded, then the natural functorial map* $\mathbf{R}\mathit{\Gamma}'_{Z\cap Z^*} \to \mathbf{R}\mathit{\Gamma}'_Z \circ \mathbf{R}\mathit{\Gamma}'_{Z^*}$ *is an isomorphism.*

(iii) $\mathbf{R}\mathit{\Gamma}'_Z\big(\mathbf{D}_{\mathrm{qc}}(X)\big) \subset \mathbf{D}_{\mathrm{qc}}(X)$.

Proof. The assertions are essentially local on X, so the first two follow from (3.1.2) and (3.1.3) respectively, and the third from (3.1.1)(2), see [H, p. 98, Prop. 4.3]. □

Assume now that X is compact or concentrated. If $\mathbf{t}^*$ is a permutation of $\mathbf{t}$ then there is an obvious functorial isomorphism $\mathcal{K}^\bullet_\infty(\mathbf{t}^*,-) \xrightarrow{\sim} \mathcal{K}^\bullet_\infty(\mathbf{t},-)$, and so by Lemma (3.1.1)(2), $\mathbf{t}^*$ is proregular $\Leftrightarrow$ so is $\mathbf{t}$. More generally:

Corollary (3.1.5). *Let* $\mathbf{t} = (t_1,\dots,t_\mu)$ *be, as before, a sequence in* $\Gamma(X,\mathcal{O})$, *with* X *compact or concentrated, and let* $\mathbf{t}^* := (t_1^*,\dots,t_\nu^*)$ *be a sequence in* $\Gamma(X,\sqrt{\mathbf{t}\mathcal{O}})$. *Then the sequence* $(\mathbf{t}^*,\mathbf{t}) := (t_1^*,\dots,t_\nu^*,t_1,\dots,t_\mu)$ *is proregular* $\Leftrightarrow$ *so is* $\mathbf{t}$. *In particular, if* $\sqrt{\mathbf{t}^*\mathcal{O}} = \sqrt{\mathbf{t}\mathcal{O}}$ *then* $\mathbf{t}^*$ *is proregular* $\Leftrightarrow$ *so is* $\mathbf{t}$.

Proof. It suffices to treat the case $\nu = 1$. Since (clearly) $\mathit{\Gamma}'_{(\mathbf{t}^*,\mathbf{t})} = \mathit{\Gamma}'_{\mathbf{t}}$, and in view of (3.1.1)(2), we need only show, with $t := t_1^*$, that for any $\mathcal{O}$-complex $\mathcal{F}$ the natural functorial map

$$\mathcal{K}^\bullet_\infty((t,t_1,\dots,t_\mu),\mathcal{F}) = \mathcal{K}^\bullet_\infty(t)\otimes\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{F}) \to \mathcal{O}\otimes\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{F}) = \mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{F})$$

induces homology isomorphisms. The kernel of this degreewise split surjective map is $\mathcal{O}_t[-1]\otimes\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{F})$, where $\mathcal{O}_t$ is the direct limit of the system $(\mathcal{O}_n)_{n>0}$ with $\mathcal{O}_n := \mathcal{O}$ for all n and with $\mathcal{O}_r \to \mathcal{O}_s$ $(r \le s)$ multiplication by t^{s-r}; and it will suffice to show that this kernel is *exact,* i.e., that *for* $j \in \mathbb{Z}$ *and* $r > 0$, *any section of* $H^j(\mathcal{K}^\bullet_\infty(\mathbf{t}^r,\mathcal{F}))$ *over an open* $U \subset X$ *is locally annihilated by a power of* t. Since $t \in \sqrt{\mathbf{t}\mathcal{O}}$ we can replace t by t_i $(1 \le i \le \mu)$ in this last statement, whereupon it becomes well-known—and easily proved by induction on μ, via (3.1.1.1). □

Corollary (3.1.6). *Let* $(X,\mathcal{O})$ *be a quasi-separated scheme and* $Z \subset X$ *a proregularly embedded subscheme. If* $X_0 \subset X$ *is a quasi-compact open subset,* $\mathcal{O}_0 := \mathcal{O}|_{X_0}$, *and* $\mathbf{t}_0$ *is a finite sequence in* $\Gamma(X_0,\mathcal{O}_0)$ *such that* $Z\cap X_0$ *is the support of* $\mathcal{O}_0/\mathbf{t}_0\mathcal{O}_0$, *then* $\mathbf{t}_0$ *is proregular.*

Proof. X_0 is covered by finitely many of the open sets $X_0 \cap X_\alpha$ with X_α as in Definition (3.0.1), and we may assume that each X_α is quasi-compact, whence so is $X_0 \cap X_\alpha$ (since X is quasi-separated). So it suffices to apply (3.1.5) to $X_0 \cap X_\alpha$, with $\mathbf{t} := \mathbf{t}_0$ and $\mathbf{t}^* := \mathbf{t}_\alpha$. □

Let $(X,\mathcal{O})$ be a scheme, let $j\colon \mathcal{A}_{\text{qc}}=\mathcal{A}_{\text{qc}}(X)\hookrightarrow\mathcal{A}$ be the inclusion of the category of quasi-coherent $\mathcal{O}$-modules into the category of all $\mathcal{O}$-modules, and let $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_{\text{qc}})\to\mathbf{D}(\mathcal{A})=:\mathbf{D}$ be the corresponding derived-category functor.

Proposition (3.1.7). *If $(X,\mathcal{O}_X)$ is a quasi-compact separated scheme and $Z\subset X$ is proregularly embedded, then the functor*

$$\varGamma_Z^{\text{qc}}:=\varGamma_Z\circ j=\varGamma'_Z\circ j\colon \mathcal{A}_{\text{qc}}\to\mathcal{A}_{\text{qc}}$$

has a derived functor

$$\mathbf{R}\varGamma_Z^{\text{qc}}\colon \mathbf{D}(\mathcal{A}_{\text{qc}})\to\mathbf{D}(\mathcal{A}_{\text{qc}});$$

and the natural functorial map $\boldsymbol{j}\circ\mathbf{R}\varGamma_Z^{\text{qc}}\to\mathbf{R}\varGamma'_Z\circ\boldsymbol{j}$ is an isomorphism.

Remark. For quasi-compact separated X, $\boldsymbol{j}$ induces an *equivalence of categories* from $\mathbf{D}(\mathcal{A}_{\text{qc}})$ to $\mathbf{D}_{\text{qc}}(X)$ [BN, p. 230, Cor. 5.5] (or see (1.3) above). Therefore any $\mathcal{F}\in\mathbf{D}_{\text{qc}}(X)$ is isomorphic to a quasi-coherent complex. In this case, then, (3.1.7) embellishes assertion (iii) in (3.1.4). (The following proof does not, however, depend on [BN] or (1.3).)

Proposition (3.1.7) is a consequence of:

Lemma (3.1.7.1). *For any inclusion $i\colon U\hookrightarrow X$ with U affine open, and any $\mathcal{J}$ which is injective in $\mathcal{A}_{\text{qc}}(U)$, the natural map $\varGamma'_Z i_*\mathcal{J}\to\mathbf{R}\varGamma'_Z i_*\mathcal{J}$ is a $\mathbf{D}$-isomorphism.*

Indeed, if $\mathcal{G}\in\mathcal{A}_{\text{qc}}$, if $(U_\alpha)_{1\le\alpha\le n}$ is an affine open cover of X, with inclusion maps $i_\alpha\colon U_\alpha\hookrightarrow X$, and if for each α, $i_\alpha^*\mathcal{G}\to\mathcal{J}_\alpha$ is a monomorphism with $\mathcal{J}_\alpha$ injective in $\mathcal{A}_{\text{qc}}(U_\alpha)$, then $i_{\alpha*}\mathcal{J}_\alpha$ is $\mathcal{A}_{\text{qc}}$-injective (since $i_{\alpha*}\colon\mathcal{A}_{\text{qc}}(U_\alpha)\to\mathcal{A}_{\text{qc}}$ has an exact left adjoint), and there are obvious monomorphisms $\mathcal{G}\to\oplus_{\alpha=1}^n i_{\alpha*}i_\alpha^*\mathcal{G}\to\oplus_{\alpha=1}^n i_{\alpha*}\mathcal{J}_\alpha$. Thus the category $\mathcal{A}_{\text{qc}}$ has enough injectives; and since, by (3.1.7.1),

$$\varGamma_Z^{\text{qc}}(\oplus_{\alpha=1}^n i_{\alpha*}\mathcal{J}_\alpha)\cong\ \oplus_{\alpha=1}^n\,\mathbf{R}\varGamma'_Z i_{\alpha*}\mathcal{J}_\alpha\,,$$

and the functor $\mathbf{R}\varGamma'_Z$ is bounded above and below (by Lemma (3.1.1)(2) and quasi-compactness of X), it follows from [H, p. 57, γb] and its proof that $\mathbf{R}\varGamma_Z^{\text{qc}}$ exists and is bounded above and below. And then the isomorphism assertion in (3.1.7) follows from [H, p. 69, (iii) and (iv)].

It remains then to prove Lemma (3.1.7.1).

Since X is concentrated, there is a finite-type $\mathcal{O}_X$-ideal $\mathcal{I}$ such that $Z=\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$. With $\mathcal{O}_U:=i^*\mathcal{O}_X$, $\mathcal{I}_U:=i^*\mathcal{I}$, we have for any $\mathcal{O}_U$-module $\mathcal{E}$,

$$\begin{aligned}\varGamma'_Z i_*\mathcal{E}&=\varinjlim\ \mathcal{H}om(\mathcal{O}_X/\mathcal{I}^n, i_*\mathcal{E})\\&=\varinjlim\ i_*\mathcal{H}om(\mathcal{O}_U/\mathcal{I}_U^n,\mathcal{E})\\&=i_*\varinjlim\mathcal{H}om(\mathcal{O}_U/\mathcal{I}_U^n,\mathcal{E})=i_*\varGamma'_{Z\cap U}\mathcal{E}\end{aligned}$$

where the interchange of $\varinjlim$ and i_* is justified by [Ke, p. 641, Prop. 6]. Since the map i is affine, and i_* takes $\mathcal{O}_U$-injectives to $\mathcal{O}_X$-injectives, and since for any $\mathcal{O}_U$-injective $\mathcal{L}$, $\varGamma'_{Z\cap U}\mathcal{L}$ is a $\varinjlim$ of flabby sheaves and hence i_*-acyclic [Ke, p. 641, Cors. 5 and 7], therefore

$$\mathbf{R}\varGamma'_Z(i_*\mathcal{J})\cong\mathbf{R}\varGamma'_Z(\mathbf{R}i_*\mathcal{J})\cong\mathbf{R}(\varGamma'_Z i_*)(\mathcal{J})=\mathbf{R}(i_*\varGamma'_{Z\cap U})(\mathcal{J})=\mathbf{R}i_*\mathbf{R}\varGamma'_{Z\cap U}(\mathcal{J}).$$

Referring again to the ring-theoretic analogue of (3.1.1)(2)$'$ [Gr, p. 24, Lemme 9, b)], we see that $\mathbf{R}\varGamma'_{Z\cap U}\mathcal{J}\cong\varGamma'_{Z\cap U}\mathcal{J}$; and since i is affine and $\varGamma'_{Z\cap U}\mathcal{J}$ is quasi-coherent, therefore

$$\mathbf{R}i_*\mathbf{R}\varGamma'_{Z\cap U}\mathcal{J}\cong\mathbf{R}i_*\varGamma'_{Z\cap U}\mathcal{J}\cong i_*\varGamma'_{Z\cap U}\mathcal{J}\cong\varGamma'_Z i_*\mathcal{J},$$

whence the desired conclusion. $\square$

(3.2). The map

$$\delta' = \delta'(\mathcal{F})\colon \mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{F} \to \mathcal{K}^{\bullet}_{\infty}(\mathbf{t},\mathcal{F}) \qquad \bigl(\mathcal{F}\in\mathbf{D}\bigr)$$

remains as in §3.1. Let Z be the support of $\mathcal{O}/\mathbf{t}\mathcal{O}$, a closed subset of X. In the following steps a)–d), *we construct a functorial map*

$$\delta = \delta(\mathcal{F})\colon \mathcal{K}^{\bullet}_{\infty}(\mathbf{t},\mathcal{F}) \to \mathbf{R}\varGamma_{Z}\mathcal{F} \qquad \bigl(\mathcal{F}\in\mathbf{D}\bigr)$$

such that $\delta\circ\delta'\colon \mathbf{R}\varGamma'_{\mathbf{t}}\mathcal{F} \to \mathbf{R}\varGamma_{Z}\mathcal{F}$ *coincides with the map induced by the obvious inclusion* $\varGamma'_{\mathbf{t}} \hookrightarrow \varGamma_{Z}$.

a) As in the definition of δ' we may assume that $\mathcal{F}$ is K-injective, and injective as well (i.e., each of its component $\mathcal{O}$-modules $\mathcal{F}^n$ $(n\in\mathbb{Z})$ is injective) [Sp, p. 138, 4.5]. If $U := (X\setminus Z) \overset{i}{\hookrightarrow} X$ is the inclusion map, then the canonical sequence of complexes $0\to\varGamma_{Z}\mathcal{F}\hookrightarrow\mathcal{F}\xrightarrow{\eta} i_*i^*\mathcal{F}\to 0$ is exact, and *there results a natural quasi-isomorphism* $\varGamma_{Z}\mathcal{F}\to\mathcal{C}_{\eta}[-1]$ *where* $\mathcal{C}_{\eta}$ *is the cone of* η.

b) Let $\mathcal{K}_{\flat}$ be the complex

$$\mathcal{K}^1_{\infty}(\mathbf{t}) \to \mathcal{K}^2_{\infty}(\mathbf{t}) \to \dots \qquad (\mathcal{K}^0_{\flat} := \mathcal{K}^1_{\infty}(\mathbf{t}),\ \mathcal{K}^1_{\flat} := \mathcal{K}^2_{\infty}(\mathbf{t}),\ \dots)$$

There is an obvious map of complexes $\mathcal{O} := \mathcal{K}^0_{\infty}(\mathbf{t}) \to \mathcal{K}_{\flat}$, inducing for any complex $\mathcal{F}$ a map $\xi = \xi(\mathcal{F})\colon \mathcal{F} = \mathcal{O}\otimes\mathcal{F} \to \mathcal{K}_{\flat}\otimes\mathcal{F}$, *whose cone* $\mathcal{C}_{\xi}$ *is* $\mathcal{K}^{\bullet}_{\infty}(\mathbf{t},\mathcal{F})[1]$.

c) Since $\mathbf{t}\mathcal{O}_U = \mathcal{O}_U$ $(\mathcal{O}_U := \mathcal{O}|_U)$, the complex $i^*\mathcal{K}^{\bullet}_{\infty}(\mathbf{t})$ is homotopically trivial at each point of U, and hence for any $\mathcal{F}$ the complex $i^*\mathcal{K}^{\bullet}_{\infty}(\mathbf{t},\mathcal{F})$ is exact. In other words, $i^*\xi(\mathcal{F})\colon i^*\mathcal{F}\to i^*\mathcal{K}_{\flat}\otimes i^*\mathcal{F}$ is a *quasi-isomorphism* for all $\mathcal{F}$.

Let $\sigma\colon i^*\mathcal{K}_{\flat}\otimes i^*\mathcal{F} \to \mathcal{L}$ be a quasi-isomorphism with $\mathcal{L}$ K-injective. Then $\sigma\circ i^*\xi\colon i^*\mathcal{F}\to\mathcal{L}$ is a quasi-isomorphism between K-injective complexes, therefore so is $\zeta := i_*(\sigma\circ i^*\xi)$, as is the induced map of cones $\epsilon\colon \mathcal{C}_{\eta}\to\mathcal{C}_{\zeta\circ\eta}$.

From the commutative diagram of complexes

(3.2.1)
$$\begin{array}{ccccc}
\mathcal{F} & \xrightarrow{\ \eta\ } & i_*i^*(\mathcal{F}) & \xrightarrow{\ \zeta\ } & i_*\mathcal{L}\\
{\scriptstyle\xi}\big\downarrow & & \big\downarrow{\scriptstyle i_*i^*\xi} & & \big\|\\
\mathcal{K}_{\flat}\otimes\mathcal{F} & \longrightarrow & i_*i^*(\mathcal{K}_{\flat}\otimes\mathcal{F}) & \xrightarrow[i_*\sigma]{} & i_*\mathcal{L}
\end{array}$$

we deduce a map of cones

(3.2.2)
$$\mathcal{C}_{\xi}\to\mathcal{C}_{\zeta\circ\eta}$$

and hence a composed $\mathbf{D}$-map

$$\delta(\mathcal{F})\colon \mathcal{K}^{\bullet}_{\infty}(\mathbf{t},\mathcal{F}) \cong \mathcal{C}_{\xi}[-1] \to \mathcal{C}_{\zeta\circ\eta}[-1] \xrightarrow{\epsilon^{-1}} \mathcal{C}_{\eta}[-1] \cong \varGamma_{Z}\mathcal{F} \cong \mathbf{R}\varGamma_{Z}\mathcal{F},$$

easily checked to be functorial in $\mathcal{F}$.

d) To check that $\delta\circ\delta'$ is as asserted above, "factor" the first square in (3.2.1) as

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\ \pi\ } & \mathcal{F}/\varGamma'_{\mathbf{t}}\mathcal{F} & \xrightarrow{\ \eta'\ } & i_*i^*(\mathcal{F}) \\ & & {\scriptstyle\xi'}\big\downarrow & & \big\downarrow \\ & & \mathcal{K}_\flat\otimes\mathcal{F} & \longrightarrow & i_*i^*(\mathcal{K}_\flat\otimes\mathcal{F})\,, \end{array}$$

derive the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\pi[-1] & \xrightarrow{\text{via }\eta'} & \mathcal{C}_\eta[-1] \\ {\scriptstyle\text{via }\xi'}\big\downarrow & & \big\downarrow{\scriptstyle\epsilon[-1]} \\ \mathcal{C}_\xi[-1] & \underset{(3.2.2)}{\longrightarrow} & \mathcal{C}_{\zeta\circ\eta}[-1]\,, \end{array}$$

and using a) and b), identify the $\mathbf{D}$-map labeled "via ξ'" (resp. "via η'") with δ' (resp. the inclusion map $\varGamma'_{\mathbf{t}}\mathcal{F}\hookrightarrow\varGamma_Z\mathcal{F}$).

The next Lemma is a derived-category version of [Gr, p. 20, Prop. 5] and [H, p. 98, Prop. 4.3, b)] (from which it follows easily if the complex $\mathcal{F}$ is bounded-below or if the functor $\varGamma_Z$ has finite homological dimension).

Lemma (3.2.3). *Let $(X,\mathcal{O})$ be a scheme, let $\mathbf{t}$ be a finite sequence in $\Gamma(X,\mathcal{O})$, and let $Z:=\operatorname{Supp}(\mathcal{O}/\mathbf{t}\mathcal{O})$. Then $\delta(\mathcal{F})\colon\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{F})\to\mathbf{R}\varGamma_Z\mathcal{F}$ is an isomorphism for all $\mathcal{F}\in\mathbf{D}_{\mathrm{qc}}(X)$.*

Proof. The question is local, so we may assume X to be affine, say $X=\operatorname{Spec}(R)$. Let $i\colon U:=(X\setminus Z)\hookrightarrow X$ be the inclusion, a quasi-compact map (since U is quasi-compact). Let $\mathcal{K}_\flat$ be as in the definition of δ, so that $\mathcal{K}_\flat=i_*i^*\mathcal{K}_\flat$. Also, the Čech resolution $i^*\xi(\mathcal{O})\colon\mathcal{O}_U\to i^*\mathcal{K}_\flat$ (see c) above) is *i_*-acyclic,* i.e., $R^pi_*(i^*\mathcal{K}^q_\flat)=0$ for all $p>0$ and $q\ge 0$: indeed, $i^*\mathcal{K}^q_\flat$ is a direct sum of sheaves of the form $j_*\mathcal{O}_V$, where $V\subset U$ is an open set of the form $\operatorname{Spec}(R_t)$ (t a product of some members of $\mathbf{t}$) and $j\colon V\hookrightarrow U$ is the inclusion map; and since V is affine, therefore

$$i_*(j_*\mathcal{O}_V)=(ij)_*\mathcal{O}_V=\mathbf{R}(ij)_*\mathcal{O}_V=\mathbf{R}i_*(\mathbf{R}j_*\mathcal{O}_V)=\mathbf{R}i_*(j_*\mathcal{O}_V),$$

whence $i_*(i^*\mathcal{K}^q_\flat)=\mathbf{R}i_*(i^*\mathcal{K}^q_\flat)$. It follows that $\mathcal{K}_\flat=i_*(i^*\mathcal{K}_\flat)\cong\mathbf{R}i_*(\mathcal{O}_U)$.

Since the bounded complex $\mathcal{K}_\flat$ is *flat,* we conclude that the bottom row of (3.2.1) is isomorphic in $\mathbf{D}$ to the canonical composition

$$\mathbf{R}i_*\mathcal{O}_U\underset{=}{\otimes}\mathcal{F}\to\mathbf{R}i_*i^*(\mathbf{R}i_*\mathcal{O}_U\underset{=}{\otimes}\mathcal{F})\xrightarrow{\ \sim\ }\mathbf{R}i_*(i^*\mathbf{R}i_*\mathcal{O}_U\underset{=}{\otimes}i^*\mathcal{F})\xrightarrow{\ \sim\ }\mathbf{R}i_*(\mathcal{O}_U\underset{=}{\otimes}i^*\mathcal{F})$$

which composition is an isomorphism for any $\mathcal{F}\in\mathbf{D}_{\mathrm{qc}}(X)$. This instance of the "projection isomorphism" of [H, p. 106] (where the hypotheses are too restrictive) is shown in [L, Prop. (3.9.4)] to hold in the necessary generality. It follows that the map $\mathcal{C}_\xi\to\mathcal{C}_{\zeta\circ\eta}$ in (3.2.2) is a $\mathbf{D}$-isomorphism, whence the assertion. □

From the implication (1) ⇒ (2) of Lemma (3.1.1)—whose proof does not need X to be concentrated—we now obtain:

Corollary (3.2.4). *If Z is a proregularly embedded subscheme of the scheme X then for all $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$, the natural map $\mathbf{R}\varGamma'_Z\mathcal{F} \to \mathbf{R}\varGamma_Z\mathcal{F}$ is an isomorphism.*

Corollary (3.2.5). *Let $(X, \mathcal{O})$ be a scheme and $Z \subset X$ a closed subscheme such that the inclusion $(X \setminus Z) \hookrightarrow X$ is quasi-compact.*

(i) *The map $\psi\colon \mathcal{E} \underset{=}{\otimes} \mathbf{R}\varGamma_Z\mathcal{F} \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_Z(\mathcal{E} \underset{=}{\otimes} \mathcal{F})$ of (2.1) is an isomorphism for all $\mathcal{E}, \mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$.*

(ii) *If $Z^* \subset X$ is a closed subscheme such that $(X \setminus Z^*) \hookrightarrow X$ is quasi-compact, then the the natural functorial map $\mathbf{R}\varGamma_{Z\cap Z^*}\mathcal{E} \to \mathbf{R}\varGamma_Z\mathbf{R}\varGamma_{Z^*}\mathcal{E}$ is an isomorphism for all $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$.*

(iii) $\mathbf{R}\varGamma_Z\big(\mathbf{D}_{\mathrm{qc}}(X)\big) \subset \mathbf{D}_{\mathrm{qc}}(X)$.

Proof. Since ψ is compatible with restriction to open subsets, we may assume that X is affine, so that $Z = \mathrm{Supp}(\mathcal{O}/\mathbf{t}\mathcal{O})$ for some finite sequence $\mathbf{t}$ in $\Gamma(X, \mathcal{O})$. We may also assume that $\mathcal{E}$ is K-flat, and then check that the following diagram—whose top row is the natural isomorphism—commutes:

$$\begin{array}{ccc} \mathcal{E} \otimes \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}) & \xrightarrow{\ \sim\ } & \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{E} \otimes \mathcal{F}) \\ {\scriptstyle \text{via } \delta(\mathcal{F})}\big\downarrow & & \big\downarrow{\scriptstyle \delta(\mathcal{E}\otimes\mathcal{F})} \\ \mathcal{E} \otimes \mathbf{R}\varGamma_Z\mathcal{F} & \xrightarrow{\ \psi\ } & \mathbf{R}\varGamma_Z(\mathcal{E} \otimes \mathcal{F}) \end{array}$$

Since both $\mathcal{E}$ and $\mathcal{F}$ are in $\mathbf{D}_{\mathrm{qc}}(X)$, so is $\mathcal{E} \underset{=}{\otimes} \mathcal{F}$: express $\mathcal{E}$ and $\mathcal{F}$ as $\varinjlim$'s of bounded-above truncations to reduce to where $\mathcal{E}, \mathcal{F} \in \mathbf{D}^-_{\mathrm{qc}}$, a case treated in [H, p. 98, Prop. 4.3]. By Lemma (3.2.3) the maps $\delta(\mathcal{F})$ and $\delta(\mathcal{E} \underset{=}{\otimes} \mathcal{F})$ are isomorphisms, and assertion (i) of the Corollary follows.

Assertion (iii) follows at once from (3.2.3), see [H, p. 98, Prop. 4.3]. And then (ii) follows from (3.2.3), since $\mathcal{K}^\bullet_\infty((\mathbf{t}, \mathbf{t}^*), -) = \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{K}^\bullet_\infty(\mathbf{t}^*, -)$. □

Remark. Actually, (i) results more directly from the triangle map (see (0.4.2.1))

$$\begin{array}{ccccccc} \mathcal{E} \underset{=}{\otimes} \mathbf{R}\varGamma_Z\mathcal{F} & \longrightarrow & \mathcal{E} \underset{=}{\otimes} \mathcal{F} & \longrightarrow & \mathcal{E} \underset{=}{\otimes} \mathbf{R}i_*i^*\mathcal{F} & \longrightarrow & \mathcal{E} \underset{=}{\otimes} \mathbf{R}\varGamma_Z\mathcal{F}[1] \\ {\scriptstyle (2.1)}\big\downarrow & & \big\| & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \text{projection}} & & \big\downarrow{\scriptstyle (2.1)} \\ \mathbf{R}\varGamma_Z(\mathcal{E} \underset{=}{\otimes} \mathcal{F}) & \longrightarrow & \mathcal{E} \underset{=}{\otimes} \mathcal{F} & \longrightarrow & \mathbf{R}i_*(i^*\mathcal{E} \underset{=}{\otimes} i^*\mathcal{F}) & \longrightarrow & \mathbf{R}\varGamma_Z(\mathcal{E} \underset{=}{\otimes} \mathcal{F})[1] \end{array}$$

with the projection isomorphism as in the proof of (3.2.3). Part (iii) also follows from (0.4.2.1), since $\mathbf{R}i_*i^*$ preserves quasi-coherence of homology (see [L, (3.9.2)] for unbounded complexes.)

As might be expected, assertion (ii) in (3.2.5) holds for *all* $\mathcal{E} \in \mathbf{D}(X)$. This is because $\mathbf{R}\varGamma_Z$ can be computed via "K-flabby" resolutions, and because for any injective K-injective complex $\mathcal{J}$, $\varGamma_{Z^*}\mathcal{J}$ is K-flabby (see e.g., [Sp, p. 146, Prop. 6.4 and p. 142, Prop. 5.15(b)], and use the natural triangle $\varGamma_{Z^*}\mathcal{J} \to \mathcal{J} \to j_*j^*\mathcal{J} \xrightarrow{+}$ where $j\colon (X \setminus Z^*) \hookrightarrow X$ is the inclusion).

Proposition (3.2.6). *Let $(X, \mathcal{O})$ be a quasi-compact separated scheme, and $Z \subset X$ a closed subscheme such that $X \setminus Z$ is quasi-compact. The following are equivalent:*

(1) *Z is proregularly embedded in X.*

(2) *The natural functorial map $\boldsymbol{j} \circ \mathbf{R}\varGamma^{\mathrm{qc}}_Z \to \mathbf{R}\varGamma_Z \circ \boldsymbol{j}$ (see Proposition (3.1.7)) is an isomorphism.*

(3) *The natural functorial maps $\boldsymbol{j} \circ \mathbf{R}\varGamma^{\mathrm{qc}}_Z \to \mathbf{R}\varGamma'_Z \circ \boldsymbol{j} \to \mathbf{R}\varGamma_Z \circ \boldsymbol{j}$ are both isomorphisms.*

Proof. (1) ⇒ (3). If Z is proregularly embedded in X then Proposition (3.1.7) says that $\boldsymbol{j}\circ\mathbf{R}\varGamma_Z^{\mathrm{qc}} \to \mathbf{R}\varGamma'_Z\circ\boldsymbol{j}$ is an isomorphism; and (3.1.1)(2) and (3.2.3) give that the natural map $\mathbf{R}\varGamma'_Z\boldsymbol{j}\mathcal{F} \to \mathbf{R}\varGamma_Z\boldsymbol{j}\mathcal{F}$ is an isomorphism for all $\mathcal{F}\in\mathbf{D}_{\mathrm{qc}}(X)$.

(3) ⇒ (2). Trivial.[11]

(2) ⇒ (1). Let $i\colon Y\hookrightarrow X$ be the inclusion of an affine open subset, so that $Y\setminus Z$ is quasi-compact, whence $Y\cap Z=\mathrm{Supp}(\mathcal{O}_Y/\mathbf{t}\mathcal{O}_Y)$ for some finite sequence $\mathbf{t}$ in $\Gamma(Y,\mathcal{O}_Y)$ ($\mathcal{O}_Y := \mathcal{O}|_Y$); and let us show for any $\mathcal{L}$ injective in $\mathcal{A}_{\mathrm{qc}}(Y)$ that the canonical map $\varGamma_{Y\cap Z}\mathcal{L}\to\mathbf{R}\varGamma_{Y\cap Z}\mathcal{L}$ is an isomorphism, i.e., by (3.2.3), that $H^n\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{L})=0$ for all $n>0$. Then (1) will follow, by the ring-theoretic analogue of the implication $(2)'\Rightarrow(1)$ in Lemma (3.1.1), cf. [Gr, p. 24, Lemme 9].

There is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{L}'$ with $i^*\mathcal{L}'=\mathcal{L}$, and an $\mathcal{A}_{\mathrm{qc}}$-injective $\mathcal{J}\supset\mathcal{L}'$. Then $\mathcal{L}\subset i^*\mathcal{J}$ is a direct summand, and so for any $n>0$, $H^n\mathbf{R}\varGamma_{Y\cap Z}\mathcal{L}$ is a direct summand of $H^n\mathbf{R}\varGamma_{Y\cap Z}i^*\mathcal{J}\cong i^*H^n\mathbf{R}\varGamma_Z\mathcal{J}$, which vanishes if (2) holds. Thus $\varGamma_{Y\cap Z}\mathcal{L}\xrightarrow{\sim}\mathbf{R}\varGamma_{Y\cap Z}\mathcal{L}$, as desired. □

Corollary (3.2.7) (cf. [Gr, p. 24, Cor. 10]). *For a concentrated scheme X, the following are equivalent:*

(1) *Every closed subscheme Z with $X\setminus Z$ quasi-compact is proregularly embedded.*

(2) *For every open immersion $i\colon U\hookrightarrow X$ with U quasi-compact, and every $\mathcal{A}_{\mathrm{qc}}$-injective $\mathcal{J}$, the canonical map $\mathcal{J}\to i_*i^*\mathcal{J}$ is surjective.*

Proof. Assuming (1), to prove (2) we may assume that X is affine. Then by [Gr, p. 16, Cor. 2.11] we have an exact sequence

$$0\to\varGamma_Z\mathcal{J}\to\mathcal{J}\to i_*i^*\mathcal{J}\to H^1\mathbf{R}\varGamma_Z\mathcal{J}\to 0,$$

and so Proposition (3.2.6) yields the conclusion.

Now assume (2) holds, so that for any $\mathcal{A}_{\mathrm{qc}}$-injective $\mathcal{J}$, any open immersion $j\colon Y\to X$ with Y affine, and any quasi-compact open $U\subset Y$, the restriction $\Gamma(Y,\mathcal{J})\to\Gamma(U,\mathcal{J})$ is surjective—in other words, $j^*\mathcal{J}$ is *quasi-flabby* [Ke, p. 640]. To prove (1) it suffices, as in proving the implication (2) ⇒ (1) in (3.2.6), to show that for any $\mathcal{A}_{\mathrm{qc}}$-injective $\mathcal{J}$ and $n>0$, $H^n\mathbf{R}\varGamma_Z\mathcal{J}=0$; and since the question is local it will be enough to show the same for any quasi-flabby $\mathcal{J}$. For $n=1$ this results from the above exact sequence, and for $n>1$ it results from the isomorphism $H^n\mathbf{R}\varGamma_Z\mathcal{J}\xrightarrow{\sim}H^{n-1}\mathbf{R}i_*i^*\mathcal{J}$ [Gr, p. 16, Cor. 2.11], whose target vanishes because $i^*\mathcal{J}$ is quasi-flabby, hence i_*-acyclic [Ke, p. 641, Cor. 5]. □

4. Local isomorphisms. This section provides the proofs which are still missing from the discussion in §2. Proposition (4.1) is a $\mathbf{D}(X)$-variant of Theorem 2.5 in [GM, p. 447], giving a local isomorphism of the homology of $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X,-)$ (called in [GM] the *local homology* of X at Z) to the left-derived functors of completion along Z. (At least this is done for quasi-coherent flat $\mathcal{O}_X$-modules, but as indicated after (2.2), Lemma (4.3) guarantees that's enough.) Corollary (4.2) allows us to conclude that on an arbitrary quasi-compact separated scheme X, these isomorphisms—defined via local Koszul complexes—patch together to a global inverse for the map $\Phi(\mathcal{F},\mathcal{O}_X)$ of (2.2).

Proposition (4.1). *Let $(X,\mathcal{O}_X)$ be a scheme, let $\mathbf{t}=(t_1,t_2,\dots,t_\mu)$ be a proregular sequence in $\Gamma(X,\mathcal{O}_X)$ (Definition (3.0.1)), and set $Z:=\mathrm{Supp}(\mathcal{O}_X/\mathbf{t}\mathcal{O}_X)$. Then for any* quasi-coherent *flat $\mathcal{O}_X$-module $\mathcal{P}$ there is a $\mathbf{D}(X)$-isomorphism*

$$\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X,\mathcal{P})\xrightarrow{\sim}\varprojlim_{r>0}\mathcal{P}/\mathbf{t}^r\mathcal{P}.$$

[11] One could also prove (1) ⇒ (2) without invoking $\varGamma'_Z$, by imitating the proof of (3.1.7).

Proof. Let $\mathcal{P} \to \mathcal{J}$ be an injective resolution. By (3.2.3),

$$\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathcal{P}) \cong \mathcal{H}om^\bullet(\mathcal{K}^\bullet_\infty(\mathbf{t}), \mathcal{J}) \cong \varprojlim \mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J});$$

and there are natural maps

$$\begin{aligned}\pi_i\colon H^i \varprojlim \mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J}) &\to \varprojlim H^i\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J})\\ &\cong \varprojlim H^i\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}),\end{aligned} \tag{4.1.1}$$

the last isomorphism holding because $\mathcal{K}^\bullet(\mathbf{t}^r)$ is a bounded complex of free finite-rank $\mathcal{O}_X$-modules.

It follows easily from the definition of $\mathcal{K}^\bullet(\mathbf{t}^r)$ that

$$H^0\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}) \cong \mathcal{P}/\mathbf{t}^r\mathcal{P};$$

and for $i \neq 0$, the implication (1) $\Rightarrow$ (3) in Lemma (3.1.1) gives

$$\varprojlim H^i\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}) = 0.$$

It suffices then that each one of the maps π_i be an isomorphism; and for that it's enough that for each affine open $U \subset X$, the natural composition

$$\begin{aligned}H^i\Gamma\big(U, \varprojlim \mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J})\big) &\cong H^i \varprojlim \mathrm{Hom}^\bullet\big(\mathcal{K}^\bullet(\mathbf{t}^r)|_U, \mathcal{J}|_U\big)\\ &\xrightarrow{\mu} \varprojlim H^i\mathrm{Hom}^\bullet\big(\mathcal{K}^\bullet(\mathbf{t}^r)|_U, \mathcal{J}|_U\big)\\ &\xrightarrow{\nu} \varprojlim \Gamma\big(U, H^i\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J})\big)\end{aligned} \tag{4.1.2}$$

be an isomorphism. (As U varies, these composed maps form a *presheaf* map whose sheafification is π_i.)

To see that ν is an isomorphism we can (for notational simplicity) replace U by X—assumed then to be affine, say $X = \mathrm{Spec}(R)$, write $\Gamma\mathcal{E}$ for $\Gamma(X, \mathcal{E})$, and note that since $\Gamma\mathcal{P} \to \Gamma\mathcal{J}$ is a quasi-isomorphism (because $\mathcal{P}$ is quasi-coherent), and since $\Gamma\mathcal{K}^\bullet(\mathbf{t}^r)$ is a finite-rank free R-complex, therefore

$$\begin{aligned}H^i\mathrm{Hom}^\bullet\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J}\big) &\cong H^i\mathrm{Hom}^\bullet_R\big(\Gamma\mathcal{K}^\bullet(\mathbf{t}^r), \Gamma\mathcal{J}\big)\\ &\cong H^i\mathrm{Hom}^\bullet_R\big(\Gamma\mathcal{K}^\bullet(\mathbf{t}^r), \Gamma\mathcal{P}\big)\\ &\cong \Gamma H^i\mathcal{H}om^\bullet\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}\big)\\ &\cong \Gamma H^i\mathcal{H}om^\bullet\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J}\big).\end{aligned} \tag{4.1.3}$$

It remains to be shown that μ is an isomorphism; and for that we can apply [EGA, p. 66, (13.2.3)]. As above we may as well assume X affine and $U = X$.

For surjectivity of μ, it is enough, by *loc. cit.,* that for each i, the inverse system

$$E_r := \mathrm{Hom}^i\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J}\big) = \prod_{0 \le p \le \mu} \mathrm{Hom}\big(\mathcal{K}^p(\mathbf{t}^r), \mathcal{J}^{p+i}\big) \qquad (r > 0)$$

satisfy the Mittag-Leffler condition (ML): *for each r there is an $s > r$ such that the images of all the maps $E_{s+n} \to E_r$ $(n \ge 0)$ are the same.*

But we have

$$\mathrm{Hom}\big(\mathcal{K}^p(\mathbf{t}^r),\,\mathcal{J}^{p+i}\big)\cong\prod_{\sigma}\mathcal{J}_{r,\sigma}$$

where σ ranges over all p-element subsets of $\{1,2,\dots,\mu\}$, and $\mathcal{J}_{r,\sigma}:=\mathcal{J}^{p+i}$ for all r and σ; and for $s>r$, the corresponding map $\prod_\sigma \mathcal{J}_{s,\sigma}\to\prod_\sigma\mathcal{J}_{r,\sigma}$ is the direct product of the maps $\mathcal{J}_{s,\sigma}\to\mathcal{J}_{r,\sigma}$ given by multiplication by t_σ^{s-r} where $t_\sigma:=\prod_{j\in\sigma}t_j$. Thus we need only show there is an N such that $t_\sigma^{N+n}\mathcal{J}_{r,\sigma}=t_\sigma^N\mathcal{J}_{r,\sigma}$ for all r, σ, and $n\ge 0$. But X being affine we have the equivalence (1) $\Leftrightarrow$ (2) in Lemma (3.1.1), which implies that any permutation of $\mathbf{t}$ is proregular. Taking $r=1$ and $i=1$ in Definition (3.0.1), and applying the following Lemma (4.1.4) with $\mathcal{I}=\mathcal{I}'=(0)$, we find then that for each r, σ, and $j=1,2,\dots,\mu$, there is an N_j such that for all $n\ge0$, $t_j^{N_j+n}\mathcal{J}_{r,\sigma}=t_j^{N_j}\mathcal{J}_{r,\sigma}$. The desired conclusion follows, with $N=\sup(N_j)$.

For bijectivity of μ, it is enough, by *loc. cit.*, that for each i, the inverse system

$$H^i\mathrm{Hom}^\bullet\big(\mathcal{K}^\bullet(\mathbf{t}^r),\mathcal{J}\big)\cong\Gamma H^i\mathcal{H}om^\bullet\big(\mathcal{K}^\bullet(\mathbf{t}^r),\mathcal{P}\big)\qquad(r>0)$$

(see (4.1.3)) satisfy (ML). For $i=0$, this is just the system $\Gamma(\mathcal{P})/\mathbf{t}^r\Gamma(\mathcal{P})$, with all maps surjective; and for $i\ne0$, the system is, by Lemma (3.1.1)(3), essentially null. □

Lemma (4.1.4). *Let $\mathcal{I}$, $\mathcal{I}'$ be $\mathcal{O}$-ideals, let $t\in\Gamma(X,\mathcal{O})$, and let $r\le s$ be integers such that $t^{s-r}(\mathcal{I}\colon t^s)\subset\mathcal{I}'$. Then for any injective $\mathcal{O}$-module $\mathcal{J}$ and any open $U\subset X$ we have, setting $\mathcal{G}_U:=\mathcal{G}|_U$ for any $\mathcal{O}$-module $\mathcal{G}$:*

$$t^{s-r}\mathrm{Hom}(\mathcal{O}_U/\mathcal{I}'_U,\mathcal{J}_U)\subset t^s\mathrm{Hom}(\mathcal{O}_U/\mathcal{I}_U,\mathcal{J}_U).$$

Proof. For any map $\rho\colon\mathcal{O}_U/\mathcal{I}'_U\to\mathcal{J}_U$, the kernel of $\mathcal{O}_U\xrightarrow{t^s}\mathcal{O}_U/\mathcal{I}_U$ annihilates $t^{s-r}\rho$ (because $(\mathcal{I}\colon t^s)t^{s-r}\subset\mathcal{I}'$), and so there is an $\mathcal{O}_U$-homomorphism

$$\psi=\psi_{r,s,\rho}\colon t^s(\mathcal{O}_U/\mathcal{I}_U)\to\mathcal{H}om(\mathcal{O}_U/\mathcal{I}'_U,\mathcal{J}_U)\subset\mathcal{J}_U$$

with $\psi(t^s+\mathcal{I}_U)=t^{s-r}\rho$. Since $\mathcal{J}_U$ is an injective $\mathcal{O}_U$-module, ψ extends to a map $\psi^0\colon\mathcal{O}_U/\mathcal{I}_U\to\mathcal{J}_U$, and then

$$t^{s-r}\rho=\psi^0(t^s+\mathcal{I}_U)=t^s\psi^0(1+\mathcal{I}_U)\in t^s\mathrm{Hom}(\mathcal{O}_U/\mathcal{I}_U,\mathcal{J}_U)\subset\Gamma(U,\mathcal{J}_U).$$

□

Corollary (4.2). *With X, $\mathbf{t}$, Z and $\mathcal{P}$ as in Proposition* (4.1), *let*

$$\Psi=\Psi(\mathcal{P})\colon\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X,\mathcal{P})\xrightarrow{\ \sim\ }\varprojlim_{r>0}\mathcal{P}/\mathbf{t}^r\mathcal{P}=\Lambda_Z\mathcal{P}=\mathbf{L}\Lambda_Z\mathcal{P}$$

be the isomorphism constructed in proving that Proposition (*easily seen to be independent of the injective resolution $\mathcal{P}\to\mathcal{J}$ used there*) *and let*

$$\Phi=\Phi(\mathcal{P},\mathcal{O}_X)\colon\mathbf{L}\Lambda_Z\mathcal{P}\longrightarrow\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X,\mathcal{P})$$

be as in (2.2). *Then $\Phi=\Psi^{-1}$, and so Φ is an isomorphism.*

Proof. We need only show that $H^0(\Psi)\circ H^0(\Phi)$ is the identity map of $\varprojlim \mathcal{P}/\mathbf{t}^r\mathcal{P}$.

Let $\chi\colon \mathcal{O}_X \to \mathcal{R}$ and $\theta\colon \mathcal{P}\otimes\mathcal{R} \to \mathcal{J}$ be quasi-isomorphisms with $\mathcal{R}$ and $\mathcal{J}$ injective complexes vanishing in all negative degrees. The composition

$$\mathcal{P} = \mathcal{P}\otimes\mathcal{O}_X \xrightarrow{1\otimes\chi} \mathcal{P}\otimes\mathcal{R} \xrightarrow{\theta} \mathcal{J}$$

is then an injective resolution of $\mathcal{P}$, which can be used to define Ψ.

We have, by Lemmas (3.1.1)(2) and (3.2.3), $\mathbf{D}(X)$-isomorphisms

$$\varGamma'_{\mathbf{t}}\mathcal{R} = \varinjlim \mathcal{H}om^\bullet(\mathcal{O}_X/\mathbf{t}^r\mathcal{O}_X, \mathcal{R}) \xrightarrow{\ \sim\ } \varinjlim \mathcal{K}^\bullet(\mathbf{t}^r, \mathcal{R}) = \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{R}) \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_Z\mathcal{O}_X,$$

whose composition "is" the natural isomorphism $\mathbf{R}\varGamma'_Z\mathcal{O}_X \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_Z\mathcal{O}_X$ of (3.2.4). The obvious commutative diagram

$$\begin{array}{ccc}
H^0\mathcal{H}om^\bullet(\varinjlim \mathcal{H}om^\bullet(\mathcal{O}_X/\mathbf{t}^r\mathcal{O}_X,\mathcal{R}), \mathcal{J}) & \xleftarrow{\ \sim\ } & H^0\mathcal{H}om^\bullet(\varinjlim \mathcal{K}^\bullet(\mathbf{t}^r,\mathcal{R}), \mathcal{J}) \\
\simeq\big\downarrow & & \big\downarrow\simeq \\
H^0\varprojlim \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{O}_X/\mathbf{t}^r\mathcal{O}_X,\mathcal{R}), \mathcal{J}) & \xleftarrow[a]{} & H^0\varprojlim \mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r,\mathcal{R}), \mathcal{J})
\end{array}$$

shows that a is an isomorphism; and it is straightforward to check that $H^0(\Psi)$ can be identified with the natural composition

$$\begin{aligned}
H^0\varprojlim \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{O}_X/\mathbf{t}^r\mathcal{O}_X,\mathcal{R}), \mathcal{J}) &\xrightarrow{a^{-1}} H^0\varprojlim \mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r,\mathcal{R}), \mathcal{J}) \\
&\xrightarrow{\ b\ } \varprojlim H^0\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r,\mathcal{R}), \mathcal{J}) \\
&\xrightarrow{\ \sim\ } \varprojlim H^0\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J}) \\
&\xrightarrow{\ \sim\ } \varprojlim H^0\mathcal{H}om^\bullet(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}) = \varprojlim \mathcal{P}/\mathbf{t}^r\mathcal{P}.
\end{aligned}$$

The map b is an isomorphism, since $H^0(\Psi)$ is. From the natural commutative diagram, in which we have abbreviated $\mathcal{H}om^\bullet$ to $\mathcal{H}$, and whose top row is $H^0(\Phi)$,

$$\begin{array}{ccccc}
\varprojlim \mathcal{P}/\mathbf{t}^r\mathcal{P} & \longrightarrow & H^0\varprojlim \mathcal{H}\big(\mathcal{H}(\mathcal{O}_X/\mathbf{t}^r\mathcal{O}_X,\mathcal{R}), \mathcal{P}\otimes\mathcal{R}\big) & \longrightarrow & H^0\varprojlim \mathcal{H}\big(\mathcal{H}(\mathcal{O}_X/\mathbf{t}^r\mathcal{O}_X,\mathcal{R}), \mathcal{J}\big) \\
{\scriptstyle c}\big\uparrow & & \big\uparrow & & \simeq\big\downarrow{\scriptstyle a^{-1}} \\
H^0\varprojlim \mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}\big) & \longrightarrow & H^0\varprojlim \mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r)\otimes\mathcal{R}, \mathcal{P}\otimes\mathcal{R}\big) & \longrightarrow & H^0\varprojlim \mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r,\mathcal{R}), \mathcal{J}\big) \\
{\scriptstyle c}\big\downarrow & & \big\downarrow & & \simeq\big\downarrow{\scriptstyle b} \\
\varprojlim H^0\mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}\big) & \longrightarrow & \varprojlim H^0\mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r)\otimes\mathcal{R}, \mathcal{P}\otimes\mathcal{R}\big) & \longrightarrow & \varprojlim H^0\mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r,\mathcal{R}), \mathcal{J}\big) \\
\big\| & & \big\downarrow & & \simeq\big\downarrow \\
\varprojlim H^0\mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}\big) & \xrightarrow{\ \sim\ } & \varprojlim H^0\mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}\otimes\mathcal{R}\big) & \xrightarrow{\ \sim\ } & \varprojlim H^0\mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{J}\big)
\end{array}$$

we find that $H^0(\Phi)\circ c = H^0(\Psi)^{-1}\circ c$; and since c has $\mathbf{t}$-adically dense image in $\varprojlim \mathcal{P}/\mathbf{t}^r\mathcal{P}$—at least after application of any functor of the form $\Gamma(U,-)$ with $U\subset X$ affine open—(because the complex $\varprojlim \mathcal{H}\big(\mathcal{K}^\bullet(\mathbf{t}^r), \mathcal{P}\big)$ is just $\mathcal{P}$ in degree 0), we conclude that $H^0(\Phi) = H^0(\Psi)^{-1}$. $\square$

And finally:

Lemma (4.3). *If X is a quasi-compact scheme and $Z \subset X$ is a proregularly embedded closed subset then the functor $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, -)\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}(X)$ is bounded above and below.*

Proof. Since X is quasi-compact the question is local, so we may assume that X is affine and that $Z = \mathrm{Supp}(\mathcal{O}_X/\mathbf{t}\mathcal{O}_X)$ for some proregular sequence $\mathbf{t} = (t_1, \dots, t_\mu)$ in $\Gamma(X, \mathcal{O}_X)$.

Lemma (3.2.3) gives a functorial isomorphism

$$\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, -) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(\mathcal{K}^\bullet_\infty(\mathbf{t}), -).$$

For any complex $\mathcal{E} \in \mathbf{D}(X)$ such that $H^i(\mathcal{E}) = 0$ whenever $i < i_0$, there is a quasi-isomorphic injective complex $\mathcal{J}$ vanishing in all degrees below i_0, and then since the complex $\mathcal{K}^\bullet_\infty(\mathbf{t})$ vanishes in all degrees outside the interval $[0, \mu]$,

$$H^i\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathcal{E}) \cong H^i\mathcal{H}om^\bullet(\mathcal{K}^\bullet_\infty(\mathbf{t}), \mathcal{J}) = 0 \quad \text{for all } i < i_0 - \mu.$$

Thus the functor $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{O}_X, -)$ is bounded below.

To establish boundedness above, suppose $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$ is such that $H^i(\mathcal{F}) = 0$ for all $i > i_0$, and let us prove that $H^i\mathbf{R}\mathcal{H}om^\bullet(\mathcal{K}^\bullet_\infty(\mathbf{t}), \mathcal{F}) = 0$ for all $i > i_0$.

By [BN, p. 225, Thm. 5.1], we may assume that $\mathcal{F}$ is actually a quasi-coherent complex, which after truncation may also be assumed to vanish in degrees $> i_0$. Let

$$f_n\colon \tau^{\geq -n}\mathcal{F} \to \mathcal{J}_n \qquad (n \geq 0)$$

be the inverse system of quasi-isomorphisms of [Sp, p. 133, Lemma 3.7], where τ is the truncation functor and $\mathcal{J}_n$ is an injective complex vanishing in degrees $< -n$. Writing $\Gamma(-)$ for $\Gamma(X, -)$, we have natural isomorphisms

$$H^{-m}\Gamma(\mathcal{F}) \xrightarrow{\sim} H^{-m}\Gamma(\tau^{\geq -n}\mathcal{F}) \xrightarrow{\sim} H^{-m}\Gamma(\mathcal{J}_n) \qquad \big(m \in \mathbb{Z},\ n > \max(m, 0)\big),$$

the second isomorphism holding since both $\tau^{\geq -n}\mathcal{F}$ and $\mathcal{J}_n$ are Γ-acyclic complexes. Further, as in the proof of [Sp, p. 134, Prop. 3.13] we have, with $\mathcal{J} := \varprojlim \mathcal{J}_n$,

$$H^{-m}\Gamma(\mathcal{J}) \xrightarrow{\sim} H^{-m}\Gamma(\mathcal{J}_n).$$

Hence *the natural map $H^{-m}\Gamma(\mathcal{F}) \to H^{-m}\Gamma(\mathcal{J})$ is an isomorphism for every m.*

Knowing that, we can argue just as in the proof of Proposition (4.1) to deduce that the maps π_i in (4.1.1)—with $\mathcal{F}$ in place of $\mathcal{P}$—are isomorphisms for all $i > i_0$, whence the conclusion. □

5. Various dualities reincarnated. Theorem (0.3) leads to sheafified generalizations ((5.1.3), respectively (5.2.3)) of the *Warwick Duality* theorem of Greenlees and the *Affine Duality* theorem of Hartshorne. In (5.3) we see how together with Grothendieck Duality, Affine Duality gives a *Formal Duality* theorem of Hartshorne. A similar argument yields the related duality theorem of [L2, p. 188], which combines local and global duality. In (5.4), using (0.3) and an [EGA] theorem on homology and completion, we establish a long exact sequence of Ext functors, which gives in particular the *Peskine-Szpiro duality sequence* (0.4.3).

Corollary (5.1.1). *Let X be a quasi-compact separated scheme and $Z \subset X$ a proregularly embedded closed subscheme. Let $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$, let $\gamma\colon \mathbf{R}\varGamma_Z\mathcal{F} \to \mathcal{F}$ be the natural map, and let $\nu\colon \mathcal{F} \to \mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}$ correspond to the natural map $\lambda\colon \mathcal{F} \to \mathbf{L}\Lambda_Z\mathcal{F}$ (see* (0.4)(a)*). Then γ and ν induce* isomorphisms

$$\text{(i)}\qquad \mathbf{L}\Lambda_Z\mathbf{R}\varGamma_Z\mathcal{F} \xrightarrow{\ \sim\ } \mathbf{L}\Lambda_Z\mathcal{F},$$

$$\text{(ii)}\qquad \mathbf{R}\varGamma_Z\mathcal{F} \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_Z\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}.$$

Proof. Recall from (3.2.5) that $\mathbf{R}\varGamma_Z\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$. Theorem (0.3) transforms the map (i) into the map

$$\mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathbf{R}\varGamma_Z\mathcal{F}) \xrightarrow{\text{via }\gamma} \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathcal{F})$$

which is, by (0.4.2), an isomorphism.

We could also proceed without recourse to Theorem (0.3), as follows. We may assume, by (1.1), that $\mathcal{F}$ is flat and quasi-coherent. The question is local, so we can replace $\mathbf{R}\varGamma_Z\mathcal{F}$ by a complex of the form $\mathcal{K}^{\bullet}_{\infty}(\mathbf{t},\mathcal{F})$ (see (3.2.3)), and then via (3.2)(c), $\gamma\colon \mathbf{R}\varGamma_Z\mathcal{F} \to \mathcal{F}$ becomes the natural map $\mathcal{C}_\xi[-1] \to \mathcal{F}$ where $\mathcal{C}_\xi$ is the cone of the map $\xi\colon \mathcal{F} \to \mathcal{K}_\flat \otimes \mathcal{F}$ of (3.2)(b). Since $\mathcal{K}_\flat \otimes \mathcal{F} = \mathbf{t}(\mathcal{K}_\flat \otimes \mathcal{F})$, therefore $\Lambda_{\mathbf{t}}(\mathcal{K}_\flat \otimes \mathcal{F}) := \varprojlim \big((\mathcal{K}_\flat \otimes \mathcal{F})/\mathbf{t}^n(\mathcal{K}_\flat \otimes \mathcal{F})\big) = 0$, and so $\mathbf{L}\Lambda_Z(\gamma)$ is an isomorphism.

As for (ii): with $\mathrm{Hom} := \mathrm{Hom}_{\mathbf{D}(X)}$ and $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$, the composition

$$\mathrm{Hom}(\mathbf{R}\varGamma_Z\mathcal{E}, \mathcal{F}) \xrightarrow[\text{via }\nu]{} \mathrm{Hom}(\mathbf{R}\varGamma_Z\mathcal{E}, \mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) \xrightarrow[(0.4)(\mathrm{a})]{\ \sim\ } \mathrm{Hom}(\mathbf{R}\varGamma_Z\mathcal{E}, \mathbf{L}\Lambda_Z\mathcal{F})$$

is an isomorphism: it is the map obtained by applying the functor $H^0\mathbf{R}\Gamma(X,-)$ to the isomorphism λ' of Theorem (0.3)(bis). (Recall that $\mathbf{R}\varGamma'_Z\mathcal{E} \cong \mathbf{R}\varGamma_Z\mathcal{E}$, (3.2.4)). Hence "via ν" is an isomorphism, and so by (0.4.2) the map

$$\mathrm{Hom}(\mathbf{R}\varGamma_Z\mathcal{E}, \mathbf{R}\varGamma_Z\mathcal{F}) \to \mathrm{Hom}(\mathbf{R}\varGamma_Z\mathcal{E}, \mathbf{R}\varGamma_Z\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F})$$

induced by ν is also an isomorphism. Taking $\mathcal{E} = \mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}$, we see then that the map (ii) has an inverse, so it is an isomorphism. □

Remark (5.1.2). We just saw that λ' an isomorphism implies that so is (5.1.1)(ii). Conversely, to show that λ' is an isomorphism, one can reduce via (0.4.2) and (5.1.1)(i) to where $\mathcal{F} = \mathbf{R}\varGamma_Z\mathcal{F}$, then use (5.1.1)(ii) to get for each open $U \subset X$ that the maps

$$\mathrm{Hom}_{\mathbf{D}(U)}\big(\mathbf{R}\varGamma_{Z\cap U}\mathcal{E}|_U, \mathcal{F}|_U[i]\big) \to \mathrm{Hom}_{\mathbf{D}(U)}\big(\mathbf{R}\varGamma_{Z\cap U}\mathcal{E}|_U, \mathbf{L}\Lambda_{Z\cap U}\mathcal{F}|_U[i]\big) \qquad (i \in \mathbb{Z})$$

induced by λ are all isomorphisms, so that λ' induces homology isomorphisms.

With the notation and relations given in Remark (0.4)(d), we find that the map (5.1.1)(ii) is an isomorphism iff the corresponding map $\mathbf{R}\Gamma_{\mathbf{t}}F \to \mathbf{R}\Gamma_{\mathbf{t}}\mathbf{L}\Lambda_{\mathbf{t}}F$ is an isomorphism for any complex of A-modules; in other words, iff Corollary (0.3.1) holds.

The next result extends Greenlees's "Warwick Duality" [Gl, p. 66, Thm. 4.1] (where $\mathcal{G} = \mathcal{O}_U$, so that $\mathrm{Ext}^n_U(\mathcal{G}, i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) = \mathbb{H}^n(X, \mathbf{R}i_*i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F})$ is the "local Tate cohomology" of $\mathcal{F}$). As before, Q is the quasi-coherator.

Proposition (5.1.3). *Let X be a quasi-compact separated scheme, let $Z \subset X$ be a proregularly embedded closed subscheme, and let $i\colon U = (X \setminus Z) \hookrightarrow X$ be the inclusion. Then for $\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(U)$ and $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$ there are natural isomorphisms*

$$\operatorname{Ext}^n_U(\mathcal{G}, i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) \xrightarrow{\sim} \operatorname{Ext}^{n+1}_X(\mathbf{R}i_*\mathcal{G}, \mathbf{R}\varGamma_Z\mathcal{F}) \qquad (n \in \mathbb{Z}).$$

Proof. Since $\mathcal{G} = i^*\mathbf{R}i_*\mathcal{G}$, there is a natural isomorphism [Sp, p. 147, Prop. 6.7, (1)]

$$(*) \qquad \mathbf{R}\mathrm{Hom}^\bullet_U(\mathcal{G}, i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{R}i_*\mathcal{G}, \mathbf{R}i_*i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}).$$

The canonical triangle $\mathbf{R}\varGamma_Z\mathbf{R}i_*\mathcal{G} \to \mathbf{R}i_*\mathcal{G} \to \mathbf{R}i_*i^*\mathbf{R}i_*\mathcal{G} \xrightarrow{+}$ (see (0.4.2.1)) implies $\mathbf{R}\varGamma_Z\mathbf{R}i_*\mathcal{G} = 0$; and $\mathbf{R}i_*\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(X)$ (see [L, (3.9.2)] for the unbounded case); hence

$$\mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{R}i_*\mathcal{G}, \mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) \cong \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{R}\varGamma_Z\mathbf{R}i_*\mathcal{G}, \mathcal{F}) = 0$$

(see (0.4)(a)), so the triangle $\mathbf{R}\varGamma_Z\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F} \to \mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F} \to \mathbf{R}i_*i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F} \xrightarrow{+}$ yields a natural isomorphism

$$(**) \quad \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{R}i_*\mathcal{G}, \mathbf{R}i_*i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_X\big(\mathbf{R}i_*\mathcal{G}, \mathbf{R}\varGamma_Z\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}[1]\big).$$

By (5.1.1)(ii) there is a natural isomorphism

$$(***) \qquad \mathbf{R}\mathrm{Hom}^\bullet_X\big(\mathbf{R}i_*\mathcal{G}, \mathbf{R}\varGamma_Z\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F}[1]\big) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_X\big(\mathbf{R}i_*\mathcal{G}, \mathbf{R}\varGamma_Z\mathcal{F}[1]\big).$$

Compose the isomorphisms $(*)$, $(**)$, $(***)$, and take homology to conclude. □

Remark. The complex $\mathcal{T}_Z\mathcal{F} := \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}i_*\mathcal{O}_U[-1], \mathbf{R}\varGamma_Z\mathcal{F})$, whose hyperhomology

$$\mathbb{T}^\bullet_Z(X,\mathcal{F}) := \mathbb{H}^\bullet(X, \mathcal{T}_Z\mathcal{F}) := H^\bullet\mathbf{R}\Gamma(X, \mathcal{T}_Z\mathcal{F}) \underset{(5.1.3)}{\cong} \mathbb{H}^\bullet(U, i^*\mathbf{R}Q\mathbf{L}\Lambda_Z\mathcal{F})$$

is the *local Tate cohomology* of $\mathcal{F}$, is the summit of a triangle based on the canonical map $\mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{O}_X, \mathbf{R}\varGamma_Z\mathcal{F}) \to \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathbf{R}\varGamma_Z\mathcal{F})$, a map isomorphic via (0.3) and (5.1.1)(i) to the natural composition $\mathbf{R}\varGamma_Z\mathcal{F} \to \mathcal{F} \to \mathbf{L}\Lambda_Z\mathcal{F}$. So there is a long exact sequence

$$\cdots \to \mathbb{H}^n_Z(X,\mathcal{F}) \to \mathbb{H}^n(X, \mathbf{L}\Lambda_Z\mathcal{F}) \to \mathbb{T}^n_Z(X,\mathcal{F}) \to \mathbb{H}^{n+1}_Z(X,\mathcal{F}) \to \cdots$$

and thus, as Greenlees points out, local Tate cohomology pastes together the right-derived functors of $\varGamma_Z$ and the left-derived functors of Λ_Z.

(5.2). Next, we derive a generalized form of Affine Duality [H2, p. 152, Thm. 4.1], see Corollary (5.2.3): "double dual = completion".

Proposition (5.2.1). *Let X be a scheme and $Z \subset X$ a closed subscheme. Then for any $\mathcal{E}, \mathcal{F} \in \mathbf{D}(X)$ there is a natural isomorphism*

$$\mathbf{R}\varGamma_Z\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}, \mathbf{R}\varGamma_Z\mathcal{F}).$$

If in addition X is quasi-compact and separated, Z is proregularly embedded, $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(X)$, and $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) \in \mathbf{D}_{\mathrm{qc}}(X)$, then there is a natural isomorphism

$$\mathbf{L}\Lambda_Z\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}, \mathbf{L}\Lambda_Z\mathcal{F}).$$

Proof. Let $i\colon (X\setminus Z)\hookrightarrow X$ be the inclusion. Since i^* has an exact left adjoint (extension by zero), therefore i^* preserves K-injectivity, and consequently there is a natural isomorphism $i^*\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F})\xrightarrow{\sim}\mathbf{R}\mathcal{H}om^\bullet(i^*\mathcal{E},i^*\mathcal{F})$. The first assertion results then from the commutative diagram, whose rows are triangles (see (0.4.2.1)):

$$\begin{array}{ccccccc}
\mathbf{R}\varGamma_Z\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathbf{R}i_*i^*\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) & \xrightarrow{+} & \\
 & & \Big\Vert & & \simeq\Big\downarrow\text{[Sp, p. 147, 6.7]} & & \\
\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathbf{R}\varGamma_Z\mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathbf{R}i_*i^*\mathcal{F}) & \xrightarrow{+} &
\end{array}$$

The second assertion is given by the sequence of natural isomorphisms

$$\begin{aligned}
\mathbf{L}\Lambda_Z\mathbf{R}\mathcal{H}om^\bullet\big(\mathcal{E},\mathcal{F}\big) &\underset{(0.3)}{\xrightarrow{\sim}} \mathbf{R}\mathcal{H}om^\bullet\big(\mathbf{R}\varGamma_Z\mathcal{O}_X,\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F})\big)\\
&\xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet\big((\mathbf{R}\varGamma_Z\mathcal{O}_X)\underset{=}{\otimes}\mathcal{E},\mathcal{F}\big) \qquad \text{[Sp, p. 147, 6.6]}\\
&\underset{(3.1.5)}{\xrightarrow{\sim}} \mathbf{R}\mathcal{H}om^\bullet\big(\mathbf{R}\varGamma'_Z\mathcal{E},\mathcal{F}\big) \underset{(0.3)}{\xrightarrow{\sim}} \mathbf{R}\mathcal{H}om^\bullet\big(\mathcal{E},\mathbf{L}\Lambda_Z\mathcal{F}\big). \qquad \square
\end{aligned}$$

Suppose further that X is noetherian. Let $\mathcal{R}\in\mathbf{D}_{\mathrm{qc}}(X)$ have *finite injective dimension* [H, p. 83, p. 134]. Then for any $\mathcal{F}\in\mathbf{D}_{\mathrm{c}}(X)$ the complex

$$\mathcal{D}(\mathcal{F}):=\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F},\mathcal{R})$$

is in $\mathbf{D}_{\mathrm{qc}}(X)$ [H, p. 91, Lemma 3.2 and p.73, Prop. 7.3], whence—by (3.2.5)—so is the "Z-dual" complex

$$\mathcal{D}_Z(\mathcal{F}):=\mathbf{R}\varGamma_Z\mathcal{D}(\mathcal{F}) \underset{(5.2.1)}{\cong} \mathbf{R}\mathcal{H}om^\bullet(\mathcal{F},\mathbf{R}\varGamma_Z\mathcal{R}).$$

For example, if $\mathcal{R}$ is a *dualizing complex* [H, p. 258], if $x\in X$ is a closed point, and $\mathcal{J}(x)$ is the injective $\mathcal{O}_X$-module vanishing except at x, where its stalk is the injective hull of the residue field of the local ring $\mathcal{O}_{X,x}$, then by [H, p. 285],

$$\mathcal{D}_{\{x\}}(\mathcal{F})=\mathcal{H}om^\bullet\big(\mathcal{F},\mathcal{J}(x)\big)[-d(x)]$$

where $d(x)$ is the integer defined in [H, p. 282].

As in the proof of the second assertion in (5.2.1), there is a natural isomorphism

$$\mathbf{L}\Lambda_Z\mathcal{D}(\mathcal{F})=\mathbf{L}\Lambda_Z\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F},\mathcal{R})\xrightarrow{\sim}\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma_Z\mathcal{F},\mathcal{R})=\mathcal{D}\mathbf{R}\varGamma_Z(\mathcal{F});$$

and so if $\mathcal{F}\in\mathbf{D}_{\mathrm{c}}(X)$, whence $\mathcal{D}(\mathcal{F})\in\mathbf{D}_{\mathrm{c}}(X)$, then there is a natural isomorphism

$$\mathbf{L}\Lambda_Z\mathcal{D}\mathcal{D}(\mathcal{F})\xrightarrow{\sim}\mathcal{D}\mathbf{R}\varGamma_Z\mathcal{D}(\mathcal{F})=\mathcal{D}\mathcal{D}_Z(\mathcal{F})\underset{(0.4.2)}{=}\mathcal{D}_Z\mathcal{D}_Z(\mathcal{F}).$$

Thus:

Corollary (5.2.2). *Let X be a noetherian separated scheme, let $Z \subset X$ be closed, and let $\mathcal{R} \in \mathbf{D}_{\mathrm{c}}(X)$ have finite injective dimension. Then for any $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$ we have, with preceding notation, canonical isomorphisms*

$$\mathcal{D}\mathbf{R}\mathit{\Gamma}_Z(\mathcal{F}) \xrightarrow{\sim} \mathbf{L}\Lambda_Z\mathcal{D}(\mathcal{F}),$$
$$\mathcal{D}_Z\mathcal{D}_Z(\mathcal{F}) \xrightarrow{\sim} \mathbf{L}\Lambda_Z\mathcal{D}\mathcal{D}(\mathcal{F}).$$

Corollary (5.2.3). *Let X be a noetherian separated scheme having a dualizing complex $\mathcal{R}$. Let $Z \subset X$ be closed, and let $\kappa\colon X_{/Z} \to X$ be the completion map. Then for $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$, and with $\mathcal{D}_Z$ as above, the natural map $\beta\colon \mathcal{F} \to \mathcal{D}_Z\mathcal{D}_Z\mathcal{F}$ factors via an isomorphism*

$$\kappa_*\kappa^*\mathcal{F} \xrightarrow{\sim} \mathcal{D}_Z\mathcal{D}_Z\mathcal{F}.$$

Proof. Since $\mathcal{R}$ is a dualizing complex, therefore $\mathcal{R} \in \mathbf{D}_{\mathrm{c}}(X)$, $\mathcal{R}$ has finite injective dimension, and the natural map $\mathcal{F} \to \mathcal{D}\mathcal{D}\mathcal{F}$ is an isomorphism [H, p. 258]. One checks then that β factors naturally as:

$$\mathcal{F} \to \kappa^*\kappa_*\mathcal{F} \underset{(0.4.1)}{\xrightarrow{\sim}} \mathbf{L}\Lambda_Z\mathcal{F} \xrightarrow{\sim} \mathbf{L}\Lambda_Z\mathcal{D}\mathcal{D}\mathcal{F} \underset{(5.2.2)}{\xrightarrow{\sim}} \mathcal{D}_Z\mathcal{D}_Z\mathcal{F}. \qquad \square$$

(5.3). Here are some applications of Theorem (0.3) involving Grothendieck Duality (abbreviated GD) and basic relations between homology and completion.

Let A be a noetherian local ring, with maximal ideal m, and let I be an injective hull of the A-module A/m. Assume that $Y := \mathrm{Spec}(A)$ has a *dualizing complex* $\mathcal{R}_Y$, which we may assume to be *normalized* [H, p. 276]; and let $f\colon X \to Y$ be a proper scheme-map, so that $\mathcal{R}_X := f^!\mathcal{R}_Y$ is a dualizing complex on X [V, p. 396, Cor. 3]. For any $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$, set

$$\mathcal{F}' := \mathcal{D}(\mathcal{F}) = \mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{R}_X) \in \mathbf{D}_{\mathrm{c}}(X).$$

Let Z be a closed subset of $f^{-1}\{m\}$, define $\mathcal{D}_Z(\mathcal{F})$ as in (5.2) to be $\mathbf{R}\mathit{\Gamma}_Z\mathcal{F}'$, and let $\kappa\colon \widehat{X} \to X$ be the canonical map to X from its formal completion along Z.

Hartshorne's Formal Duality theorem [H3, p. 48, Prop. (5.2)] is a quite special instance of the following composed isomorphism, for $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$:[12]

$$\begin{aligned}
\mathbf{R}\Gamma(\widehat{X}, \kappa^*\mathcal{F}) = \mathbf{R}\Gamma(X, \kappa_*\kappa^*\mathcal{F}) &\xrightarrow{\sim} \mathbf{R}\Gamma(X, \mathcal{D}_Z\mathcal{D}_Z\mathcal{F}) && (5.2.3)\\
&\xrightarrow{\sim} \mathbf{R}\Gamma(X, \mathcal{D}\mathcal{D}_Z\mathcal{F}) && (5.2.1), (0.4.2)\\
&=\!= \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{R}\mathit{\Gamma}_Z\mathcal{F}', \mathcal{R}_X) &&\\
&\xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}_Z\mathcal{F}', \mathcal{R}_Y) && (\mathrm{GD})\\
&\xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}_Z\mathcal{F}', \mathbf{R}\mathit{\Gamma}_{\{m\}}\mathcal{R}_Y) && (0.4.2)\\
&\xrightarrow{\sim} \mathrm{Hom}_A(\mathbf{R}\Gamma_Z\mathcal{F}', I) && [\mathrm{H, p. 285}]
\end{aligned}$$

where $\Gamma_Z(-) := \Gamma\big(X, \mathit{\Gamma}_Z(-)\big)$. The last isomorphism follows from (0.4.4) because $\widetilde{I} \cong \mathbf{R}\mathit{\Gamma}_{\{m\}}\mathcal{R}_Y$ [H, p. 285], and $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}_Z\mathcal{F}' \cong \widetilde{\mathbf{R}\Gamma_Z\mathcal{F}'}$.[13]

[12] Hartshorne requires Z, but not necessarily X, to be proper over A. Assuming f separated and finite-type, we can reduce that situation to the present one by compactifying f [Lü].

[13] Some technical points here need attention, especially when $\mathcal{F}$ is unbounded. First, GD holds

Taking homology, we get isomorphisms

$$H^q(\widehat{X}, \kappa^*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Ext}_Z^{-q}(\mathcal{F}, \mathcal{R}_X), I) \qquad (\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X),\ q \in \mathbb{Z}). \tag{5.3.1}$$

(The functor $\mathrm{Ext}_Z^{\bullet}$ is reviewed in §5.4 below).

For example, if A is Gorenstein and f is a Cohen-Macaulay map of relative dimension n, then $\mathcal{R}_Y \cong \mathcal{O}_Y$, $\mathcal{R}_X \cong \omega[n]$ for some coherent $\mathcal{O}_X$-module ω (the *relative dualizing sheaf*), and (5.3.1) becomes

$$H^q(\widehat{X}, \kappa^*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Ext}_Z^{n-q}(\mathcal{F}, \omega), I).$$

Assume now that $Z = f^{-1}\{m\}$. For $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$ the following Lemma (with $J = m$), and the preceding composition yield isomorphisms

$$\mathbf{R}\mathrm{Hom}_X^{\bullet}(\mathcal{F}, \mathcal{R}_X) \otimes_A \hat{A} \cong \mathbf{R}\Gamma(X, \mathcal{F}') \otimes_A \hat{A} \cong \mathbf{R}\Gamma(\widehat{X}, \kappa^*\mathcal{F}') \cong \mathrm{Hom}_A(\mathbf{R}\Gamma_Z\mathcal{F}'', I).$$

Thus (since $\mathcal{F}'' = \mathcal{F}$) there is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}_X^{\bullet}(\mathcal{F}, \mathcal{R}_X) \otimes_A \hat{A} \xrightarrow{\sim} \mathrm{Hom}_A(\mathbf{R}\Gamma_Z\mathcal{F}, I) \qquad \big(\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)\big).$$

Since $\mathbf{R}\mathrm{Hom}_X^{\bullet}(\mathcal{F}, \mathcal{R}_X)$ has noetherian homology modules therefore $\mathbf{R}\Gamma_Z\mathcal{F}$ has artinian homology modules, and Matlis dualization produces a natural isomorphism

$$\mathbf{R}\Gamma_Z\mathcal{F} \xrightarrow{\sim} \mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_X^{\bullet}(\mathcal{F}, \mathcal{R}_X), I) \qquad \big(\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)\big). \tag{5.3.2}$$

For bounded $\mathcal{F}$, this isomorphism is [L2, p. 188, Theorem], deduced there directly from GD and Local Duality (which is the case $X = Y$, $f =$ identity map).

Lemma (5.3.3). *Let A be a noetherian ring, J an A-ideal, $\hat{A}$ the J-completion, $f\colon X \to \mathrm{Spec}(A)$ a finite-type map, $Z := f^{-1}\mathrm{Spec}(A/J)$, and $\kappa\colon \widehat{X} = X_{/Z} \to X$ the canonical flat map.*

(a) *If $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$ has proper support (i.e., $\mathcal{E}$ is exact outside a subscheme Y of X which is proper over* $\mathrm{Spec}(A)$), *then there is a natural isomorphism*

$$\mathbf{R}\Gamma(X, \mathcal{E}) \otimes_A \hat{A} \xrightarrow{\sim} \mathbf{R}\Gamma(\widehat{X}, \kappa^*\mathcal{E}).$$

(b) *Let $\mathcal{E} \in \mathbf{D}_{\mathrm{c}}(X)$, $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}^{+}(X)$, and suppose either that $\mathcal{E} \in \mathbf{D}_{\mathrm{c}}^{-}(X)$ or that $\mathcal{F}$ has finite injective dimension. Suppose further that $\mathbf{R}\mathcal{H}om_X^{\bullet}(\mathcal{E}, \mathcal{F})$ has proper support. Then there is a natural isomorphism*

$$\mathbf{R}\mathrm{Hom}_X^{\bullet}(\mathcal{E}, \mathcal{F}) \otimes_A \hat{A} \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\widehat{X}}^{\bullet}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F})$$

Hence, by $(0.3)_{\mathrm{c}}$, *if moreover $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^{+}(X)$ then there is a natural isomorphism*

$$\mathbf{R}\mathrm{Hom}_X^{\bullet}(\mathcal{E}, \mathcal{F}) \otimes_A \hat{A} \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_X^{\bullet}(\mathbf{R}\varGamma_Z\mathcal{E}, \mathcal{F}).$$

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for unbounded $\mathcal{F}$, see [N]. Next, since $\mathbf{R}\varGamma_Z\mathcal{F}' \in \mathbf{D}_{\mathrm{qc}}(X)$ (3.2.5), therefore $\mathbf{R}f_*\mathbf{R}\varGamma_Z\mathcal{F}' \in \mathbf{D}_{\mathrm{qc}}(Y)$ [L, (3.9.2)]; and so by (1.3), $\mathbf{R}f_*\mathbf{R}\varGamma_Z\mathcal{F}' \cong \big(\mathbf{R}\Gamma(Y, \mathbf{R}f_*\mathbf{R}\varGamma_Z\mathcal{F}')\big)^{\sim}$. Finally, using [Sp, 6.4 and 6.7] and the fact that f_* and $\varGamma_Z$ preserve K-flabbiness (see Remark following (3.2.5) above), one checks that $\mathbf{R}\Gamma(Y, \mathbf{R}f_*\mathbf{R}\varGamma_Z\mathcal{F}') \cong \mathbf{R}\Gamma(X, \mathbf{R}\varGamma_Z\mathcal{F}') \cong \mathbf{R}\Gamma_Z\mathcal{F}'$.

Proof. (a) For bounded-below $\mathcal{E}$, way-out reasoning [H, p. 68, Prop. 7.1] brings us to where $\mathcal{E} = \mathcal{G}$, a single quasi-coherent $\mathcal{O}_X$-module supported in Y. Since $\mathcal{G}$ is the $\varinjlim$ of its coherent submodules, and since homology on the noetherian spaces X and $\widehat{X}$ commutes with $\varinjlim$, as does κ^*, we can conclude via [EGA, p. 129, 4.1.10].

There is an integer d such that $H^n(X, \mathcal{G}) = 0$ for all $n > d$ and all such $\mathcal{G}$; so the same holds for $H^n(\widehat{X}, \kappa^*\mathcal{G})$, and hence the method (deriving from [Sp]) used to prove [L, Prop. (3.9.2)] gets us from the bounded to the unbounded case.

(b) By (a), and since $\mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E}, \mathcal{F}) \in \mathbf{D}^+_{\mathrm{qc}}(X)$, [H, p. 92, 3.3], it suffices to show that the natural map $\kappa^*\mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E}, \mathcal{F}) \to \mathbf{R}\mathcal{H}om^\bullet_{\widehat{X}}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F})$ is an isomorphism. The question is local, so we can assume X affine and, κ being flat, we can use [H, p. 68, Prop. 7.1] to reduce to the trivial case $\mathcal{E} = \mathcal{O}^n_X$. □

(5.4). The exact sequence (0.4.3) is a special case of the last sequence in the following Proposition (5.4.1) (which also generalizes the last assertion in (5.3.3)(b)).

When W is a locally closed subset of a ringed space X, and $\mathcal{E}, \mathcal{F} \in \mathbf{D}(X)$, then following [Gr, Exposé VI] one sets

$$\mathrm{Ext}^n_W\big(\mathcal{E}, \mathcal{F}\big) := H^n\big(\mathbf{R}\Gamma_W \mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E}, \mathcal{F})\big) = H^n\big(\mathbf{R}(\Gamma_W \mathcal{H}om_X)(\mathcal{E}, \mathcal{F})\big) \quad (n \in \mathbb{Z})$$

where $\Gamma_W(-) := \Gamma\big(X, \mathit{\Gamma}_W(-)\big)$ is the functor of global sections supported in W, and the second equality is justified by [Sp, p. 146, 6.1(iii) and 6.4] (which uses the preparatory results 4.5, 5.6, 5.12, and 5.22). It also holds, via (5.2.1), that

$$\mathrm{Ext}^n_W\big(\mathcal{E}, \mathcal{F}\big) = H^n\big(\mathbf{R}\mathrm{Hom}^\bullet_X(\mathcal{E}, \mathbf{R}\mathit{\Gamma}_W \mathcal{F})\big).$$

With $U := X \setminus W$ there is a canonical triangle (cf. (0.4.2.1))

$$\mathbf{R}\Gamma_W \mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E}, \mathcal{F}) \to \mathbf{R}\Gamma_X \mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E}, \mathcal{F}) \to \mathbf{R}\Gamma_U \mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{E}, \mathcal{F}) \xrightarrow{+}$$

whence a long exact cohomology sequence

$$\cdots \to \mathrm{Ext}^n_W(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^n_X(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^n_U(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^{n+1}_W(\mathcal{E}, \mathcal{F}) \to \cdots$$

Proposition (5.4.1). *Let X be a noetherian separated scheme, let $Z \subset X$ be a closed subscheme, and let $\kappa\colon \widehat{X} = X_{/Z} \to X$ be the canonical map. Let $\mathcal{E} \in \mathbf{D}(X)$ and $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$. Let $W \subset X$ be closed, so that $W \cap Z$ is closed in $\widehat{X}$. Then there are natural isomorphisms*

$$\mathrm{Ext}^n_{W\cap Z}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F}) \xrightarrow{\ \sim\ } \mathrm{Ext}^n_X(\mathbf{R}\mathit{\Gamma}'_Z\mathcal{E}, \mathbf{R}\mathit{\Gamma}_{W\cap Z}\mathcal{F}) \qquad (n \in \mathbb{Z}),$$

and so with $U := X \setminus W$ and $\widehat{U} := U_{/Z\cap U}$ there is a long exact sequence

$$\cdots \to \mathrm{Ext}^n_X(\mathbf{R}\mathit{\Gamma}'_Z\mathcal{E}, \mathbf{R}\mathit{\Gamma}_{W\cap Z}\mathcal{F}) \to \mathrm{Ext}^n_{\widehat{X}}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F}) \to \mathrm{Ext}^n_{\widehat{U}}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F}) \to \cdots$$

Hence under the assumptions of Lemma (5.3.3)(b) *there is an exact sequence*

$$\cdots \to \mathrm{Ext}^n_X(\mathbf{R}\mathit{\Gamma}'_Z\mathcal{E}, \mathbf{R}\mathit{\Gamma}_{W\cap Z}\mathcal{F}) \to \mathrm{Ext}^n_X(\mathcal{E}, \mathcal{F}) \otimes_A \hat{A} \to \mathrm{Ext}^n_{\widehat{U}}(\kappa^*\mathcal{E}, \kappa^*\mathcal{F}) \to \cdots$$

Proof. There are natural isomorphisms

$$\begin{aligned}\kappa_*\mathbf{R}\varGamma_{W\cap Z}\mathbf{R}\mathcal{H}om^\bullet_{\widehat{X}}(\kappa^*\mathcal{E},\kappa^*\mathcal{F}) &\xrightarrow{\ \sim\ } \mathbf{R}\varGamma_W\kappa_*\mathbf{R}\mathcal{H}om^\bullet_{\widehat{X}}(\kappa^*\mathcal{E},\kappa^*\mathcal{F})\\ &\xrightarrow{\ \sim\ } \mathbf{R}\varGamma_W\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma'_Z\mathcal{E},\mathcal{F})\\ &\xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma'_Z\mathcal{E},\mathbf{R}\varGamma_W\mathcal{F})\\ &\xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{R}\varGamma'_Z\mathcal{E},\mathbf{R}\varGamma_{W\cap Z}\mathcal{F}).\end{aligned}$$

The first isomorphism results from the equality $\kappa_*\varGamma_{W\cap Z}=\varGamma_W\kappa_*$, since κ_* preserves K-flabbiness [Sp, p. 142, 5.15(b) and p. 146, 6.4]. The second comes from $(0.3)_{\mathrm{c}}$. The third comes from (5.2.1). The last comes from (0.4.2) and (3.2.5)(ii).

To conclude, apply the functor $\mathbf{R}\Gamma_X$ and take homology. □

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