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ADJOINTS AND POLARS OF SIMPLE COMPLETE IDEALS IN TWO-DIMENSIONAL REGULAR LOCAL RINGS

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Introduction. In the study of the singularity of a complex plane curve germ $(C,0)$, given locally, say, by $f(x,y)=0$, important roles are played by the *adjoint* curves (those having a local equation $g=0$ such that g restricts on C to an element in the conductor ideal of the one-dimensional local ring $\mathcal{O}_{C,0}$ of germs of holomorphic functions), and by the *polar* curves (those given locally by an equation of the form $af_x+bf_y+cf=0$ with $a,b,c\in\mathbb{C}$). In this paper, we introduce corresponding complete ideals $\mathfrak{C}$ and $\mathfrak{P}$ in the convergent power series ring $\mathbb{C}\{x,y\}$, and indeed in any two-dimensional regular local ring.

The classical theory of base points of linear systems of curves on smooth surfaces, exposed at length in [4, book 4] and summarized in [16, Chaps. 1 and 2], inspired Zariski to create the theory of complete (=integrally closed) ideals in two-dimensional regular local rings [15, pp. 199–201]. Over the past half century, this theory has been further developed, by Zariski himself and others [17, Appendix 5], [6, Chaps. II and V], [7], [5], [12], [8].

A complete ideal $\mathfrak{p}$ in $\mathbb{C}\{x,y\}$ encodes local "base conditions" to be satisfied by certain complete linear systems of curves C through the origin; the *adjoint ideal* $\mathfrak{C}_\mathfrak{p}$ (defined in (2.1)) encodes the conditions which are then satisfied by the adjoints of a generic such C, and the *polar ideal* $\mathfrak{P}_\mathfrak{p}$ (defined in (5.1)) does the same for curves whose multiplicities at the infinitely near base points of the linear system are at least as big as those of the generic polar of C. These "polar multiplicities" were worked out in [4, pp. 374–381] (and for an extensive modern treatment of local polars of plane curves, see [1], [2], [3]). They are just the integers which make up the *point basis* of $\mathfrak{P}$, i.e., the s_i prescribed in Theorem (5.2) below in terms of the point basis $\{r_i\}$ of $\mathfrak{p}$, which $\{r_i\}$ may also be thought of as the sequence

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of infinitely near multiplicities of the generic curve C (weighted by the degrees of certain field extensions when we work with fields which are not algebraically closed). The factorization of $\mathfrak{P}$ into simple complete ideals is related to the number of irreducible branches of the generic polar, cf. (5.3). The basic results on the adjoint ideal are given in (2.2) and (3.1), and further interesting properties appear in §4.

When our two-dimensional regular local ring, with maximal ideal $\mathfrak{m}$, has an algebraically closed residue field, then $\mathfrak{C} = \mathfrak{P} : \mathfrak{m}$. The general case is treated in (2.3).

Though the preceding loosely-described background is geometric, all our considerations will be purely algebraic and precise (except for some remarks at the very end). We assume throughout that the complete ideal $\mathfrak{p}$ is *simple.* The extension to more general complete ideals—corresponding to the extension from irreducible plane curve germs to reducible ones—is left to the sufficiently motivated reader.

1. Preliminaries. We first fix some notation and terminology, and recall a few basic facts. More details and references can be found in [7].

(1.1) Let K be a field. Greek letters α, β, γ, ... will denote two-dimensional regular local rings with fraction field K; and such objects will be called "points." A *quadratic transform* of a point α is a localization $\alpha[x^{-1}\mathfrak{m}_\alpha]_\mathfrak{n}$ where x is an element of the maximal ideal $\mathfrak{m}_\alpha$ of α, $x \notin \mathfrak{m}_\alpha^2$, and $\mathfrak{n}$ is a maximal ideal in the ring $\alpha[x^{-1}\mathfrak{m}_\alpha]$. Any such quadratic transform is itself a point, in which $\mathfrak{m}_\alpha$ generates a principal prime ideal.

A point β is said to be *infinitely near* to a point α if $\beta \supset \alpha$. There exists then a unique sequence of the form

$$\alpha =: \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n := \beta \qquad (n \geq 0) \tag{1.1.1}$$

where for each $i < n$, α_{i+1} is a quadratic transform of α_i. The maximal ideal $\mathfrak{m}_\beta$ contains $\mathfrak{m}_\alpha$, and the residue field extension $\alpha/\mathfrak{m}_\alpha \hookrightarrow \beta/\mathfrak{m}_\beta$ has finite degree, denoted $[\beta : \alpha]$.

(1.2) For an α-ideal I of finite colength (i.e., I contains some power of $\mathfrak{m}_\alpha$), the *transform* I^β of I in a point $\beta \supset \alpha$ is the ideal $I(I\beta)^{-1}$, which is $\mathfrak{m}_\beta$-primary unless $I\beta$ is a principal ideal, in which case $I^\beta = \beta$. If β is a quadratic transform of α, then the ideal $\mathfrak{m}_\alpha\beta$ is principal, and $I^\beta = I(\mathfrak{m}_\alpha\beta)^{-r}$, where $r := \mathrm{ord}_\alpha(I)$ is the largest among those integers s such that $I \subset \mathfrak{m}_\alpha^s$. Transform is transitive: if $\gamma \supset \beta$ then $(I^\beta)^\gamma = I^\gamma$. Transform preserves products: $(IJ)^\beta = I^\beta J^\beta$.

An α-ideal I is *complete* (i.e., integrally closed) if $IR \cap \alpha = I$ for every valuation ring R such that $K \supset R \supset \alpha$. Any product of complete ideals is complete. I is *simple* if it is not the product of two other ideals. If the finite-colength ideal I is complete (resp. complete and simple) then so is I^β.

The *point basis* of I is the family of integers $\{\mathrm{ord}_\beta(I^\beta)\}_{\beta\supset\alpha}$, where ord_β is the unique discrete valuation of K satisfying

$$\mathrm{ord}_\beta(x) = \max\{\, n \mid x \in \mathfrak{m}_\beta^n \,\} \qquad (0 \neq x \in \beta). \tag{1.2.1}$$

Two complete finite-colength α-ideals coincide iff they have the same point basis.

The Hoskin-Deligne formula for the colength of an $\mathfrak{m}_\alpha$-primary complete ideal I with point basis $\{r_\beta\}_{\beta\supset\alpha}$ is

$$\lambda(\alpha/I) = \sum_{\beta\supset\alpha} [\beta:\alpha] r_\beta(r_\beta+1)/2 \tag{1.2.2}$$

(λ denotes "length"). For such an I, then, there are only finitely many points β with $r_\beta \neq 0$. (In fact $r_\beta \neq 0$ iff there is a point $\gamma \supset \beta$ such that $\operatorname{ord}_\gamma$ is a Rees valuation of I.)

Using (1.2.2)—or otherwise—one can show that

$$\lambda(I/\mathfrak{m}_\alpha I) = \operatorname{ord}_\alpha(I) + 1 \tag{1.2.3}$$

and

$$\lambda\big((I:\mathfrak{m}_\alpha)/I\big) = \operatorname{ord}_\alpha(I), \tag{1.2.4}$$

cf. [7, §3].

(1.3) A valuation v of K *dominates* α if $v(x) \geq 0$ for all $x \in \alpha$ and $v(x) > 0$ for all x in the maximal ideal $\mathfrak{m}_\alpha$ of α; in other words, the valuation ring R_v contains α, and its maximal ideal $\mathfrak{m}_v$ contains $\mathfrak{m}_\alpha$. For such a v, a *v-ideal* in α is an ideal of the form $J \cap \alpha$ where J is a non-zero ideal in R_v. The *v-successor* of a v-ideal I in α is the ideal

$$I'_v := \{\, x \in \alpha \mid xR_v \subsetneqq IR_v \,\} = \{\, x \in \alpha \mid v(x) > v(I) \,\}$$

where $v(I) := \min\{\, v(y) \mid y \in I \,\}$ (which exists since I is finitely generated). The principal ideal IR_v strictly contains $I'_v R_v$, and it follows that $I'_v = (I'_v R_v) \cap \alpha$, so that I' is a v-ideal, clearly the largest one properly contained in I.

Associated to any v we have then the sequence of v-ideals

$$\alpha =: I_0 \supset I_1\,(= \mathfrak{m}_\alpha) \supset I_2 \supset \cdots \supset I_n \supset \dots \tag{1.3.1}$$

in which I_{j+1} is the v-successor of I_j for every $j \geq 0$. (We also say that I_j is the *v-predecessor* of I_{j+1}.) For $j > 0$, the ideal I_j is $\mathfrak{m}_\alpha$-primary: clearly $I_j \supset \mathfrak{m}_\alpha I_{j-1}$ whence, by induction, $I_j \supset \mathfrak{m}_\alpha^j$. If v is rank-one discrete, then *every* v-ideal appears somewhere in (1.3.1).

(1.4) We'll call a valuation v of K a *prime divisor of* α if v dominates α and the residue field extension $\alpha/\mathfrak{m}_\alpha \hookrightarrow R_v/\mathfrak{m}_v$ is transcendental. For example, if $\beta \supset \alpha$ then ord_β is a prime divisor of α, cf. (1.2.1); and in fact every prime divisor of α is of this form for a unique β.

Moreover, there is a one-one correspondence $\mathfrak{p} \leftrightarrow v_\mathfrak{p}$ between the set of simple $\mathfrak{m}_\alpha$-primary complete ideals and the set of prime divisors of α [17, p. 391, (E)]. (Actually, $v_\mathfrak{p}$ is the unique Rees valuation of $\mathfrak{p}$, cf. [6, p. 245, (21.3)] or [5, p. 333, Thm. 4.2]).

Thus we have three sets in one-one correspondence:

$$\begin{gathered}\{\text{prime divisors of } \alpha\}\\ \updownarrow \\ \{\text{points infinitely near to } \alpha\}\\ \updownarrow \\ \{\text{simple } \mathfrak{m}_\alpha\text{-primary complete ideals}\}\end{gathered}$$

The infinitely near point $\beta_{\mathfrak{p}}$ corresponding to a simple $\mathfrak{m}_\alpha$-primary complete ideal $\mathfrak{p}$ (i.e., $\mathrm{ord}_{\beta_{\mathfrak{p}}} = v_{\mathfrak{p}}$) is characterized by any one of the following properties:

—$\beta_{\mathfrak{p}}$ is the largest among those $\beta \supset \alpha$ such that $\mathfrak{p}^\beta \neq \beta$ [17, p. 389, (B)].

—$\beta_{\mathfrak{p}}$ is the unique point infinitely near to α in which the transform of $\mathfrak{p}$ is the maximal ideal [17, p. 389, (B)].

—A valuation v dominating α dominates $\beta_{\mathfrak{p}}$ iff $\mathfrak{p}$ is a v-ideal. (This follows from [17, p. 390, (D)].)

If $\beta_{\mathfrak{p}} \subset \beta_{\mathfrak{p}'}$ then $\mathfrak{p} \supset \mathfrak{p}'$. (The converse doesn't hold.)

Furthermore, for any $v_{\mathfrak{p}}$-ideal I in α, the ideal $I\beta_{\mathfrak{p}}$ is principal iff $\mathfrak{p}$ does not divide I (i.e., $I \neq \mathfrak{p}(I\colon \mathfrak{p})$) [17, p. 392, (F), (2)].

Lemma (1.5). *Let $\beta \supset \alpha$, let v be a valuation dominating β, and let I be a v-ideal such that the ideal $I\beta$ is principal. Then the v-successor of I is the ideal $I' := (\mathfrak{m}_\beta I) \cap \alpha$. Moreover, the length of the α-module I/I' is $\leq [\beta : \alpha]$.*

Proof. Let $z \in I$ be such that $I\beta = z\beta$, and map I α-linearly to $\beta/\mathfrak{m}_\beta$ by sending $x \in I$ to $(x/z) + \mathfrak{m}_\beta$. It is immediate that the kernel of this map is I', and that the kernel of its composition with the injective map $\beta/\mathfrak{m}_\beta \to R_v/\mathfrak{m}_v$ is I'_v. Thus $I'_v = I'$; and we have an $(\alpha/\mathfrak{m}_\alpha)$-linear injection $I/I' \to \beta/\mathfrak{m}_\beta$, whence the last assertion. □

Corollary (1.5.1).[1] *For any v dominating $\beta_{\mathfrak{p}}$ (i.e., such that $\mathfrak{p}$ is a v-ideal) the sequence* (1.3.1) *of v-ideals in α coincides up to and including $\mathfrak{p}$ with the corresponding sequence for $v_{\mathfrak{p}}$.*

(1.6) Various proofs of the following useful relation can be found in [6, p. 247, Prop. (21.4)], [13, §7], [5, p. 334, Thm. 4.3], and [8, Cor. (4.8)]:

$$\mathrm{ord}_\alpha(\mathfrak{p}) = [\beta_{\mathfrak{p}} : \alpha]\, v_{\mathfrak{p}}(\mathfrak{m}_\alpha). \tag{1.6.1}$$

Here is one application. Define the *intersection number* of two $\mathfrak{m}_\alpha$-primary ideals I and J, with respective point bases $\{r_\beta\}$ and $\{s_\beta\}$, to be

$$(I \cdot J) := \sum_{\beta \supset \alpha} [\beta : \alpha] r_\beta s_\beta.$$

[1]Even for "polynomial ideals" this strengthens [15, p. 171, Thm. 6.2.], where only 0-dimensional valuations occur [*ibid.,* bottom of p. 155].

Lemma (1.6.2). *For any $\mathfrak{m}_\alpha$-primary ideals $\mathfrak{p}$, J, with $\mathfrak{p}$ complete and simple, we have*

$$(\mathfrak{p}\cdot J)=[\beta_{\mathfrak{p}}:\alpha]v_{\mathfrak{p}}(J).$$

Proof. Since for $\beta\supset\alpha$, $\mathfrak{p}^\beta=\beta$ unless $\beta_{\mathfrak{p}}\supset\beta$, therefore

$$\begin{aligned}(\mathfrak{p}\cdot J)&=\sum_{\beta_{\mathfrak{p}}\supset\beta\supset\alpha}[\beta:\alpha]\mathrm{ord}_\beta(\mathfrak{p}^\beta)\mathrm{ord}_\beta(J^\beta)\\&\overset{(1.6.1)}{=}\sum_{\beta_{\mathfrak{p}}\supset\beta\supset\alpha}[\beta:\alpha][\beta_{\mathfrak{p}}:\beta]v_{\mathfrak{p}}(\mathfrak{m}_\beta)\mathrm{ord}_\beta(J^\beta)\\&=[\beta_{\mathfrak{p}}:\alpha]v_{\mathfrak{p}}(J),\end{aligned}$$

the last equality by an easy induction on the number of points between $\beta_{\mathfrak{p}}$ and α, cf. [7, pp. 209–210, Lemma (1.11)]. □

We mention in passing a more general form of (1.6.1):

Corollary (1.6.3). (Reciprocity) *For any two simple $\mathfrak{m}_\alpha$-primary ideals $\mathfrak{p}$ and $\mathfrak{q}$,*

$$[\beta_{\mathfrak{p}}:\alpha]v_{\mathfrak{p}}(\mathfrak{q})=[\beta_{\mathfrak{q}}:\alpha]v_{\mathfrak{q}}(\mathfrak{p}).$$

Geometrically, (1.6.1) can be understood thus: the left-hand side is the intersection number of the proper transform of a general element of $\mathfrak{p}$ with the closed fiber $f^{-1}\{\mathfrak{m}_\alpha\}$ of any map $f\colon X\to\mathrm{Spec}(\alpha)$ obtained by a succession of point blowups, while the right hand side is that intersection number when f is the map obtained by successively blowing up all the points between α and $\beta_{\mathfrak{p}}$, inclusive. A more precise intersection-theoretic interpretation of (1.6.3) is given in the proof of [6, p. 247, Prop. (21.4)].

2. The adjoint ideal. We fix a point α, and denote its maximal ideal by $\mathfrak{m}$. The length of an α-module M will be denoted by $\lambda(M)$. For any simple $\mathfrak{m}$-primary complete ideal $\mathfrak{p}$ in α, we set

$$f_{\mathfrak{p}}:=[\beta_{\mathfrak{p}}:\alpha],$$

the degree of the residue field extension associated with $\beta_{\mathfrak{p}}\supset\alpha$.

By (1.5) and the statement immediately preceding it, if I is a $v_{\mathfrak{p}}$-ideal not divisible by $\mathfrak{p}$, and I' is the $v_{\mathfrak{p}}$-successor of I, then $\lambda(I/I')\le f_{\mathfrak{p}}$. In particular, if the residue field $\alpha/\mathfrak{m}$ is algebraically closed then $\lambda(I/I')=f_{\mathfrak{p}}=1$.

Definition (2.1). The *adjoint ideal* $\mathfrak{C}_{\mathfrak{p}}$ of a simple $\mathfrak{m}$-primary complete ideal $\mathfrak{p}$ is the largest $v_{\mathfrak{p}}$-ideal in α containing $\mathfrak{p}$ such that whenever I is a $v_{\mathfrak{p}}$-ideal with $v_{\mathfrak{p}}$-successor I' and $\mathfrak{C}_{\mathfrak{p}}\supset I\supset I'\supset\mathfrak{p}$, then

(i) $v_{\mathfrak{p}}(I')=v_{\mathfrak{p}}(I)+1$, and
(ii) $\lambda(I/I')=f_{\mathfrak{p}}$.

Remark. Corollary (2.2.1) below entails that the "conductor property" (i) holds for *any* successive pair $I\supset I'$ of $v_{\mathfrak{p}}$-ideals such that $\mathfrak{C}_{\mathfrak{p}}\supset I$. Property (ii), however, doesn't—consider the case $\mathfrak{p}=\mathfrak{m}$. (For more information, see [12, Remark 3.3].)

Theorem (2.2). *Let $\mathfrak{p}$ be a simple $\mathfrak{m}$-primary complete ideal, and let $\mathfrak{q}$ be the largest among those $v_{\mathfrak{p}}$-ideals $\mathfrak{q}'$ such that* $\operatorname{ord}_\alpha(\mathfrak{q}') = \operatorname{ord}_\alpha(\mathfrak{p})$. *Then* $\mathfrak{C}_{\mathfrak{p}} = \mathfrak{q} : \mathfrak{m}$ *and* $\mathfrak{m}\mathfrak{C}_{\mathfrak{p}} = \mathfrak{q}$. *In particular,* $\operatorname{ord}_\alpha(\mathfrak{C}_{\mathfrak{p}}) = \operatorname{ord}_\alpha(\mathfrak{p}) - 1$.

Corollary (2.2.1). *For every integer $n \ge n_0 := v_{\mathfrak{p}}(\mathfrak{C}_{\mathfrak{p}})$, there exists an $x \in \alpha$ such that $v_{\mathfrak{p}}(x) = n$.*

Proof. Since $v_{\mathfrak{p}}(z\mathfrak{m}) = v_{\mathfrak{p}}(z) + v_{\mathfrak{p}}(\mathfrak{m})$ for any $z \in \alpha$, it suffices to show that (2.2.1) holds for $n = n_0,\ n_0 + 1, \dots,\ n_0 + v_{\mathfrak{p}}(\mathfrak{m}) - 1$, which by (2.1)(i) follows easily from the relation $\mathfrak{m}\mathfrak{C}_{\mathfrak{p}} = \mathfrak{q} \supset \mathfrak{p}$.

Proof of (2.2). Setting $v := v_{\mathfrak{p}}$, we have, by (1.2.4) and (1.6.1),

$$\lambda\big((\mathfrak{q} : \mathfrak{m})/\mathfrak{q}\big) = \operatorname{ord}_\alpha(\mathfrak{q}) = \operatorname{ord}_\alpha(\mathfrak{p}) = f_{\mathfrak{p}} v(\mathfrak{m}). \tag{2.2.2}$$

In view of the paragraph preceding (2.1), there must then be *at least* $v(\mathfrak{m})$ v-ideals contained in $\mathfrak{q} : \mathfrak{m}$ and strictly containing $\mathfrak{q}$. On the other hand, $\mathfrak{m}(\mathfrak{q} : \mathfrak{m}) \subset \mathfrak{q}$ implies that $v(\mathfrak{q}) - v(\mathfrak{q} : \mathfrak{m}) \le v(\mathfrak{m})$, and so there are *at most* $v(\mathfrak{m})$ such ideals. Hence there are *precisely* $v(\mathfrak{m})$ such ideals; and furthermore if I is any one of them, with v-successor I', then (i) and (ii) in (2.1) must hold. Thus $\mathfrak{q} : \mathfrak{m} \subset \mathfrak{C}_{\mathfrak{p}}$, and $v(\mathfrak{q} : \mathfrak{m}) = v(\mathfrak{q}) - v(\mathfrak{m})$. (Incidentally, the same argument applies to any $\mathfrak{q}'$ in (2.2)).

Now let $\mathfrak{P}$ be the v-predecessor of $\mathfrak{q}$, so that $v(\mathfrak{P}) = v(\mathfrak{q}) - 1 = v(\mathfrak{q} : \mathfrak{m}) + v(\mathfrak{m}) - 1$ and $\lambda(\mathfrak{P}/\mathfrak{q}) = f_{\mathfrak{p}}$. Then $\mathfrak{P} : \mathfrak{m}$ is a v-ideal containing $\mathfrak{q} : \mathfrak{m}$, and again by (1.2.4) and (1.6.1) we have

$$\lambda\big((\mathfrak{P} : \mathfrak{m})/\mathfrak{P}\big) = \operatorname{ord}_\alpha(\mathfrak{P}) < \operatorname{ord}_\alpha(\mathfrak{p}) = f_{\mathfrak{p}} v(\mathfrak{m}),$$

whence $\lambda\big((\mathfrak{P} : \mathfrak{m})/(\mathfrak{q} : \mathfrak{m})\big) < f_{\mathfrak{p}}$ (cf. (2.2.2)). So if $\mathfrak{q} : \mathfrak{m} \subsetneqq \mathfrak{C}_{\mathfrak{p}}$, then by (ii) in (2.1) we must have $\mathfrak{P} : \mathfrak{m} = \mathfrak{q} : \mathfrak{m}$, and further, by (i), the v-predecessor I of $\mathfrak{q} : \mathfrak{m}$ must satisfy $v(I) = v(\mathfrak{q} : \mathfrak{m}) - 1$. But this is impossible, since then

$$v(\mathfrak{m}I) = v(\mathfrak{m}) + v(I) = v(\mathfrak{m}) + v(\mathfrak{q} : \mathfrak{m}) - 1 = v(\mathfrak{P})$$

(see above), whence $\mathfrak{m}I \subset \mathfrak{P}$, i.e., $I \subset \mathfrak{P} : \mathfrak{m} = \mathfrak{q} : \mathfrak{m}$. Thus $\mathfrak{q} : \mathfrak{m} = \mathfrak{C}_{\mathfrak{p}}$.

Now $\operatorname{ord}_\alpha(\mathfrak{C}_{\mathfrak{p}}) < \operatorname{ord}_\alpha(\mathfrak{p})$ (since $\mathfrak{C}_{\mathfrak{p}} \supsetneqq \mathfrak{q}$); and

$$\mathfrak{m}\mathfrak{C}_{\mathfrak{p}} \subset \mathfrak{q} \implies 1 + \operatorname{ord}_\alpha(\mathfrak{C}_{\mathfrak{p}}) \ge \operatorname{ord}_\alpha(\mathfrak{q}) = \operatorname{ord}_\alpha(\mathfrak{p}).$$

So $\operatorname{ord}_\alpha(\mathfrak{C}_{\mathfrak{p}}) = \operatorname{ord}_\alpha(\mathfrak{p}) - 1$. Finally,

$$\lambda\big((\mathfrak{q} : \mathfrak{m})/\mathfrak{m}\mathfrak{C}_{\mathfrak{p}}\big) = \lambda(\mathfrak{C}_{\mathfrak{p}}/\mathfrak{m}\mathfrak{C}_{\mathfrak{p}}) = \operatorname{ord}_\alpha(\mathfrak{C}_{\mathfrak{p}}) + 1 = \operatorname{ord}_\alpha(\mathfrak{p}) = \lambda\big((\mathfrak{q} : \mathfrak{m})/\mathfrak{q}\big),$$

the second equality by (1.2.3) and the fourth by (2.2.2), so that $\mathfrak{m}\mathfrak{C}_{\mathfrak{p}} = \mathfrak{q}$. $\square$

Some peculiarities of the case where $\alpha/\mathfrak{m}_\alpha$ is not algebraically closed are illustrated by:

Corollary (2.3). *With notation as in the proof of* (2.2), *and* $\mathfrak{C} := \mathfrak{C}_\mathfrak{p}$, *the following conditions are equivalent:*

(i) $f_\mathfrak{p} > 1$.
(ii) $\mathfrak{P} : \mathfrak{m} \neq \mathfrak{C}$.
(iii) $v(\mathfrak{P} : \mathfrak{m}) = v(\mathfrak{C}) - 1$.
(iv) *There exists* $z \in \alpha$ *with* $v(z) = v(\mathfrak{C}) - 1$.

Proof. (i)$\Leftrightarrow$(ii). Since $\mathfrak{q} \subset \mathfrak{P} \subset \mathfrak{C} = \mathfrak{q} : \mathfrak{m}$, therefore

$$\begin{aligned} f_\mathfrak{p} = \lambda(\mathfrak{P}/\mathfrak{q}) &= \lambda(\mathfrak{P}/\mathfrak{m}\mathfrak{C}) \\ &\geq \lambda(\mathfrak{C}/\mathfrak{m}\mathfrak{C}) - \lambda(\mathfrak{C}/\mathfrak{P}) - \lambda\big((\mathfrak{P} : \mathfrak{m})/\mathfrak{C}\big) \\ &= \lambda(\mathfrak{C}/\mathfrak{m}\mathfrak{C}) - \lambda\big((\mathfrak{P} : \mathfrak{m})/\mathfrak{P}\big) \\ &= \operatorname{ord}_\alpha(\mathfrak{C}) + 1 - \operatorname{ord}_\alpha(\mathfrak{P}) \qquad \text{(cf. (1.2.3), (1.2.4))} \\ &= 1, \end{aligned}$$

with equality throughout iff $\mathfrak{P} : \mathfrak{m} = \mathfrak{C}$.

(ii)$\Leftrightarrow$(iii) follows easily (since $\mathfrak{P} : \mathfrak{m} \supset \mathfrak{C} = \mathfrak{q} : \mathfrak{m}$) from

$$v(\mathfrak{m}) + v(\mathfrak{P} : \mathfrak{m}) = v\big(\mathfrak{m}(\mathfrak{P} : \mathfrak{m})\big) \geq v(\mathfrak{P}) = v(\mathfrak{q}) - 1 = v(\mathfrak{m}) + v(\mathfrak{C}) - 1.$$

(iii)$\Rightarrow$(iv)$\Rightarrow$(i). The first implication is obvious. As for the second, with $\mathfrak{P}'$ the v-ideal

$$\mathfrak{P}' := \{\, x \in \alpha \mid v(x) \geq v(z) \,\}$$

we have $v(\mathfrak{P}') = v(\mathfrak{C}) - 1$, whence by (2.1) and the paragraph preceding it, $f_\mathfrak{p} > \lambda(\mathfrak{P}'/\mathfrak{C}) \geq 1$. $\square$

3. Covariance of the adjoint.

Notation remains as in §2. This section is devoted to the proof of the following key result, to the effect that *adjoint commutes with transform.* Immediate consequences appear in §4.

Theorem (3.1). *Assume* $\mathfrak{p} \neq \mathfrak{m}$, *so that there is a unique quadratic transform* β *of* α *dominated by* $v_\mathfrak{p}$. *Denote the transform* I^β *of an* $\mathfrak{m}$*-primary ideal* I *by* I'. *Then* $\mathfrak{p}'$ *is a simple* $\mathfrak{m}_\beta$*-primary complete ideal, and*

$$\mathfrak{C}_{\mathfrak{p}'} = (\mathfrak{C}_\mathfrak{p})'.$$

Proof. That $\mathfrak{p}'$ is a simple $\mathfrak{m}_\beta$-primary complete ideal is proved e.g., in [17, p. 381, Prop. 5 and p. 386, Lemma 6].

Let $\mathfrak{q} = \mathfrak{m}\mathfrak{C}_\mathfrak{p}$ be as in (2.2), so that, $\mathfrak{m}\beta$ being principal, $\mathfrak{q}' = (\mathfrak{C}_\mathfrak{p})'$. We show first that $\mathfrak{q}' \subset \mathfrak{C}_{\mathfrak{p}'}$. Set $v := v_\mathfrak{p} = v_{\mathfrak{p}'}$, and let

$$\mathfrak{q} = \mathfrak{q}_0 > \mathfrak{q}_1 > \mathfrak{q}_2 > \cdots > \mathfrak{q}_n = \mathfrak{p}$$

be the sequence of all the v-ideals between $\mathfrak{q}$ and $\mathfrak{p}$, cf. (1.3.1). Since all these ideals have the same order, say r, their transforms—obtained by extending to β and then dividing by the principal ideal $(\mathfrak{m}\beta)^{-r}$—form a descending sequence

$$\mathfrak{q}' = \mathfrak{q}'_0 > \mathfrak{q}'_1 > \mathfrak{q}'_2 > \cdots > \mathfrak{q}'_n = \mathfrak{p}'$$

whose members are v-ideals in β [17, p. 390, (D) (1)]. A similar argument holds for any quadratic transform γ of α; and so if $\gamma \neq \beta$ then $\mathfrak{q}_i^\gamma \supset \mathfrak{p}^\gamma = \gamma$ for all i. From (1.2.2) and "transitivity of transform" we find then that

$$\lambda_\alpha(\alpha/\mathfrak{q}_i) = [\beta : \alpha]\big(\tfrac{1}{2}r(r+1) + \lambda_\beta(\beta/\mathfrak{q}'_i)\big)$$

$$\lambda_\alpha(\alpha/\mathfrak{q}_{i+1}) = [\beta : \alpha]\big(\tfrac{1}{2}r(r+1) + \lambda_\beta(\beta/\mathfrak{q}'_{i+1})\big).$$

Subtracting the first of these equations from the second, and by (2.1)(ii), we get

$$[\beta:\alpha]\lambda_\beta(\mathfrak{q}'_i/\mathfrak{q}'_{i+1}) = \lambda_\alpha(\mathfrak{q}_i/\mathfrak{q}_{i+1}) = f_\mathfrak{p},$$

so that

$$\lambda_\beta(\mathfrak{q}'_i/\mathfrak{q}'_{i+1}) = [\beta:\alpha]^{-1}f_\mathfrak{p} = f_{\mathfrak{p}'}.$$

Also,

$$v(\mathfrak{q}_i) = v(\mathfrak{q}_i\beta) = v(\mathfrak{q}'_i) + rv(\mathfrak{m})$$
$$v(\mathfrak{q}_{i+1}) = v(\mathfrak{q}_{i+1}\beta) = v(\mathfrak{q}'_{i+1}) + rv(\mathfrak{m})$$

and so

$$v(\mathfrak{q}'_{i+1}) - v(\mathfrak{q}'_i) = v(\mathfrak{q}_{i+1}) - v(\mathfrak{q}_i) = 1.$$

It follows, by (2.1), that indeed $\mathfrak{q}' \subset \mathfrak{C}_{\mathfrak{p}'}$, i.e., $(\mathfrak{C}_\mathfrak{p})' \subset \mathfrak{C}_{\mathfrak{p}'}$.

If $\mathfrak{q}' = \beta$ we are done. Otherwise, let $\mathfrak{P}' \subset \beta$ be the v-predecessor of $\mathfrak{q}'$, and let $\mathfrak{P} \subset \alpha$ be the *inverse transform* of $\mathfrak{P}'$, i.e., the largest ideal in α whose transform in β is $\mathfrak{P}'$ [17, p. 390]. Clearly, $\mathfrak{P}$ is not divisible by $\mathfrak{m}$ and $\mathfrak{P} = (\mathfrak{m}^s\mathfrak{P}') \cap \alpha$ where $s := \operatorname{ord}_\alpha(\mathfrak{P})$.

Now $s < r := \operatorname{ord}_\alpha(\mathfrak{q})$; for if not, then we would have

$$\mathfrak{m}^{s-r+1}\mathfrak{C}_\mathfrak{p} = \mathfrak{m}^{s-r}\mathfrak{q} \subset (\mathfrak{m}^{s-r}\mathfrak{q}\beta)\cap\alpha = (\mathfrak{m}^s\mathfrak{q}')\cap\alpha \subset (\mathfrak{m}^s\mathfrak{P}')\cap\alpha = \mathfrak{P};$$

and since $\operatorname{ord}_\alpha(\mathfrak{m}^{s-r+1}\mathfrak{C}_\mathfrak{p}) = s = \operatorname{ord}_\alpha(\mathfrak{P})$, the initial forms of elements in $\mathfrak{P}$ with order s would form an $\alpha/\mathfrak{m}_\alpha$-vector space of dimension > 1, contradicting [17, p. 368, Prop. 3] since $\mathfrak{P}$ is not divisible by $\mathfrak{m}$. So $t := r-1-s \geq 0$, and

$$\begin{aligned} v\big((\mathfrak{C}_\mathfrak{p})'\big) - v\big(\mathfrak{P}'\big) &= \big(v(\mathfrak{C}_\mathfrak{p}) - (r-1)v(\mathfrak{m})\big) - \big(v(\mathfrak{m}^t\mathfrak{P}) - (r-1)v(\mathfrak{m})\big)\\ &= v\big(\mathfrak{C}_\mathfrak{p}\big) - v\big(\mathfrak{m}^t\mathfrak{P}\big). \end{aligned}$$

If $v((\mathfrak{C}_\mathfrak{p})') - v(\mathfrak{P}') > 1$, then by (2.1)(i), $(\mathfrak{C}_\mathfrak{p})' = \mathfrak{C}_{\mathfrak{p}'}$, q.e.d. So suppose that $v(\mathfrak{C}_\mathfrak{p}) - v(\mathfrak{m}^t\mathfrak{P}) = 1$, whence, by (2.1) again, the v-predecessor I of $\mathfrak{C}_\mathfrak{p}$ satisfies $\lambda(I/\mathfrak{C}_\mathfrak{p}) < f_\mathfrak{p}$. Note that $\mathfrak{m}^t\mathfrak{P} = (\mathfrak{m}^t\mathfrak{P}\beta)\cap\alpha$ [17, p. 369, Cor. 2], and that

$$\mathfrak{C}_\mathfrak{p}\beta = \mathfrak{m}^{r-1}\mathfrak{q}' \subset \mathfrak{m}^{r-1}\mathfrak{P}' = \mathfrak{m}^t\mathfrak{P}\beta,$$

so that $\mathfrak{C}_\mathfrak{p} \subset \mathfrak{m}^t\mathfrak{P} \subset I$. Thus $\lambda(\mathfrak{m}^t\mathfrak{P}/\mathfrak{C}_\mathfrak{p}) < f_\mathfrak{p}$. Moreover, $\mathfrak{m}^t\mathfrak{P}$ and $\mathfrak{C}_\mathfrak{p}$ have the same order $r-1$, and since as before $(\mathfrak{C}_\mathfrak{p})^\gamma = \mathfrak{q}^\gamma = \gamma$ for any quadratic transform γ of α other than β, therefore $(\mathfrak{m}^t\mathfrak{P})^\gamma = \gamma$ too. Applying (1.2.2) to $\mathfrak{C}_\mathfrak{p}$ and to $\mathfrak{m}^t\mathfrak{P}$,[2] we conclude as above that

$$\begin{aligned} \lambda_\beta\big(\mathfrak{P}'/(\mathfrak{C}_\mathfrak{p})'\big) = \lambda_\beta\big((\mathfrak{m}^t\mathfrak{P})'/(\mathfrak{C}_\mathfrak{p})'\big) &= [\beta:\alpha]^{-1}\lambda_\alpha\big(\mathfrak{m}^t\mathfrak{P}/\mathfrak{C}_\mathfrak{p}\big)\\ &< [\beta:\alpha]^{-1}f_\mathfrak{p} = f_{\mathfrak{p}'}. \end{aligned}$$

So once again we have by (2.1) that $(\mathfrak{C}_\mathfrak{p})' = \mathfrak{C}_{\mathfrak{p}'}$. $\square$

[2] Here we use completeness of $\mathfrak{m}^t\mathfrak{P}$; but $\mathfrak{m}^t\mathfrak{P} = (\mathfrak{m}^t\mathfrak{P}\beta)\cap\alpha$ suffices, cf. [7, p. 223, (3.1) etc.].

4. More properties of the adjoint.

Denote the point basis of the simple $\mathfrak{m}_\alpha$-primary complete ideal $\mathfrak{p}$ by $\{r_\beta\}$. From (2.2) and (3.1) we get:

Corollary (4.1). *The point basis of* $\mathfrak{C}_\mathfrak{p}$ *is* $\{r_\beta - 1\}$.

Hence, by (1.2.2):

Corollary (4.2).

$$\lambda(\alpha/\mathfrak{C}_\mathfrak{p}) = \sum_{\beta\supset\alpha}[\beta:\alpha]r_\beta(r_\beta-1)/2.$$

Recall that $f_\mathfrak{p} := [\beta_\mathfrak{p} : \alpha]$.

Corollary (4.3). (Gorenstein property)

$$f_\mathfrak{p} v_\mathfrak{p}(\mathfrak{C}_\mathfrak{p}) = 2\lambda(\alpha/\mathfrak{C}_\mathfrak{p}).$$

Proof. By (1.6.2), the left-hand side is $(\mathfrak{p}\cdot\mathfrak{C}_\mathfrak{p}) = \sum_{\beta\supset\alpha}[\beta:\alpha]r_\beta(r_\beta-1)$.

Corollary (4.4). (Noh, [11, Thm. 1]). *If* $f_\mathfrak{p} = 1$ (*for example, if* $\alpha/\mathfrak{m}_\alpha$ *is algebraically closed*) *then the semigroup*

$$S_\mathfrak{p} := \{\, v_\mathfrak{p}(x) \mid 0 \ne x \in \alpha \,\}$$

is symmetric.

Proof. If $f_\mathfrak{p} = 1$ then by (2.1) and (2.2.1), $c := v_\mathfrak{p}(\mathfrak{C}_\mathfrak{p})$ is the *conductor* of $S_\mathfrak{p}$, i.e., $S_\mathfrak{p}$ contains every integer $\ge c$, but $c-1 \notin S_\mathfrak{p}$. *Symmetry* means, by definition, that $c-1-s \in S_\mathfrak{p}$ for every $s \in S_\mathfrak{p}$, or, equivalently, that $S_\mathfrak{p}$ contains exactly half of the integers in the interval $[0, c-1]$. But when $f_\mathfrak{p} = 1$, we have by (1.5) and the statement preceding it that $\lambda(I/I') = 1$ for any two successive $v_\mathfrak{p}$-ideals containing $\mathfrak{C}_\mathfrak{p}$. Hence the symmetry of $S_\mathfrak{p}$ results from (4.3). □

(4.5). Let $\beta \supset \alpha$, and set $\mathfrak{p} := \mathfrak{p}_\beta$, $\mathfrak{C} := \mathfrak{C}_\mathfrak{p}$. The quadratic sequence

$$\alpha =: \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n := \beta \qquad (n \ge 0)$$

in (1.1) gives rise to the sequence of regular surfaces

$$\operatorname{Spec}(\alpha) =: X_0 \underset{f_0}{\longleftarrow} X_1 \underset{f_1}{\longleftarrow} \cdots \underset{f_n}{\longleftarrow} X_{n+1} =: X$$

where $f_i\colon X_{i+1} \to X_i$ $(0 \le i \le n)$ is obtained by blowing up the geometric point $x_i \in X_i$ whose local ring is α_i. Set $\mathfrak{m}_i := \mathfrak{m}_{\alpha_i}$. Note that $\mathfrak{m}_i\mathcal{O}_X$ is an invertible $\mathcal{O}_X$-ideal. Also $\mathfrak{p}\mathcal{O}_X$ and $\mathfrak{C}\mathcal{O}_X$ are invertible; indeed, an easy induction yields

$$\mathfrak{p}\mathcal{O}_X = \prod_{0\le i\le n}\mathfrak{m}_i^{r_i}\mathcal{O}_X \qquad \big(r_i := \operatorname{ord}_{\alpha_i}(\mathfrak{p}^{\alpha_i})\big)$$

and similarly, in view of (4.1),

$$\mathfrak{C}\mathcal{O}_X = \prod_{0\le i\le n}\mathfrak{m}_i^{r_i-1}\mathcal{O}_X\,.$$

Let E_i' be the curve $E_i' := f_i^{-1}\{x_i\}$, and let E_i be the proper transform of E_i' on X, i.e., the unique curve on X mapped isomorphically onto E_i' by the composed map $f_{i+1} \circ f_{i+2} \circ \cdots \circ f_n$. Let $\iota\colon E_i \hookrightarrow X$ be the inclusion map. For any invertible $\mathcal{O}_X$-module $\mathcal{L}$, $(\mathcal{L} \cdot E_i)$ denotes the degree, over $\alpha/\mathfrak{m}_\alpha$, of the invertible sheaf $\iota^*\mathcal{L}$ on the curve E_i. For any divisor D on X, the intersection number $(D \cdot E_i)$ is, by definition, $(\mathcal{O}_X(D) \cdot E_i)$.

Proposition (4.5.1). *Set*

$$\mathcal{K} := \mathfrak{p}(\mathfrak{C}\mathcal{O}_X)^{-1} = \prod_{0 \le i \le n} \mathfrak{m}_i \mathcal{O}_X.$$

Then

$$(\mathcal{K} \cdot E_j) = (E_j \cdot E_j) + 2[\alpha_j : \alpha] \qquad (0 \le j \le n).$$

Proof.[3] Standard intersection theory (cf. e.g., [6, §13]) yields the following facts.

—If $i < j$ then the $\mathcal{O}_X$-module $\mathfrak{m}_i\mathcal{O}_X$ is free (of rank one) in a neighborhood of E_j, and hence $(\mathfrak{m}_i\mathcal{O}_X \cdot E_j) = 0$.

—If $i = j$ then

$$(\mathfrak{m}_i\mathcal{O}_X \cdot E_j) = (\mathfrak{m}_j\mathcal{O}_{X_{j+1}} \cdot E_j') = -(E_j' \cdot E_j') = [\alpha_j : \alpha].$$

—If $i > j$ then $(\mathfrak{m}_i\mathcal{O}_X \cdot E_j) = -[\alpha_i : \alpha]$ if x_i lies on the proper transform of E_j' on X_i (in which case we say that α_i is *proximate to* α_j, and write $\alpha_i \succ \alpha_j$); and otherwise $(\mathfrak{m}_i\mathcal{O}_X \cdot E_j) = 0$.

Thus

$$(\mathcal{K} \cdot E_j) = \sum_{j \le i \le n} (\mathfrak{m}_i \cdot E_j) = [\alpha_j : \alpha]\big(1 - \sum_{\alpha_i \succ \alpha_j} [\alpha_i : \alpha_j]\big).$$

Now let E_j^i be the proper transform of E_j' on X_i $(i > j)$. Then

$$\begin{aligned}(E_j^{i+1} \cdot E_j^{i+1}) &= (E_j^i \cdot E_j^i) - [\alpha_i : \alpha] &&\text{if } \alpha_i \succ \alpha_j\\ &= (E_j^i \cdot E_j^i) &&\text{otherwise.}\end{aligned}$$

Since

$$(E_j^{j+1} \cdot E_j^{j+1}) = (E_j' \cdot E_j') = -[\alpha_j : \alpha],$$

we conclude that

$$(E_j \cdot E_j) = -[\alpha_j : \alpha] - \sum_{\alpha_i \succ \alpha_j} [\alpha_i : \alpha] = [\alpha_j : \alpha]\big(-1 - \sum_{\alpha_i \succ \alpha_j} [\alpha_i : \alpha_j]\big),$$

and (4.5.1) results. $\square$

[3] Alternatively, $[\alpha_j : \alpha]$ is the Euler characteristic of $\mathcal{O}_{E_j}$; and hence (4.5.1) amounts to saying that $\mathcal{K} = \omega^{-1}$ where ω is the *relative canonical sheaf* for $X \to \operatorname{Spec}(\alpha)$, an assertion proved, in essence, in [10, p. 111] or [9, p. 202 and p. 206, (2.3)]. In particular, $\mathfrak{C} = H^0(X, \mathfrak{p}\omega)$.

Corollary (4.5.2). *With $E_\gamma := E_i$ when $\gamma = \alpha_i$, the adjoint ideal $\mathfrak{C}$ factors as*

$$\mathfrak{C} = \prod_{\alpha \subset \gamma \subsetneqq \beta} \mathfrak{p}_\gamma^{c_\gamma}$$

where

$$c_\gamma = -[\gamma : \alpha]^{-1}(E_\gamma \cdot E_\gamma) - 2$$

is one less than the number of $\delta \subset \beta$ proximate to γ, each such δ being counted with multiplicity $[\delta : \gamma]$.

Proof. The treatment of factorization of complete ideals given in [6, §§18–19] gives $\mathfrak{C} = \prod_{\alpha \subset \gamma \subsetneqq \beta} \mathfrak{p}_\gamma^{c_\gamma}$ with

$$c_\gamma = [\gamma : \alpha]^{-1}(\mathfrak{C}\mathcal{O}_X \cdot E_\gamma) \overset{(4.5.1)}{=} (\mathfrak{p} \cdot E_\gamma) - (\mathcal{K} \cdot E_\gamma) = 0 - [\gamma : \alpha]^{-1}(E_\gamma \cdot E_\gamma) - 2.$$

The rest is contained in the proof of (4.5.1).[4] $\square$

5. The polar ideal.

Notation remains as in §2.

Definition (5.1). The *polar ideal* $\mathfrak{P}_\mathfrak{p}$ of a simple $\mathfrak{m}$-primary complete ideal $\mathfrak{p}$ is the smallest among those $\mathfrak{m}$-primary $v_\mathfrak{p}$-ideals $\mathfrak{P}$ satisfying $\mathrm{ord}_\alpha(\mathfrak{P}) = \mathrm{ord}_\alpha(\mathfrak{p}) - 1$, i.e., $\mathfrak{P}_\mathfrak{p}$ is the $v_\mathfrak{p}$-predecessor of the ideal $\mathfrak{q} = \mathfrak{m}\mathfrak{C}_\mathfrak{p}$ of Theorem (2.2).

Corollary (5.1.1).

$$v_\mathfrak{p}(\mathfrak{P}_\mathfrak{p}) = v_\mathfrak{p}(\mathfrak{C}_\mathfrak{p}) + f_\mathfrak{p}^{-1}\mathrm{ord}_\alpha(\mathfrak{p}) - 1,$$
$$\lambda(\alpha/\mathfrak{P}_\mathfrak{p}) = \lambda(\alpha/\mathfrak{C}_\mathfrak{p}) + \mathrm{ord}_\alpha(\mathfrak{p}) - f_\mathfrak{p}.$$

Proof. Since $\mathfrak{P}_\mathfrak{p} \subset \mathfrak{C}_\mathfrak{p}$, therefore, by (2.1) and (2.2),

$$v_\mathfrak{p}(\mathfrak{P}_\mathfrak{p}) = v_\mathfrak{p}(\mathfrak{q}) - 1 = v_\mathfrak{p}(\mathfrak{C}_\mathfrak{p}) + v_\mathfrak{p}(\mathfrak{m}) - 1 \overset{(1.6.1)}{=} v_\mathfrak{p}(\mathfrak{C}_\mathfrak{p}) + f_\mathfrak{p}^{-1}\mathrm{ord}_\alpha(\mathfrak{p}) - 1,$$

and also,

$$\lambda(\alpha/\mathfrak{P}_\mathfrak{p}) = \lambda(\alpha/\mathfrak{C}_\mathfrak{p}) + \lambda(\mathfrak{C}_\mathfrak{p}/\mathfrak{m}\mathfrak{C}_\mathfrak{p}) - f_\mathfrak{p} \overset{(1.2.3)}{=} \lambda(\alpha/\mathfrak{C}_\mathfrak{p}) + \mathrm{ord}_\alpha(\mathfrak{p}) - f_\mathfrak{p}.$$

$\square$

Henceforth, we will write $\mathfrak{P}$ (resp. $\mathfrak{C}$, v) for $\mathfrak{P}_\mathfrak{p}$ (resp. $\mathfrak{C}_\mathfrak{p}$, $v_\mathfrak{p}$) when there is no possibility of confusion.

Next we describe how to derive the point basis $\{s_\gamma\}_{\gamma \supset \alpha}$ of $\mathfrak{P}$ from the point basis $\{r_\gamma\}_{\gamma \supset \alpha}$ of $\mathfrak{p}$. As mentioned in the Introduction, this will explain the appellation "polar ideal."

[4] Another justification—not using geometric methods—of this description of c_γ in terms of proximity appears in [8, Example (3.2)].

Let $\beta \supset \alpha$ be the point corresponding to $\mathfrak{p}$ (so that $v = \operatorname{ord}_\beta$, cf. (1.4)) and let

$$\alpha =: \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n := \beta$$

be the corresponding quadratic sequence (1.1.1). Then r_γ and s_γ both vanish unless $\gamma = \alpha_i$ for some i. (Such vanishing holds for the point basis of *any* v-ideal, cf. [17, p. 392, (F)].) So setting $r_i := r_{\alpha_i}$, $s_i := s_{\alpha_i}$, we need only look at the sequences $(r_i)_{0 \le i \le n}$, $(s_i)_{0 \le i \le n}$.

We say, as in the proof of (4.5.1), that α_i is proximate to α_j, and write $\alpha_i \succ \alpha_j$, if $i > j$ and if the valuation ring of $\operatorname{ord}_{\alpha_j}$ contains α_i. When $i > 0$, α_i is proximate to α_{i-1} and to *at most* one other α_j (because, for example, the closed points on X_i in (4.5) form a normal-crossing divisor; or, one can use (5.2.2) below). If there is such an α_j, we say that α_i is a *satellite,* and otherwise that α_i is *free,* with respect to (w.r.t.) α.

Set $\mathfrak{m}_i := \mathfrak{m}_{\alpha_i}$. The integers $v(\mathfrak{m}_i) = [\beta : \alpha_i]^{-1} r_i$ (cf. (1.6)) appearing in the following Theorem may be thought of, informally, as representing the infinitely near multiplicities of a general element of $\mathfrak{p}$. At least that's what they do when $\alpha/\mathfrak{m}_\alpha$ is algebraically closed. We also let $\mathfrak{P}'_i$ be the transform $\mathfrak{P}^{\alpha_i}$ (which is a v-ideal in α_i [17, p. 390, (1)]); and we set $\mathfrak{p}_i := \mathfrak{p}^{\alpha_i}$, $\mathfrak{P}_i :=$ polar ideal—in α_i—of $\mathfrak{p}_i$.

Theorem (5.2). *With preceding notation, we have the following inductive prescription for determining* $r_i - s_i$ *:*

(i) $\mathfrak{P}_i \supset \mathfrak{P}'_i \supset \mathfrak{p}_i$, *so that* $r_i - s_i = 0$ or 1 $\quad (0 \le i \le n)$.

(ii) $r_0 - s_0 = 1$.

(iii) *If* $v(\mathfrak{m}_{i+1}) = v(\mathfrak{m}_i)$ *then* $r_{i+1} - s_{i+1} = r_i - s_i$ unless *either* $i + 1 = n$ *or* α_{i+1} *is a satellite and* α_{i+2} *is free, in which cases* $r_{i+1} - s_{i+1} = 1$ (*whether or not* $v(\mathfrak{m}_{i+1}) = v(\mathfrak{m}_i)$).

(iv) *If* $v(\mathfrak{m}_{i+1}) < v(\mathfrak{m}_i)$ *then* $r_{i+1} - s_{i+1} \ne r_i - s_i$.

Moreover, if $r_i - s_i = 1$ *then* $\mathfrak{P}'_i = \mathfrak{P}_i$. *And* $r_i - s_i = 1$ *whenever* α_{i+1} *is free.*

Proof. Condition (ii) holds by the definition of $\mathfrak{P}$. For the rest, we proceed by induction on the number of points between α and β, everything being obvious if $\alpha = \beta$. Suppose $\alpha \ne \beta$ and that we can find a $k > 0$ such that $\mathfrak{P}'_k = \mathfrak{P}_k$. The inductive hypothesis establishes $(5.2)_k$, the statement (5.2) with the underlying pair $\alpha \subset \beta$ replaced by $\alpha_k \subset \beta$. Such a replacement does not affect $\mathfrak{p}_i$ or $\mathfrak{P}'_i$ for any $i \ge k$ (use transitivity of transform (1.2)), and hence does not affect r_i, s_i, $\mathfrak{P}_i$, or $v(\mathfrak{m}_i) = \operatorname{ord}_\beta(\mathfrak{m}_i)$. For $i \ge k$ then, assertions (i) and (iv) in (5.2), as well as the assertions following (iv), are clearly implied by the corresponding assertions in $(5.2)_k$. So is assertion (iii), as we see after making the following observation: if w.r.t. α_k, α_{i+1} is a satellite and α_{i+2} is free, then the same is true w.r.t. α_0—for if $\alpha_{i+2} \succ \alpha_j$ for some $j < k$, i.e., α_{i+2} is contained in the valuation ring R_j of $\operatorname{ord}_{\alpha_j}$, then also $\alpha_{i+1} \subset R_j$, so that α_{i+1} is proximate to α_j as well as to two other points containing α_k, contradiction.

Thus, to establish (5.2) it will suffice to prove its assertions for $i < k$, and hence, to prove the following Lemma.

Lemma (5.2.1). *Assume $\alpha \neq \beta$. Let $a \geq 0$ and b be the unique integers such that*

$$v(\mathfrak{m}_0) = av(\mathfrak{m}_1) + b \qquad \big(0 < b \leq v(\mathfrak{m}_1)\big).$$

Then:

(i) $\{\alpha_1, \alpha_2, \ldots, \alpha_{a+1}\}$ *is the set of* all *points proximate to* α_0 *and contained in* β*; and so for* $2 \leq i \leq a+1$, α_i *is a satellite* (*w.r.t.* α).

(ii) $v(\mathfrak{m}_1) = v(\mathfrak{m}_2) = \cdots = v(\mathfrak{m}_a) \geq v(\mathfrak{m}_{a+1}) = b$.

(iii) *If* $b = v(\mathfrak{m}_1)$ *and* $1 \leq a \leq n-2$, *then* α_{a+1} *is a satellite and* α_{a+2} *is free.*

(iv) *For* $1 \leq i \leq a+1$, $\mathfrak{P}_i \supset \mathfrak{P}'_i \supset \mathfrak{p}_i$, *and*

$$v(\mathfrak{P}'_i) - v(\mathfrak{P}_i) = \sum_{i<j\leq a+1} v(\mathfrak{m}_j).$$

Thus $\mathfrak{P}'_{a+1} = \mathfrak{P}_{a+1}$, $r_{a+1} - s_{a+1} = 1$, *and for* $1 \leq i \leq a$, $\mathfrak{P}_i \neq \mathfrak{P}'_i$, *so that* $r_i - s_i = 0$.

Proof. If α_{a+2} is proximate to α_i for some $i \leq a$, then so is α_{a+1}, and therefore $i = 0$ or $i = a$. The first possibility is excluded by (i). So is the second under the hypotheses of (iii), since then by (ii), $v(\mathfrak{m}_a) = v(\mathfrak{m}_{a+1})$, and we can replace $\mathfrak{m}_0$ by $\mathfrak{m}_\alpha$ in (i) to conclude. Hence (iii) follows from (i) and (ii).

Now (i) and (ii) follow readily from the next result (applied with $\gamma := \alpha$, δ the largest point between α and β proximate to α, and $w := v$).

Lemma (5.2.2). *Let $\gamma \prec \delta$ be two points, with corresponding quadratic sequence*

$$\gamma =: \gamma_0 \subset \gamma_1 \subset \cdots \subset \gamma_d := \delta.$$

If w is any valuation dominating δ, then, with $\mathfrak{m}_i := \mathfrak{m}_{\gamma_i}$,

(5.2.2.1) $$w(\mathfrak{m}_\gamma) \geq w(\mathfrak{m}_1) + w(\mathfrak{m}_2) + \cdots + w(\mathfrak{m}_{d-1}) + w(\mathfrak{m}_\delta),$$

with equality iff w does not dominate that quadratic transform of δ which is proximate to γ; and consequently[5]

(5.2.2.2) $$w(\mathfrak{m}_\gamma) \geq w(\mathfrak{m}_1) = w(\mathfrak{m}_2) = \cdots = w(\mathfrak{m}_{d-1}) \geq w(\mathfrak{m}_\delta).$$

Moreover, $[\gamma_i : \gamma_1] = 1$ for $i = 1, 2, \ldots, d$.

Proof. Let $\mathfrak{m}_0\gamma_1 = t\gamma_1$, and let $u \in \gamma_1$ be such that $\{t, u\}$ is a regular system of parameters in γ_1 (u exists because $\gamma_1/\mathfrak{m}_0\gamma_1$ is regular). The localization of γ_1 at the prime ideal $\mathfrak{m}_0\gamma_1$ is the valuation ring of ord_γ, and so $\mathrm{ord}_\gamma(t) = 1$, $\mathrm{ord}_\gamma(u) = 0$.

We show next, by induction on i, that for $1 \leq i \leq d$, $\{t/u^{i-1}, u\}$ is a regular system of parameters in γ_i, and $[\gamma_i : \gamma_1] = 1$; and that for $1 \leq i < d$, $\mathfrak{m}_i\gamma_{i+1} = u\gamma_{i+1}$. Indeed, if $i < d$ and if $\{t/u^{i-1}, u\}$ is a regular system of parameters in γ_i, then since ord_γ is non-negative on δ (by definition of the condition $\gamma \prec \delta$),

[5] (5.2.2.2) can be deduced from the preceding statement with γ_i in place of γ_0 ($1 \leq i \leq d-2$), since γ_{i+2}, being proximate to γ_0 and γ_{i+1}, can't be proximate to γ_i.

hence on γ_{i+1}, and since $\operatorname{ord}_\gamma(u^i/t) < 0$, therefore γ_{i+1} cannot be a localization of the ring $\gamma_i[(t/u^{i-1})^{-1}\mathfrak{m}_i]$, so γ_{i+1} must be the localization of $\gamma_i[u^{-1}\mathfrak{m}_i]$ at its maximal ideal $(t/u^i, u)$.

Now for $i \le i < d$, we have $w(\mathfrak{m}_i) = w(\mathfrak{m}_i\gamma_{i+1}) = w(u)$, which is independent of i, proving (5.2.2.2). Furthermore,

$$\begin{aligned} w(\mathfrak{m}_\gamma) = w(t) &= (d-1)w(u) + w(t/u^{d-1}) \\ &= w(\mathfrak{m}_1) + w(\mathfrak{m}_2) + \cdots + w(\mathfrak{m}_{d-1}) + w(t/u^{d-1}). \end{aligned}$$

As above, the unique quadratic transform δ' of δ which is proximate to γ is the localization of the ring $\delta[t/u^d, u]$ at its maximal ideal $(t/u^d, u)$; and w dominates this δ' iff $w(t/u^d) > 0$, i.e., iff $w(t/u^{d-1}) > w(u)$. Since

$$w(\mathfrak{m}_\delta) = w\big((t/u^{d-1}, u)\delta\big) = \min\{w(t/u^{d-1}), w(u)\},$$

the rest of (5.2.2) follows. $\square$

It remains to prove (iv) in Lemma (5.2.1). We proceed by induction, beginning with $i = 1$. Since $\mathfrak{P}$ contains the inverse transform $\mathfrak{p}_0$ of $\mathfrak{p}_1$, and since $\mathfrak{P}$ is contained in the inverse transform of $\mathfrak{P}'_1$ [17, p. 390, (3)], therefore [*ibid,* (4)] gives $\mathfrak{P}'_1 \supset \mathfrak{p}_1$. Let $\mathfrak{C}_i$ be the adjoint ideal in α_i of $\mathfrak{p}_i$. From (2.2) and (3.1) we get $\mathfrak{m}_0^{r_0-1}\mathfrak{C}_1 = \mathfrak{C}_0\alpha_1$. Also, $\mathfrak{m}_0^{r_0-1}\mathfrak{P}'_1 = \mathfrak{P}_0\alpha_1$. So

$$\begin{aligned} v(\mathfrak{m}_1\mathfrak{C}_1) &= v(\mathfrak{m}_1) + v(\mathfrak{C}_0) - (r_0-1)v(\mathfrak{m}_0), \\ v(\mathfrak{P}'_1) = v(\mathfrak{P}_0) - (r_0-1)v(\mathfrak{m}_0) &= v(\mathfrak{m}_0) + v(\mathfrak{C}_0) - 1 - (r_0-1)v(\mathfrak{m}_0) \end{aligned}$$

(cf. proof of (5.1.1)). Hence

$$v(\mathfrak{P}'_1) - v(\mathfrak{P}_1) = v(\mathfrak{P}'_1) - v(\mathfrak{m}_1\mathfrak{C}_1) + 1 = v(\mathfrak{m}_0) - v(\mathfrak{m}_1) = (a-1)v(\mathfrak{m}_1) + b.$$

Having already proved (ii) in (5.2.1), we then deduce (iv) for $i = 1$.

If $n = 1$ we are done. Otherwise, $v(\mathfrak{m}_0) > v(\mathfrak{m}_1)$, so $\mathfrak{P}_1 \supsetneqq \mathfrak{P}'_1 \supset \mathfrak{p}_1$, so $\mathfrak{m}_1\mathfrak{C}_1 \supset \mathfrak{P}'_1$ and $\operatorname{ord}_{\alpha_1}(\mathfrak{P}'_1) = \operatorname{ord}_{\alpha_1}(\mathfrak{p}_1)$. Since as in the proof of (5.1.1), $v(\mathfrak{P}_i) = v(\mathfrak{m}_i\mathfrak{C}_i) - 1$, the rest follows from the next Lemma.

Lemma (5.2.3). *Let L be a v-ideal in α_i $(1 \le i < n)$ such that $\mathfrak{m}_i\mathfrak{C}_i \supset L \supset \mathfrak{p}_i$. Then $L^{\alpha_{i+1}} \supset \mathfrak{p}_{i+1}$, and*

$$v(L^{\alpha_{i+1}}) - v(\mathfrak{m}_{i+1}\mathfrak{C}_{i+1}) = v(L) - v(\mathfrak{m}_i\mathfrak{C}_i) - v(\mathfrak{m}_{i+1}).$$

Proof. From (2.2) and (3.1), we see that

$$\mathfrak{m}_{i+1}\mathfrak{C}_{i+1} = \mathfrak{m}_{i+1}(\mathfrak{m}_i\mathfrak{C}_i)^{\alpha_{i+1}} = \mathfrak{m}_{i+1}\mathfrak{m}_i\mathfrak{C}_i(\mathfrak{m}_i\alpha_{i+1})^{-r_i}$$

where

$$r_i := \operatorname{ord}_{\alpha_i}(\mathfrak{p}_i) = \operatorname{ord}_{\alpha_i}(\mathfrak{m}_i\mathfrak{C}_i) = \operatorname{ord}_{\alpha_i}(L),$$

so that $L^{\alpha_{i+1}} = L(\mathfrak{m}_i\alpha_{i+1})^{-r_i}$. The conclusion results. $\square$

This completes the proof of Lemma (5.2.1) and of Theorem (5.2).

Remarks. (A) Let

$$\mathfrak{m} := \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n := \mathfrak{p}$$

be the sequence of simple complete ideals in α corresponding to the quadratic sequence (1.1.1) (where $\beta = \beta_{\mathfrak{p}}$, cf. (1.4)). The v-ideal $\mathfrak{P} := \mathfrak{P}_{\mathfrak{p}}$ factors uniquely as

$$\mathfrak{P} = \prod_{i=1}^{n} \mathfrak{p}_i^{a_i},$$

cf. [17, p. 392, (2)]. Here $a_n = 0$ since $\mathfrak{P} \supsetneqq \mathfrak{p}_n = \mathfrak{p}$. The remaining exponents a_i can be characterized in terms of the point basis $\{s_i\}_{1 \le i \le n}$ of $\mathfrak{P}$, by means of proximity:

$$a_i = s_i - \sum_{\alpha_j \succ \alpha_i} [\alpha_j : \alpha_i] s_j,$$

cf. [8, Cor. (3.1)]. Hence [2, §1] yields the following geometric interpretation when α is the local ring of a smooth point on an algebraic surface over an algebraically closed characteristic 0 field:

Let f be a generic element of $\mathfrak{p}$, and let ζ be a generic polar of the curve germ $f = 0$. Then ζ has $\sum_{i=1}^{n} a_i$ irreducible branches, of which precisely a_i pass through the infinitely near points[6] $\alpha_1, \alpha_2, \ldots, \alpha_i$ but not through α_{i+1}.

(B) Now observe, keeping in mind that transform preserves products (1.2) and consequently "point basis" takes products to sums, that for any finite-colength α-ideal L,

$$(\mathfrak{P} \cdot L) = \sum_{i=1}^{n} a_i (\mathfrak{p}_i \cdot L) \overset{(1.6.2)}{=} \sum_{i=1}^{n} a_i [\alpha_i : \alpha] \operatorname{ord}_{\alpha_i}(L).$$

Assume, for simplicity, that $[\beta : \alpha] = 1$ (so that $[\alpha_i : \alpha] = 1$ for all i). Set $\mathfrak{m} := \mathfrak{m}_\alpha$. Define:

$$\begin{aligned} \overline{e}_i &:= \operatorname{ord}_{\alpha_i}(\mathfrak{p}) - \operatorname{ord}_{\alpha_i}(\mathfrak{m}) \qquad && (1 \le i \le n), \\ \overline{m}_i &:= \operatorname{ord}_{\alpha_i}(\mathfrak{m}) && (1 \le i \le n). \end{aligned}$$

The integers $\overline{e}_i, \overline{m}_i$ are the same as the integers e_q, m_q associated by Teissier [14, p. 270] to any one of the a_i branches ζ_q of ζ which part company at α_i with the curve $f = 0$. To prove this, one needs to know that except for the α_i, all other infinitely near points on ζ are nonsingular on ζ and free w.r.t. α, cf. [2, p. 4, Thm. 2.1].

With $v = \operatorname{ord}_\beta$ as before, we have then

$$\sum_{i=1}^{n} a_i \overline{m}_i = (\mathfrak{P} \cdot \mathfrak{m}) = \operatorname{ord}_\alpha(\mathfrak{P}) = \operatorname{ord}_\alpha(\mathfrak{p}) - 1,$$

[6] A curve germ "passes through" a point α_j if it has a defining equation $g = 0$ ($g \in \alpha$) with $g\alpha_j \ne \mathfrak{m}^t \alpha_j$ ($t := \operatorname{ord}_\alpha(g)$), i.e., the proper transform of g in α_j is a non-unit.

and

$$\begin{aligned}\sum_{i=1}^{n} a_i\overline{e}_i = (\mathfrak{P}\cdot\mathfrak{p}) - (\mathfrak{P}\cdot\mathfrak{m}) &\overset{(1.6.2)}{=} v(\mathfrak{P}) - \operatorname{ord}_\alpha(\mathfrak{P})\\ &\overset{(5.1)}{=} \big(v(\mathfrak{m}) + v(\mathfrak{C}) - 1\big) - \big(\operatorname{ord}_\alpha(\mathfrak{p}) - 1\big)\\ &\overset{(1.6.2)}{=} v(\mathfrak{C})\\ &\overset{(4.3)}{=} 2\lambda(\alpha/\mathfrak{C})\\ &\overset{(4.2)}{=} \sum_{i=1}^{n} r_i(r_i-1).\end{aligned}$$

These relations are in accord with the equations near the top of p. 270 in [14].

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