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Non-noetherian Grothendieck Duality

Joseph Lipman

Abstract. For any separated map $f : X \to Y$ of quasi-compact quasi-separated schemes, $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qc}}(X) \to \mathbf{D}^+(Y)$ has a right adjoint $f^!$. If f is proper and pseudo-coherent (e.g., finitely-presented and flat) then Duality and tor-independent Base Change hold for $f^!$.

Preface

This is a research summary written early in 1991, concerning results obtained by the author during a stay at MSRI in Berkeley during 1989–90. The intention then—and now—is to include details in the notes [**Li**] (whose completion has been delayed by the development of the other papers in this volume). The methods still seem relevant, though in the meantime fresh ideas have been brought to the subject by Neeman [**N**], who has given, in particular, new proofs of existence and sheafification (= open base change) for the Grothendieck duality functor.

Introduction

The fundamental results of scheme-theoretic Grothendieck Duality were first treated systematically in [**H**] (see also [**Co**]). Another approach is indicated in [**V**]. In these sources, heavy use is made of noetherian hypotheses. (See, e.g., the discussion on p. 12 of [**H**].) Also, in [**H**] and [**V**] (though not in [**Co**]) short shrift is made of "compatibilities," i.e., the commutativity of certain functorial diagrams, see e.g., [**H,** p. 118].

In the intervening years several developments have made possible substantial improvements in this theory, notably with regard to *elimination of noetherian hypotheses.* Also, many of the definitions and supporting lemmas can now be more simply formulated in the language of categories, thus acquiring meaning in larger contexts and revealing more of their interrelationships. In particular, the compatibilities problem can be better understood in terms of *coherence in categories.*

In §1, we review the intervening developments just alluded to. The category-theoretic formalism is sketched in §2. Some preliminary isomorphisms appear in §3,

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while the main new results (Existence, Duality, and tor-independent Base Change, in a not-necessarily-noetherian setting) are described in §4.

It should be noted that we have not de-noetherianized all of the main results of duality theory. For example, we are as yet unable to show the existence and uniqueness of a pseudofunctor $f^!$ defined for *all* separated pseudo-coherent maps f of quasi-compact quasi-separated schemes (1.2), right adjoint to $\mathbf{R}f_*$ when f is proper, and isomorphic to f^* when f is étale. (Existence, at least, can be shown for Cohen-Macaulay maps, by descent to the noetherian case and the fact that they satisfy *universal* Base Change, i.e., (4.3) without the assumption that u has finite tor-dimension.)

1. Background

(1.1) On a quasi-compact separated scheme X, the natural functor from the derived category of (homologically) bounded-below quasi-coherent $\mathcal{O}_X$-complexes to the derived category $\mathbf{D}^+_{\mathrm{qc}}(X)$ of bounded-below complexes with quasi-coherent homology, is an *equivalence of categories.* (This is due to Verdier, [**I,** p. 189, Proposition 3.5]. The same holds true on any locally noetherian scheme [**H,** p. 133, 7.19].)

(1.2) *Pseudo-coherence* of complexes and maps will replace noetherianness of schemes in our principal results. On a scheme X, a bounded $\mathcal{O}_X$-complex F is pseudo-coherent if each $x \in X$ has a neighborhood over which F is isomorphic, in the derived category, to a bounded-above complex of *finite-rank* free $\mathcal{O}_X$-modules [**I,** p. 175, 2.2.10]. For unbounded complexes the idea is similar, but a little more involved [**I,** p. 98, Définition 2.3; p. 108, 2.15]. If $\mathcal{O}_X$ is coherent, pseudo-coherence means simply that F has coherent homology [**I,** p. 115, Corollary 3.5 b)].

A map $f\colon X \to Y$ is said to be pseudo-coherent if it factors locally as $f = p \circ i$ where $i\colon U \to Z$ (U open in X) is a closed immersion such that $i_*\mathcal{O}_U$ is pseudo-coherent on Z, and $p\colon Z \to Y$ is smooth [**I,** p. 228, Déf. 1.2]. For example, if Y is noetherian then any finite-type $f\colon X \to Y$ is pseudo-coherent. Pseudo-coherence of maps is preserved by tor-independent base change [**I,** p. 233, Cor. 1.10]. It follows, by reduction to the noetherian case [**GD,** (11.2.7), proof], that *any flat finitely-presentable scheme-map is pseudo-coherent.*

Kiehl's Finiteness Theorem [**K**][1] generalizes preservation of coherence by higher direct images under proper maps of noetherian schemes: *If $f\colon X \to Y$ is a proper pseudo-coherent map of arbitrary schemes, then $\mathbf{R}f_*$ preserves pseudo-coherence of complexes.*

(1.3) (Derived functors for unbounded complexes.) In [**Sp**], Spaltenstein has proved existence of the standard derived functors $\mathbf{R}f_*$, $\mathbf{L}f^*$, $\mathbf{R}\mathcal{H}om^\bullet$, and $\underline{\underline{\otimes}}$, for *all* $\mathcal{O}_X$-complexes (X a scheme, or more generally any ringed space). This elimination of the usual boundedness conditions is a technical godsend. For instance, Spaltenstein establishes a natural functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X)}\big(A \underline{\underline{\otimes}} B,\, C\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(X)}\big(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,C)\big)\,,$$

making the derived category $\mathbf{D}(X)$ of $\mathcal{O}_X$-modules into a *monoidal closed category,* so that all the facts about categories with that structure become available (cf. [**EK**], or [**So**] and its references). This would not be so under boundedness restrictions.

[1] conjectured by Illusie, and proved by him for projective maps [**I,** p. 236, Thm. 2.2]

Moreover, given a scheme-map $f\colon X \to Y$, the functors $\mathbf{L}f^*$ and $\mathbf{R}f_*$ are *adjoint;* and they are *monoidal* [**EK,** pp. 473 and 513]: there are natural functorial maps (corresponding by adjunction, the second an isomorphism)

$$\mathbf{R}f_*(A) \underset{=}{\otimes} \mathbf{R}f_*(B) \to \mathbf{R}f_*(A \underset{=}{\otimes} B), \qquad \mathbf{L}f^*(C \underset{=}{\otimes} D) \xrightarrow{\ \sim\ } \mathbf{L}f^*(C) \underset{=}{\otimes} \mathbf{L}f^*(D),$$

respecting the monoidal structures on $\mathbf{D}(X)$ and $\mathbf{D}(Y)$, in that certain obvious diagrams commute. We also have, for a pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, natural isomorphisms $\mathbf{R}(gf)_* \xrightarrow{\ \sim\ } \mathbf{R}g_*\mathbf{R}f_*$ and (by adjointness) $\mathbf{L}f^*\mathbf{L}g^* \xrightarrow{\ \sim\ } \mathbf{L}(gf)^*$, making derived direct and inverse image into adjoint pseudofunctors [**Li,** §3.6].

(1.4) Beginning in the 1960's, Mac Lane and others have studied the problem, in structured categories, of deducing commutativity of complicated functorial diagrams from simple ones (cf. [**So,** references]). This subject is called "Coherence in Categories" (where "coherence" has nothing to do with sheaves). Its relevance for duality theory will be illustrated in §2.

2. Formal definitions

To reduce clutter, we will use the following abbreviations, for a scheme-map h or a scheme Z:

$$h_* := \mathbf{R}h_*\,, \quad h^* := \mathbf{L}h^*\,, \quad \mathcal{H}_Z := \mathbf{R}\mathcal{H}om^{\bullet}{}_Z\,, \quad \otimes_Z := \underset{=}{\otimes}{}_Z\,.$$

(2.1) Except for the right adjoint $f^\times$ of f_*, whose existence is asserted in (4.0) below, all the functorial maps to be considered, and the relations among them, can be derived formally from the data in (1.3). The basic results in §§3–4 state that under suitable conditions certain such maps are isomorphisms.

For example, given $f\colon X \to Y$, the bifunctorial *projection map*

$$p = p(F,G)\colon F \otimes_Y f_*G \to f_*(f^*F \otimes_X G) \qquad \big(F \in \mathbf{D}(Y),\ G \in \mathbf{D}(X)\big)$$

is defined to be the natural composition

$$F \otimes_Y f_*G \xrightarrow{\eta\otimes 1} f_*f^*F \otimes_Y f_*G \longrightarrow f_*(f^*F \otimes_X G).$$

(Here $\eta\colon F \to f_*f^*F$ corresponds to the identity map of f^*F via adjointness of f_* and f^*.)

For fixed $F \in \mathbf{D}(Y)$, the functors (from $\mathbf{D}(Y)$ to $\mathbf{D}(X)$)

$$H \mapsto f^*(H \otimes_Y F) \quad \text{and} \quad H \mapsto f^*H \otimes_X f^*F$$

both have right adjoints, namely

$$G \mapsto \mathcal{H}_Y(F,\, f_*G) \quad \text{and} \quad G \mapsto f_*\mathcal{H}_X(f^*F,\, G).$$

Hence there is a functorial isomorphism

$$\alpha(F,G)\colon f_*\mathcal{H}_X(f^*F,\, G) \xrightarrow{\ \sim\ } \mathcal{H}_Y(F,\, f_*G) \qquad \big(F \in \mathbf{D}(Y),\ G \in \mathbf{D}(X)\big)$$

conjugate to the natural functorial isomorphism $f^*(H \otimes_Y F) \xrightarrow{\ \sim\ } f^*H \otimes_X f^*F$.

We deduce functorial maps

$$\begin{aligned}\nu(E,G)\colon f_*\mathcal{H}_X(E,\,G) &\longrightarrow f_*\mathcal{H}_X(f^*f_*E,\,G)\\ &\xrightarrow[\alpha]{\sim} \mathcal{H}_Y(f_*E,\,f_*G) \qquad \big(E,G\in\mathbf{D}(X)\big)\end{aligned}$$

and

$$\begin{aligned}\rho(F,H)\colon f^*\mathcal{H}_Y(F,\,H) &\longrightarrow f^*\mathcal{H}_Y(F,\,f_*f^*H)\\ &\xrightarrow[\alpha^{-1}]{\sim} f^*f_*\mathcal{H}_X(f^*F,\,f^*H) \longrightarrow \mathcal{H}_X(f^*F,\,f^*H)\\ &\qquad\qquad\big(F,H\in\mathbf{D}(Y)\big).\end{aligned}$$

To any commutative square

$$\begin{array}{ccc} X' & \xrightarrow{\;v\;} & X\\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f}\\ Y' & \xrightarrow[\;u\;]{} & Y \end{array} \tag{2.1.1}$$

there is associated a functorial map

$$\theta(G)\colon u^*f_*G \to g_*v^*G \qquad \big(G\in\mathbf{D}(X)\big)$$

which corresponds by adjunction to either of the functorial compositions

$$f_* \longrightarrow f_*v_*v^* \xrightarrow{\;\sim\;} u_*g_*v^*, \qquad g^*u^*f_* \xrightarrow{\;\sim\;} v^*f^*f_* \longrightarrow v_*\,.$$

(2.2) Such functorial maps are related by various commutative diagrams. Here are two useful examples, with the maps p,θ,ν,ρ as above. (Commutativity is proved formally, i.e., category-theoretically—see e.g., [**Li,** Proposition (3.7.3)].)

(2.2.1) In (2.1.1), set $h = fv = ug$. Then the following natural diagram commutes, for all $F\in\mathbf{D}(Y)$ and $G\in\mathbf{D}(X)$:

$$\begin{array}{ccccc} u^*F\otimes u^*f_*G & \xleftarrow{\;\sim\;} & u^*(F\otimes f_*G) & \xrightarrow{\;u^*(p)\;} & u^*f_*(f^*F\otimes G)\\ {\scriptstyle 1\otimes\theta}\big\downarrow & & & & \big\downarrow{\scriptstyle \theta}\\ u^*F\otimes g_*v^*G & & & & g_*v^*(f^*F\otimes G)\\ {\scriptstyle p}\big\downarrow & & & & \big\downarrow{\scriptstyle \simeq}\\ g_*(g^*u^*F\otimes v^*G) & \xrightarrow{\;\sim\;} & g_*(h^*F\otimes v^*G) & \xleftarrow{\;\sim\;} & g_*(v^*f^*F\otimes v^*G) \end{array}$$

(2.2.2) Given a commutative square (2.1.1), let $E \in \mathbf{D}(X)$, $G \in \mathbf{D}(X)$. Then the following diagram commutes:

$$
\begin{array}{ccccc}
u^*f_*\mathcal{H}_X(E,\,G) & & \xrightarrow{\quad\nu\quad} & & u^*\mathcal{H}_Y(f_*E,\,f_*G) \\
{\scriptstyle\theta}\big\downarrow & & & & \big\downarrow{\scriptstyle\rho} \\
g_*v^*\mathcal{H}_X(E,\,G) & & & & \mathcal{H}_{Y'}(u^*f_*E,\,u^*f_*G) \\
{\scriptstyle\rho}\big\downarrow & & & & \big\downarrow{\scriptstyle\theta} \\
g_*\mathcal{H}_{X'}(v^*E,\,v^*G) & \underset{\nu}{\longrightarrow} & \mathcal{H}_{Y'}(g_*v^*E,\,g_*v^*G) & \underset{\theta}{\longrightarrow} & \mathcal{H}_{Y'}(u^*f_*E,\,g_*v^*G)
\end{array}
$$

(2.3) Typically, one will want to prove that one part of a functorial diagram is an isomorphism, knowing that another part is. Thus the question will be whether the diagram commutes—in other words, can it be decomposed into the elementary diagrams expressing the axioms of monoidal closed categories, monoidal functors, and adjointness and pseudofunctoriality of f^* and f_*? Rather than attacking diagrams one by one, it would be better to prove theorems showing that whole classes of diagrams must commute. This is the stuff of "Coherence in Categories." Such theorems have been proved for situations in which there is a functor f_*, but without the adjoint f^*, cf. [**Le**]. A corresponding investigation including f^* would surely be useful for our present purposes. (To see why, write down a detailed proof of commutativity in (2.2.1) and (2.2.2)!)

3. Minor results

(3.1) Some conditions under which the projection map $p(F,G)$ in (2.1) is an isomorphism are given in [**I,** p. 247, Proposition 3.7] and [**Li,** Proposition (3.9.4)]. Usually, the complex F has to have quasi-coherent homology.

(3.2) The map $\rho(F,H)$ in (2.1) is an isomorphism if f is an open immersion; or if f has finite tor-dimension, F is pseudo-coherent, and H is bounded-below.

(3.3) (A kind of Künneth isomorphism, see [**Li,** Proposition (3.10.3)].) Suppose (2.1.1) is a fibre square (i.e., cartesian), with f, g quasi-compact quasi-separated, and u, v of finite tor-dimension. If f and u are *tor-independent,* i.e., if for all $y \in Y$, $y' \in Y'$, and $x \in X$ such that $y = u(y') = f(x)$, we have, with $R = \mathcal{O}_{Y,y}$, $R' = \mathcal{O}_{Y',y'}$, $S = \mathcal{O}_{X,x}$ that

$$\mathrm{Tor}_n^R(R', S) = 0 \quad \text{for all } n > 0,$$

then for any $G \in \mathbf{D}^+_{\mathrm{qc}}(X)$, the map $\theta(G)$ in (2.1) is an isomorphism; and the converse is true if the scheme X is quasi-separated.

4. Major results

Recall the abbreviations introduced in §2. All schemes are now assumed to be quasi-compact and quasi-separated, and all scheme-maps separated.

Theorem (4.0) (Existence). *For any $f\colon X \to Y$ (X, Y quasi-compact quasi-separated, f separated), the functor $f_*\colon \mathbf{D}^+_{\mathrm{qc}}(X) \to \mathbf{D}^+(Y)$ has a right adjoint.*

The proof is based on ideas of Verdier [**G**] and Deligne [**H,** appendix], and also uses (1.1). (It is similar to the one in §4 of the paper "Duality … on formal schemes" in this volume. See also [**Li,** Chapter 4].)

Denote the right adjoint in question by $f^{\times}$, or by $f^!$ when f is proper; and denote by $\tau\colon f_*f^{\times}\to\mathbf{1}$ the natural functorial map.

THEOREM (4.1). *Suppose* (2.1.1) *is a fibre square, with f proper and pseudo-coherent, u of finite tor-dimension, and f and u tor-independent, see* (3.3). *Then for any bounded $G'\in\mathbf{D}^+_{\mathrm{qc}}(X')$ and any $H\in\mathbf{D}^+_{\mathrm{qc}}(Y)$, the composition*

$$\begin{aligned} g_*\mathcal{H}_{X'}(G',\,v^*f^!H) &\xrightarrow{\ \nu\ } \mathcal{H}_{Y'}(g_*G',\,g_*v^*f^!H)\\ &\underset{(3.3)}{\xrightarrow{\ \sim\ }} \mathcal{H}_{Y'}(g_*G',\,u^*f_*f^!H) \underset{\tau}{\longrightarrow} \mathcal{H}_{Y'}(g_*G',\,u^*H) \end{aligned}$$

is an isomorphism.

In (4.1), if u and v are identity maps then so is the map labeled (3.3), and the resulting composition

$$\delta(G,H)\colon f_*\mathcal{H}_X(G,\,f^!H)\xrightarrow{\nu}\mathcal{H}_Y(f_*G,\,f_*f^!H)\xrightarrow{\tau}\mathcal{H}_Y(f_*G,H)$$

is called the *duality map.* Thus:

COROLLARY (4.2) (Duality). *Let $f\colon X\to Y$ be proper and pseudo-coherent. Then for any bounded $G\in\mathbf{D}^+_{\mathrm{qc}}(X)$ and any $H\in\mathbf{D}^+_{\mathrm{qc}}(Y)$, the duality map $\delta(G,H)$ is an isomorphism.*

Moreover:

COROLLARY (4.3) (Base Change). *With assumptions as in* (4.1), *the map of functors* (*from $\mathbf{D}^+_{\mathrm{qc}}(Y)$ to $\mathbf{D}^+_{\mathrm{qc}}(X)$*) *corresponding by* (4.0) *to the composition*

$$g_*v^*f^!\underset{(3.3)}{\xrightarrow{\ \sim\ }}u^*f_*f^!\underset{\text{via }\tau}{\longrightarrow}u^*$$

is an isomorphism

$$\beta\colon v^*f^!\xrightarrow{\ \sim\ }g^!u^*.$$

To deduce (4.3) from (4.1), consider the next diagram, whose commutativity follows from the definition of β:

$$\begin{array}{ccc} g_*\mathcal{H}_{X'}(G',\,v^*f^!H) & \xrightarrow{\ \beta\ } & g_*\mathcal{H}_{X'}(G',\,g^!u^*H)\\ {\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle\nu}\\ \mathcal{H}_{Y'}(g_*G',\,g_*v^*f^!H) & \xrightarrow{\ \beta\ } & \mathcal{H}_{Y'}(g_*G',\,g_*g^!u^*H)\\ {\scriptstyle(3.3)}\big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\tau}\\ \mathcal{H}_{Y'}(g_*G',\,u^*f_*f^!H) & \underset{\tau}{\longrightarrow} & \mathcal{H}_{Y'}(g_*G',\,u^*H) \end{array} \tag{4.3.1}$$

By (4.1), $\tau\circ(3.3)\circ\nu$ is an isomorphism; and by (4.2), the right column is an isomorphism too. It follows that the top row is an isomorphism, and applying the functor $H^0\mathbf{R}\Gamma_{Y'}$, we conclude that for all $G'\in\mathbf{D}^+_{\mathrm{qc}}(X')$,

$$\mathrm{Hom}_{\mathbf{D}(X')}(G',\,v^*f^!H)\xrightarrow{\text{via }\beta}\mathrm{Hom}_{\mathbf{D}(X')}(G',\,g^!u^*H)$$

is an isomorphism. Consequently β itself must be an isomorphism. □

Remark (4.3.2). Conversely, the commutativity of (4.3.1) shows that (4.2) and (4.3) together imply (4.1).

In describing the organization of the *proof of* (4.1), we will attach symbols to labels of the form (4.x) to refer to special cases of (4.x):
$(4.1)^*_{\mathrm{pc}}$:= (4.1) with $G' = v^*G$, where $G \in \mathbf{D}(X)$ is pseudo-coherent.
$(4.2)_{\mathrm{pc}}$:= (4.2) with G pseudo-coherent := $(4.1)^*_{\mathrm{pc}}$ with $u = v = \mathrm{id}$.
$(4.3)^{\mathrm{o}}$:= (4.3) with the map u an open immersion.
$(4.3)^{\mathrm{af}}$:= (4.3) with the map u affine.
Our strategy is to prove $(4.2)_{\mathrm{pc}}$ and the chain of implications

$$(4.2)_{\mathrm{pc}} \Leftrightarrow (4.1)^*_{\mathrm{pc}} \Rightarrow \big((4.3)^{\mathrm{o}} + (4.3)^{\mathrm{af}}\big) \Rightarrow (4.3) \Rightarrow (4.3)^{\mathrm{o}} \Rightarrow (4.2).$$

By (4.3.2) then, (4.1) results. Here are some brief indications.

Proof of $(4.2)_{\mathrm{pc}}$ (with roots in [**V,** p. 404, proof of Proposition 3]): Formally, the source and target of the duality map δ are right-adjoint to the target and source of the projection map p, and δ is conjugate to p; so if p is an isomorphism .... This approach has pitfalls, because of the boundedness restriction in (4.0); but the idea can be pushed through, as follows. Let τ_m ($m \in \mathbb{Z}$) be the functor taking a complex $A^\bullet$ to the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{coker}(A^{m-1} \to A^m) \longrightarrow A^{m+1} \longrightarrow A^{m+2} \longrightarrow \cdots \quad .$$

One shows, for any $F \in \mathbf{D}^-_{\mathrm{qc}}(Y)$, that for $m \ll 0$ the map

$$\mathrm{Hom}_{\mathbf{D}(Y)}\big(F,\, f_*\mathcal{H}_X(G,\, f^!H)\big) \xrightarrow{\text{via } \delta} \mathrm{Hom}_{\mathbf{D}(Y)}\big(F,\, \mathcal{H}_Y(f_*G,\, H)\big)$$

is naturally isomorphic to the map

$$\mathrm{Hom}_{\mathbf{D}(Y)}\big(f_*\tau_m(f^*F \otimes G),\, H\big) \xrightarrow{p_m} \mathrm{Hom}_{\mathbf{D}(Y)}\big(F \otimes f_*G,\, H\big)$$

derived from p and the natural map $(f^*F \otimes G) \to \tau_m(f^*F \otimes G)$. A variant of (3.1) yields that for $m \ll 0$, the latter map is an isomorphism, whence so is the former, and hence so is δ itself if its source and target have quasi-coherent homology. When G is pseudo-coherent, then it is easily seen that the source does indeed have quasi-coherent homology; and if further the map f is pseudo-coherent and proper, then it follows from the Finiteness Theorem in (1.2) that the target does too.

Similarly, for *any* $f\colon X \to Y$, and pseudo-coherent G, we have an *isomorphism*

$$\mathcal{H}_X(f^*G, f^\times H) \xrightarrow{\sim} f^\times\mathcal{H}_Y(G, H)$$

adjoint to the composition $f_*\mathcal{H}_X(f^*G, f^\times H) \underset{\alpha}{\longrightarrow} \mathcal{H}_X(f^*G, f_*f^\times H) \underset{\tau}{\longrightarrow} \mathcal{H}_Y(G, H)$.

$(4.2)_{\mathrm{pc}} \Leftrightarrow (4.1)^*_{\mathrm{pc}}$: The implication $\Leftarrow$ is obvious. Now in (2.2.2), replace E by G and G by $f^!H$, and derive the commutative diagram

$$\begin{array}{ccc} u^*f_*\mathcal{H}_X(G,\, f^!H) & \xrightarrow{u^*(\delta)} & u^*\mathcal{H}_Y(f_*G,\, H) \\ \big\downarrow & & \big\downarrow \\ g_*\mathcal{H}_{X'}(v^*G,\, v^*f^!H) & \underset{(4.1)}{\longrightarrow} & \mathcal{H}_{Y'}(g_*v^*G,\, u^*H) \end{array} \tag{4.3.3}$$

If G—and hence, by the Finiteness Theorem, f_*G—is pseudo-coherent, then by §3, the columns are isomorphisms, whence the implication $\Rightarrow$.

$(4.1)^*_{\mathrm{pc}} \Rightarrow (4.3)^{\mathrm{o}} + (4.3)^{\mathrm{af}}$: Since u (hence v) is an open immersion or affine, it suffices that $v_*\beta$ be an isomorphism. Let G, and hence v^*G, be pseudo-coherent. Using (4.3.1), we derive a commutative diagram

$$\begin{array}{ccc}
f_*\mathcal{H}_X(G,\, v_*v^*f^!H) & \xrightarrow{v_*\beta} & f_*\mathcal{H}_X(G,\, v_*g^!u^*H) \\
{\scriptstyle\alpha^{-1}}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle\alpha^{-1}} \\
f_*v_*\mathcal{H}_{X'}(v^*G,\, v^*f^!H) & \xrightarrow{\beta} & f_*v_*\mathcal{H}_{X'}(v^*G,\, g^!u^*H) \\
\big\downarrow\simeq & & \simeq\big\downarrow \\
u_*g_*\mathcal{H}_{X'}(v^*G,\, v^*f^!H) & \underset{(4.1)^*_{\mathrm{pc}}}{\xrightarrow{\ \sim\ }} & u_*\mathcal{H}_{Y'}(g_*v^*G,\, u^*H)
\end{array}$$

in which α is as in (2.1), and the right column is an isomorphism by $(4.2)_{\mathrm{pc}}$ (for g). The desired implication results then from the following:

Key Fact (4.4). *If $f\colon X \to Y$ is a finitely presented scheme-map, then a map ϕ in $\mathbf{D}^+_{\mathrm{qc}}(X)$ is an isomorphism iff so is the map $f_*\mathcal{H}_X(G,\, \phi)$ (in $\mathbf{D}(Y)$) for every pseudo-coherent $G \in \mathbf{D}(X)$.*

$(4.3)^{\mathrm{o}} + (4.3)^{\mathrm{af}} \Rightarrow (4.3)$: Follows from a simple "transitivity" property of θ (hence β) with respect to horizontal composition of fibre squares.

$(4.3)^{\mathrm{o}} \Rightarrow (4.2)$: When u is an open immersion, the columns of (4.3.3) are isomorphisms. And applying the functor $\mathbf{R}\Gamma_{Y'}$ to the bottom row of (4.3.3) produces an isomorphism: to see this, apply $\mathbf{R}\Gamma_{Y'}$ to (4.3.1), and note that then the right hand column induces homology isomorphisms (by definition of $g^!$), hence is itself an isomorphism, and that the top row is an isomorphism by assumption. We conclude that $\mathbf{R}\Gamma_{Y'}u^*(\delta)$ is an isomorphism for all open immersions $u\colon Y' \to Y$; and (4.2) follows. (Taking $G = v_*G'$, we get the converse implication.)

Remark (4.5). Let $F \in \mathbf{D}^+_{\mathrm{qc}}(Y)$ be isomorphic to a bounded flat complex, and let $H \in \mathbf{D}^+_{\mathrm{qc}}(Y)$. We have a map

$$f^*F \otimes_X f^\times H \to f^\times(F \otimes_Y H) \tag{4.5.1}$$

adjoint to

$$f_*(f^*F \otimes_X f^\times H) \underset{(3.1)}{\xrightarrow{\ \sim\ }} F \otimes_Y f_*f^\times H \underset{\tau}{\to} F \otimes_Y H\,.$$

If f is proper and pseudo-coherent, then (4.5.1) is an *isomorphism.* To see this, one reduces to where F is a single quasi-coherent flat $\mathcal{O}_Y$-module, and applies (4.3) with $Y' = \mathrm{Spec}(\mathcal{O}_Y \oplus F)$ (where $F^2 = 0$). Alternatively, one can deduce this isomorphism from (4.2) and (4.4), and use it to prove (4.3).

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Department of Mathematics, Purdue University, W. Lafayette IN 47907
E-mail address: `lipman@math.purdue.edu`
URL: `www.math.purdue.edu/~lipman`