Relation between two twisted inverse image pseudofunctors in duality theory

Srikanth B. Iyengar, Joseph Lipman and Amnon Neeman

Abstract

Grothendieck duality theory assigns to essentially-finite-type maps f of noetherian schemes a pseudofunctor $f^{\times}$ right-adjoint to $\mathsf{R}f_*$, and a pseudofunctor $f^!$ agreeing with $f^{\times}$ when f is proper, but equal to the usual inverse image f^* when f is étale. We define and study a canonical map from the first pseudofunctor to the second. This map behaves well with respect to flat base change, and is taken to an isomorphism by "compactly supported" versions of standard derived functors. Concrete realizations are described, for instance for maps of affine schemes. Applications include proofs of reduction theorems for Hochschild homology and cohomology, and of a remarkable formula for the fundamental class of a flat map of affine schemes.

Introduction

The relation in the title is given by a canonical pseudofunctorial map $\psi\colon (-)^{\times} \to (-)^!$ between "twisted inverse image" pseudofunctors with which Grothendieck duality theory is concerned. These pseudofunctors on the category $\mathcal{E}$ of essentially-finite-type separated maps of noetherian schemes take values in bounded-below derived categories of complexes with quasi-coherent homology, see 1.1 and 1.2. The map ψ, derived from the pseudofunctorial "fake unit map" $\mathrm{id} \to (-)^! \circ \mathsf{R}(-)_*$ of Proposition 2.1, is specified in Corollary 2.1.4. A number of concrete examples appear in §3. For instance, if f is a map in $\mathcal{E}$, then $\psi(f)$ is an isomorphism if f is proper; but if f is, say, an open immersion, so that $f^!$ is the usual inverse image functor f^* whereas $f^{\times}$ is right-adjoint to $\mathsf{R}f_*$, then $\psi(f)$ is usually quite far from being an isomorphism (see e.g., 3.1.2, 3.1.3 and 3.3).

After some preliminaries are covered in §1, the definition of the pseudofunctorial map ψ is worked out at the beginning of §2. Its good behavior with respect to flat base change is given by Proposition 2.2.

The rest of Section 2 shows that under suitable "compact support" conditions, various operations from duality theory take ψ to an isomorphism. To wit:

Let $\mathbf{D}_{\mathsf{qc}}(X)$ be the derived category of $\mathcal{O}_X$-complexes with quasi-coherent homology, and let $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(-,-)$ be the internal hom in the closed category $\mathbf{D}_{\mathsf{qc}}(X)$ (§1.5). Proposition 2.3.2 says:

2010 Mathematics Subject Classification Primary: 14F05, 13D09. Secondary: 13D03

Keywords: Grothendieck duality, twisted inverse image pseudofunctors, Hochschild derived functors, relative perfection, relative dualizing complex, fundamental class.

This article is based on work supported by the National Science Foundation under Grant No. 0932078000, while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring semester of 2013. The first author was partly supported by NSF grant DMS 1201889 and a Simons Fellowship.

If $f\colon X \to Y$ is a map in $\mathcal{E}$, if W is a union of closed subsets of X to each of which the restriction of f is proper, and if $E \in \mathbf{D}_{\mathsf{qc}}(X)$ has support contained in W, then each of the functors $\mathsf{R}\varGamma_W(-)$, $E \otimes^{\mathsf{L}}_X (-)$ and $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(E,-)$ takes the map $\psi(f)\colon f^\times \to f^!$ to an isomorphism.

The proof uses properties of a bijection between subsets of X and "localizing tensor ideals" in $\mathbf{D}_{\mathsf{qc}}(X)$, reviewed in Appendix A. A consequence is that even for nonproper f, $f^!$ still has dualizing properties for complexes having support in such a W (Corollary 2.3.3); and there results, for $d = \sup\{\, \ell \mid H^\ell f^! \mathcal{O}_Y \neq 0\,\}$ and ω_f a relative dualizing sheaf, a "generalized residue map"

$$\int_W \colon H^d \mathsf{R} f_* \mathsf{R}\varGamma_W(\omega_f) \to \mathcal{O}_Y.$$

Proposition 2.3.5 says that *for $\mathcal{E}$-maps $W \xrightarrow{g} X \xrightarrow{f} Y$ of noetherian schemes such that fg is proper, and any $F \in \mathbf{D}_{\mathsf{qc}}(X)$, $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$, the functors $\mathsf{L}g^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F,-)$ and $g^\times \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F,-)$ both take the map $\psi(f)G\colon f^\times G \to f^! G$ to an isomorphism.*

Section 3 gives some concrete realizations of ψ. Besides the examples mentioned above, one has that if R is a noetherian ring, S a flat essentially-finite-type R-algebra, $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ the corresponding scheme-map, and M an R-module, with sheafification $\mathcal{M}$, then with $S^{\mathsf{e}} := S \otimes_R S$, the map $\psi(f)(\mathcal{M})\colon f^\times \mathcal{M} \to f^! \mathcal{M}$ is the sheafification of a simple $\mathbf{D}(S)$-map

$$\mathsf{R}\operatorname{Hom}_R(S, M) \to S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\operatorname{Hom}_R(S, S \otimes_R M), \tag{0.0.1}$$

described in Proposition 3.2.9. So if $S \to T$ is an R-algebra map with T module-finite over R, then, as above, the functors $T \otimes^{\mathsf{L}}_S -$ and $\mathsf{R}\operatorname{Hom}_S(T,-)$ take (0.0.1) to an isomorphism.

In the case where R is a field, more information about the map (0.0.1) appears in Proposition 3.3: the map is represented by a split S-module surjection with an enormous kernel.

In §4, there are two applications of the map ψ. The first is to a "reduction theorem" for the Hochschild homology of flat $\mathcal{E}$-maps that was stated in [AILN10, Theorem 4.6] in algebraic terms (see (4.1.1) below), with only an indication of proof. The scheme-theoretic version appears here in 4.1.8.

The paper [AILN10] also treats the nonflat algebraic case, where S^{e} becomes a derived tensor product. In fact, we conjecture that the natural home of the reduction theorems is in a more general derived-algebraic-geometry setting.

The special case (4.1.1)′ of (4.1.1) gives a canonical description of the relative dualizing sheaf $f^!\mathcal{O}_Y$ of a flat $\mathcal{E}$-map $f\colon X \to Y$ between affine schemes. The proof is based on the known theory of $f^!$, which is constructed using arbitrary choices, such as a compactification of f or a factorization of f as smooth∘finite; but the choice-free formula (4.1.1)′ might be a jumping-off point for a choice-free redevelopment of the underlying theory.

The second application is to a simple formula for the *fundamental class* of a flat map f of affine schemes. The fundamental class of a flat $\mathcal{E}$-map $g\colon X \to Y$—a globalization of the Grothendieck residue map—goes from the Hochschild complex of g to the relative dualizing complex $g^!\mathcal{O}_Y$. This map is defined in terms of sophisticated abstract notions from duality theory (see (4.2.1)). But for maps $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ as above, Theorem 4.2.4 says that, with $\mu\colon S \to \operatorname{Hom}_R(S,S)$ the S^{e}-homomorphism taking $s \in S$ to multiplication by s, *the fundamental class is isomorphic to the sheafification of the natural composite map*

$$S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} S \xrightarrow{\mathrm{id}\otimes\mu} S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \operatorname{Hom}_R(S,S) \longrightarrow S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\operatorname{Hom}_R(S,S).$$

1. Preliminaries: twisted inverse image functors, essentially finite-type compactification, conjugate maps

1.1. For a scheme X, $\mathbf{D}(X)$ is the derived category of $\mathcal{O}_X$-modules, and $\mathbf{D}_{\mathsf{qc}}(X)$ ($\mathbf{D}^+_{\mathsf{qc}}(X)$) is the full subcategory spanned by the complexes with quasi-coherent cohomology modules (vanishing in all but finitely many negative degrees). We will use freely some standard functorial maps, for instance the projection isomorphism associated to a map $f\colon X \to Y$ of noetherian schemes (see, e.g., [L09, 3.9.4]):

$$\mathsf{R}f_*E \otimes^{\mathsf{L}}_Y F \xrightarrow{\ \sim\ } \mathsf{R}f_*(E \otimes^{\mathsf{L}}_X \mathsf{L}f^*F) \qquad \big(E \in \mathbf{D}_{\mathsf{qc}}(X),\ F \in \mathbf{D}_{\mathsf{qc}}(Y)\big).$$

Denote by $\mathcal{E}$ the category of *separated essentially-finite-type maps of noetherian schemes.* By [Nk09, 5.2 and 5.3], there is a contravariant $\mathbf{D}^+_{\mathsf{qc}}$-valued pseudofunctor $(-)^!$ over $\mathcal{E}$, determined up to isomorphism by the properties:

(i) The pseudofunctor $(-)^!$ restricts over the subcategory of proper maps in $\mathcal{E}$ to a right adjoint of the derived direct-image pseudofunctor.

(ii) The pseudofunctor $(-)^!$ restricts over the subcategory of formally étale maps in $\mathcal{E}$ to the usual inverse-image pseudofunctor $(-)^*$.

(iii) For any fiber square in $\mathcal{E}$:

$$\begin{array}{ccc} \bullet & \xrightarrow{\ v\ } & \bullet \\ {\scriptstyle g}\big\downarrow & \Xi & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array}$$

with f, g proper and u, v formally étale, the base-change map β_Ξ, defined to be the adjoint of the natural composition

$$\mathsf{R}g_*v^*f^! \xrightarrow{\ \sim\ } u^*\mathsf{R}f_*f^! \longrightarrow u^*, \tag{1.1.1}$$

is equal to the natural composite isomorphism

$$v^*f^! = v^!f^! \xrightarrow{\ \sim\ } (fv)^! = (ug)^! \xrightarrow{\ \sim\ } g^!u^! = g^!u^*. \tag{1.1.2}$$

There is in fact a family of base-change *isomorphisms*

$$\beta_\Xi\colon v^*f^! \xrightarrow{\ \sim\ } g^!u^*, \tag{1.1.3}$$

indexed by *all* commutative $\mathcal{E}$-squares

$$\begin{array}{ccc} \bullet & \xrightarrow{\ v\ } & \bullet \\ {\scriptstyle g}\big\downarrow & \Xi & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array}$$

that are such that in the associated diagram (which exists in $\mathcal{E}$, see [Nk09, §2.2])

$$\begin{array}{ccccc} \bullet & \xrightarrow{\ i\ } & \bullet & \xrightarrow{\ w\ } & \bullet \\ & & {\scriptstyle h}\big\downarrow & \Xi' & \big\downarrow{\scriptstyle f} \\ & & \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array}$$

it holds that Ξ' is a fiber square, $wi = v$ and $hi = g$, the map u is flat and i is formally étale, a family that is the unique such one that behaves transitively with respect to vertical and horizontal composition of such Ξ (cf. [L09, (4.8.2)(3)]), and satisfies:

(iv) if Ξ is a fiber square with f proper then the map β_Ξ is adjoint to the composite map (1.1.1);

(v) if f—hence g—is formally étale, so that $f^! = f^*$ and $g^! = g^*$, then β_Ξ is the natural isomorphism $v^*f^* \xrightarrow{\sim} g^*u^*$; and

(vi) if u—hence v—is formally étale, so that $u^* = u^!$ and $v^* = v^!$, then β_Ξ is the natural isomorphism (1.1.2).

(For further explanation see [L09, Thm. 4.8.3] and [Nk09, §5.2].)

Remark. With regard to (vi), if Ξ is *any* commutative $\mathcal{E}$-diagram with u and v formally étale, then in the associated diagram i is necessarily formally étale ([GrD67, (17.1.3(iii) and 17.1.4]), so that β_Ξ exists (and can be identified with the canonical isomorphism $v^!f^! \xrightarrow{\sim} g^!u^!$).

1.2. For *any* $\mathcal{E}$-map $f\colon X \to Y$, there exists a functor $f^\times\colon \mathbf{D}(Y) \to \mathbf{D}_{\mathsf{qc}}(X)$ that is bounded below and right-adjoint to $\mathsf{R}f_*$. There results a $\mathbf{D}_{\mathsf{qc}}$-valued pseudofunctor $(-)^\times$ on $\mathcal{E}$, for which the said adjunction is pseudofunctorial [L09, Corollary (4.1.2)]. Obviously, the restriction of $(-)^\times$ to $\mathbf{D}^+_{\mathsf{qc}}$ over proper maps in $\mathcal{E}$ is isomorphic to that of $(-)^!$. Accordingly, we will identify these two restricted pseudofunctors.

1.3. Nayak's construction of $(-)^!$ is based on his extension [Nk09, p. 536, Thm. 4.1] of Nagata's compactification theorem, to wit, that any map f in $\mathcal{E}$ factors as pu where p is proper and u is a localizing immersion (see below). Such a factorization is called a *compactification of* f.

A *localizing immersion* is an $\mathcal{E}$-map $u\colon X \to Y$ for which every $y \in u(X)$ has a neighborhood $V = \operatorname{Spec} A$ such that $u^{-1}V = \operatorname{Spec} A_M$ for some multiplicatively closed subset $M \subseteq A$, see [Nk09, p. 532, 2.8.8]. For example, *finite-type* localizing immersions are just open immersions [Nk09, p. 531, 2.8.3].

Any localizing immersion u is formally étale, so that $u^! = u^*$.

1.4. Any localizing immersion $u\colon X \to Y$ is a flat monomorphism, whence *the natural map* $\epsilon_1\colon u^*\mathsf{R}u_* \xrightarrow{\sim} \mathrm{id}_X$ *is an isomorphism*: associated to the fiber square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ X & \xrightarrow[u]{} & Y \end{array}$$

there is the flat base-change isomorphism $u^*\mathsf{R}u_* \xrightarrow{\sim} \mathsf{R}p_{2*}p_1^*$, and since u is a monomorphism, p_1 and p_2 are equal isomorphisms, so that $\mathsf{R}p_{2*}p_1^* = \mathrm{id}_X$.

That ϵ_1 is an isomorphism means that the natural map is an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X)}(E,F) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathsf{R}u_*E, \mathsf{R}u_*F) \qquad (E, F \in \mathbf{D}(X)),$$

which implies that the natural map $\eta_2\colon \mathrm{id}_X \to u^\times\mathsf{R}u_*$ is an isomorphism.

Conversely, *any flat monomorphism* f *in* $\mathcal{E}$ *is a localizing immersion,* which can be seen as follows. Using [Nk09, 2.7] and [GrD67, 8.11.5.1 and 17.6.1] one reduces to where f is a map of affine schemes, corresponding to a composite ring map $A \to B \to B_M$ with $A \to B$ étale and M a multiplicative submonoid of B. The kernel of multiplication $B \otimes_A B \to B$ is generated by an idempotent e, and

$B_M \otimes_A B_M \to B_M$ is an isomorphism, so e is annihilated by an element of the form $m \otimes m$ ($m \in M$). Consequently, $B[1/m] \otimes B[1/m] \to B[1/m]$ is an isomorphism, and so replacing B_M by $B[1/m]$ reduces the problem further to the case where $A \to B_M$ is a finite-type algebra. Finally, localizing A with respect to its submonoid of elements that are sent to units in B_M, one may assume further that f is surjective, in which case [GrD67, 17.9.1] gives that f is an isomorphism.

1.5. For a noetherian scheme X, the functor $\mathrm{id}_X^\times$ specified in §1.2 is right-adjoint to the inclusion $\mathbf{D}_{\mathsf{qc}}(X) \hookrightarrow \mathbf{D}(X)$. It is sometimes called the *derived quasi-coherator.*

For any $C \in \mathbf{D}_{\mathsf{qc}}(X)$, the unit map is an isomorphism $C \xrightarrow{\sim} \mathrm{id}_X^\times C$.

For any complexes A and B in $\mathbf{D}_{\mathsf{qc}}(X)$, set

$$\mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}(A,B) := \mathrm{id}_X^\times \mathsf{R}\mathcal{H}om_X(A,B) \in \mathbf{D}_{\mathsf{qc}}(X). \tag{1.5.1}$$

Then for A in $\mathbf{D}_{\mathsf{qc}}(X)$, the functor $\mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}(A,-)$ is right-adjoint to the endofunctor $-\otimes_X^{\mathsf{L}} A$ of $\mathbf{D}_{\mathsf{qc}}(X)$. Thus, $\mathbf{D}_{\mathsf{qc}}(X)$ is a closed category with multiplication given by $\otimes_X^{\mathsf{L}}$ and internal hom given by $\mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}$.

As above, the canonical $\mathbf{D}(X)$-map $\mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}(A,B) \to \mathsf{R}\mathcal{H}om_X(A,B)$ is an *isomorphism* whenever $\mathsf{R}\mathcal{H}om_X(A,B) \in \mathbf{D}_{\mathsf{qc}}(X)$—for example, whenever $B \in \mathbf{D}^+_{\mathsf{qc}}(X)$ and the cohomology sheaves H^iA are coherent for all i, vanishing for $i \gg 0$ [H66, p. 92, 3.3].

1.6. For categories P and Q, let $\mathrm{Fun}(P,Q)$ be the category of functors from P to Q, and let $\mathrm{Fun}^{\mathrm{L}}(P,Q)$ (resp. $\mathrm{Fun}^{\mathrm{R}}(P,Q)$) be the full subcategory spanned by the objects that have right (resp. left) adjoints. There is a contravariant isomorphism of categories

$$\xi\colon \mathrm{Fun}^{\mathrm{L}}(P,Q) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{R}}(P,Q)$$

that takes any map of functors to the *right-conjugate* map between the respective right adjoints (see e.g., [L09, 3.3.5–3.3.7]). The image under ξ^{-1} of a map of functors is its *left-conjugate* map. The functor ξ (resp. ξ^{-1}) takes isomorphisms of functors to isomorphisms.

For instance, for any $\mathcal{E}$-map $f\colon X \to Z$ there is a bifunctorial *sheafified duality* isomorphism, with $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and $F \in \mathbf{D}_{\mathsf{qc}}(Z)$:

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}(E, f^\times F) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Z^{\mathsf{qc}}(\mathsf{R}f_*E, F), \tag{1.6.1}$$

right-conjugate, for each fixed E, to the projection isomorphism

$$\mathsf{R}f_*(\mathsf{L}f^*G \otimes_X^{\mathsf{L}} E) \xleftarrow{\sim} G \otimes_Z^{\mathsf{L}} \mathsf{R}f_*E.$$

Likewise, there is a functorial isomorphism

$$\mathsf{R}\mathcal{H}om_W^{\mathsf{qc}}(\mathsf{L}f^*G, f^\times H) \xrightarrow{\sim} f^\times \mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}(G,H) \tag{1.6.2}$$

right-conjugate to the projection isomorphism $\mathsf{R}f_*(E \otimes_X^{\mathsf{L}} \mathsf{L}f^*G) \xleftarrow{\sim} \mathsf{R}f_*E \otimes_Z^{\mathsf{L}} G$.

2. The basic map

In this section we construct a pseudofunctorial map $\psi\colon (-)^\times \to (-)^!$. The construction is based on the following "fake unit" map.

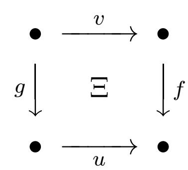
Proposition 2.1. *Over $\mathcal{E}$ there is a unique pseudofunctorial map*

$$\eta\colon\ \mathrm{id}\to(-)^!\circ\mathsf{R}(-)_*$$

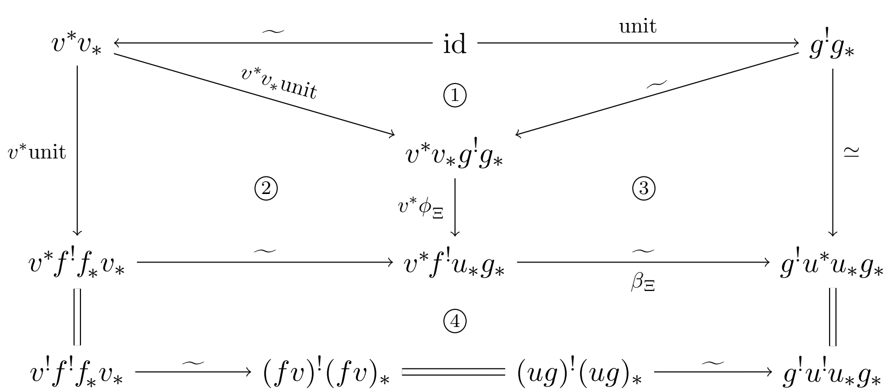
whose restriction to the subcategory of proper maps in $\mathcal{E}$ is the unit of the adjunction between $\mathsf{R}(-)_*$ and $(-)^!$, and such that if u is a localizing immersion then $\eta(u)$ is inverse to the isomorphism $u^!\mathsf{R}u_*=u^*\mathsf{R}u_*\xrightarrow{\sim}\mathrm{id}$ in 1.4.

The proof uses the next result—in which the occurrence of β_Ξ is justified by the remark at the end of §1.1. As we are dealing only with functors between derived categories, we will reduce clutter by writing h_* for $\mathsf{R}h_*$ (h any map in $\mathcal{E}$).

Lemma 2.1.1. *Let Ξ be a commutative square in $\mathcal{E}$:*

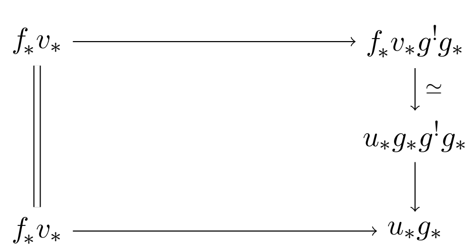


with f,g proper and u,v localizing immersions. Let $\phi_\Xi\colon v_*g^!\to f^!u_*$ be the functorial map adjoint to the natural composite map $f_*v_*g^!\xrightarrow{\sim}u_*g_*g^!\to u_*$. Then the following natural diagram commutes.



Proof. Commutativity of subdiagram ① is clear.

For commutativity of subdiagram ②, drop v^* and note the obvious commutativity of the following adjoint of the resulting diagram:



Showing commutativity of subdiagram ③ is similar to working out [L09, Exercise 3.10.4(b)]. (Details are left to the reader.)

Commutativity of subdiagram ④ is given by (vi) in section 1.1. □

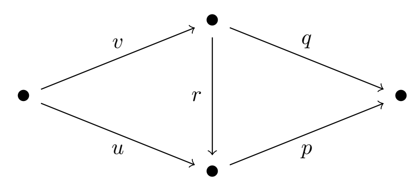
Proof of Proposition 2.1. As before, for any map h in $\mathcal{E}$ we abbreviate $\mathsf{R}h_*$ to h_*. Let f be a map in $\mathcal{E}$, and $f = pu$ a compactification. If η exists, then $\eta(f)\colon \mathrm{id} \to f^!f_*$ must be given by the natural composition

$$\mathrm{id} \xrightarrow{\ \sim\ } u^*u_* \xrightarrow{\text{via unit}} u^*p^!p_*u_* \xrightarrow{\ \sim\ } f^!f_*, \tag{2.1.2}$$

so that uniqueness holds.

Let us show now that this composite map does not depend on the choice of compactification.

A morphism $r\colon (f = qv) \to (f = pu)$ from one compactification of f to another is a commutative diagram of scheme-maps

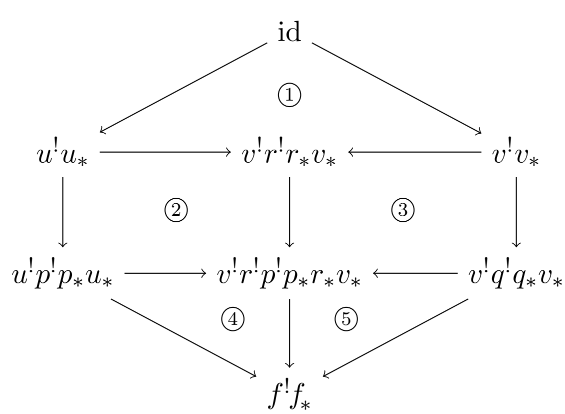


(2.1.3)

If such a map r—necessarily proper—exists, we say that the compactification $f = qv$ *dominates* $f = pu$.

Any two compactifications $X \xrightarrow{u_1} Z_1 \xrightarrow{p_1} Y$, $X \xrightarrow{u_2} Z_2 \xrightarrow{p_2} Y$ *of a given* $f\colon X \to Y$ *are dominated by a third one.* Indeed, let $v\colon X \to Z_1 \times_Y Z_2$ be the map corresponding to the pair (u_1, u_2), let $Z \subseteq Z_1 \times_Y Z_2$ be the schematic closure of v—so that $v\colon X \to Z$ has schematically dense image—and let $r_i : Z \to Z_i$ $(i = 1, 2)$ be the maps induced by the two canonical projections. Since $u = r_iv$ is a localizing immersion, therefore, by [Nk09, p. 533, 3.2], so is v. Thus $f = (p_ir_i)v$ is a compactification, not depending on i, mapped by r_i to the compactification $f = p_iu_i$.

So to show that (2.1.2) gives the same result for any two compactifications of f, it suffices to do so when one of the compactifications dominates the other. Thus with reference to the diagram (2.1.3), and keeping in mind that $u^* = u^!$ and $v^* = v^!$, one need only show that the following natural diagram commutes.



Commutativity of subdiagram ① is given by Lemma 2.1.1, with $f := r$ and $g := \mathrm{id}_X$.

Commutativity of ② is clear.

Commutativity of ③ holds because over proper maps, $(-)^!$ and $(-)_*$ are *pseudofunctorially* adjoint (see [L09, Corollary (4.1.2)].

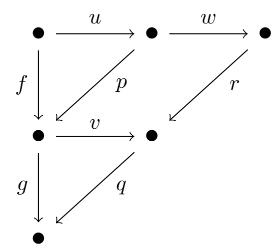
Commutativity of ④ and ⑤ results from the pseudofunctoriality of $(-)^!$ and $(-)_*$.

Thus (2.1.2) is indeed independent of choice of compactification, so that $\eta(f)$ is well-defined.

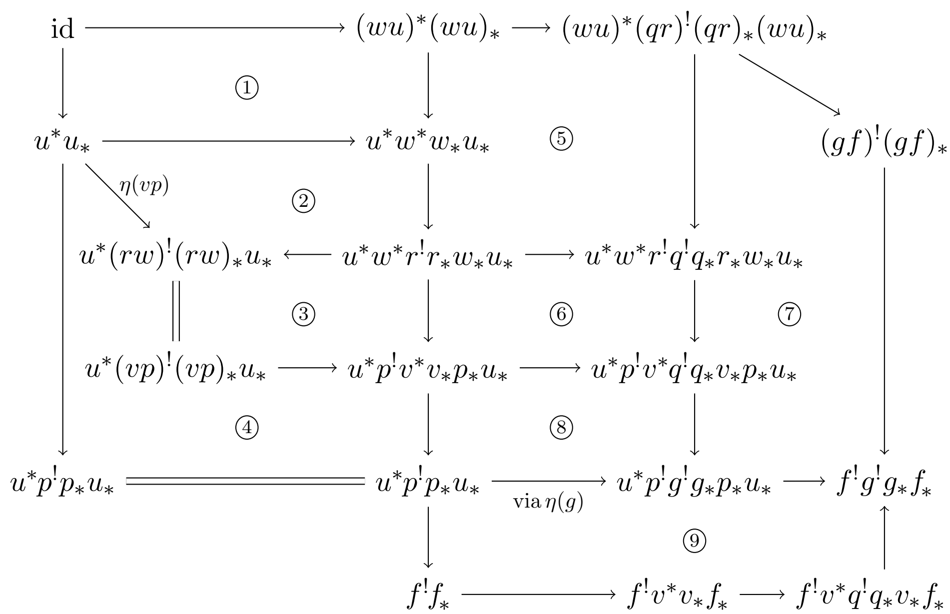
Finally, it must be shown that η is pseudofunctorial, i.e., for any composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$, the next diagram commutes:

$$\begin{array}{ccc} \mathrm{id} & \xrightarrow{\eta(gf)} & (gf)^!(gf)_* \\ {\scriptstyle \eta(f)}\big\downarrow & & \big\downarrow{\scriptstyle \simeq} \\ f^!f_* & \xrightarrow[f^!\eta(g)]{} & f^!g^!g_*f_* \end{array}$$

Consider therefore a diagram



where pu is a compactification of f, qv of g, and rw of vp—so that $(qr)(wu)$ is a compactification of gf. The problem then is to show commutativity of (the border of) the following natural diagram.



That subdiagram ① commutes is shown, e.g., in [L09, §3.6, up to (3.6.5)]. (In other words, the adjunction between $(-)^*$ and $(-)_*$ is pseudofunctorial, see *ibid.*, (3.6.7)(d).)

Commutativity of ② is the definition of $\eta(vp)$ via the compactification rw.

Commutativity of ③ holds by definition of the vertical arrow on its right.

Commutativity of ④ (omitting u^* and u_*) is the case $(f,g,u,v) := (r,p,v,w)$ of Lemma 2.1.1.

Commutativity of ⑤ holds because of pseudofunctoriality of the adjunction between $(-)_*$ and $(-)^!$ over *proper* maps (see §1.2).

Commutativity of ⑥ is clear.

Commutativity of ⑦ results from pseudofunctoriality of $(-)^!$ and $(-)_*$.

Commutativity of ⑧ is the definition of $\eta(g)$ via the compactification qv.

Commutativity of ⑨ is simple to verify.

This completes the proof of Proposition 2.1. □

Theorem 2.1.4. *There is a unique pseudofunctorial map $\psi\colon (-)^\times \to (-)^!$ whose restriction over the subcategory of proper maps in $\mathcal{E}$ is the identity, and such that for every localizing immersion u, $\psi(u)\colon u^\times \to u^!$ is the natural composition*

$$u^\times \xrightarrow{\ \sim\ } u^*\mathsf{R}u_*u^\times \longrightarrow u^* = u^!.$$

Proof. Let f be an $\mathcal{E}$-map, and $f = pu$ a compactification. If ψ exists, then $\psi(f)\colon f^\times \to f^!$ must be given by the natural composition

$$f^\times \xrightarrow{\ \sim\ } u^\times p^\times = u^\times p^! \to u^! p^! \xrightarrow{\ \sim\ } f^!, \tag{2.1.4.1}$$

so that uniqueness holds.

As for existence, using 2.1 we can take $\psi(f)$ to be the natural composition

$$f^\times \xrightarrow{\text{via}\,\eta} f^!\mathsf{R}f_*f^\times \longrightarrow f^!.$$

This is as required when f is proper or a localizing immersion, and it behaves pseudofunctorially, because both η and the counit map $\mathsf{R}f_*f^\times \to \mathrm{id}$ do. □

Remark 2.1.5. Conversely, one can recover η from ψ: it is simple to show that for any $\mathcal{E}$-map $f\colon X \to Y$ and $E \in \mathbf{D}_{\mathsf{qc}}(X)$, and with $\eta_2\colon \mathrm{id}_X \to f^\times\mathsf{R}f_*$ the unit map of the adjunction $f^\times \dashv \mathsf{R}f_*$, one has

$$\eta(E) = \psi(f)(\mathsf{R}f_*E)\circ\eta_2(E). \tag{2.1.5.1}$$

(*Notation*: $\mathsf{F} \dashv \mathsf{G}$ signifies that the functor F is left-adjoint to the functor G.)

Remark 2.1.6. If $u\colon X \to Y$ is a localizing immersion, then the map

$$u^\times\mathsf{R}u_* \xrightarrow{\psi(u)} u^*\mathsf{R}u_* \xrightarrow{\ \sim\ } \mathrm{id}\,.$$

is an *isomorphism,* inverse to the isomorphism η_2 in §1.4. (A proof is left to the reader.)

Remark 2.1.7. *The map $\mathsf{R}u_*\psi(u)\colon \mathsf{R}u_*u^\times \to \mathsf{R}u_*u^*$ is equal to the composite*

$$\mathsf{R}u_*u^\times \xrightarrow{\epsilon_2} \mathrm{id} \xrightarrow{\eta_1} \mathsf{R}u_*u^*,$$

where ϵ_2 is the counit of the adjunction $\mathsf{R}u_* \dashv u^\times$ and η_1 is the unit of the adjunction $u^* \dashv \mathsf{R}u_*$.

Indeed, by §1.4 the counit ϵ_1 of the adjunction $u^* \dashv \mathsf{R}u_*$ is an isomorphism; and since the composite

$$\mathsf{R}u_* \xrightarrow{\eta_1\mathsf{R}u_*} \mathsf{R}u_*u^*\mathsf{R}u_* \xrightarrow{\mathsf{R}u_*\epsilon_1} \mathsf{R}u_*$$

is the identity map, therefore $\mathsf{R}u_*\epsilon_1^{-1} = \eta_1\mathsf{R}u_*$, as both are the (unique) inverse of $\mathsf{R}u_*\epsilon_1$; so the next diagram commutes, giving the assertion:

$$\begin{array}{ccc} \mathsf{R}u_*u^\times & \xrightarrow{\mathsf{R}u_*\epsilon_1^{-1}u^\times = \eta_1\mathsf{R}u_*u^\times} & \mathsf{R}u_*u^*\mathsf{R}u_*u^\times \\ {\scriptstyle\epsilon_2}\big\downarrow & & \big\downarrow{\scriptstyle\mathsf{R}u_*u^*\epsilon_2} \\ \mathrm{id} & \xrightarrow[\eta_1]{} & \mathsf{R}u_*u^* \end{array}$$

Also, using the isomorphism $\epsilon_1\colon u^*\mathsf{R}u_* \xrightarrow{\,\sim\,} \mathrm{id}$ (resp., its right conjugate $\eta_2\colon \mathrm{id} \xrightarrow{\,\sim\,} u^\times\mathsf{R}u_*$), one can recover $\psi(u)$ from $\mathsf{R}u_*\psi(u)$ by applying the functor u^* (resp. $u^\times$), thereby obtaining alternate definitions of $\psi(u)$.

The next Proposition asserts compatibility of ψ with the flat base-change maps for $(-)^!$ (see (1.1.3)) and for $(-)^\times$.

Proposition 2.2. *Let $f\colon X\to Z$ and $g\colon Y\to Z$ be maps in $\mathcal{E}$, with g flat. Let $p\colon X\times_Z Y\to X$ and $q\colon X\times_Z Y\to Y$ be the projections. Let $\beta\colon p^*f^\times\to q^\times g^*$ be the map adjoint to the natural composite map*

$$\mathsf{R}q_*p^*f^\times \xrightarrow{\,\sim\,} g^*\mathsf{R}f_*f^\times \to g^*.$$

Then the following diagram commutes.

$$\begin{array}{ccc}
p^*f^\times & \xrightarrow{p^*\psi(f)} & p^*f^! \\
{\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle(1.1.3)} \\
q^\times g^* & \xrightarrow[\psi(q)]{} & q^!g^*
\end{array}$$

Proof. Let $f=\bar f u$ be a compactification, so that there is a composite cartesian diagram (with h flat and with $\bar q v$ a compactification of q):

$$\begin{array}{ccc}
X\times_Z Y & \xrightarrow{\;p\;} & X \\
{\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle u} \\
W\times_Z Y & \xrightarrow{\;h\;} & W \\
{\scriptstyle \bar q}\big\downarrow & & \big\downarrow{\scriptstyle \bar f} \\
Y & \xrightarrow[\;g\;]{} & Z
\end{array}$$

In view of the pseudofunctoriality of ψ, what needs to be shown is commutativity of the following natural diagram.

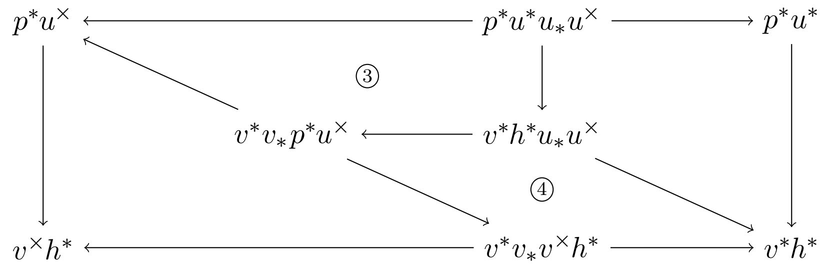
$$\begin{array}{ccccccc}
p^*f^\times & \xrightarrow{\;\sim\;} & p^*u^\times\bar f^\times & \longrightarrow & p^*u^*\bar f^! & \xrightarrow{\;\sim\;} & p^*f^! \\
\big\downarrow & & \big\downarrow & ① & \big\downarrow & & \big\downarrow \\
\big\downarrow & & v^\times h^*\bar f^\times & \longrightarrow & v^*h^*\bar f^! & & \big\downarrow \\
\big\downarrow & & \big\downarrow & ② & \big\downarrow & & \big\downarrow \\
q^\times g^* & \xrightarrow{\;\sim\;} & v^\times\bar q^\times g^* & \longrightarrow & v^*\bar q^! g^* & \xrightarrow{\;\sim\;} & q^!g^*
\end{array}$$

Commutativity of each of the unlabeled subdiagrams is an instance of transitivity of the appropriate base-change map (see e.g., [L09, Thm. (4.8.3)]).

Commutativity of ② is straightforward to verify.

Subdiagram ①, without $\bar f^\bullet$, expands naturally as follows (where we have written u_* (resp. v_*)

for $\mathsf{R}u_*$ (resp. $\mathsf{R}v_*$)):



Here the unlabeled diagrams clearly commute.

Commutativity of ③ results from the fact that the natural isomorphism $h^*u_* \to v_*p^*$ is adjoint to the natural composition $v^*h^*u_* \xrightarrow{\sim} p^*u^*u_* \to p^*$ (see [L09, 3.7.2(c)]).

Commutativity of ④ results from the fact that the base-change map $p^*u^\times \to v^\times h^*$ is adjoint to $v_*p^* \xrightarrow{\sim} h^*u_*u^\times \to h^*$.

Thus ① commutes; and Proposition 2.2 is proved. $\square$

2.3. Next we treat the interaction of the map ψ with standard derived functors. Our approach involves the notion of *support,* reviewed in Appendix A.

Lemma 2.3.1. *Let $u\colon X \to Z$ be a localizing immersion, $\epsilon_2\colon \mathsf{R}u_*u^\times \to \mathrm{id}$ the counit of the adjunction $\mathsf{R}u_* \dashv u^\times$, and $\eta_1\colon \mathrm{id} \to \mathsf{R}u_*u^*$ the unit of the adjunction $u^* \dashv \mathsf{R}u_*$. For all $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and $F \in \mathbf{D}_{\mathsf{qc}}(Z)$, the maps $\mathsf{R}u_*E \otimes^{\mathsf{L}}_Z \eta_1(F)$ and $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z\big(\mathsf{R}u_*E, \epsilon_2(F)\big)$ are isomorphisms.*

Proof. Projection isomorphisms make the map $\mathsf{R}u_*E \otimes^{\mathsf{L}}_Z \eta_1$ isomorphic to

$$\mathsf{R}u_*(E \otimes^{\mathsf{L}}_X u^*) \xrightarrow{\text{via } u^*\eta_1} \mathsf{R}u_*(E \otimes^{\mathsf{L}}_X u^*\mathsf{R}u_*u^*).$$

Since $u^*\eta_1$ is an isomorphism (with inverse the isomorphism $u^*\mathsf{R}u_*u^* \xrightarrow{\sim} u^*$ from 1.4), therefore so is $\mathsf{R}u_*E \otimes^{\mathsf{L}}_Z \eta_1$.

Similarly, to show that $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z\big(\mathsf{R}u_*E, \epsilon_2\big)$ is an isomorphism, one can use the duality isomorphism (1.6.1) to reduce to noting that $u^\times\epsilon_2$ is an isomorphism because it is right-conjugate to the inverse of the isomorphism $\mathsf{R}u_*\eta_1 : \mathsf{R}u_*u^*\mathsf{R}u_* \xrightarrow{\sim} \mathsf{R}u_*$. $\square$

Proposition 2.3.2. *Let $f\colon X \to Y$ be a map in $\mathcal{E}$, W a union of closed subsets of X to each of which the restriction of f is proper, and $E \in \mathbf{D}_{\mathsf{qc}}(X)$ a complex with support* $\mathrm{supp}(E)$ *contained in W. Then the functors $\mathsf{R}\varGamma_W(-)$, $E \otimes^{\mathsf{L}}_X (-)$ and $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(E, -)$ take $\psi(f)\colon f^\times \to f^!$ to an isomorphism.*

Proof. By A.3(ii), it is enough to prove that Proposition 2.3.2 holds for one E with $\mathrm{supp}(E) = W$, like $E = \mathsf{R}\varGamma_W\mathcal{O}_X$ (see A.4). For such an E, A.3 shows it enough to prove that $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(E, \psi(f))$ is an isomorphism.

Let $X \xrightarrow{u} Z \xrightarrow{p} Y$ be a compactification of f (§1.3). In view of (2.1.4.1), we need only treat the case $f = u$. In this case it suffices to show, with ϵ_2 and η_1 as in Remark 2.1.7, that

$$\begin{aligned}\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z\big(\mathsf{R}u_*E, \eta_1\epsilon_2\big) &\cong \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z\big(\mathsf{R}u_*E, \mathsf{R}u_*\psi(u)\big)\\ &\cong \mathsf{R}u_*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X\big(u^*\mathsf{R}u_*E, \psi(u)\big)\\ &\cong \mathsf{R}u_*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X\big(E, \psi(u)\big)\end{aligned}$$

is an isomorphism.

Lemma 2.3.1 gives that $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}u_*E,\epsilon_2)$ is an isomorphism. It remains to be shown that $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}u_*E,\eta_1)$ is an isomorphism.

The localizing immersion u maps X homeomorphically onto $u(X)$ [Nk09, 2.8.2], so we can regard X as a topological subspace of Z. Let $i\colon V\hookrightarrow X$ be the inclusion into X of a subscheme such that the restriction $fi = pui$ is proper. Then ui is proper, and so V is a closed subset of Z. Thus $W = \operatorname{supp}_X(E) = \operatorname{supp}_Z(\mathsf{R}u_*E)$ (see Remark A.5.1) is a union of subsets of X that are closed in Z. So Proposition A.3 can be applied to show that, since, by Lemma 2.3.1, $\mathsf{R}u_*E\otimes^{\mathsf{L}}_Z\eta_1$ is an isomorphism, therefore $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}u_*E,\eta_1)$ is an isomorphism, as required. ☐

Let $W\subseteq X$ be as in Proposition 2.3.2. Let $\mathbf{D}_{\mathsf{qc}}(X)_W\subseteq\mathbf{D}_{\mathsf{qc}}(X)$ be the essential image of $\mathsf{R}\varGamma_W(X)$—the full subcategory spanned by the complexes that are exact outside W. By Lemma A.1, any $E\in\mathbf{D}_{\mathsf{qc}}(X)_W$ satisfies $\operatorname{supp}E\subseteq W$. Arguing as in [AJS04, §2.3] one finds that the two natural maps from $\mathsf{R}\varGamma_W\mathsf{R}\varGamma_W$ to $\mathsf{R}\varGamma_W$ are *equal isomorphisms;* and deduces that the natural map is an isomorphism

$$\operatorname{Hom}_{\mathbf{D}(X)}(E,\mathsf{R}\varGamma_W F)\xrightarrow{\ \sim\ }\operatorname{Hom}_{\mathbf{D}(X)}(E,F)\quad\big(E\in\mathbf{D}_{\mathsf{qc}}(X)_W,\ F\in\mathbf{D}_{\mathsf{qc}}(Y)\big),$$

with inverse the natural composition

$$\operatorname{Hom}_{\mathbf{D}(X)}(E,F)\to\operatorname{Hom}_{\mathbf{D}(X)}(\mathsf{R}\varGamma_W E,\mathsf{R}\varGamma_W F)\xrightarrow{\ \sim\ }\operatorname{Hom}_{\mathbf{D}(X)}(E,\mathsf{R}\varGamma_W F).$$

Corollary 2.3.3. *With the preceding notation,* $\mathsf{R}f_*\colon\mathbf{D}_{\mathsf{qc}}(X)_W\to\mathbf{D}_{\mathsf{qc}}(Y)$ *has as right adjoint the functor* $\mathsf{R}\varGamma_W f^{\times}$*. When restricted to* $\mathbf{D}^+_{\mathsf{qc}}(Y)$*, this right adjoint is isomorphic to* $\mathsf{R}\varGamma_W f^{!}$*.*

Proof. For $E\in\mathbf{D}_{\mathsf{qc}}(X)_W$ and $G\in\mathbf{D}_{\mathsf{qc}}(Y)$, there are natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbf{D}(Y)}(\mathsf{R}f_*E,G)&\cong\operatorname{Hom}_{\mathbf{D}(X)}(E,f^{\times}G)\\&\cong\operatorname{Hom}_{\mathbf{D}(X)}(E,\mathsf{R}\varGamma_W f^{\times}G)\underset{2.3.2}{\xrightarrow{\ \sim\ }}\operatorname{Hom}_{\mathbf{D}(X)}(E,\mathsf{R}\varGamma_W f^{!}G).\end{aligned}$$

☐

Remark 2.3.4. The preceding Corollary entails the existence of a counit map

$$\bar{\int_W}\colon\mathsf{R}f_*\mathsf{R}\varGamma_W f^{!}\mathcal{O}_Y\to\mathcal{O}_Y.$$

Factoring f over suitable affine open subsets U as $U\xrightarrow{i_U}Z\xrightarrow{h_U}Y$ where i_U is finite and h_U is essentially smooth, one gets that $i_{U*}f^{!}\mathcal{O}_Y|_U$ is of the form $\mathsf{R}\mathcal{H}om_Z(\mathsf{R}i_{U*}\mathcal{O}_X,\Omega^n_{h_U}[n])$ for some $n=n_U$ such that the sheaf $\Omega^n_{h_U}$ of relative n-forms is free of rank 1; and hence local depth considerations imply that there is an integer d such that $H^{-e}f^{!}\mathcal{O}_Y=0$ for all $e>d$, while $\omega_f := H^{-d}f^{!}\mathcal{O}_Y\neq 0$. This ω_f, determined up to isomorphism by f, is a *relative dualizing sheaf* (or *relative canonical sheaf*) of f.

There results a natural composite map of $\mathcal{O}_Y$-modules

$$\begin{aligned}\int_W\colon H^d\mathsf{R}f_*\mathsf{R}\varGamma_W(\omega_f)&=H^0\mathsf{R}f_*\mathsf{R}\varGamma_W(\omega_f[d])\\&\longrightarrow H^0(\mathsf{R}f_*\mathsf{R}\varGamma_W f^{!}\mathcal{O}_Y)\xrightarrow{\text{via }\bar{\int}_W}H^0\mathcal{O}_Y=\mathcal{O}_Y,\end{aligned}$$

that generalizes the map denoted “res_Z” in [S04, §3.1].

A deeper study of this map involves the realization of ω_f, for certain f, in terms of regular differential forms, and the resulting relation of $\int_W$ with residues of differential forms, cf. [HK90a] and [HK90b]. See also §4.2 below.

Proposition 2.3.5. *Let* $W \xrightarrow{g} X \xrightarrow{f} Y$ *be* $\mathcal{E}$*-maps such that* fg *is proper. For any* $F \in \mathbf{D}_{\mathsf{qc}}(X)$ *and* $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$*, the maps*

$$\mathsf{L}g^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^\times G) \xrightarrow{\text{via}\,\psi} \mathsf{L}g^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^! G) \tag{2.3.5.1}$$

$$g^\times\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^\times G) \xrightarrow{\text{via}\,\psi} g^\times\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^! G) \tag{2.3.5.2}$$

$$g^\times\mathsf{R}\mathcal{H}om_X(F, f^\times G) \xrightarrow{\text{via}\,\psi} g^\times\mathsf{R}\mathcal{H}om_X(F, f^! G) \tag{2.3.5.2$'$}$$

are isomorphisms.

Proof. Since $g^\times \mathrm{id}_X^\times \cong (\mathrm{id}_X \circ g)^\times = g^\times$, therefore (2.3.5.2) is an isomorphism if and only if so is (2.3.5.2)$'$. (Recall that $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X = \mathrm{id}_X^\times \mathsf{R}\mathcal{H}om_X$.)

As for (2.3.5.2) and (2.3.5.1), note first that the proper map g induces a surjection g_2 of W onto a closed subscheme V of X; so $g = g_1g_2$ with g_1 a closed immersion and g_2 surjective.

Let $X \xrightarrow{u} Z \xrightarrow{p} Y$ be a compactification of f. Since pug_1g_2 is proper, so is ug_1g_2, whence ug_1 maps $V = g_2(g_2^{-1}V)$ homeomorphically onto a closed subset of Z, and for each $x \in V$ the natural map $\mathcal{O}_{Z,ug_1x} \to \mathcal{O}_{V,x}$ is a surjection (see [Nk09, 2.8.2]); thus ug_1 is a closed immersion, and therefore $fg_1 = pug_1$ is of finite type, hence, by [GrD61, 5.4.3], proper (since fg_1g_2 is).

Since $\mathsf{L}g^* = \mathsf{L}g_2^*\mathsf{L}g_1^*$ and $g^\times = g_2^\times g_1^\times$, it suffices that the Proposition hold when $g = g_1$, i.e., we may assume that $g\colon W \to X$ *is a closed immersion.* It's enough then to show that (2.3.5.1) and (2.3.5.2) become isomorphisms after application of the functor g_*.

Via projection isomorphisms, the map

$$g_*\mathsf{L}g^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^\times G) \xrightarrow{g_*(2.3.5.1)} g_*\mathsf{L}g^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^! G)$$

is isomorphic to the map

$$g_*\mathcal{O}_W \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^\times G) \xrightarrow{\text{via}\,\psi} g_*\mathcal{O}_W \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^! G); \tag{2.3.5.3}$$

and making the substitution

$$\big(f\colon X \to Z, E, F\big) \mapsto \big(g\colon W \to X, \mathcal{O}_W, \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^\times G)\big)$$

in the isomorphism (1.6.1) leads to an isomorphism between the map

$$g_*g^\times\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^\times G) \xrightarrow{g_*(2.3.5.2)} g_*g^\times\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^! G)$$

and the map

$$\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(g_*\mathcal{O}_W, \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^\times G)) \xrightarrow{\text{via}\,\psi} \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(g_*\mathcal{O}_W, \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(F, f^! G)). \tag{2.3.5.4}$$

Via adjunction and projection isomorphisms, (2.3.5.4) is isomorphic to

$$\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(g_*\mathsf{L}g^*F, f^\times G) \xrightarrow{\text{via}\,\psi} \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(g_*\mathsf{L}g^*F, f^! G). \tag{2.3.5.5}$$

By Lemma A.1, $\mathrm{supp}(g_*\mathsf{L}g^*F) \subseteq \mathrm{Supp}(g_*\mathsf{L}g^*F) \subseteq W$, so 2.3.2 gives that (2.3.5.5) is an isomorphism, whence so is (2.3.5.4).

Thus (2.3.5.2) is an isomorphism. Also, $\mathrm{supp}(g_*\mathcal{O}_W) = \mathrm{Supp}(\mathcal{O}_W) = W$, so A.3 shows that (2.3.5.3) is an isomorphism, whence so is (2.3.5.1). $\square$

Remark 2.3.6. (Added in proof.) For an $\mathcal{E}$-map f, with compactification $f = pu$, set

$$(p,u)^! G := u^! p^! G \qquad \big(G \in \mathbf{D}_{\mathsf{qc}}(Y)\big).$$

It is shown in [Nm14b, Section 4] that $(p,u)^!$ depends only on f, in the sense that up to canonical isomorphism $(p,u)^!$ is independent of the factorization $f = pu$. (When this paper was written it was known only that $(p,u)^! G$ is canonically isomorphic to $f^! G$ when $G \in \mathbf{D}^+_{\mathsf{qc}}(Y)$.) Likewise, for all $G \in \mathbf{D}_{\mathsf{qc}}(Y)$ the functorial map

$$\psi(p,u)(G)\colon f^\times G \xrightarrow{\ \sim\ } u^\times p^\times G = u^\times p^! G \xrightarrow{\psi(u)p^!} u^! p^! G = (p,u)^! G$$

depends only on f [Nm14b, Section 8]. So one may set $f^! := (p,u)^!$ and $\psi(f) := \psi(p,u)$; and then the preceding proof of Proposition 2.3.5 works for *all* $G \in \mathbf{D}_{\mathsf{qc}}(Y)$.

3. Examples

Corollaries 3.1.1–3.1.3 provide concrete interpretations of the map $\psi(u)$ for certain localizing immersions u.

Proposition 3.2.9 gives a purely algebraic expression for $\psi(f)$ when f is a *flat* $\mathcal{E}$-map between affine schemes. An elaboration for when the target of f is the Spec of a field is given in Proposition 3.3. The scheme-theoretic results 2.1.4, 2.2 and 2.3.5 tell us some facts about the pseudofunctorial behavior of $\psi(f)$; but how to prove these facts by purely algebraic arguments is left open.

LEMMA 3.1. *Let $f\colon X \to Z$ be an $\mathcal{E}$-map, and let $F \in \mathbf{D}_{\mathsf{qc}}(Z)$. The functorial isomorphism $\zeta(F)$ inverse to that gotten by setting $E = \mathcal{O}_X$ in (1.6.1) makes the following, otherwise natural, functorial diagram commute*:

$$\begin{array}{ccc}
\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}f_*\mathcal{O}_X, F) & \xrightarrow{\quad\zeta(F)\quad} & \mathsf{R}f_* f^\times F \\
\big\downarrow & & \big\downarrow \\
\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathcal{O}_Z, F) & \xrightarrow{\quad\sim\quad} & F
\end{array}$$

Proof. Abbreviating $\mathsf{R}f_*$ to f_* and $\mathsf{L}f^*$ to f^*, one checks that the diagram in question is right-conjugate to the natural diagram, functorial in $G \in \mathbf{D}_{\mathsf{qc}}(Z)$,

$$\begin{array}{ccccc}
G \otimes^{\mathsf{L}}_Z f_*\mathcal{O}_X & \xleftarrow{\text{projection}} & f_*(f^*G \otimes^{\mathsf{L}}_X \mathcal{O}_X) & \xleftarrow{\ \sim\ } & f_* f^* G \\
\big\uparrow & & & & \big\uparrow \\
G \otimes^{\mathsf{L}}_Z \mathcal{O}_Z & & \xleftarrow{\quad\sim\quad} & & G,
\end{array}$$

whose commutativity is given by [L09, 3.4.7(ii)]. □

COROLLARY 3.1.1. *For any localizing immersion $u\colon X \to Z$ and $F \in \mathbf{D}_{\mathsf{qc}}(Z)$, the map $\psi(u)\big(F\big)$ from 2.1.4 is isomorphic to the natural composite map*

$$u^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}u_*\mathcal{O}_X, F) \longrightarrow u^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathcal{O}_Z, F) \xrightarrow{\ \sim\ } u^*F.$$

Proof. This is immediate from 3.1 (with $f = u$). □

For the next Corollary recall that, when $Z = \operatorname{Spec} R$, the sheafification functor ${}^\sim = {}^{\sim_R}$ is an isomorphism from $\mathbf{D}(R)$ to the derived category of quasi-coherent $\mathcal{O}_Z$-modules, whose inclusion into $\mathbf{D}_{\mathsf{qc}}(Z)$ is an equivalence of categories [BN93, p. 230, 5.5].

Corollary 3.1.2. *In 3.1.1, if $X = \operatorname{Spec} S$ and $Z = \operatorname{Spec} R$ are affine—so that u corresponds to a flat epimorphic ring homomorphism $R \to S$—and $M \in \mathbf{D}(R)$, then $\psi(u)(M^\sim)$ is the sheafification of the natural $\mathbf{D}(S)$-map*

$$\mathsf{R}\mathrm{Hom}_R(S, M) \cong S \otimes_R \mathsf{R}\mathrm{Hom}_R(S, M) \to S \otimes_R (\mathsf{R}\mathrm{Hom}_R(R, M)) = S \otimes_R M.$$

Proof. Use the following well-known facts:

1. $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(A^\sim, B^\sim) \cong \mathsf{R}\mathrm{Hom}_R(A, B)^\sim \qquad \big(A, B \in \mathbf{D}(R)\big).$

This results from the sequence of natural isomorphisms, for any $C \in \mathbf{D}(R)$:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(Z)}(C^\sim, \mathsf{R}\mathrm{Hom}_R(A, B)^\sim) &\cong \mathrm{Hom}_{\mathbf{D}(R)}(C, \mathsf{R}\mathrm{Hom}_R(A, B))\\ &\cong \mathrm{Hom}_{\mathbf{D}(R)}(C \otimes^{\mathsf{L}}_R A, B)\\ &\cong \mathrm{Hom}_{\mathbf{D}(Z)}((C \otimes^{\mathsf{L}}_R A)^\sim, B^\sim)\\ &\cong \mathrm{Hom}_{\mathbf{D}(Z)}(C^\sim \otimes^{\mathsf{L}}_Z A^\sim, B^\sim)\\ &\cong \mathrm{Hom}_{\mathbf{D}(Z)}(C^\sim, \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(A^\sim, B^\sim)).\end{aligned}$$

2. $\mathsf{R}\mathrm{Hom}_R(S, M)^{\sim_R} = u_*\mathsf{R}\mathrm{Hom}_R(S, M)^{\sim_S}$.
3. $u^*(A^{\sim_R}) = (S \otimes_R A)^{\sim_S} \qquad \big(A \in \mathbf{D}(R)\big)$.
4. For any $N \in \mathbf{D}(S)$, the natural $\mathbf{D}(Z)$-map $u^*\mathsf{R}u_*N^{\sim_S} \to N^{\sim_S}$ is the sheafification of the natural $\mathbf{D}(S)$-map $S \otimes_R N \to N$. □

Corollary 3.1.3. *Let R be a noetherian ring that is complete with respect to the topology defined by an ideal I, let $p\colon Z \to \operatorname{Spec} R$ be a proper map, and let $X := (Z \setminus p^{-1} \operatorname{Spec} R/I) \overset{u}{\hookrightarrow} Z$ be the inclusion. For any $F \in \mathbf{D}_{\mathsf{qc}}(Z)$ whose cohomology modules are all coherent, $u^\times F = 0$.*

Proof. Since $u^*\mathsf{R}u_*u^\times F \cong u^\times F$ (§1.4), it suffices that $\mathsf{R}u_*u^\times F = 0$, that is, by 3.1.1, that $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}u_*\mathcal{O}_X, F) = 0$.

Set $W := p^{-1} \operatorname{Spec}(R/I)$. There is a natural triangle

$$\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}u_*\mathcal{O}_X, F) \to \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathcal{O}_Z, F) \xrightarrow{\alpha} \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}\varGamma_W\mathcal{O}_Z, F) \xrightarrow{+}$$

It is enough therefore to show that α is an isomorphism.

Let $\kappa\colon Z_{/W} \to Z$ be the formal completion of Z along W. For any $\mathcal{O}_Z$-module G let $G_{/W}$ be the completion of G—an $\mathcal{O}_{Z_{/W}}$-module; and let Λ_W be the functor given objectwise by $\kappa_*G_{/W}$. The composition of α with the “Greenlees-May” isomorphism

$$\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Z(\mathsf{R}\varGamma_W\mathcal{O}_Z, F) \xrightarrow{\ \sim\ } \mathrm{id}^\times_Z \mathsf{L}\Lambda_W F,$$

given by [AJL97, 0.3] is, by *loc. cit.*, the map $\mathrm{id}^\times_Z\lambda$, where $\lambda\colon F \to \mathsf{L}\Lambda_W F$ is the unique map whose composition with the canonical map $\mathsf{L}\Lambda_W F \to \Lambda_W F$ is the completion map $F \to \Lambda_W F$. So we need $\mathrm{id}^\times_Z\lambda$ to be an isomorphism. Hence, the isomorphisms $F = \mathrm{id}^\times_Z F \xrightarrow{\ \sim\ } \mathrm{id}^\times_Z \kappa_*\kappa^*F$ in [AJL99, 3.3.1(2)] (where $\mathrm{id}^\times_Z$ is denoted $\mathsf{R}Q$) and $\lambda^*_*\colon \kappa_*\kappa^*F \xrightarrow{\ \sim\ } \mathsf{L}\Lambda_W F$ in [AJL97, 0.4.1] (which requires coherence of the cohomology of F) reduce the problem to showing that *the natural composite map*

$$F \to \kappa_*\kappa^*F \xrightarrow{\lambda^*_*} \mathsf{L}\Lambda_W F \to \Lambda_W F$$

is the completion map.

By the description of λ_*^* preceding [AJL97, 0.4.1], this amounts to commutativity of the border of the following natural diagram:

$$\begin{array}{ccccc} F & \longrightarrow & \Lambda_W F & = & \kappa_* F_{/W} \\ \downarrow & & \downarrow & & \downarrow\uparrow \\ \kappa_*\kappa^* F & \longrightarrow & \kappa_*\kappa^*\Lambda_W F & = & \kappa_*\kappa^*\kappa_* F_{/W} \end{array}$$

Verification of the commutativity is left to the reader. □

3.2. Next we generalize Corollary 3.1.2, replacing u by an arbitrary flat map $f\colon X = \mathrm{Spec}(S) \to \mathrm{Spec}(R) = Z$ in $\mathcal{E}$, corresponding to a flat ring-homomorphism $\sigma\colon R \to S$. Lemma 3.2.1 gives an expression for $\psi(f)$ for an *arbitrary* flat $\mathcal{E}$-map f, that in the foregoing affine case implies, as shown in Lemma 3.2.8, that for $M \in \mathbf{D}(R)$, and $S^{\mathsf{e}} := S \otimes_R S$, $\psi(f)M$ is (naturally isomorphic to) the sheafification of the natural composite $\mathbf{D}(S)$-map

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_R(S, M) &\xrightarrow{\sim} S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} (S^{\mathsf{e}} \otimes_S \mathsf{R}\mathrm{Hom}_R(S, M)) \\ &\xrightarrow{\sim} S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} (S \otimes_R \mathsf{R}\mathrm{Hom}_R(S, M)) \longrightarrow S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_R(S, S \otimes_R M), \end{aligned}$$

or, more simply, (see Proposition 3.2.9),

$$\mathsf{R}\mathrm{Hom}_R(S, M) \to \mathsf{R}\mathrm{Hom}_R(S, S \otimes_R M) \to S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_R(S, S \otimes_R M).$$

(The expanded notation in 3.2.8 and 3.2.9 indicates the S-actions involved.)

So let $f : X \to Z$ be a *flat* $\mathcal{E}$-map, let $\delta : X \to X \times_Z X$ be the diagonal, and let π_1, π_2 be the projections from $X \times_Z X$ to X. There is a base-change isomorphism $\beta' = \pi_2^* f^! \xrightarrow{\sim} \pi_1^! f^*$, as in 1.1.3. There is also a base-change map $\beta\colon \pi_2^* f^\times \to \pi_1^\times f^*$ as in Proposition 2.2 (with $g = f$, $p = \pi_2$, $q = \pi_1$); this β need not be an isomorphism.

The next Lemma concerns functors from $\mathbf{D}^+_{\mathsf{qc}}(Z)$ to $\mathbf{D}^+_{\mathsf{qc}}(X)$.

LEMMA 3.2.1. *With preceding notation, there is an isomorphism of functors $\nu\colon \mathsf{L}\delta^*\pi_1^\times f^* \xrightarrow{\sim} f^!$ such that the map $\psi(f) : f^\times \to f^!$ from 2.1.4 is the composite*

$$f^\times = \mathrm{id}_X^* f^\times \cong \mathsf{L}\delta^*\pi_2^* f^\times \xrightarrow{\mathsf{L}\delta^*\beta} \mathsf{L}\delta^*\pi_1^\times f^* \xrightarrow{\nu} f^!.$$

Proof. Consider the diagram, where θ and θ' are the natural isomorphisms,

$$\begin{array}{ccccc} f^\times & \xrightarrow[\theta]{\sim} & \mathsf{L}\delta^*\pi_2^* f^\times & \xrightarrow{\mathsf{L}\delta^*\beta} & \mathsf{L}\delta^*\pi_1^\times f^* \\ {\scriptstyle \psi(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{L}\delta^*\pi_2^*\psi(f)} & & \downarrow{\scriptstyle \mathsf{L}\delta^*\psi(\pi_1)} \\ f^! & \xrightarrow[\theta']{\sim} & \mathsf{L}\delta^*\pi_2^* f^! & \xrightarrow[\mathsf{L}\delta^*\beta']{} & \mathsf{L}\delta^*\pi_1^! f^* \end{array}$$

The left square obviously commutes, and the right square commutes by Proposition 2.2. Since $\pi_1\delta = \mathrm{id}_X$ is proper, Proposition 2.3.5 guarantees that $\mathsf{L}\delta^*\psi(\pi_1)$ is an isomorphism, while $\mathsf{L}\delta^*\beta'$ is an isomorphism since β' is.

The Lemma results, with $\nu := (\theta')^{-1} \circ (\mathsf{L}\delta^*\beta')^{-1} \circ \mathsf{L}\delta^*\psi(\pi_1)$. □

COROLLARY 3.2.2. *The map $\psi(f)$ in Lemma 3.2.1 factors as*

$$f^\times \xrightarrow{\eta} \mathsf{R}\pi_{2*}\pi_2^* f^\times \xrightarrow{\mathsf{R}\pi_{2*}\beta} \mathsf{R}\pi_{2*}\pi_1^\times f^* \xrightarrow{\eta} \mathsf{R}\pi_{2*}\mathsf{R}\delta_*\mathsf{L}\delta^*\pi_1^\times f^* \xrightarrow{\sim} \mathsf{L}\delta^*\pi_1^\times f^* \xrightarrow{\nu} f^!,$$

where the maps labeled η are induced by units of adjunction, and the isomorphism obtains because $\pi_2\delta = \mathrm{id}_X$.

Proof. By Lemma 3.2.1 it suffices that the following diagram commute.

$$
\begin{array}{ccccc}
f^{\times} & \xrightarrow{\quad\eta\quad} & \mathsf{R}\pi_{2*}\pi_2^*f^{\times} & \xrightarrow{\ \mathsf{R}\pi_{2*}\beta\ } & \mathsf{R}\pi_{2*}\pi_1^{\times}f^* \\
\simeq\big\downarrow & \text{①} & \eta\big\downarrow & & \big\downarrow\eta \\
\mathsf{L}\delta^*\pi_2^*f^{\times} & \longrightarrow & \mathsf{R}\pi_{2*}\mathsf{R}\delta_*\mathsf{L}\delta^*\pi_2^*f^{\times} & \xrightarrow{\ \text{via}\,\beta\ } & \mathsf{R}\pi_{2*}\mathsf{R}\delta_*\mathsf{L}\delta^*\pi_1^{\times}f^* \\
& \searrow_{\mathsf{L}\delta^*\beta} & & \nearrow & \\
& & \mathsf{L}\delta^*\pi_1^{\times}f^* & &
\end{array}
$$

Commutativity of the unlabeled subdiagrams is clear.

Subdiagram ① (without $f^{\times}$) expands as

$$
\begin{array}{ccccc}
\mathrm{id} & \xrightarrow{\qquad\qquad\eta\qquad\qquad} & & & \mathsf{R}\pi_{2*}\pi_2^* \\
\Big\| & \searrow^{\eta} & & & \\
(\pi_2\delta)^* & = & (\pi_2\delta)_*(\pi_2\delta)^* & \text{②} & \big\downarrow\eta \\
\simeq\big\downarrow & & \big\downarrow\simeq & & \\
\mathsf{L}\delta^*\pi_2^* & = & (\pi_2\delta)_*\mathsf{L}\delta^*\pi_2^* & \xrightarrow{\ \sim\ } & \mathsf{R}\pi_{2*}\mathsf{R}\delta_*\mathsf{L}\delta^*\pi_2^*
\end{array}
$$

Commutativity of subdiagram ② is given by [L09, (3.6.2)]. Verification of commutativity of the remaining two subdiagrams is left to the reader. □

3.2.3. We now concretize the preceding results in case $X = \operatorname{Spec}(S)$ and $Z = \operatorname{Spec}(R)$ are affine, so that the flat map $f\colon X \to Z$ corresponds to a flat homomorphism $\sigma\colon R \to S$ of noetherian rings.

First, some notation. For a ring P, $\mathbf{M}(P)$ will denote the category of P-modules. Forgetting for the moment that σ is flat, let $\tau\colon R \to T$ be a flat ring-homomorphism. If

$$\operatorname{Hom}_{\sigma,\tau}\colon \mathbf{M}(S)^{\mathsf{op}} \times \mathbf{M}(T) \to \mathbf{M}(T \otimes_R S)$$

is the obvious functor such that

$$\operatorname{Hom}_{\sigma,\tau}(A, B) := \operatorname{Hom}_R(A, B),$$

then, since (by flatness of τ) any K-injective T-complex is K-injective over R, there is a derived functor

$$\mathsf{R}\operatorname{Hom}_{\sigma,\tau}\colon \mathbf{D}(S)^{\mathsf{op}} \times \mathbf{D}(T) \to \mathbf{D}(T \otimes_R S)$$

such that, with $(F \to J_F)_{F\in\mathbf{D}(T)}$ a family of K-injective T-resolutions, and $E \in \mathbf{D}(S)$,

$$\mathsf{R}\operatorname{Hom}_{\sigma,\tau}(E, F) := \operatorname{Hom}_{\sigma,\tau}(E, J_F).$$

Set $\operatorname{Hom}_\sigma := \operatorname{Hom}_{\sigma,\mathrm{id}_R}$.

Let $p_1\colon T \to T \otimes_R S$ be the R-algebra homomorphism with $p_1(t) = t \otimes 1$. There is a natural functorial isomorphism in $\mathbf{D}(T \otimes_R S)$:

$$\mathsf{R}\operatorname{Hom}_{\sigma,\tau}(E, F) \xrightarrow{\ \sim\ } \mathsf{R}\operatorname{Hom}_{p_1}(T \otimes_R E, F) \quad (F \in \mathbf{D}(T)). \tag{3.2.4}$$

(For this, just replace F by a K-injective T-resolution.)

Let $p_2\colon S \to T \otimes_R S$ be the R-algebra map with $p_2(s) = 1 \otimes s$. Let $\rho_\tau\colon \mathbf{D}(T) \to \mathbf{D}(R)$ be the restriction-of-scalars functor induced by τ; and define ρ_{p_2} analogously. Then, in $\mathbf{D}(S)$,

$$\mathsf{R}\operatorname{Hom}_\sigma(E, \rho_\tau F) = \rho_{p_2}\mathsf{R}\operatorname{Hom}_{\sigma,\tau}(E, F) \qquad (E \in \mathbf{D}(S),\ F \in \mathbf{D}(T)).$$

There results a "multiplication" map in $\mathbf{D}(T\otimes_R S)$:

$$\mu\colon (T\otimes_R S)\otimes_S \mathsf{R}\mathrm{Hom}_\sigma(E,\rho_\tau F)\to \mathsf{R}\mathrm{Hom}_{\sigma,\tau}(E,F),$$

and hence a natural composition in $\mathbf{D}(S)$

$$\begin{aligned}\mathsf{R}\mathrm{Hom}_\sigma(E,\rho_\tau F) &\xrightarrow{\ \sim\ } S\otimes^{\mathsf{L}}_{T\otimes_R S}\big((T\otimes_R S)\otimes_S \mathsf{R}\mathrm{Hom}_\sigma(E,\rho_\tau F)\big)\\ &\xrightarrow{S\otimes^{\mathsf{L}}_{T\otimes_R S}\mu} S\otimes^{\mathsf{L}}_{T\otimes_R S}\mathsf{R}\mathrm{Hom}_{\sigma,\tau}(E,F).\end{aligned}\tag{3.2.5}$$

Now, assuming σ to be flat, we derive algebraic expressions for $f^\times$ and $f^!$.

Application of the functor $\mathsf{R}\Gamma(Z,-)=\mathsf{R}\mathrm{Hom}(\mathcal{O}_Z,-)$ to item 1 in the proof of Corollary 3.1.2, gives $\mathsf{R}\mathrm{Hom}_Z(A^\sim,B^\sim)=\mathsf{R}\mathrm{Hom}_R(A,B)$. Since $(-)^{\sim_S}\colon \mathbf{D}(S)\to\mathbf{D}_{\mathsf{qc}}(X)$ is an equivalence of categories [BN93, p. 230, 5.5], it results from the canonical isomorphism (with $E\in\mathbf{D}(S)$, $M\in\mathbf{D}(R)$ and $\sigma_*\colon\mathbf{D}(S)\to\mathbf{D}(R)$ the functor given by restricting scalars)

$$\mathrm{Hom}_{\mathbf{D}(S)}\big(E,\mathsf{R}\mathrm{Hom}_\sigma(S,M)\big)\xrightarrow{\ \sim\ }\mathrm{Hom}_{\mathbf{D}(R)}(\sigma_*E,M)$$

that there is a functorial isomorphism

$$\varrho(M)\colon \big(\mathsf{R}\mathrm{Hom}_\sigma(S,M)\big)^{\sim_S}\cong f^\times\big(M^{\sim_R}\big)\qquad (M\in\mathbf{D}(R))\tag{3.2.6}$$

such that $f_*\varrho(M)$ is the isomorphism $\zeta(M^{\sim_R})$ in Lemma 3.1.

Next, let $\pi_i\colon X\times_Z X\to X$ $(i=1,2)$ be the projection maps, and let $\delta\colon X\to X\times_Z X$ be the diagonal map. Set $S^{\mathsf{e}}:=S\otimes_R S$. Note that if $A\to B$ is a homomorphism of rings, corresponding to $g\colon\operatorname{Spec}B\to\operatorname{Spec}A$, and if $N\in\mathbf{D}(A)$, then

$$\mathsf{L}g^*\big(N^{\sim_A}\big)=\big(B\otimes^{\mathsf{L}}_A N\big)^{\sim_B}.\tag{3.2.7}$$

This follows easily from the fact that the functor $(-)^{\sim_A}$ preserves both quasi-isomorphisms and K-flatness of complexes.

Lemma 3.2.8. *There is a natural functorial isomorphism of the map*

$$\psi(f)M^{\sim_R}: f^\times M^{\sim_R}\to f^! M^{\sim_R}\qquad \big(M\in\mathbf{D}(R)\big)$$

with the sheafification of the natural composite $\mathbf{D}(S)$*-map*

$$\begin{aligned}\psi(\sigma)M\colon \mathsf{R}\mathrm{Hom}_\sigma(S,M) &\xrightarrow{\ \sim\ } S\otimes^{\mathsf{L}}_{S^{\mathsf{e}}}(S^{\mathsf{e}}\otimes_S \mathsf{R}\mathrm{Hom}_\sigma(S,M))\\ &\xrightarrow{\ \sim\ } S\otimes^{\mathsf{L}}_{S^{\mathsf{e}}}(S\otimes_R \mathsf{R}\mathrm{Hom}_R(S,M))\\ &\longrightarrow S\otimes^{\mathsf{L}}_{S^{\mathsf{e}}}\mathsf{R}\mathrm{Hom}_{\sigma,\sigma}(S,S\otimes_R M).\end{aligned}$$

Proof. Using (3.2.6) and (3.2.7), and the fact that sheafification is an equivalence of categories from $\mathbf{D}(S)$ to $\mathbf{D}(\operatorname{Spec}S)$ ([BN93, p. 230, 5.5]), one translates the definition of the base-change map β in 2.2 to the commutative-algebra context, and finds that

$$\beta(M^{\sim_R}):\pi_2^*f^\times M^{\sim_R}\to\pi_1^\times f^*M^{\sim_R}$$

is naturally isomorphic to the sheafification of the natural composite $\mathbf{D}(S^{\mathsf{e}})$-map

$$S\otimes_R \mathsf{R}\mathrm{Hom}_\sigma(S,M)\to\mathsf{R}\mathrm{Hom}_{\sigma,\sigma}(S,S\otimes_R M)\xrightarrow{\ \sim\ }\mathsf{R}\mathrm{Hom}_{p_1}(S^{\mathsf{e}},S\otimes_R M)$$

where the isomorphism comes from (3.2.4) (with $T=S$).

Lemma 3.2.1 gives that $\psi(f)$ is naturally isomorphic to the composite

$$f^\times\cong\mathsf{L}\delta^*\pi_2^*f^\times\xrightarrow{\mathsf{L}\delta^*\beta}\mathsf{L}\delta^*\pi_1^\times f^*,$$

whence the conclusion. ∎

Here is a neater description of $\psi(\sigma)M$—and hence of $\psi(f)M^{\sim_R}$.

PROPOSITION 3.2.9. *The map $\psi(\sigma)M$ in 3.2.8 factors as*

$$\mathsf{R}\mathrm{Hom}_\sigma(S,M) \xrightarrow{\vartheta} \varpi\mathsf{R}\mathrm{Hom}_\sigma(S,S\otimes_R M) \xrightarrow{(3.2.5)} S\otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{\sigma,\sigma}(S,S\otimes_R M),$$

where ϑ is induced by the natural $\mathbf{D}(R)$-map $M\to S\otimes_R M$.

Proof. Note that ϑ is the natural composite $\mathbf{D}(S)$-map

$$\mathsf{R}\mathrm{Hom}_\sigma(S,M)\to S\otimes_R \mathsf{R}\mathrm{Hom}_\sigma(S,M)\to \mathsf{R}\mathrm{Hom}_\sigma(S,S\otimes_R M),$$

recall the description in the proof of 3.2.8 of the map β, refer to the factorization of $\psi(f)M^{\sim_R}$ coming from 3.2.2, and fill in the details. $\square$

From 3.2.9 and (2.1.5.1) it follows easily that:

COROLLARY 3.2.10. *For any $N\in\mathbf{D}(S)$, the map $\eta(N^{\sim_S})$ from 2.1 sheafifies the natural composite $\mathbf{D}(S)$-map*

$$\begin{aligned}N \xrightarrow{\vartheta'} \mathrm{Hom}_\sigma(S,S\otimes_R N) &\longrightarrow \mathsf{R}\mathrm{Hom}_\sigma(S,S\otimes_R N)\\ &\xrightarrow{(3.2.5)} S\otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{\sigma,\sigma}(S,S\otimes_R N),\end{aligned}$$

where ϑ' takes $n\in N$ to the map $s\mapsto s\otimes n$. $\square$

Using Proposition 2.3.2, we now develop more information about the above map $\psi(\sigma)M$ when $\sigma\colon k\to S$ is an essentially-finite-type algebra over a field k, and $M=k$.

For any $\mathfrak{p}\in\operatorname{Spec}S$, let $I(\mathfrak{p})$ be the injective hull of the residue field $\kappa(\mathfrak{p}):=S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. Let $D^\sigma\in\mathbf{D}(S)$ be a *normalized residual complex,* thus a complex of the form

$$D^\sigma := \cdots 0\to I^{-n}\to I^{-n+1}\to\cdots\to I^0\to 0\cdots$$

where for each integer m, I^{-m} is the direct sum of the $I(\mathfrak{p})$ as $\mathfrak{p}$ runs through the primes such that $S/\mathfrak{p}$ has dimension m. The sheafification of D^σ is $f^!k$, where $f:=\operatorname{Spec}\sigma$ and where we identify k with the structure sheaf of $\operatorname{Spec}k$, see [H66, Chapter VI, §1].

PROPOSITION 3.3. *Under the preceding circumstances, there exists a split exact sequence of S-modules*

$$0\longrightarrow \bigoplus_{\mathfrak{p}\text{ nonmaximal}} J(\mathfrak{p})\longrightarrow \mathrm{Hom}_\sigma(S,k)\xrightarrow{\psi^0} I^0\longrightarrow 0,$$

such that for each nonmaximal prime $\mathfrak{p}$, $J(\mathfrak{p})$ is a direct sum of uncountably many copies of $I(\mathfrak{p})$, and in $\mathbf{D}(S)$, $\psi(\sigma)k$ is the composition

$$\mathsf{R}\mathrm{Hom}_\sigma(S,k)=\mathrm{Hom}_\sigma(S,k)\xrightarrow{\psi^0} I^0\hookrightarrow \mathbf{D}^\sigma.$$

Proof. Since $\mathrm{Hom}_\sigma(S,k)$ is an injective S-module, there is a decomposition

$$\mathrm{Hom}_\sigma(S,k)\cong\bigoplus_{\mathfrak{p}\in\operatorname{Spec}S} I(\mathfrak{p})^{\mu(\mathfrak{p})}$$

where, $\sigma_\mathfrak{p}$ being the natural composite map $k\xrightarrow{\sigma} S\twoheadrightarrow S/\mathfrak{p}$, $\mu(\mathfrak{p})$ is the dimension of the $\kappa(\mathfrak{p})$-vector space

$$\begin{aligned}\mathrm{Hom}_{S_\mathfrak{p}}\big(\kappa(\mathfrak{p}),\mathrm{Hom}_k(S,k)_\mathfrak{p}\big) &= \mathrm{Hom}_S\big(S/\mathfrak{p},\mathrm{Hom}_\sigma(S,k)\big)\otimes_S S_\mathfrak{p}\\ &\cong \mathrm{Hom}_{\sigma_\mathfrak{p}}(S/\mathfrak{p},k)\otimes_{S/\mathfrak{p}}\kappa(\mathfrak{p}).\end{aligned}$$

In particular, if $\mathfrak{p}$ is maximal (so that $S/\mathfrak{p} = \kappa(\mathfrak{p})$) then $\mu(\mathfrak{p}) = 1$. Thus $\operatorname{Hom}_\sigma(S,k)$ has a direct summand J^0 isomorphic to I^0. (This J^0 does not depend on the foregoing decomposition: it consists of all $h \in \operatorname{Hom}_\sigma(S,k)$ such that the S-submodule Sh has finite length.)

Now since D^σ is a bounded injective complex, the $\mathbf{D}(S)$-map $\psi(\sigma)$ is represented by an ordinary map of S-complexes $\operatorname{Hom}_\sigma(S,k) \to D^\sigma$, that is, by a map of S-modules $\psi^0\colon \operatorname{Hom}_\sigma(S,k) \to I^0$. By 3.2.8, the sheafification of $\psi(\sigma)$ is $\psi(f)k\colon f^\times k \to f^! k$, and hence Proposition 2.3.2 implies that ψ^0 maps J^0 isomorphically onto I^0. Thus ψ^0 has a right inverse, unique up to automorphisms of I^0; and $\operatorname{Hom}_\sigma(S,k)$ is the direct sum of J^0 and $\ker(\psi^0)$, whence

$$\ker(\psi^0) \cong \bigoplus_{\mathfrak{p}\text{ nonmaximal}} I(\mathfrak{p})^{\mu(\mathfrak{p})}.$$

Last, in [Nm14a, Theorem 1.11] it is shown that for nonmaximal $\mathfrak{p}$,

$$\mu_\mathfrak{p} = \dim_{\kappa(\mathfrak{p})}\big(\operatorname{Hom}_{\sigma_\mathfrak{p}}(S/\mathfrak{p},k) \otimes_{S/\mathfrak{p}} \kappa(\mathfrak{p})\big) \geqslant |\#k|^{\aleph_0},$$

with equality if S is finitely generated over k. □

4. Applications

4.1. (Reduction Theorems.) At least for flat maps, $\psi\colon (-)^\times \to (-)^!$ can be used to prove one of the main results in [AILN10], namely Theorem 4.6 (for which only a hint of a proof is given there). With notation as in §3.2, and again, $S^\mathsf{e} := S \otimes_R S$, that Theorem 4.6 asserts the existence of a complex $D^\sigma \in \mathbf{D}(S)$, depending only on σ, and for all σ-perfect $M \in \mathbf{D}(S)$ (i.e., M is isomorphic in $\mathbf{D}(R)$ to a bounded complex of flat R-modules, the cohomology modules of M are all finitely generated over S, and all but finitely many of them vanish), and all $N \in \mathbf{D}(S)$, a functorial $\mathbf{D}(S)$-isomorphism

$$\boxed{\mathsf{R}\operatorname{Hom}_S(M, D^\sigma) \otimes^{\mathsf{L}}_S N \cong S \otimes^{\mathsf{L}}_{S^e} \mathsf{R}\operatorname{Hom}_{\sigma,\sigma}(M,N).} \tag{4.1.1}$$

In particular,

$$D^\sigma \cong S \otimes^{\mathsf{L}}_{S^e} \mathsf{R}\operatorname{Hom}_{\sigma,\sigma}(S,S). \tag*{(4.1.1)$'$}$$

This explicit description is noteworthy in that the sheafification $\widetilde{D^\sigma}$ is a *relative dualizing complex* $f^!\mathcal{O}_Y$, where $f := \operatorname{Spec}\sigma\colon \operatorname{Spec} S \to \operatorname{Spec} R$ (see [AIL11, Example 2.3.2] or Lemma 3.2.8 above); and otherwise-known definitions of $f^!$ involve choices, of which $f^!$ must be proved independent.

The present proof will be based on the isomorphism in Lemma (4.1.5) below,[1] which is similar to (and more or less implied by) the isomorphism in [AILN10, 6.6].

Let $f\colon X \to Z$ be an arbitrary map in $\mathcal{E}$. Let $Y := X \times_Z X$, and let π_1 and π_2 be the projections from Y to X. For $M, N \in \mathbf{D}_{\mathsf{qc}}(X)$ there are natural maps

$$\begin{aligned}\pi_1^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(M, f^!\mathcal{O}_Z) \otimes^{\mathsf{L}}_Y \pi_2^* N &\longrightarrow \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y(\pi_1^*M, \pi_1^* f^!\mathcal{O}_Z) \otimes^{\mathsf{L}}_Y \pi_2^*N \\ &\longrightarrow \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y(\pi_1^*M, \pi_1^* f^!\mathcal{O}_Z \otimes^{\mathsf{L}}_Y \pi_2^*N).\end{aligned} \tag{4.1.2}$$

The first of these is the unique one making the following otherwise natural diagram (whose top left entry is in $\mathbf{D}_{\mathsf{qc}}(Y)$) commute:

[1] After [AILN10] appeared, Leo Alonso and Ana Jeremías informed us that Lemma (4.1.5) is an instance of [G71, p. 123, (6.4.2)]—whose proof, however, is not given in detail.

$$\begin{array}{ccc} \pi_1^*\mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}(M, f^!\mathcal{O}_Z)\otimes^{\mathsf{L}}_Y \pi_2^*N & \longrightarrow & \mathsf{R}\mathcal{H}om_Y^{\mathsf{qc}}(\pi_1^*M, \pi_1^*f^!\mathcal{O}_Z)\otimes^{\mathsf{L}}_Y \pi_2^*N \\ \downarrow & & \downarrow \\ \pi_1^*\mathsf{R}\mathcal{H}om_X(M, f^!\mathcal{O}_Z)\otimes^{\mathsf{L}}_Y \pi_2^*N & \longrightarrow & \mathsf{R}\mathcal{H}om_Y(\pi_1^*M, \pi_1^*f^!\mathcal{O}_Z)\otimes^{\mathsf{L}}_Y \pi_2^*N \end{array} \tag{4.1.3}$$

In [AJL11, §5.7] it is shown that for *perfect* $\mathcal{E}$-maps $e\colon X \to Z$ (that is, e has finite flat dimension), the functor $e^!\colon \mathbf{D}^+_{\mathsf{qc}}(Z) \to \mathbf{D}^+_{\mathsf{qc}}(X)$ extends pseudofunctorially to a functor—still denoted $e^!$—from $\mathbf{D}_{\mathsf{qc}}(Z)$ to $\mathbf{D}_{\mathsf{qc}}(X)$ such that

$$e^!F = e^!\mathcal{O}_Z \otimes^{\mathsf{L}}_X \mathsf{L}e^*F \qquad (F \in \mathbf{D}_{\mathsf{qc}}(X). \tag{4.1.4}$$

For proper e, the extended $e^!$ is still right-adjoint to $\mathsf{R}e_*$ (see [AJL11, proof of Prop. 5.9.3]).

The complex $M \in \mathbf{D}(X)$ is *perfect relative to* f (or simply f*-perfect*) if M has coherent cohomology and has finite flat dimension over Z. In particular, the map f is perfect if and only if $\mathcal{O}_X$ is f-perfect.

Lemma 4.1.5. *If the $\mathcal{E}$-map $f\colon X \to Z$ is flat and $M \in \mathbf{D}(X)$ is f-perfect, then for all $N \in \mathbf{D}_{\mathsf{qc}}(X)$, the composite map (4.1.2) is an isomorphism.*

Proof. It holds that $\mathsf{R}\mathcal{H}om_X(M, f^!\mathcal{O}_Z) \in \mathbf{D}_{\mathsf{qc}}(X)$ and $\mathsf{R}\mathcal{H}om_Y(\pi_1^*M, \pi_1^*f^!\mathcal{O}_Z) \in \mathbf{D}_{\mathsf{qc}}(Y)$ (see proof of [AILN10, 6.6]); and so the vertical arrows in (4.1.3) are isomorphisms. So is the bottom arrow in (4.1.3) (see e.g., [L09, (4.6.6)]). Hence the first map in (4.1.2) is an isomorphism.

As for the second, from the flatness of f it follows that π_1^*M is π_2-perfect, and that there is a base-change isomorphism (cf. (1.1.3))

$$\pi_1^*f^!\mathcal{O}_Z \xrightarrow{\ \sim\ } \pi_2^!\mathcal{O}_X. \tag{4.1.6}$$

The conclusion follows then from [AILN10, 6.6] (with $g := \pi_2$, $E' = \mathcal{O}_X$, $F' = N$, and with $\mathsf{R}\mathcal{H}om$ replaced throughout by $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}$), in whose proof we can replace the duality isomorphism (5.9.1) there by (1.6.1) in this paper, and use the definition (4.1.4) of $e^!$ for any finite-flat-dimensional map e in $\mathcal{E}$ (for instance g, h and i in *loc. cit.*), thereby rendering unnecessary the boundedness condition in *loc. cit.* on the complex F'. (In this connection, note that if $e = hi$ with h smooth and i a closed immersion then i is perfect [I71, p. 246, 3.6].) $\square$

For $f\colon X \to Z$ a flat $\mathcal{E}$-map and $M \in \mathbf{D}_{\mathsf{qc}}(X)$ set

$$M^\vee := \mathsf{R}\mathcal{H}om_X^{\mathsf{qc}}(M, f^!\mathcal{O}_Z),$$

and consider the composite map, with $N \in \mathbf{D}^+_{\mathsf{qc}}(X)$,

$$\mathsf{R}\mathcal{H}om_Y^{\mathsf{qc}}(\pi_1^*M, \pi_2^\times N) \longrightarrow \mathsf{R}\mathcal{H}om_Y^{\mathsf{qc}}(\pi_1^*M, \pi_2^!N) \xrightarrow{\ \sim\ } \pi_1^*M^\vee \otimes^{\mathsf{L}}_Y \pi_2^*N \tag{4.1.7}$$

where the first map is induced by $\psi(\pi_2)$, and the isomorphism on the right is gotten by inverting the one given by 4.1.5 and then replacing $\pi_1^*f^!\mathcal{O}_Z \otimes^{\mathsf{L}}_Y \pi_2^*N$ by the isomorphic object $\pi_2^!N$ (see (4.1.4) and (4.1.6)). Remark 6.2 in [AILN10] authorizes replacement in (4.1.7) of M by $M^\vee$, and recalls that the natural map is an isomorphism $M \xrightarrow{\ \sim\ } M^{\vee\vee}$; thus one gets the composite map

$$\mathsf{R}\mathcal{H}om_Y^{\mathsf{qc}}(\pi_1^*M^\vee, \pi_2^\times N) \longrightarrow \mathsf{R}\mathcal{H}om_Y^{\mathsf{qc}}(\pi_1^*M^\vee, \pi_2^!N) \xrightarrow{\ \sim\ } \pi_1^*M \otimes^{\mathsf{L}}_Y \pi_2^*N. \tag{4.1.7$^\vee$}$$

Theorem 4.1.8. *If $M \in \mathbf{D}_{\mathsf{qc}}(X)$ is f-perfect, and $N \in \mathbf{D}^+_{\mathsf{qc}}(X)$, then application of* $\mathsf{L}\delta^*$ *(resp. $\delta^!$) to the composite* (4.1.7) *(resp.* $(4.1.7)^\vee$*) produces an isomorphism*

$$\mathsf{L}\delta^*\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_Y(\pi_1^*M, \pi_2^\times N) \xrightarrow{\sim} M^\vee \otimes^{\mathsf{L}}_X N$$
$$(resp.) \qquad \delta^!(\pi_1^*M \otimes^{\mathsf{L}}_Y \pi_2^*N) \xleftarrow{\sim} \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(M^\vee, N).$$

Proof. By 2.3.5, application of $\mathsf{L}\delta^*$ to the first map in (4.1.7) produces an isomorphism. Similarly, in view of (1.6.2), applying $\delta^\times$ $(= \delta^!)$ to the first map in $(4.1.7)^\vee$ produces an isomorphism. □

Remark 4.1.9. Using Remark 2.3.6 for the first map in (4.1.7), *one can extend Theorem 4.1.8 to all* $N \in \mathbf{D}_{\mathsf{qc}}(X)$. This results immediately from the fact, given by [Nm14b, Proposition 7.11], that *if $e = pu$ is a compactification of a perfect $\mathcal{E}$-map $e\colon X \to Z$ then the following natural map is an isomorphism*:

$$e^!N \underset{(4.1.4)}{=\!=} e^!\mathcal{O}_Z \otimes^{\mathsf{L}}_X \mathsf{L}e^*N \cong u^*(p^\times\mathcal{O}_Z \otimes^{\mathsf{L}}_X \mathsf{L}p^*N) \to u^*p^\times N \qquad (N \in \mathbf{D}_{\mathsf{qc}}(Z)).$$

Remark 4.1.10. The first isomorphism in Theorem 4.1.8 is a globalization (for flat f and cohomologically bounded-below N) of [AILN10, Theorem 4.6]. Indeed, let $\sigma\colon R \to S$ be an essentially-finite-type flat homomorphism of noetherian rings, $f = \operatorname{Spec}(\sigma)$, $S^{\mathsf{e}} := S \otimes_R S$ and $p_i\colon S \to S^{\mathsf{e}}$ $(i = 1, 2)$ the canonical maps. Let $M, N, D^\sigma \in \mathbf{D}(S)$, where M is σ-perfect and D^σ is a relative dualizing complex, sheafifying to $\widetilde{D^\sigma} = f^!\mathcal{O}_Z$ [AIL11, Example 2.3.2]. Set $X := \operatorname{Spec} S$, $Z := \operatorname{Spec} R$, $Y := X \times_Z X$, and let $\delta\colon X \to Y$ be the diagonal. Then (as the cohomology of M is bounded and finitely generated over S) $\delta_*\big(\widetilde{M^\vee} \otimes^{\mathsf{L}}_X \widetilde{N}\big)$ sheafifies $\mathsf{R}\mathrm{Hom}_S(M, D^\sigma) \otimes^{\mathsf{L}}_S N \in \mathbf{D}(S^{\mathsf{e}})$, and, with notation as in §3.2, $\delta_*\mathsf{L}\delta^*\mathsf{R}\mathcal{H}om_Y(\pi_1^*M, \pi_2^\times N)$ sheafifies

$$\begin{aligned} S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{S^{\mathsf{e}}}(M \otimes_S S^{\mathsf{e}}, \mathsf{R}\mathrm{Hom}_{p_2}(S^{\mathsf{e}}, N)) &\cong S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{p_2}(M \otimes_S S^{\mathsf{e}}, N) \\ &\cong S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{p_2}(M \otimes_R S, N) \\ &\cong S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{\sigma,\sigma}(M, N). \end{aligned}$$

Thus in this situation, application of δ_* to (4.1.8) gives the existence of a functorial isomorphism (4.1.1) (that should be closely related to—if not identical with—the one in [AILN10, 4.6]).

Remark 4.1.11. Let f be as in 4.1.5, and let $\delta\colon X \to Y := X \times_Z X$ be the diagonal map. Keeping in mind the last paragraph of section 1.5 above, one checks that the reduction isomorphism [AILN10, Corollary 6.5]

$$\boxed{\delta^!(\pi_1^*M \otimes^{\mathsf{L}}_X \pi_2^*N) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(M^\vee, N)} \tag{4.1.11.1}$$

is inverse to the second isomorphism in 4.1.8. (In [AILN10], see the proof of Corollary 6.5, and the last four lines of the proof of Theorem 6.1 with $(X', Y', Y, Z) = (Y, X, Z, X)$, $E = \mathcal{O}_Y$, and $(g, u, f, v) = (\pi_2, f, f, \pi_1)$, so that $\nu = \gamma = \mathrm{id}_X$.)

In the affine case, with assumptions on σ, M and N as above, "desheafification" of (i.e., applying derived global sections to) (4.1.11.1) produces a functorial isomorphism

$$\boxed{\mathsf{R}\mathrm{Hom}_{S^{\mathsf{e}}}(S, M \otimes^{\mathsf{L}}_R N) \xrightarrow{\sim} \mathsf{R}\mathrm{Hom}_S(\mathsf{R}\mathrm{Hom}_S(M, D^\sigma), N)}$$

with the same source and target as the one in [AILN10, p. 736, Theorem 1]. (We suspect, but don't know, that the two isomorphisms are the same—at least up to sign.)

4.2. In this section we review, from the perspective afforded by results in this paper, some known basic facts about *integrals, residues and fundamental classes.* The description is mostly in abstract terms. What will be new is a direct *concrete* description of the fundamental class of a flat essentially-finite-type homomorphism $\sigma\colon R \to S$ of noetherian rings (Theorem 4.2.4).

Let $I \subset S$ be an ideal such that S/I is a finite R-module, and let Γ_I be the subfunctor of the identity functor on S-modules M given objectwise by

$$\Gamma_I(M) := \{\, m \in M \mid I^n m = 0 \text{ for some } n > 0 \,\}.$$

There is an obvious map from the derived functor $\mathsf{R}\Gamma_I$ to the identity functor on $\mathbf{D}(S)$.

In view of the isomorphism (3.2.6), one can apply derived global sections in Remark 2.3.4 to get, in the present context, the following diagram, whose rectangle commutes. In this diagram, $\sigma_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ is the functor given by restricting scalars; and ω_σ is a canonical module of σ (that is, an S-module whose sheafification is a relative dualizing sheaf of $f := \operatorname{Spec}\sigma$, as in Remark 2.3.4, where the integer d is defined as well); and D^σ is, as in 4.1.10, a relative dualizing complex.

$$\begin{array}{ccccc}
\sigma_*\mathsf{R}\Gamma_I\,\mathsf{R}\mathrm{Hom}_\sigma(S,R) & \xrightarrow[\text{via}\,\psi]{\sim} & \sigma_*\mathsf{R}\Gamma_I D^\sigma & \longleftarrow & \sigma_*\mathsf{R}\Gamma_I\,\omega_\sigma[d] \\
\downarrow & & \Big\downarrow {\scriptstyle \bar{\int}_I} & & \\
\sigma_*\mathsf{R}\mathrm{Hom}_\sigma(S,R) & & & & \\
\Vert & & & & \\
\mathsf{R}\mathrm{Hom}_R(S,R) \xrightarrow[\text{via}\,\sigma]{} \mathsf{R}\mathrm{Hom}_R(R,R) & = & R & &
\end{array}$$

If σ is Cohen-Macaulay and equidimensional, the natural map is an isomorphism $\omega_\sigma[d] \xrightarrow{\sim} D^\sigma$; and application of H^0 to the preceding diagram produces a commutative diagram of R-modules

$$\begin{array}{ccc}
\Gamma_I\,\mathrm{Hom}_\sigma(S,R) & \xrightarrow[\text{via}\,\psi]{\sim} & \mathrm{H}^d_I\,\omega_\sigma \\
\Big\downarrow & & \Big\downarrow {\scriptstyle \int_I} \\
\mathrm{Hom}_R(S,R) & \xrightarrow[\text{evaluation at } 1]{} & R
\end{array}$$

This shows that *an explicit description of* $(\text{via}\,\psi)^{-1}$ *is more or less the same as an explicit description of* $\int_I$*—and so, when* I *is a maximal ideal, of residues.* "Explicit" includes the realization of the relative canonical module ω_σ in terms of regular differential forms (cf. Remark 2.3.4).

Such a realization comes out of the theory of the *fundamental class* $\mathbf{c}_f$ of a flat $\mathcal{E}$-map f, as indicated below. This $\mathbf{c}_f$ is a key link between the abstract duality theory of f and its canonical reification via differential forms. It may be viewed as an orientation, compatible with essentially étale base change, in a suitable bivariant theory on the category of flat $\mathcal{E}$-maps [AJL14].

Given a flat $\mathcal{E}$-map $f\colon X \to Z$, with π_1 and π_2 the projections from $Y := X \times_Z X$ to X, and $\delta\colon X \to Y$ the diagonal map, let $\mathbf{c}_f$ be, as in [AJL14, Example 2.3], the natural composite $\mathbf{D}(X)$-map

$$\mathsf{L}\delta^*\delta_*\mathcal{O}_X \xrightarrow{\sim} \mathsf{L}\delta^*\delta_*\delta^!\pi_1^!\mathcal{O}_X \longrightarrow \mathsf{L}\delta^*\pi_1^!\mathcal{O}_X \underset{(1.1.3)}{\xrightarrow{\sim}} \mathsf{L}\delta^*\pi_2^*f^!\mathcal{O}_Z \xrightarrow{\sim} f^!\mathcal{O}_Z. \tag{4.2.1}$$

Let $\mathcal{J}$ be the kernel of the natural surjection $\mathcal{O}_Y \twoheadrightarrow \delta_*\mathcal{O}_X$. Using a flat $\mathcal{O}_Y$-resolution of $\delta_*\mathcal{O}_X$ one gets a natural isomorphism of $\mathcal{O}_X$-modules

$$\Omega^1_f = \mathcal{J}/\mathcal{J}^2 \cong \mathcal{T}or_1^{\mathcal{O}_Y}(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X) = H^{-1}\mathsf{L}\delta^*\delta_*\mathcal{O}_X,$$

whence a map of graded-commutative $\mathcal{O}_X$-algebras, with $\Omega^i_f := \wedge^i \Omega^1_f$,

$$\oplus_{i\geqslant 0}\ \Omega^i_f \to \oplus_{i\geqslant 0}\, \mathcal{T}or_i^{\mathcal{O}_Y}(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X) = \oplus_{i\geqslant 0}\, H^{-i}\mathsf{L}\delta^*\delta_*\mathcal{O}_X\,. \tag{4.2.2}$$

In particular one has, with d as above, a natural composition

$$\gamma_f\colon \Omega^d_f \to H^{-d}\mathsf{L}\delta^*\delta_*\mathcal{O}_X \xrightarrow{\text{via } \mathbf{c}_f} H^{-d}f^!\mathcal{O}_Z =: \omega_f.$$

(In the literature, the term "fundamental class" often refers to this γ_f rather than to $\mathbf{c}_f$.) When f is essentially smooth, this map is an isomorphism, as is $\omega_f \to f^!\mathcal{O}_Z$. (The proof uses the known fact that there exists an isomorphism $\Omega^d_f \xrightarrow{\sim} f^!\mathcal{O}_Z$, but does not reveal the relation between that isomorphism and γ_f, see [AJL14, 2.4.2, 2.4.4].) It follows that if f is just *generically* smooth, then γ_f is a generic isomorphism. For example, if X is a reduced algebraic variety over a field k, of pure dimension d, with structure map $f\colon X \to \operatorname{Spec} k$, then one deduces that ω_f is canonically represented by a coherent sheaf of meromorphic d-forms—the sheaf of regular d-forms—containing the sheaf Ω^d_f of holomorphic d-forms, with equality over the smooth part of X.

From γ_f and the above $\int_I$ one deduces a map

$$\mathrm{H}^d_I\Omega^d_\sigma \to R$$

that generalizes the classical residue map.

Theorem 4.2.4 below provides a direct concrete definition of the fundamental class of a flat essentially-finite-type homomorphism $\sigma\colon R\to S$ of noetherian rings.

First, some preliminaries. As before, set $S^{\mathsf{e}} := S\otimes_R S$, let $p_1\colon S\to S^{\mathsf{e}}$ be the homomorphism such that for $s\in S$, $p_1(s) = s\otimes 1$, and $p_2\colon S\to S^{\mathsf{e}}$ such that $p_2(s) = 1\otimes s$.

Let $f\colon \operatorname{Spec} S =: X \to Z := \operatorname{Spec} R$ be the scheme-map corresponding to σ. Let π_1 and π_2 be the projections (corresponding to p_1 and p_2) from $X\times_Z X$ to X.

Let $\operatorname{Hom}_{\sigma,\sigma}$ and Hom_{p_1} be as in §3.2.3.

For an S-complex F, considered as an S^{e}-complex via the multiplication map $S^{\mathsf{e}}\to S$, let $\mu_F\colon F\to \operatorname{Hom}_{\sigma,\sigma}(S,F)$ be the S^{e}-homomorphism taking $f\in F$ to the map $s\mapsto sf$.

For an S-complex E, there is an obvious S^{e}-isomorphism

$$\operatorname{Hom}_{\sigma,\sigma}(S,E) \xrightarrow{\sim} \operatorname{Hom}_{p_1}(S^{\mathsf{e}},E).$$

Taking E to be a K-injective resolution of F (over S, and hence, since σ is flat, also over R), one gets the isomorphism in the following statement.

Lemma 4.2.3. *Let $F\in\mathbf{D}(S)$ have sheafification $F^\sim\in\mathbf{D}(X)$. The sheafification of the natural composite $\mathbf{D}(S^{\mathsf{e}})$-map*

$$\xi(F)\colon F \xrightarrow{\mu_F} \operatorname{Hom}_{\sigma,\sigma}(S,F) \longrightarrow \mathsf{R}\mathrm{Hom}_{\sigma,\sigma}(S,F) \xrightarrow{\sim} \mathsf{R}\mathrm{Hom}_{p_1}(S^{\mathsf{e}},F)$$

is the natural composite (with ϵ_2 the counit map)

$$\delta_*F^\sim \xrightarrow{\sim} \delta_*\delta^\times\pi_1^\times F^\sim \xrightarrow{\epsilon_2} \pi_1^\times F^\sim. \tag{4.2.3.1}$$

Proof. The sheafification of $\mathsf{R}\mathrm{Hom}_{p_1}(S^{\mathsf{e}}, F)$ is $\pi_1^{\times}F^{\sim}$, see (3.2.6). Likewise, with $m\colon S^{\mathsf{e}} \to S$ the multiplication map, and $G \in \mathbf{D}(S^{\mathsf{e}})$, one has that $\delta^{\times}G^{\sim_{S^{\mathsf{e}}}}$ is the sheafification of $\mathsf{R}\mathrm{Hom}_m(S, G)$; and Lemma 3.1 implies that ϵ_2 is the sheafification of the "evaluation at 1" map

$$\mathrm{ev}\colon \mathsf{R}\mathrm{Hom}_m(S, \mathsf{R}\mathrm{Hom}_{p_1}(S^{\mathsf{e}}, F)) \to \mathsf{R}\mathrm{Hom}_{p_1}(S^{\mathsf{e}}, F).$$

Moreover, one checks that the isomorphism $\delta_*F^{\sim} \xrightarrow{\,\sim\,} \delta_*\delta^{\times}\pi_1^{\times}F^{\sim}$ is the sheafification of the natural isomorphism $F \xrightarrow{\,\sim\,} \mathsf{R}\mathrm{Hom}_m(S, \mathsf{R}\mathrm{Hom}_{p_1}(S^{\mathsf{e}}, F))$.

Under the allowable assumption that F is K-injective, one finds then that (4.2.3.1) is the sheafification of the map $\xi'(F)\colon F \to \mathrm{Hom}_{p_1}(S^{\mathsf{e}}, F)$ that takes $f \in F$ to the map $[s \otimes s' \mapsto ss'f]$. It is simple to check that $\xi'(F) = \xi(F)$. $\square$

Theorem 4.2.4. *Let $\sigma\colon R \to S$ be a flat essentially-finite-type map of noetherian rings, and $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ the corresponding scheme-map. Let $\mu\colon S \to \mathrm{Hom}_{\sigma,\sigma}(S, S)$ be the S^{e}-homomorphism taking $s \in S$ to multiplication by s. Then the fundamental class $\mathbf{c}_f$ given by* (4.2.1) *is naturally isomorphic to the sheafification of the natural composite map*

$$S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} S \xrightarrow{\mathrm{id}\otimes\mu} S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathrm{Hom}_{\sigma,\sigma}(S, S) \longrightarrow S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathsf{R}\mathrm{Hom}_{\sigma,\sigma}(S, S).$$

Proof. It suffices to show that the map in Theorem 4.2.4 sheafifies to a map isomorphic to the canonical composite map

$$\mathsf{L}\delta^*\delta_*\mathcal{O}_X \xrightarrow{\,\sim\,} \mathsf{L}\delta^*\delta_*\delta^!\pi_1^!\mathcal{O}_X \longrightarrow \mathsf{L}\delta^*\pi_1^!\mathcal{O}_X \tag{4.2.4.1}$$

(see (4.2.1), in which the last two maps are isomorphisms).

Applying pseudofunctoriality of ψ (Corollary 2.1.4) to $\mathrm{id}_X = \pi_1\delta$, one sees that the map in (4.2.4.1) factors as

$$\mathsf{L}\delta^*\delta_*\mathcal{O}_X \to \mathsf{L}\delta^*\delta_*\delta^{\times}\pi_1^{\times}\mathcal{O}_X \to \mathsf{L}\delta^*\pi_1^{\times}\mathcal{O}_X \xrightarrow{\,\sim\,} \mathsf{L}\delta^*\pi_1^!\mathcal{O}_X,$$

where the isomorphism is from Proposition 2.3.5. Thus the conclusion follows from Lemma 4.2.3. $\square$

Example 4.2.5. Let T be a finite étale R-algebra. The (desheafified) fundamental class $\mathbf{c}_{R\to T}$ is the $\mathbf{D}(T)$-isomorphism from $T \otimes^{\mathsf{L}}_{T^{\mathsf{e}}} T = T$ to $T \otimes^{\mathsf{L}}_{T^{\mathsf{e}}} \mathrm{Hom}_R(T, T) \cong \mathrm{Hom}_R(T, R)$ taking 1 to the trace map. (Cf. [AJL14, Example 2.6].)

If S is an essentially étale T-algebra (for instance, a localization of T), then there is a canonical identification of $\mathbf{c}_{R\to S}$ with $(\mathbf{c}_{R\to T}) \otimes_T S$. (This fact results from [AJL14, 2.5 and 3.1], but can be proved more directly.) However, $\mathbf{c}_{R\to S}$ depends only on $R \to S$, not on T.

Appendix A. Supports

The goal of this appendix is to establish some basic facts—used repeatedly in §2.3—about the relation between subsets of a noetherian scheme X and "localizing tensor ideals" in $\mathbf{D}_{\mathsf{qc}}(X)$.

Notation: Let X be a noetherian scheme. For any $x \in X$, let $\mathcal{O}_x$ be the stalk $\mathcal{O}_{X,x}$, let $\kappa(x)$ be the residue field of $\mathcal{O}_x$, let $\widetilde{\kappa(x)}$ be the corresponding sheaf on $X_x := \operatorname{Spec} \mathcal{O}_x$—a quasi-coherent, *flasque* sheaf, let $\iota_x\colon X_x \to X$ be the canonical (flat) map—a localizing immersion, and let

$$k(x) := \iota_{x*}\widetilde{\kappa(x)} = \mathsf{R}\iota_{x*}\widetilde{\kappa(x)},$$

a quasi-coherent flasque $\mathcal{O}_X$-module whose stalk at a point y is $\kappa(x)$ if y is a specialization of x, and 0 otherwise.

For $E \in \mathbf{D}(X)$, we consider two notions of the *support of* E:

$$\operatorname{supp}(E) := \{\, x \in X \mid E \otimes^{\mathsf{L}}_X k(x) \neq 0 \in \mathbf{D}(X)\,\},$$
$$\operatorname{Supp}(E) := \{\, x \in X \mid E_x \neq 0 \in \mathbf{D}(\mathcal{O}_x)\,\}.$$

Let $\mathbf{D}_{\mathsf{c}}(X)$ ($\mathbf{D}^+_{\mathsf{c}}(X)$) be the full subcategory of $\mathbf{D}(X)$ spanned by the complexes with coherent cohomology modules (vanishing in all but finitely many negative degrees). For affine X and $E \in \mathbf{D}^+_{\mathsf{c}}(X)$ the next Lemma appears in [F79, top of page 158].

LEMMA A.1. *For any* $E \in \mathbf{D}_{\mathsf{qc}}(X)$,

$$\operatorname{supp}(E) \subseteq \operatorname{Supp}(E);$$

and equality holds whenever $E \in \mathbf{D}_{\mathsf{c}}(X)$.

Proof. For $E \in \mathbf{D}_{\mathsf{qc}}(X)$, there is a projection isomorphism

$$E \otimes^{\mathsf{L}}_X k(x) \cong \mathsf{R}\iota_{x*}\big(\iota_x^* E \otimes^{\mathsf{L}}_{X_x} \widetilde{\kappa(x)}\big).$$

Applying ι_x^* to this isomorphism, and recalling from §1.4 that $\iota_x^*\mathsf{R}\iota_{x*}$ is isomorphic to the identity, we get

$$\iota_x^*\big(E \otimes^{\mathsf{L}}_X k(x)\big) \cong \iota_x^* E \otimes^{\mathsf{L}}_{X_x} \widetilde{\kappa(x)}.$$

These two isomorphisms tell us that $E \otimes^{\mathsf{L}}_X k(x)$ vanishes in $\mathbf{D}_{\mathsf{qc}}(X)$ if and only if $\iota_x^* E \otimes^{\mathsf{L}}_{X_x} \widetilde{\kappa(x)}$ vanishes in $\mathbf{D}_{\mathsf{qc}}(X_x)$.

Moreover, $\iota_x^* E \otimes^{\mathsf{L}}_{X_x} \widetilde{\kappa(x)}$ is the sheafification of $E_x \otimes^{\mathsf{L}}_{\mathcal{O}_x} \kappa(x) \in \mathbf{D}(\mathcal{O}_x)$, and so its vanishing in $\mathbf{D}(X_x)$ (i.e., its being exact) is equivalent to that of $E_x \otimes^{\mathsf{L}}_{\mathcal{O}_x} \kappa(x)$ in $\mathbf{D}(\mathcal{O}_x)$. Thus

$$x \in \operatorname{supp}(E) \iff E_x \otimes^{\mathsf{L}}_{\mathcal{O}_x} \kappa(x) \neq 0.$$

It follows that if $x \in \operatorname{supp}(E)$, then $E_x \neq 0$, that is to say, $x \in \operatorname{Supp}(E)$. So for all $E \in \mathbf{D}_{\mathsf{qc}}(X)$ we have $\operatorname{supp}(E) \subseteq \operatorname{Supp}(E)$.

Now suppose $E \in \mathbf{D}_{\mathsf{c}}(X)$ and $x \notin \operatorname{supp}(E)$, i.e., $E_x \otimes^{\mathsf{L}}_{\mathcal{O}_x} \kappa(x) = 0$. Let K be the Koszul complex on a finite set of generators for the maximal ideal of the local ring $\mathcal{O}_x$. It is easy to check that the full subcategory of $\mathbf{D}(\mathcal{O}_x)$ consisting of complexes C such that $E_x \otimes^{\mathsf{L}}_{\mathcal{O}_x} C = 0$ is a thick subcategory. It contains $\kappa(x)$, and hence also K, since the $\mathcal{O}_x$-module $\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(K)$ has finite length, see [DGI06, 3.5]. Thus $E_x \otimes^{\mathsf{L}}_{\mathcal{O}_x} K = 0$ in $\mathbf{D}(\mathcal{O}_x)$; and since the cohomology of E_x is finitely generated in all degrees, [FI03, 1.3(2)] gives $E_x = 0$. Thus, $x \notin \operatorname{Supp}(E)$; and so $\operatorname{supp}(E) \supseteq \operatorname{Supp}(E)$. □

A *localizing tensor ideal* $\mathcal{L} \subseteq \mathbf{D}_{\mathsf{qc}}(X)$ is a full triangulated subcategory of $\mathbf{D}_{\mathsf{qc}}(X)$, closed under arbitrary direct sums, and such that for all $G \in \mathcal{L}$ and $E \in \mathbf{D}_{\mathsf{qc}}(X)$, it holds that $G \otimes^{\mathsf{L}}_X E \in \mathcal{L}$.

The next Proposition is proved in [Nm92, §2] in the affine case; and in [AJS04] (where localizing tensor ideals are called *rigid localizing subcategories*) the proof is extended to noetherian schemes. (Use e.g., *ibid.*, Corollary 4.11 and the bijection in Theorem 4.12, as described at the beginning of its proof.)

PROPOSITION A.2. *Let* $\mathcal{L} \subseteq \mathbf{D}_{\mathsf{qc}}(X)$ *be a localizing tensor ideal. A complex* $E \in \mathbf{D}_{\mathsf{qc}}(X)$ *is in* $\mathcal{L}$ *if and only if so is* $k(x)$ *for all* x *in* $\operatorname{supp}(E)$.

For closed subsets of affine schemes the next result is part of [DG02, Proposition 6.5].

PROPOSITION A.3. *Let $E \in \mathbf{D}_{\mathsf{qc}}(X)$ be such that $W := \operatorname{supp}(E)$ is a union of closed subsets of X.*

(i) *For any $F \in \mathbf{D}_{\mathsf{qc}}(X)$,*

$$\begin{aligned} E \otimes^{\mathsf{L}}_X F = 0 &\iff \mathsf{R}\mathcal{H}om_X(E,F) = 0 \\ &\iff \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(E,F) = 0 \\ &\iff \mathsf{R}\varGamma_W F = 0. \end{aligned}$$

(ii) *For any morphism $\phi \in \mathbf{D}_{\mathsf{qc}}(X)$,*

$$\begin{aligned} E \otimes^{\mathsf{L}}_X \phi \text{ is an isomorphism} &\iff \mathsf{R}\mathcal{H}om_X(E,\phi) \text{ is an isomorphism} \\ &\iff \mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X(E,\phi) \text{ is an isomorphism} \\ &\iff \mathsf{R}\varGamma_W \phi \text{ is an isomorphism.} \end{aligned}$$

Proof. Let $\mathcal{L} \subseteq \mathbf{D}_{\mathsf{qc}}(X)$ (resp. $\mathcal{L}' \subseteq \mathbf{D}_{\mathsf{qc}}(X)$) be the full subcategory spanned by the complexes C such that $C \otimes^{\mathsf{L}}_X F = 0$ (resp. $\mathsf{R}\mathcal{H}om_X(C,F) = 0$). It is clear that $\mathcal{L}$ is a localizing tensor ideal; and using the natural isomorphisms (with $G \in \mathbf{D}_{\mathsf{qc}}(X)$),

$$\begin{aligned} \mathsf{R}\mathcal{H}om_X\big(\oplus_{i\in I} C_i, F\big) &\cong \prod_{i\in I} \mathsf{R}\mathcal{H}om_X(C_i, F), \\ \mathsf{R}\mathcal{H}om_X(G \otimes^{\mathsf{L}}_X C, F) &\cong \mathsf{R}\mathcal{H}om_X(G, \mathsf{R}\mathcal{H}om_X(C,F)) \end{aligned} \tag{A.3.1}$$

one sees that $\mathcal{L}'$ is a localizing tensor ideal too.

We claim that when E is in $\mathcal{L}$ it is also in $\mathcal{L}'$. For this it's enough, by Proposition A.2, that for any $x \in W$, $k(x)$ be in $\mathcal{L}'$. By [T07, Lemma 3.4], there is a perfect $\mathcal{O}_X$-complex C such that $\operatorname{Supp}(C)$ is the closure $\overline{\{x\}}$. We have

$$\operatorname{supp}(C) = \operatorname{Supp}(C) = \overline{\{x\}} \subseteq W,$$

where the first equality holds by Lemma A.1 and the inclusion holds because W is a union of closed sets. Thus A.2 yields $C \in \mathcal{L}$; and the dual complex $C' := \mathsf{R}\mathcal{H}om_X(C, \mathcal{O}_X)$ is in $\mathcal{L}'$, because $\mathsf{R}\mathcal{H}om_X(C', F) \cong C \otimes^{\mathsf{L}}_X F = 0$. Since

$$x \in \operatorname{supp}(C) = \operatorname{Supp}(C) = \operatorname{Supp}(C') = \operatorname{supp}(C'),$$

therefore A.2 gives that, indeed, $k(x) \in \mathcal{L}'$.

Similarly, if $E \in \mathcal{L}'$ then $E \in \mathcal{L}$, proving the first part of (i).

The same argument holds with $\mathsf{R}\mathcal{H}om^{\mathsf{qc}}_X$ in place of $\mathsf{R}\mathcal{H}om_X$. (After that replacement, the isomorphisms (A.3.1) still hold if $\prod$ is prefixed by $\mathrm{id}^{\times}_X$: this can be checked by applying the functors $\operatorname{Hom}_X(H, -)$ for all $H \in \mathbf{D}_{\mathsf{qc}}(X)$.)

As for the rest, recall that $\mathsf{R}\varGamma_W \mathcal{O}_X \in \mathbf{D}_{\mathsf{qc}}(X)$: when W itself is closed, this results from the standard triangle (with $w\colon X \setminus W \hookrightarrow X$ the inclusion)

$$\mathsf{R}\varGamma_W \mathcal{O}_X \to \mathcal{O}_X \to \mathsf{R}w_* w^* \mathcal{O}_X \xrightarrow{+} (\mathsf{R}\varGamma_W \mathcal{O}_X)[1]$$

(or from the local representation of $\mathsf{R}\varGamma_W \mathcal{O}_X$ by a $\varinjlim$ of Koszul complexes); and then for the general case, use that $\varGamma_W = \varinjlim \varGamma_Z$ where Z runs through all closed subsets of W.

By the following Lemma, $\operatorname{supp}(\mathsf{R}\varGamma_W \mathcal{O}_X) = W$, so Proposition A.2 implies that $E \in \mathcal{L}$ if and only if $\mathsf{R}\varGamma_W \mathcal{O}_X \in \mathcal{L}$, i.e., $E \otimes^{\mathsf{L}}_X F = 0$ if and only if $\mathsf{R}\varGamma_W \mathcal{O}_X \otimes^{\mathsf{L}}_X F = 0$.

The last part of (i) results then from the standard isomorphism $\mathsf{R}\varGamma_W\mathcal{O}_X \otimes^{\mathsf{L}}_X F \cong \mathsf{R}\varGamma_W F$ (for which see, e.g., [AJL97, 3.1.4(i) or 3.2.5(i)] when W itself is closed, then pass to the general case using $\varGamma_W = \varinjlim \varGamma_Z$). And applying (i) to the third vertex of a triangle based on ϕ gives (ii). ☐

Lemma A.4. *If W is a union of closed subsets of X, then* $\operatorname{supp}(\mathsf{R}\varGamma_W\mathcal{O}_X) = W$.

Proof. As seen a few lines back, $(\mathsf{R}\varGamma_W\mathcal{O}_X)\otimes^{\mathsf{L}}_X k(x) \cong \mathsf{R}\varGamma_W k(x)$ for any $x \in X$. As $k(x)$ is flasque, the canonical map $\varGamma_W k(x) \to \mathsf{R}\varGamma_W k(x)$ is an isomorphism. The assertion is then that $\varGamma_W k(x) \neq 0 \iff x \in W$ (i.e., $\overline{\{x\}} \subset W$), which is easily verified since $k(x)$ is constant on $\overline{\{x\}}$ and vanishes elsewhere. ☐

Lemma A.5. *Let $u\colon W \to X$ be a localizing immersion, and $F \in \mathbf{D}_{\mathsf{qc}}(X)$. The following conditions are equivalent.*

(i) $\operatorname{supp}(F) \subseteq W$.

(ii) *The canonical map is an isomorphism* $F \xrightarrow{\sim} \mathsf{R}u_*u^*F$.

(iii) $F \cong \mathsf{R}u_*G$ *for some* $G \in \mathbf{D}_{\mathsf{qc}}(W)$.

Proof. As in Remark 2.1.7 , the canonical map $\mathsf{R}u_*G \to \mathsf{R}u_*u^*\mathsf{R}u_*G$ is an isomorphism, whence (iii)⇒(ii); and the converse implication is trivial.

Next, if $x \notin W$ then $\overline{\{x\}} \cap W = \phi$: to see this, one reduces easily to the case in which u is the natural map $\operatorname{Spec} A_M \to \operatorname{Spec} A$, where M is a multiplicatively closed subset of the noetherian ring A (see §1.3). Since $k(x)$ vanishes outside $\overline{\{x\}}$, it follows that $u^*k(x) = 0$ whenever $x \notin W$. Using the projection isomorphism $\mathsf{R}u_*G \otimes^{\mathsf{L}}_X k(x) \cong \mathsf{R}u_*(G \otimes^{\mathsf{L}}_W u^*k(x))$, one sees then that (iii)⇒(i).

The complexes $F \in \mathbf{D}_{\mathsf{qc}}(X)$ satisfying (i) span a localizing tensor ideal. So do those F satisfying (iii): the full subcategory $\mathbf{D}_3 \subseteq \mathbf{D}_{\mathsf{qc}}(X)$ spanned by them is triangulated, as one finds by applying $\mathsf{R}u_*u^*$ to a triangle based on a $\mathbf{D}_{\mathsf{qc}}(X)$-map $\mathsf{R}u_*G_1 \to \mathsf{R}u_*G_2$; $\mathbf{D}_3$ is closed under direct sums (since $\mathsf{R}u_*$ respects direct sums, see [Nm96, Lemma 1.4], whose proof—in view of the equivalence of categories mentioned above just before 3.1.2—applies to $\mathbf{D}_{\mathsf{qc}}(X)$); and $\mathbf{D}_3$ is a tensor ideal since $\mathsf{R}u_*G \otimes^{\mathsf{L}}_X E \cong \mathsf{R}u_*(G \otimes^{\mathsf{L}}_W u^*E)$ for all $E \in \mathbf{D}_{\mathsf{qc}}(X)$. So A.2 shows that for the implication (i)⇒(iii) we need only treat the case $F = k(x)$.

Since $\operatorname{supp}(k(x)) = x$ (see, e.g., [AJS04, 4.6, 4.7]), it suffices now to note that if $x \in W$ then $\mathcal{O}_{W,x} = \mathcal{O}_{X,x}$, so the canonical map $\iota_x : W_x = X_x \to X$ in the definition of $k(x)$ (near the beginning of this Appendix) factors as $X_x \to W \xrightarrow{u} X$, whence $k(x) = \mathsf{R}\iota_{x*}\widetilde{\kappa(x)}$ satisfies (iii). ☐

Remark A.5.1. With u as in A.5, one checks that if $x \in W$ then (with self-explanatory notation) $u^*k(x)_X = k(x)_W$. Also, as above, if $x \notin W$ then $u^*k(x)_X = 0$. So for $E \in \mathbf{D}_{\mathsf{qc}}(W)$,

$$\mathsf{R}u_*E \otimes^{\mathsf{L}}_X k(x)_X \cong \mathsf{R}u_*\big(E \otimes^{\mathsf{L}}_W u^*k(x)_X\big) \cong \begin{cases} 0 & \text{if } x \notin W, \\ \mathsf{R}u_*\big(E \otimes^{\mathsf{L}}_W k(x)_W\big) & \text{if } x \in W; \end{cases}$$

and since for $F \in \mathbf{D}_{\mathsf{qc}}(W)$, $[0 = F \cong u^*\mathsf{R}u_*F] \iff [\mathsf{R}u_*F = 0]$, therefore

$$\operatorname{supp}_X(\mathsf{R}u_*E) = \operatorname{supp}_W(E).$$

References

AIL11 L. L. Avramov, S. B. Iyengar, J. Lipman, Reflexivity and rigidity for complexes, II: Schemes. *Algebra Number Theory,* **5** (2011), 379–429. 20, 22

AILN10 ______, S. Nayak, Reduction of derived Hochschild functors over commutative algebras and schemes. *Advances in Math.,* **223** (2010), 735–772. 2, 20, 21, 22

AJL97 L. Alonso Tarrío, A. Jeremías López, J. Lipman, Local homology and cohomology on schemes. *Ann. Scient. École Normale Sup.,* **30** (1997), 1–39. 15, 16, 28

AJL99 ______, Duality and flat base change on formal schemes. *Contemporary Math.,* Vol. **244**, Amer. Math. Soc., Providence, R.I. (1999), 3–90. 15

AJL11 ______, Bivariance, Grothendieck duality and Hochschild homology I: Construction of a bivariant theory. *Asian J. Math.,* **15** (2011), 451–497. 21

AJL14 ______, Bivariance, Grothendieck duality and Hochschild homology II: the fundamental class of a flat scheme-map. *Advances in Mathematics* **257** (2014), 365– 461. 23, 24, 25

AJS04 L. Alonso Tarrío, A. Jeremías López, M. J. Souto Salorio, Bousfield localization on formal schemes. *J. Algebra,* **278** (2004), 585–610. 12, 26, 28

BN93 M. Bökstedt, A. Neeman, Homotopy limits in triangulated categories. *Compositio Math.,* **86** (1993), 209–234 15, 18

DG02 W. G. Dwyer, J. P. C. Greenlees, Complete modules and torsion modules. *Amer. J. Math.,* **124** (2002), 199–220. 26

DGI06 ______, S. B. Iyengar, Finiteness in derived categories of local rings. *Comment. Math. Helv.,* **81** (2006), 383–432. 26

F79 H. B. Foxby, Bounded complexes of flat modules. *J. Pure Appl. Algebra,* **15** (1979), 149–172. 26

FI03 ______, S. B. Iyengar, Depth and amplitude for unbounded complexes. *Commutative algebra and its interaction with algebraic geometry (Grenoble-Lyon, 2001)*, Contemp. Math. Vol. **331**, Amer. Math. Soc., Providence, R.I. (2003), 119–137. 26

G71 A. Grothendieck, Formule de Lefschetz. *Cohomologie l-adique et Fonctions L* (SGA 5), Lecture Notes in Math., **589**, Springer-Verlag, Berlin-New York, 1971, 73–137. 20

GrD61 ______, J. Dieudonné, *Eléments de Géométrie Algébrique II, Étude gobale élémentaire de quelques classes de morphisme.* Publications Mathématiques, **8**, Institut des Hautes Études Scientifiques, Paris, 1961. 13

GrD67 ______, J. Dieudonné, *Eléments de Géométrie Algébrique IV, Étude locale des schémas et des morphismes of schémas.* Publications Mathématiques, **28**, **32**, Institut des Hautes Études Scientifiques, Paris, 1966, 1967. 4, 5

H66 R. Hartshorne, *Residues and Duality.* Lecture Notes in Math., **20**. Springer-Verlag, Berlin-New York, 1966. 5, 19

HK90a R. Hübl, E. Kunz, Integration of differential forms on schemes. *J. Reine Angew. Math.,* **410** 1990, 53–83. 12

HK90b ______, Regular differential forms and duality for projective morphisms. *J. Reine Angew. Math.,* **410** (1990), 84–108. 12

I71 L. Illusie, Conditions de finitude relative, *Théorie des Intersections et Théorème de Riemann-Roch (SGA 6).* Lecture Notes in Math. **225**, Springer-Verlag, New York, 1971, 222–273. 21

L09 J. Lipman, Notes on derived categories and Grothendieck Duality. *Foundations of Grothendieck duality for diagrams of schemes.* Lecture Notes in Math., **1960**, Springer-Verlag, Berlin-New York, 2009, 1–259. 3, 4, 5, 6, 7, 8, 10, 11, 14, 17, 21

Nk09 S. Nayak, Compactification for essentially finite type maps. *Adv. Math.* **222**, (2009), 527–546. 3, 4, 7, 12, 13

Nm92 A. Neeman, The chromatic tower for $\mathbf{D}(R)$. *Topology* **31** (1992), 519–532. 26

Nm96 ———, The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Amer. Math. Soc.* **9** (1996), 205–236. 28

Nm14a ———, The decomposition of $\mathrm{Hom}_k(S,k)$ into indecomposable injectives. *Acta Mathematica Vietnamica*. To appear. 20

Nm14b ———, An improvement on the base-change theorem and the functor $f^!$. arXiv:1406.7599. 14, 22

S04 P. Sastry, Base change and Grothendieck duality for Cohen-Macaulay maps. *Compositio Math.*, **140** (2004), 729–777. 12

T07 R. Thomason, The classification of triangulated subcategories. *Compositio Math.* **105** (1997), 1–27. 27

Srikanth B. Iyengar s.b.iyengar@unl.edu
Department of Mathematics, University of Nebraska, Lincoln, NE 68588, U.S.A.

Joseph Lipman jlipman@purdue.edu
Department of Mathematics, Purdue University, West Lafayette IN 47907, U.S.A.

Amnon Neeman Amnon.Neeman@anu.edu.au
Centre for Mathematics and its Applications, Mathematical Sciences Institute Australian National University, Canberra, ACT 0200, Australia.