GROUP-THEORETIC AXIOMS FOR PROJECTIVE GEOMETRY

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ABSTRACT. We show that a certain category $\mathbb{G}$, whose objects are pairs $G \supset H$ of groups subject to simple axioms, is equivalent to the category of ≥ 2-dimensional vector spaces and injective semi-linear maps; and deduce via the "Fundamental Theorem of Projective Geometry" that the category of ≥ 2-dimensional projective spaces is equivalent to the quotient of a suitable subcategory of $\mathbb{G}$ by the least equivalence relation which identifies conjugation by any element of H with the identity automorphism of G.

Introduction. Let V be a left vector space of dimension ≥ 2 over a (not necessarily commutative) field F. For any pair $0 \neq a \in F$, $v \in V$, let $[a,v]\colon V \to V$ be the map given by

$$[a,v](w) = aw + v \qquad (w \in V).$$

The set G of all such maps is closed under composition:

$$[a,v] \circ [a',v'] = [aa',\, av' + v]\,;$$

and each $[a,v]$ is bijective, with inverse

$$[a,v]^{-1} = [a^{-1},\, -a^{-1}v]\,.$$

So G is a group of transformations of V, with identity element $e = [1,0]$.

Denote by H the subgroup of G consisting of all maps of the form $[a,0]$. H is isomorphic to the multiplicative group of non-zero elements in F.

Denote by T the subgroup of G consisting of all maps of the form $[1,v]$. T is a *normal subgroup* of G, isomorphic to the additive group V. Every element of G is uniquely of the form th, with $t \in T$, $h \in H$:

$$[a,v] = [1,v][a,0]\,.$$

The elements of T are called *translations*. One checks that:

(0.1) A non-identity element $g \in G$ is a translation if and only if no conjugate of g lies in H.

Recall that for $g \in G$, the double coset HgH is the set

$$HgH = \{\, h_1 g h_2 \mid g \in G \text{ and } h_1, h_2 \in H \,\}\,.$$

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One verifies the following properties of the pair (G, H).

(GH1) For all $g \in G$,

$$G \neq HgH \cup H.$$

(In other words H has at least three distinct double cosets in G.)

(GH2) For all $g \in G$,

$$gHg^{-1} \subset HgH \cup \{e\}.$$

(GH3) For all $g \in G$,

$$HgH = Hg^{-1}H.$$

Remarks. (i) With regard to (GH2), note that for $g \in G$ and $h \in H$,

$$ghg^{-1} = e \iff h = e.$$

(ii) The property (GH3) follows formally from (GH2), except when $H = \{e\}$, in which case (GH2) says nothing at all.

Indeed, if $g \in G$ and $e \neq h \in H$, then applying (GH2) twice we get

$$gh^{-1}g^{-1} \in HgH,$$
$$gh^{-1}g^{-1} = (ghg^{-1})^{-1} \in (HgH)^{-1} = Hg^{-1}H;$$

so the double cosets HgH and $Hg^{-1}H$ meet, and hence they are equal.

(iii) We will see later (Remark 1.11) that given (GH1), conditions (GH2) and (GH3) together are group-theoretically equivalent to :

(GH2)* $$\bigcap_{g \in G} gHg^{-1} = \{e\}, \text{ and}$$

(GH3)* for each $g \in G$, $HgH \cup H$ is a subgroup of G.

Now let $\mathbb{V}$ be the category whose objects are all pairs (V, F) as above, and whose arrows are *injective semi-linear maps*

$$(\zeta, \theta) \colon (V, F) \to (V', F').$$

More specifically:

— $\theta \colon F \to F'$ is a homomorphism of fields.

— $\zeta \colon V \to V'$ is an injective map satisfying

$$\zeta(v_1 + v_2) = \zeta(v_1) + \zeta(v_2) \qquad (v_1, v_2 \in V),$$
$$\zeta(av) = \theta(a)\zeta(v) \qquad (a \in F,\ v \in V).$$

Let $\mathbb{G}$ be the category whose objects are pairs (G,H) consisting of a group G and a subgroup H satisfying (GH1)–(GH3) above, and whose arrows $(G,H)\to(G',H')$ are those group homomorphisms $f\colon G\to G'$ satisfying

(0.2.1) $f(H)\subset(H')$, and

(0.2.2) if $g\in G$ has no conjugate lying in H, then $f(g)\in G'$ has no conjugate lying in H'.

We define a *functor* $\Gamma\colon\mathbb{V}\to\mathbb{G}$ as follows:

- For any object (V,F), $\Gamma(V,F)$ is the pair (G,H) described at the beginning of this Introduction.
- For any arrow $(\zeta,\theta)\colon(V,F)\to(V',F')$, the arrow
$$\Gamma(\zeta,\theta)\colon\Gamma(V,F)\to\Gamma(V',F')$$
is given by the formula
$$\Gamma(\zeta,\theta)[a,v]=[\theta(a),\zeta(v)]\,.$$

To verify that Γ is indeed a functor, check that it respects categorical identities and composition, and that $\Gamma(\zeta,\theta)$ is a group homomorphism satisfying (0.2.1) and (0.2.2)—cf. (0.1).

Our main result is:

Theorem 1. Γ *is an equivalence of categories.*

In other words, there exists a functor $\Theta\colon\mathbb{G}\to\mathbb{V}$ together with isomorphisms of functors
$$\Gamma\Theta\ \xrightarrow{\sim}\ \mathbf{1}_{\mathbb{G}}\,,\qquad \Theta\Gamma\ \xrightarrow{\sim}\ \mathbf{1}_{\mathbb{V}}$$
(where $\mathbf{1}$ denotes an identity functor). Such a functor Θ is called a *pseudo-inverse* of Γ.

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The title of this paper refers to Theorem 2 below. By way of explanation, we first reformulate the Fundamental Theorem of Projective Geometry [A, p. 88] in the language of categories.

The *projective space* $P=P(V,F)$ is, by definition, the set of one-dimensional subspaces of V. A *projective subspace* of P is a subset consisting of all the one-dimensional subspaces of some vector subspace of V.

A sequence $(x_1,x_2,\dots,x_n)$ in P is *linearly independent* if for each $i=1,2,\dots,n$, x_i lies outside some projective subspace $P_i\subset P$ which contains x_j for all $j\neq i$;[1] and *linearly dependent* otherwise. Three points $x_1,x_2,x_3\in P$ are *collinear* if (x_1,x_2,x_3) is linearly dependent.

A map
$$\pi\colon P(V,F)=P\to P'=P(V',F')$$
is a *collineation* if for any three collinear points $x_1,x_2,x_3\in P$ the points $\pi(x_1),\pi(x_2),\pi(x_3)\in P'$ are also collinear. A rather simple induction shows that this is equivalent to π mapping every linearly dependent sequence in P to a linearly dependent sequence in P'. We say that π is *linearly faithful* if it satisfies the following conditions, which are (exercise) equivalent:

- A sequence $(x_1,x_2,\dots,x_n)$ in P is linearly independent if and only if the sequence $(\pi(x_1),\pi(x_2),\dots,\pi(x_n))$ in P' is linearly independent.
- Every projective subspace $P_1\subset P$ is of the form $\pi^{-1}(P_1')$ for some projective subspace $P_1'\subset P'$.

[1] i.e., if the subspace $x_1+x_2+\cdots+x_n\subset V$ spanned by the x_i has dimension n.

Any linearly faithful map is an injective collineation. A bijective collineation π is linearly faithful if and only if π^{-1} is a collineation.

We define $\mathbb{P}_2$, *the category of projective spaces of dimension* ≥ 2, as follows:

— The objects of $\mathbb{P}_2$ are the vector spaces (V, F) of dimension ≥ 3.

— The arrows $(V, F) \to (V', F')$ in $\mathbb{P}_2$ are the *linearly faithful* maps $\pi\colon P(V, F) \to P(V', F')$.

Let $\mathbb{V}_3$ be the subcategory of $\mathbb{V}$ with objects the vector spaces (V, F) of dimension ≥ 3, and with arrows the semi-linear maps

$$(\zeta, \theta)\colon (V, F) \to (V', F')$$

such that ζ is *linearly faithful* in the sense that it satisfies the following equivalent conditions:

— A sequence $(v_1, v_2, \dots, v_n)$ in V is linearly independent (over F) if and only if the sequence $(\zeta(x_1), \zeta(x_2), \dots, \zeta(x_n))$ in V' is linearly independent (over F').

— Every linear subspace $V_1 \subset V$ is of the form $\zeta^{-1}(V_1')$ for some linear subspace $V_1' \subset V'$.

Remarks. (i) If ζ is linearly faithful then ζ is injective.

(ii) If ζ is injective and θ is bijective then ζ is linearly faithful.[2]

(iii) If $\dim V \geq 3$, and if a is any non-zero element of F, then the automorphism $(\zeta_a, \theta_a)\colon (V, F) \to (V, F)$ given by

$$\zeta_a(v) = av \qquad (v \in V)$$
$$\theta_a(b) = aba^{-1} \qquad (b \in F)$$

is an arrow in $\mathbb{V}_3$.

We define a functor $\Pi\colon \mathbb{V}_3 \to \mathbb{P}_2$ by

$$\Pi(V, F) = (V, F)\,,$$
$$\Pi\big((\zeta, \theta)\colon (V, F) \to (V', F') \big) = \pi_\zeta$$

where for each non-zero $v \in V$, π_ζ takes the subspace $Fv \in P(V, F)$ to the subspace $F'\zeta(v) \in P(V', F')$.

Next we form a quotient category of $\mathbb{V}_3$ through which Π factors. Let $\mathbf{R}$ be the equivalence relation under which two arrows in V_3

$$(\zeta_1, \theta_1),\ (\zeta_2, \theta_2)\colon (V, F) \to (V', F')$$

are equivalent if there exists a non-zero $a \in F'$ such that

$$(\zeta_2, \theta_2) = (\zeta_a, \theta_a) \circ (\zeta_1, \theta_1)\,.$$

The *quotient category* $\mathbb{V}_3/\mathbf{R}$ has the same objects as $\mathbb{V}_3$; but for two objects (V, F), (V', F'), the $\mathbb{V}_3/\mathbf{R}$-arrows $(V, F) \to (V', F')$ are the *equivalence classes* under $\mathbf{R}$ of arrows in $\mathbb{V}_3$ between these two objects, composition being defined in the natural way.

There is a canonical functor $\rho\colon \mathbb{V}_3 \to \mathbb{V}_3/\mathbf{R}$ taking any object to itself and taking any arrow to its equivalence class; and it is easily checked that *there is a unique functor*

$$\overline{\Pi}\colon \mathbb{V}_3/\mathbf{R} \longrightarrow \mathbb{P}_2$$

such that

$$\Pi = \overline{\Pi} \circ \rho\,.$$

Now, at last, we can state:

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[2] More generally, ζ is linearly faithful $\iff$ the map $F' \otimes_F V \to V'$ induced by (ζ, θ) is injective.

Fundamental Theorem of Projective Geometry. *The functor $\overline{\Pi}$ is an isomorphism of categories.*

In other words, $\overline{\Pi}$ is bijective on objects (clearly) and on arrows: every arrow in $\mathbb{P}_2$ is of the form π_ζ, and

$$\pi_\zeta = \pi_{\zeta'} \iff \zeta \equiv \zeta' \pmod{\mathbf{R}}.$$

The proof is essentially given by E. Artin in [A, pp. 88–91]. Artin restricts his attention to finite-dimensional spaces and to arrows which are *isomorphisms*; but his arguments are easily modified to cover the present statement.

The Fundamental Theorem in some sense reduces Projective Geometry to Linear Algebra. Consequently, using Theorem 1, *we can reduce Projective Geometry to Group Theory.* Here is a precise formulation; proofs are provided in §2.

Let $\mathbb{G}_3$ be the subcategory of $\mathbb{G}$ whose objects are pairs (G, H) satisfying (GH2), (GH3)—or (GH2)*, (GH3)*—and

(GH1)* For all $g_1, g_2 \in G$,

$$(*) \qquad G \neq (Hg_1 \cup H)(Hg_2 \cup H)\,;$$

and whose arrows $f\colon (G,H) \to (G',H')$ are those of $\mathbb{G}$ which further satisfy:

(0.3) Every subgroup G_1 of G containing H is of the form $f^{-1}(G_1')$ for some subgroup G_1' of G' containing H'.

If Θ is, as above, a pseudo-inverse of Γ, then (GH1)* is equivalent to the vector space $\Theta(G,H)$ having dimension ≥ 3 (cf. Corollary (2.2), which also shows that if $(*)$ holds for some two elements $g_1, g_2 \in G$ such that

$$H \neq Hg_1H \neq Hg_2H \neq H$$

then it holds for all $g_1, g_2 \in G$); and (0.3) is equivalent to $\Theta(f)$ being linearly faithful. In fact *the above functors Γ and Θ induce pseudo-inverse equivalences between the categories $\mathbb{V}_3 \subset \mathbb{V}$ and $\mathbb{G}_3 \subset \mathbb{G}$.*

Note that if $(G,H) \in \mathbb{G}_3$ then for any $h \in H$, the inner automorphism $\gamma_h\colon G \to G$, given by

$$\gamma_h(g) = hgh^{-1} \qquad (g \in G)$$

is an arrow in $\mathbb{G}_3$.

On $\mathbb{G}_3$ we consider the equivalence relation $\mathbf{R}^*$ under which two arrows

$$f_1, f_2\colon (G,H) \to (G',H')$$

are equivalent if there exists an $h \in H'$ such that $f_2 = \gamma_h \circ f_1$. As above, we have a quotient category $\mathbb{G}_3/\mathbf{R}^*$, together with a canonical functor $\rho^*\colon \mathbb{G}_3 \to \mathbb{G}_3/\mathbf{R}^*$. And there is a unique functor $\overline{\Theta}\colon \mathbb{G}_3/\mathbf{R}^* \to \mathbb{V}_3/\mathbf{R}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{G}_3 & \xrightarrow{\ \Theta\ } & \mathbb{V}_3 \\ {\scriptstyle \rho^*}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathbb{G}_3/\mathbf{R}^* & \xrightarrow[\ \overline{\Theta}\ ]{} & \mathbb{V}_3/\mathbf{R} \end{array}$$

This $\overline{\Theta}$ is also an equivalence of categories.

Following $\overline{\Theta}$ by the isomorphism $\overline{\Pi}$, we obtain the above-indicated *group-theoretic foundations for projective geometry*:

Theorem 2. *The category $\mathbb{P}_2$ of projective spaces of dimension ≥ 2 is equivalent to the quotient category $\mathbb{G}_3/\mathbf{R}^*$ just described.*

1. Proof of Theorem 1. We first define a category $\mathbb{S}$ which will serve as an intermediary in the proof of the equivalence of $\mathbb{G}$ and $\mathbb{V}$.

The objects of $\mathbb{S}$, called "pointed geometries," are triples (G, S, p) with S a set, $p \in S$, and G a group acting faithfully on S, i.e., there is a map $G \times S \to S$—for which the image of a pair (g, s) is denoted gs—such that

$$(gh)s = g(hs) \qquad (g, h \in G; s \in S)$$

and such that

$$gs = s \text{ for all } s \in S \iff g = e, \text{ the identity.}$$

(For motivation, consider the triple $(G, V, 0)$ described at the beginning of the Introduction; or cf. [L]). A *translation* of (G, S, p) is defined as in [L, p. 272] to be an element $g \in G$ such that either $g = e$ or g has no fixed points (i.e., $gs \neq s$ for all $s \in S$). A *line* of (G, S, p) is by definition a subset of S of the form

$$s_1 + s_2 = \{\, s \in S \mid s = s_1 \text{ or } \exists g \in G \text{ with } \ gs_1 = s_1,\ gs_2 = s \,\}$$

where s_1, s_2 are distinct points in S. It is assumed further that *G acts doubly transitively on lines, but not on all of S.* (Cf. [L, p. 268, AXIOM 1, and p. 271, AXIOM 2]; recall that a group acts doubly transitively on a set if for any s_1, s_2, s_3, s_4 in the set with $s_1 \neq s_2$, $s_3 \neq s_4$, there is a g in the group such that $gs_1 = s_3$ and $gs_2 = s_4$.) Note that then G acts *transitively* on S, i.e., for any s_1, s_2 in S there is a g in G with $gs_1 = s_2$. (There is even such a g which furthermore satisfies $gs_2 = s_1$: this is obvious if $s_1 = s_2$, and otherwise holds because s_1 and s_2 both lie on the line $s_1 + s_2$).

The arrows of $\mathbb{S}$ are pairs

$$(\phi, \psi)\colon (G, S, p) \to (G', S', p')$$

where $\phi\colon G \to G'$ is a group homomorphism and $\psi\colon S \to S'$ is a map of sets with $\psi(p) = p'$, such that

$$\psi(gs) = \phi(g)\psi(s) \qquad (g \in G,\ s \in S) \tag{1.1}$$

and such that

(1.2) ϕ takes non-identity translations of (G, S, p) to non-identity translations of (G', S', p').

Composition of arrows $(\phi, \psi)\colon (G, S, p) \to (G', S', p')$ and $(\phi', \psi')\colon (G', S', p') \to (G'', S'', p'')$ is defined in the obvious way:

$$(\phi', \psi') \circ (\phi, \psi) = (\phi' \circ \phi,\ \psi' \circ \psi)\colon (G, S, p) \to (G'', S'', p'')\,.$$

We observe in passing the following facts.

Remarks (1.3). *Let (ϕ, ψ) be a pair as above satisfying* (1.1).

1. *If ψ is bijective then* (1.2) *holds.*
2. *If* (1.2) *holds then both ϕ and ψ are injective.*

Proof. (1) Assuming ψ to be bijective, let g be a non-identity translation of (G, S, p), so that $gs \neq s$ for any $s \in S$. Since ψ is injective, therefore

$$\phi(g)\psi(s) = \psi(gs) \neq \psi(s)\,;$$

and since ψ is surjective, this means that $\phi(g)s' \neq s'$ for any $s' \in S'$, i.e., $\phi(g)$ is a non-identity translation of (G', S', p'). Thus (1.2) holds.

(2) Assuming (1.2), let us show first that ψ is injective. Let s_1, s_2 be distinct points of S. By [L, p. 272, Thm. 8] there exists a translation g (obviously non-identity) with $gs_1 = s_2$. Then

$$\psi(s_2) = \psi(gs_1) = \phi(g)\psi(s_1) \neq \psi(s_1)$$

because $\phi(g)$ is a non-identity translation. So ψ is indeed injective.

Finally, if $g \in G$ is such that $\phi(g) = e'$, the identity in G', then for every $s \in S$ we have

$$\psi(gs) = \phi(g)\psi(s) = \psi(s)\,,$$

and since ψ is injective, therefore $gs = s$. So $g = e$, and ϕ is injective, as asserted. □

We show now that:

Theorem 1a. *The categories $\mathbb{G}$ and $\mathbb{S}$ are equivalent.*

Proof. The asserted equivalence is induced by a well-known equivalence

$$\Phi\colon \overline{\mathbb{G}} \to \overline{\mathbb{S}}$$

where the categories $\overline{\mathbb{G}} \supset \mathbb{G}$ and $\overline{\mathbb{S}} \supset \mathbb{S}$ are as follows:

The objects of $\overline{\mathbb{G}}$ are pairs (G, H) with G a group and H a subgroup of G; and the arrows $(G, H) \to (G', H')$ are the group homomorphisms $f\colon G \to G'$ for which $f(H) \subset H'$.

The objects of $\overline{\mathbb{S}}$ are triples (G, S, p) with S a set, $p \in S$, and G a group acting transitively on S; and the arrows $(G, S, p) \to (G', S', p')$ are pairs of maps (ϕ, ψ) as above, satisfying (1.1) (but not necessarily (1.2)).

For $(G, H) \in \overline{\mathbb{G}}$, let G/H be the set consisting of all the left cosets of H in G. G acts transitively on G/H by left multiplication.

The above equivalence Φ is the functor given by

$$\Phi(G, H) = (G,\, G/H,\, H)$$
$$\Phi\big(\, f : (G, H) \to (G', H')\,\big) = (f, \psi_f)\colon (G,\, G/H,\, H) \to (G',\, G'/H',\, H')$$

where

$$\psi_f(gH) = f(g)H' \qquad (g \in G).$$

For $(G, S, p) \in \overline{\mathbb{S}}$, let $G_p \subset G$ be the *stabilizer* of p, i.e., the subgroup

$$G_p = \{\, g \in G \mid gp = p\,\}\,.$$

It is straightforward to verify that a pseudo-inverse of Φ is the functor $\Psi\colon \overline{\mathbb{S}} \to \overline{\mathbb{G}}$ given by

$$\Psi(G, S, p) = (G, G_p)$$
$$\Psi\big(\,(\phi, \psi)\colon (G, S, p) \to (G', S', p')\,\big) = \phi\colon (G, G_p) \to (G', G'_{p'})\,.$$

In fact $\Psi\Phi = \mathbf{1}_{\overline{\mathbb{G}}}$; and a functorial isomorphism $\Phi\Psi \xrightarrow{\sim} \mathbf{1}_{\overline{\mathbb{S}}}$ is given for $(G,S,p) \in \overline{\mathbb{S}}$ by

$$(1,\psi)\colon \Phi\Psi(G,S,p) = (G, G/G_p, G_p) \xrightarrow{\sim} (G,S,p)$$

where $\psi(gG_p) = gp$.

We will show that Φ maps the subcategory $\mathbb{G}$ of $\overline{\mathbb{G}}$ into the subcategory $\mathbb{S}$ of $\overline{\mathbb{S}}$, and that Ψ maps $\mathbb{S}$ into $\mathbb{G}$. It is easily checked that an isomorphism in $\overline{\mathbb{G}}$ between two objects of $\mathbb{G}$ is actually an isomorphism in $\mathbb{G}$; and similarly for $\mathbb{S} \subset \overline{\mathbb{S}}$. It will follow then that *the restrictions of* Φ *and* Ψ *to the respective subcategories* $\mathbb{G}$, $\mathbb{S}$ *are pseudo-inverse equivalences*, proving Theorem 1a.

From the simple fact that $g \in G$ is a non-identity translation of $(G,S,p) \in \overline{\mathbb{G}}$ (i.e., $gs \ne s$ for all $s \in S$) if and only if no conjugate of g lies in G_p, it follows easily that Φ *takes arrows of* $\mathbb{G}$ *to arrows of* $\mathbb{S}$, and that Ψ *takes arrows of* $\mathbb{S}$ *to arrows of* $\mathbb{G}$. So it remains to examine the effect of Φ (resp. Ψ) on *objects* of $\mathbb{G}$ (resp. $\mathbb{S}$).

Let us show that:

$$\Phi(G,H) = (G, G/H, H) \in \mathbb{S} \quad \text{for } (G,H) \in \mathbb{G}. \tag{1.4}$$

(1.4.1) *G acts faithfully on G/H.*

We must show: if $g' \in G$ is such that $g'gH = gH$ for all $gH \in G/H$, i.e., $g' \in gHg^{-1}$ for all $g \in G$, then $g' = e$. But by (GH1) there is a $g \in G$ such that the double cosets HgH and H are distinct, whence $HgH \cap H = \emptyset$; and from (GH2) we then get

$$g' \in gHg^{-1} \cap eHe^{-1} \subset (HgH \cup \{e\}) \cap H = \{e\}\,.$$

(1.4.2) *G does not act doubly transitively on G/H.*

By (GH1) there exist three distinct double cosets $H, HgH, Hg'H$. Then $H \ne g^{-1}H$ and $g'H \ne H$, but there is no $j \in G$ such that $jH = g'H$ and $jg^{-1}H = H$, since such a j would lie in $Hg'H \cap HgH = \emptyset$.

(1.4.3) *G acts doubly transitively on lines.*

We first give a condition for $cH \in G/H$ to lie on a line $aH + bH$.

Lemma(1.5). *If $aH \ne bH$ and $aH \ne cH$ are in G/H, then*

$$\begin{aligned} cH \in aH + bH \ &\underset{\text{(i)}}{\iff} \ a^{-1}c \in Ha^{-1}bH \\ &\underset{\text{(ii)}}{\iff} \ (aH)^{-1}(cH) = (aH)^{-1}(bH)\,. \end{aligned}$$

Proof. Since $cH \ne aH$, we have

$$\begin{aligned} cH \in aH + bH &\iff \exists g \in G \text{ with } gaH = aH,\ gbH = cH \\ &\iff aHa^{-1} \cap cHb^{-1} \ne \emptyset \\ &\iff Ha^{-1}b \cap a^{-1}cH \ne \emptyset \\ &\iff a^{-1}c \in Ha^{-1}bH. \end{aligned}$$

This proves the logical equivalence (i); and (ii) is obvious. □

Now let $g_1H, g_2H, g_3H, g_4H \in aH + bH$, $g_1H \neq g_2H$, $g_3H \neq g_4H$. By (1.5),

$$a^{-1}g_i \in H \cup Ha^{-1}bH \qquad (i = 1, 2, 3, 4)\,.$$

Note that $a^{-1}g_i \in H \iff g_iH = aH$. In particular, $a^{-1}g_1$ and $a^{-1}g_2$ can't both lie in H. It follows that

$$g_1^{-1}g_2 = (g_1^{-1}a)(a^{-1}g_2) \in Hb^{-1}aH \cup Ha^{-1}bH \cup (Hb^{-1}aH)(Ha^{-1}bH)\,.$$

By (GH3),

$$Hb^{-1}aH = Ha^{-1}bH,$$

and by (GH2),

$$(Hb^{-1}aH)(Ha^{-1}bH) = H(b^{-1}aHa^{-1}b)H \subset Hb^{-1}aH \cup H;$$

since $g_1H \neq g_2H$, therefore $g_1^{-1}g_2 \notin H$, and we conclude then that $g_1^{-1}g_2 \in Ha^{-1}bH$.

Similarly, $g_3^{-1}g_4 \in Ha^{-1}bH$. So

$$Hg_1^{-1}g_2H = Ha^{-1}bH = Hg_3^{-1}g_4H,$$

i.e., there exist $h_1, h_2 \in H$ such that

$$h_1g_1^{-1}g_2h_2 = g_3^{-1}g_4.$$

Thus, for g_1H, g_2H, g_3H, g_4H on one line, $g_1H \neq g_2H$, $g_3H \neq g_4H$, there is an element of G, (namely $g_3h_1g_1^{-1} = g_4h_2^{-1}g_2^{-1}$) which sends g_1H to g_3H and g_2H to g_4H.

This completes the proof of (1.4) □

Let us show finally that

(1.6) $$\Psi(G, S, p) = (G, G_p) \in \mathbb{G} \quad \text{for } (G, S, p) \in \mathbb{S}.$$

For convenience, set $G_p = H$.

(1.6.1) *$G \neq HgH \cup H$ for any $g \in G$.*

For, were $G = HgH \cup H$, then for any point $p \neq p_1 \in S$ we would have (by transitivity, see the definition of $\mathbb{S}$), for some $g_1 \in G$,

$$p_1 = g_1p = h_1gp \qquad (h_1 \in H),$$

and since $h_1p = p$ therefore $p_1 \in p + gp$. Thus the line $p + gp$ would be all of S; and since G acts doubly transitively on lines, but not on all of S, we would have a contradiction.

(1.6.2) *$HgH = Hg^{-1}H$ for any $g \in G$.*

We may assume that $g \notin H$, i.e., $gp \neq p$. Since G acts doubly transitively on the line $p + gp$, there exists $g_1 \in G$ such that

$$g_1p = gp, \qquad g_1gp = p.$$

It follows that there exist $h_1, h_2 \in H$ such that

$$g_1 = gh_1 = h_2g^{-1}.$$

Hence $g \in Hg^{-1}H$, i.e., $HgH = Hg^{-1}H$.

(1.6.3) $gHg^{-1} \subset HgH \cup \{e\}$ *for any* $g \in G$.

We may assume that $g \notin H$. Let $h \in H$, with $h \notin g^{-1}Hg$. Then p, $g^{-1}p$, and $hg^{-1}p$ are three distinct points on the line $L = p + g^{-1}p$. Since G acts doubly transitively on L, there is a $j \in G$ such that

$$jp = g^{-1}p \quad \text{and} \quad jg^{-1}p = hg^{-1}p.$$

Then we have $j = g^{-1}h_1$ for some $h_1 \in H$, and $g^{-1}h_1g^{-1}p = hg^{-1}p$, so that $hg^{-1} \in g^{-1}h_1g^{-1}H$, whence

$$ghg^{-1} \in Hg^{-1}H \overset{(1.6.2)}{=} HgH.$$

It remains to consider elements of the form ghg^{-1} with $h \in H \cap g^{-1}Hg$. Such an h has the two distinct points $p, g^{-1}p \in S$ as fixed points, and so by [L, p. 271, Thm. 5], $h = e$, and $ghg^{-1} = e$.

This completes the proof of (1.6), and of Theorem 1a. $\square$

Our next task is to describe a functor $\Theta\colon \mathbb{G} \to \mathbb{V}$ pseudo-inverse to Γ. This is basically an elaboration of [L, §7].

For $(G, H) \in \mathbb{G}$, set

$$\begin{aligned} T = T_{G,H} &= \{\,\text{translations of } (G,H)\,\} \\ &\overset{\text{def}}{=} \{e\} \cup \{\, g \in G \mid \text{no conjugate of } g \text{ lies in } H\,\}. \end{aligned}$$

The set T is closed under conjugation:

$$jTj^{-1} = T \qquad (j \in G).$$

As above, $\Phi(G, H) = (G, G/H, H) \in \mathbb{S}$; and it is immediate that T is the set of all translations of $(G, G/H, H)$—cf. beginning of this §1. Hence, by [L, p. 274, Thm. 11], *T is an abelian (normal) subgroup of G.* We also note, for later use, the following consequence of [L, p. 272, Thm. 8]:

(1.7) *Each left coset gH contains exactly one translation.*

Let E be the ring of additive endomorphisms of T, with zero-element $\mathbf{0} = \mathbf{0}_E$. We define a map $\gamma\colon H \to E$ by

$$[\gamma(h)](t) = hth^{-1} \qquad (h \in H,\ t \in T).$$

According to [L, p. 276], *γ is injective and $F = \gamma(H) \cup \{\mathbf{0}\}$ is a subfield of E.* Thus T is a vector space over F.

Extend γ to $\widetilde{\gamma}\colon H \cup \{\mathbf{0}\} \to E$ sending $\mathbf{0}$ to $\mathbf{0}$. Since $\widetilde{\gamma}$ is injective, there is a unique field structure on $\widetilde{H} = H \cup \{\mathbf{0}\}$ such that $\widetilde{\gamma}$ is a field isomorphism. For this structure, the product of two elements $h_1, h_2 \in H$ is the same as their product in G. In particular, the multiplicative identity $1_{\widetilde{H}}$ is e. The additive inverse $-h_1 \in \widetilde{H}$ is the unique $h \in H$ such that

$$hth^{-1} = (h_1th_1^{-1})^{-1} = h_1t^{-1}h^{-1} \quad \text{for all} \quad t \in T.$$

If $h_2 \neq -h_1$, then the sum $h_1 + h_2 \in \widetilde{H}$ is the unique $h \in H$ such that for all $t \in T$

$$hth^{-1} = h_1th_1^{-1}h_2th_2^{-1}. \tag{1.8}$$

All this follows from [L, p. 276]. We leave it to the reader to explicate sums and products in $\widetilde{H}$ involving $\mathbf{0}$.

Accordingly, we can regard T as a vector space over $\widetilde{H}$, with scalar multiplication

$$h \cdot t = hth^{-1} \qquad (h \in H,\ t \in T).$$

Remark (1.9). Given $h_1, h_2 \in H$, with $h_1 \neq -h_2$, if $h \in H$ is such that (1.8) holds for *one* non-identity $t \in T$, then $h = h_1 + h_2$.

This follows from [L, p. 275, Cor. 12.1] and the injectivity of γ.

The above-mentioned functor Θ is specified by:

$$\Theta(G, H) = (T_{G,H},\ \widetilde{H})$$
$$\Theta\big(f\colon (G,H) \to (G', H')\big) = (\zeta_f, \theta_f)\colon (T_{G,H},\ \widetilde{H}) \to (T_{G',H'},\ \widetilde{H'})$$

where

$$\begin{aligned} \zeta_f(t) &= f(t) \qquad (t \in T_{G,H}) \\ \theta_f(h) &= f(h) \qquad (h \in H) \\ \theta_f(\mathbf{0}) &= \mathbf{0}'. \end{aligned}$$

To see that Θ is indeed a functor from $\mathbb{G}$ to $\mathbb{V}$ we need to prove that (ζ_f, θ_f) is an injective semi-linear map.

The injectivity of ζ_f follows from that of f, which in turn follows from the fact that $\Phi(f) = (f, \psi_f)$ is a map in $\mathbb{S}$ (cf. (2) in (1.3) above.)

The conditions

$$\begin{aligned} \zeta_f(t_1 + t_2) &= \zeta_f(t_1) + \zeta_f(t_2) \qquad && (t_1, t_2 \in T_{G,H}) \\ \zeta_f(ht) &= \theta_f(h)\zeta_f(t) && (h \in \widetilde{H},\ t \in T) \\ \theta_f(h_1 h_2) &= \theta_f(h_1)\theta_f(h_2) && (h_1, h_2 \in \widetilde{H}) \\ \theta_f(1_{\widetilde{H}}) &= 1_{\widetilde{H'}} \end{aligned}$$

are trivial to verify. It remains then to show that

$$\theta_f(h_1 + h_2) = \theta_f(h_1) + \theta_f(h_2) \qquad (h_1, h_2 \in \widetilde{H}). \tag{1.10}$$

If any one of h_1, h_2, or $h_1 + h_2$ is $\mathbf{0}$, then (1.10) is obvious. Otherwise, we can apply f to (1.8), where $h = h_1 + h_2$ and t is a non-identity translation (which exists by (1.7), since $G \neq H$, cf. (GH1)) to get

$$f(h)f(t)f(h)^{-1} = f(h_1)f(t)f(h_1)^{-1}f(h_2)f(t)f(h_2)^{-1}.$$

By (0.2.2), no conjugate of $f(t)$ lies in H', i.e., $f(t)$ is a non-identity translation of (G', H'). Hence by (1.9),

$$f(h_1 + h_2) = f(h) = f(h_1) + f(h_2)\,,$$

proving (1.10).

Now that we have the functor Θ, let us show that it is a pseudo-inverse of Γ.

For $(G,H)\in\mathbb{G}$, and $T=T_{G,H}$, definitions yield

$$\Gamma\Theta(G,H)=\big(\{\,[h,t]\,\},\ \{\,[h,0]\,\}\big)\qquad (h\in H,\ t\in T)$$

where $[h,t]\colon T\to T$ is the map defined by

$$[h,t](\tau)=(\gamma(h)\tau)t=h\tau h^{-1}t\qquad (\tau\in T).$$

We define a map

$$\alpha=\alpha_{G,H}\colon \Gamma\Theta(G,H)\to(G,H)$$

by

$$\alpha[h,t]=th.$$

Clearly $\alpha\{\,[h,0]\,\}=H$; and one checks (using the fact that T is an abelian normal subgroup of G) that

$$\alpha\big(\,[h,t]\circ[h',t']\,\big)=tht'h'=\alpha[h,t]\alpha[h',t']\,,$$

so that α is a map in the category $\overline{\mathbb{G}}$. Furthermore, α is an *isomorphism* in $\overline{\mathbb{G}}$ (hence, as previously remarked, in $\mathbb{G}$) because, by (1.7), every element in G is uniquely of the form th.[3] Finally, α is *functorial*, i.e., for any arrow $f\colon(G,H)\to(G',H')$ in $\mathbb{G}$, the resulting diagram

$$\begin{array}{ccc}\Gamma\Theta(G,H) & \xrightarrow{\ \alpha_{G,H}\ } & (G,H)\\ {\scriptstyle\Gamma\Theta(f)}\big\downarrow & & \big\downarrow{\scriptstyle f}\\ \Gamma\Theta(G',H') & \xrightarrow[\ \alpha_{G',H'}\ ]{} & (G',H')\end{array}$$

commutes, as follows easily from definitions. Thus we have an isomorphism of functors $\Gamma\Theta\ \xrightarrow{\sim}\ \mathbf{1}_{\mathbb{G}}$.

Next, for $(V,F)\in\mathbb{V}$, let $(G,H)=\Gamma(V,F)$ be as in the Introduction, and let T be the subgroup of G consisting of all the translations $[1,v]$ with $v\in V$, so that with $\widetilde{H}$ as above we have

$$\Theta\Gamma(V,F)=(T,\widetilde{H})\,.$$

We define a map

$$(\zeta,\theta)=(\zeta_{V,F},\ \theta_{V,F})\colon \Theta\Gamma(V,F)\to(V,F)$$

by

$$\begin{aligned}\zeta[1,v]&=v &&(v\in V)\\ \theta[a,0]&=a &&(0\ne a\in F)\\ \theta(\mathbf{0})&=0_F\,.\end{aligned}$$

[3]The reader who so desires can rephrase this argument in terms of semi-direct products.

Both ζ and θ are bijective. We leave it to the reader to check (mechanically, via definitions) that (ζ, θ) is *semi-linear*, and hence is an isomorphism in $\mathbb{V}$. For functoriality, we need to check that for any arrow $(\xi, \eta)\colon (V, F) \to (V', F')$ in $\mathbb{V}$, the following diagram commutes:

$$\begin{array}{ccc} \Theta\Gamma(V,F) & \xrightarrow{(\zeta_{V,F},\,\theta_{V,F})} & (V,F) \\ {\scriptstyle \Theta\Gamma(\xi,\eta)}\big\downarrow & & \big\downarrow{\scriptstyle (\xi,\eta)} \\ \Theta\Gamma(V',F') & \xrightarrow[(\zeta_{V',F'},\,\theta_{V',F'})]{} & (V',F') \end{array}$$

This again is a mechanical exercise. So we have an isomorphism of functors $\Theta\Gamma \xrightarrow{\sim} \mathbf{1}_{\mathbb{V}}$; and Θ is indeed a pseudo-inverse of Γ.

This completes the proof of Theorem 1. $\square$

Remark (1.11). Let us verify Remark (iii) in the Introduction. Assuming (GH1)–(GH3), we can prove (GH2)* as in (1.4.1); and we find that

$$\begin{aligned}(HgH \cup H)(HgH \cup H)^{-1} &= (HgH \cup H)(Hg^{-1}H \cup H) \\ &= HgHg^{-1}H \cup HgH \cup Hg^{-1}H \cup H \\ &\subset HgH \cup H,\end{aligned}$$

yielding (GH3)*. Conversely, (GH3)* easily implies (GH3) and that for any $g \in G$,

$$gHg^{-1} \subset (HgH \cup H)(HgH \cup H)^{-1} \subset (HgH \cup H)\,.$$

So to prove (GH2), we need to show that if $g \notin H$ then

$$gHg^{-1} \cap H = \{e\}\,.$$

For this, observe that (GH3)* alone is enough for the proof of (1.4.3), (GH1) is enough for (1.4.2), and (GH2)* is enough for (1.4.1). So we can apply [L, p. 271, Thm. 5] to any $h \in gHg^{-1} \cap H$: since $hH = H$ and $hgH = gH$, therefore $h = e$. $\square$

Remark (1.12). Here is a variation on the theme of Theorem 1.

Recall the definition of "faithful group action" given near the beginning of §1. Let H be a group acting faithfully on a group T, via *automorphisms*, i.e.,

$$h(t_1 \cdot t_2) = ht_1 \cdot ht_2 \qquad (h \in H;\ t_1, t_2 \in T)$$

where $\cdot$ is the group operation in T. The *orbit* $\langle t \rangle$ of $t \in T$ is the set

$$\langle t \rangle = \{\, ht \mid h \in H \,\}\,.$$

For example, the orbit of the identity element e_T consists of e_T alone.

Theorem 1b. *Let H be a group acting faithfully via automorphisms on a group T. Then the following conditions (1) and (2) are equivalent:*

(1) *(i) There are at least three distinct orbits in T, and*
(ii) for any $t \in T$, $\langle t\rangle \cup \{e_T\}$ is a subgroup of T.

(2) *There is a vector space V of dimension ≥ 2 over a field F, such that H is the multiplicative group of F, T is the additive group of V, and the action $H \times T \to T$ is induced by the scalar multiplication $F \times V \to V$.*

Proof. Let G be the semi-direct product of H by T with respect to the given action: G is the set $T \times H$ with multiplication

$$(t, h)(t', h') = (t \cdot ht', hh')\,.$$

H can be identified with the subgroup $\{\,(e_T, h) \mid h \in H\,\}$ of G, T can be identified with the *normal* subgroup $\{\,(t, e_H) \mid t \in T\,\}$, and then the action of H on T is given by conjugation inside G. Note that:

$(*)$ every element $g \in G$ is uniquely of the form $g = th \quad (t \in T,\ h \in H)$.

In view of Remark (1.11) and the proof of Theorem 1, we can prove the implication $(1) \Rightarrow (2)$ by showing that the pair $G \supset H$ satisfies (GH1), (GH2)* and (GH3)*; and that T is the set of translations of (G, H), i.e., a non-identity element $g \in G$ is in T if and only if no conjugate of g is in H.

For (GH1), choose $t, t' \in T$ such that the orbits $\langle t\rangle$, $\langle t'\rangle$, and $\langle e_T\rangle$ are distinct. Inside G, if t' were in HtH we would have

$$t' = hth' = (hth^{-1})(hh') \qquad (h, h' \in H)$$

whence, by $(*)$, $t' = hth^{-1} \in \langle t\rangle$. So $t' \notin HtH$, and the three double cosets HtH, $Ht'H$ and H are distinct, proving (GH1).

For (GH2)*, let $t \in T$ and consider an element $h \in H \cap tHt^{-1}$. We have

$$eh = h = (th_1t^{-1}h_1^{-1})h_1 \qquad (h_1 \in H)$$

whence, by $(*)$, $e = th_1t^{-1}h_1^{-1}$ and $h = h_1$, so that $hth^{-1} = t$. Thus if $h \in \bigcap_{t\in T} tHt^{-1}$, then $hth^{-1} = t$ for all $t \in T$, and since H acts *faithfully* on T, therefore $h = e$, proving (GH2)*.

For (GH3)*, note first that for any $t \in T$ we have, inside G,

$$(**) \qquad H\langle t\rangle = \langle t\rangle H,$$

as follows from the identities

$$\begin{aligned} h_1hth^{-1} &= (h_1hth^{-1}h_1^{-1})h_1 \\ hth^{-1}h_1 &= h_1(h_1^{-1}hth^{-1}h_1) \qquad (h, h_1 \in H). \end{aligned}$$

For any $g \in G$, writing $g = th$, cf. $(*)$, we see then that

$$HgH \cup H = HtH \cup H = H\langle t\rangle \cup H,$$

and we deduce easily from (ii) in (1) and from $(**)$ that $HgH \cup H$ is a subgroup of G, proving (GH3)*.

Finally, since the normal subgroup T equals any of its conjugates in G, and since $T \cap H = \{e\}$, it is clear that every $t \in T$ is a translation. Conversely, if $g = th$ (cf. $(*)$) is a translation, then (1.7) implies that $g = t \in T$.

This completes the proof that (1) $\Rightarrow$ (2); and the implication (2) $\Rightarrow$ (1) is clear. $\square$

There is an obvious functor Γ^* from the category $\mathbb{V}$ to the category whose objects are pairs (T, H) satisfying the conditions in Theorem 1b, and whose arrows $(T, H) \to (T', H')$ are pairs (ξ, η) where $\xi\colon T \to T'$ is an *injective* group homomorphism and $\eta\colon H \to H'$ is a group homomorphism such that

$$\xi(ht) = \eta(h)\xi(t) \qquad (h \in H,\ t \in T).$$

We leave it to the reader to verify that: Γ^* *is an equivalence of categories.*

2. Proof of Theorem 2.

Proposition (2.1). *Let (V, F) and $\Gamma(V, F) = (G, H)$ be as in the Introduction. To each subspace V' of V associate the subgroup $\Gamma_0(V')$ of G given by*

$$\Gamma_0(V') = \left\{ [a, v] \in G \;\middle|\; 0 \neq a \in F,\ v \in V' \right\}.$$

(1) Γ_0 *is an inclusion preserving bijective map from the set of subspaces of V onto the set of subgroups of G containing H.*

(2) *For any element $0 \neq v \in V$ and any non-zero $c \in F$ we have*

$$\Gamma_0(Fv) = (H[c, v]H) \cup H.$$

(3) *For any two subspaces $V_1 \subset V$, $V_2 \subset V$, we have*

$$\Gamma_0(V_1 + V_2) = \Gamma_0(V_1)\Gamma_0(V_2) = \Gamma_0(V_2)\Gamma_0(V_1).$$

Proof. (1) First of all it is clear that $\Gamma_0(V')$ is a subgroup of G containing H.

We show that Γ_0 is bijective by constructing an inverse map. For any subgroup $G' \subset G$ with $G' \supset H$ set

$$\Theta_0(G') = \left\{ v \in V \;\middle|\; [1, v] \in G' \right\}.$$

Since

$$[1, v_1][1, v_2] = [1, v_1 + v_2],$$

therefore $\Theta_0(G')$ is closed under addition; and since for non-zero $a \in F$ we have $[a, 0] \in H \subset G'$ and

$$[a, 0][1, v][a, 0]^{-1} = [1, av],$$

therefore $\Theta_0(G')$ is closed under scalar multiplication; so $\Theta_0(G')$ is a subspace of V.

By definition

$$v \in \Theta_0\Gamma_0(V') \iff [1, v] \in \Gamma_0(V') \iff v \in V',$$

i.e., $\Theta_0\Gamma_0(V') = V'$.

Moreover, if $[1, v] \in G'$ and $0 \neq a \in F$, then

$$[a, v] = [1, v][a, 0] \in G' \,;$$

and conversely if $[a, v] \in G'$, then

$$[1, v] = [a, v][a, 0]^{-1} \in G'.$$

It follows easily that $\Gamma_0\Theta_0(G') = G'$, and (1) is proved.

(2) We have

$$\begin{aligned}\Gamma_0(Fv) &= \big\{\, [a, bv] \;\big|\; 0 \neq a \in F,\ 0 \neq b \in F \,\big\} \cup \big\{\, [a, 0] \;\big|\; 0 \neq a \in F \,\big\} \\ &= \big\{\, [b, 0][c, v][c^{-1}b^{-1}a, 0] \;\big|\; 0 \neq a \in F,\ 0 \neq b \in F \,\big\} \cup H \\ &= (H[c, v]H) \cup H.\end{aligned}$$

(3) The inclusions

$$\Gamma_0(V_2)\Gamma_0(V_1) \subset \Gamma_0(V_1 + V_2) \supset \Gamma_0(V_1)\Gamma_0(V_2)$$

are obvious; and the opposite inclusions follow from the equalities

$$[1, v_2][a, v_1] = [a, v_1 + v_2] = [1, v_1][a, v_2]\,. \quad \square$$

Corollary (2.2). *Given vectors* $v_0, v_1, \ldots, v_n \in V$, *and non-zero elements* $c_i \in F$ $(0 \leq i \leq n)$, *with* $g_i = [c_i, v_i]$ *we have that*

$$\begin{aligned}&v_0 \text{ is a linear combination of } v_1, v_2, \ldots, v_n \\ &\iff g_0 \in (Hg_1H \cup H)(Hg_2H \cup H)\cdots(Hg_nH \cup H)\,.\end{aligned}$$

Corollary (2.3). Γ_0 *induces a one-one correspondence between the set of 1-dimensional subspaces of* V *(i.e., the points of the projective space* $P(V, F)$*) and the set of double cosets* $HgH \neq H$.

Corollary (2.4). *Let* $(\zeta, \theta)\colon (V, F) \to (V', F')$ *be a semi-linear map, and let*

$$f = \Gamma(\zeta, \theta)\colon \Gamma(V, F) = (G, H) \to (G', H') = \Gamma(V', F')\,.$$

Then ζ *is linearly faithful* $\iff$ (0.3) *holds.*

Proof. Recall from the Introduction that ζ linearly faithful means that every subspace $V_1 \subset V$ is of the form $\zeta^{-1}(V_1')$ for some subspace $V_1' \subset V'$. The conclusion follows easily from (1) in (2.1). $\square$

Let us show now that, as asserted in the Introduction, *the functors* Γ *and* Θ *induce pseudo-inverse equivalences between the categories* $\mathbb{V}_3 \subset \mathbb{V}$ *and* $\mathbb{G}_3 \subset \mathbb{G}$.

It will suffice to show that (a): $\Gamma(\mathbb{V}_3) \subset \mathbb{G}_3$, and (b): $\Theta(\mathbb{G}_3) \subset \mathbb{V}_3$. Then any pair of isomorphisms $\Gamma\Theta \xrightarrow{\sim} \mathbf{1}$, $\Theta\Gamma \xrightarrow{\sim} \mathbf{1}$, will induce similar isomorphisms for the restrictions of

Γ and Θ to $\mathbb{V}_3$ and $\mathbb{G}_3$ respectively, because any isomorphism in $\mathbb{V}$ between objects of $\mathbb{V}_3$ is linearly faithful, and hence is an isomorphism in $\mathbb{V}_3$; and similarly any isomorphism in $\mathbb{G}$ between objects of $\mathbb{G}_3$ is an isomorphism in $\mathbb{G}_3$.

Assertion (a) follows easily for objects from (2.2) and for arrows from (2.4).

As for (b), let us first consider an object $(G, H) \in \mathbb{G}_3$, and set $\Theta(G, H) = (V, F)$. Then (G, H) is isomorphic to $\Gamma(V, F)$ (in $\mathbb{G}$). By definition of $\mathbb{G}_3$,

$$G \neq (Hg_1H \cup H)(Hg_2H \cup H) \quad \text{for all} \quad g_1, g_2 \in G,$$

and it follows from (2.2) that V is not spanned by two 1-dimensional subspaces, i.e., $\dim V_3 \geq 3$, i.e., $(V, F) \in \mathbb{V}_3$.

Let $f\colon (G, H) \to (G', H')$ be a map in $\mathbb{G}_3$, and let $(V, F) = \Theta(G, H)$, $(V', F') = \Theta(G', H')$. Then the functorial isomorphism $\Gamma\Theta \xrightarrow{\sim} \mathbf{1}$ gives a commutative diagram

$$\begin{array}{ccc} \Gamma(V,F) & \xrightarrow{\sim} & (G,H) \\ {\scriptstyle \Gamma\Theta(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \Gamma(V',F') & \xrightarrow{\sim} & (G',H') \end{array}$$

where the horizontal arrows are isomorphisms. Then $\Gamma\Theta(f)$ satisfies condition (0.3) since f does; and hence by (2.4), $\Theta(f)$ is linearly faithful. This completes the proof of (b). $\square$

Next let us note that Γ and Θ respect the equivalence relations $\mathbf{R}$ and $\mathbf{R}^*$ defined in the Introduction. Indeed, if $(G, H) = \Gamma(V, F)$, and if $0 \neq a \in F$, so that $h = [a, 0] \in H$, then

$$\Gamma(\zeta_a, \theta_a) = \gamma_h\,.$$

Conversely, if $(V, F) = \Theta(G, H)$, then there is an $a \in F$ such that

$$\Theta(\gamma_h) = (\zeta_a, \theta_a)\,.$$

To see this, recall that there is a functorial isomorphism

$$f\colon (G, H) \xrightarrow{\sim} \Gamma\Theta(G, H)$$

and hence $\Gamma\Theta(\gamma_h) = \gamma_{f(h)}$; but as we just saw, there is an a such that $\Gamma(\zeta_a, \theta_a) = \gamma_{f(h)}$; and since the equivalence Γ acts injectively on arrows, this a is as desired.

The existence of the functor $\overline{\Theta}$ defined near the end of the Introduction follows easily, as does the existence of a unique functor $\overline{\Gamma}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{G}_3 & \xleftarrow{\ \Gamma\ } & \mathbb{V}_3 \\ {\scriptstyle \rho^*}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathbb{G}_3/\mathbf{R}^* & \xleftarrow[\ \overline{\Gamma}\ ]{} & \mathbb{V}_3/\mathbf{R} \end{array}$$

Finally, $\overline{\Theta}$ *and* $\overline{\Gamma}$ *are pseudo-inverse equivalences.* For example to get an isomorphism $\overline{\Gamma}\,\overline{\Theta} \xrightarrow{\sim} \mathbf{1}_{\mathbb{G}_3/\mathbf{R}^*}$ we need for each $A = (G, H) \in \mathbb{G}_3/\mathbf{R}^*$ an isomorphism $\overline{\Gamma}\,\overline{\Theta}(A) \xrightarrow{\sim} A$. But $\overline{\Gamma}\,\overline{\Theta}(A) = \Gamma\Theta(A)$, and we have an isomorphism $f\colon \Gamma\Theta(A) \xrightarrow{\sim} A$ in $\mathbb{G}_3$. The equivalence class of f under $\mathbf{R}^*$ gives what we want.

Theorem 2 should now be clear.

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