Lesson 11: The chain rule; Derivatives of the natural log.

Chain rule

$$y = g(h(x))$$

$$\frac{dy}{dx} = \frac{dy}{dh} \cdot \frac{dh}{dx} \qquad y' = g'(h(x)) \cdot h'(x)$$

Examples

① $f(x) = (3x+1)^3 (x^4 - 3x)^{4/3}$

$$f'(x) = \frac{d}{dx}\left[(3x+1)^3\right] \cdot (x^4 - 3x)^{4/3} + (3x+1)^3 \cdot \frac{d}{dx}\left[(x^4-3x)^{4/3}\right]$$

$$\frac{d}{dx}\left((3x+1)^3\right) = 3(3x+1)^2 \cdot 3$$

$$= 9(3x+1)^2$$

$$\frac{d}{dx}\left[(x^4 - 3x)^{4/3}\right] = \frac{4}{3}(x^4 - 3x)^{\overbrace{4/3 - 1}^{= 1/3}} \cdot (4x^3 - 3)$$

$$= \frac{4}{3}(x^4 - 3x)^{1/3}(4x^3 - 3)$$

$$f'(x) = 9(3x+1)^2 \cdot (x^4 - 3x)^{4/3} + (3x+1)^3 \cdot \frac{4}{3}(x^4 - 3x)^{1/3}(4x^3 - 3)$$

② $y = 3\sin(2x)$

out: $g(x) = 3\sin(x)$

$g'(x) = 3\cos x$

in: $h(x) = 2x$

$h'(x) = 2$

$$g(h(x)) = 3\sin(2x) = y \quad \checkmark$$

$$y' = g'(h(x)) \cdot h'(x)$$

$$= 3\cos(2x) \cdot 2$$

$$= 6\cos(2x) \quad \checkmark$$

③ $y = 3\csc(5x^3 - 2x^2 + 7)$

out: $g(x) = 3\csc(x)$

$g'(x) = -3\csc x \cdot \cot x$

in: $h(x) = 5x^3 - 2x^2 + 7$

$h'(x) = 15x^2 - 4x + 0$

$$y' = g'(h(x)) \cdot h'(x)$$

$$= -3\csc(5x^3 - 2x^2 + 7)\cot(5x^3 - 2x^2 + 7) \cdot (15x^2 - 4x) \quad \checkmark$$

(4) $y = 6\tan^2(4x)$ $\qquad y'\left(\frac{\pi}{12}\right) = ?$

$= 6\left(\tan(4x)\right)^2$

out: $g(x) = 6x^2$ $\qquad h(x) = \tan(4x)$

$g'(x) = 12x$ $\qquad h'(x) = \sec^2(4x) \cdot 4$

$= 4\sec^2(4x)$

$$y' = g'(h(x)) \cdot h'(x)$$

$$= 12(\tan(4x)) \cdot 4 \cdot \sec^2(4x)$$

$$= 48\sec^2(4x)\tan(4x)$$

$$y'\left(\frac{\pi}{12}\right) = 48\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)$$

$$= 48\left(\sec\left(\frac{\pi}{3}\right)\right)^2 \cdot \frac{\sin(\pi/3)}{\cos(\pi/3)}$$

$$= 48\left(\frac{1}{\cos(\pi/3)}\right)^2 \cdot \frac{\sin(\pi/3)}{\cos(\pi/3)}$$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\qquad = 48\left(\frac{1}{1/2}\right)^2 \cdot \frac{\sqrt{3}/2}{1/2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$ $\qquad = 48(2)^2 \cdot \frac{\sqrt{3}}{1}$

$$= 48 \cdot 4\sqrt{3}$$

$$= 192\sqrt{3}$$

Derivative of natural log.

$$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$$

⑤ $f(x) = e^{3x}\ln(x^2+3)$

$$f'(x) = \frac{d}{dx}\left[e^{3x}\right]\cdot\ln(x^2+3) + e^{3x}\frac{d}{dx}\left[\ln(x^2+3)\right]$$

$$\frac{d}{dx}(e^{3x})$$

out $g(x) = e^x$ $\quad g'(x) = e^x$

in $h(x) = 3x$ $\quad h'(x) = 3$

$$\| \quad g'(h(x))\cdot h'(x) = e^{3x}\cdot 3 = 3e^{3x}$$

$$\frac{d}{dx}\left[\ln(x^2+3)\right]$$

out $g(x) = \ln x$ $\quad g'(x) = \frac{1}{x}$

in $h(x) = x^2+3$ $\quad h'(x) = 2x$

$$\| \quad g'(h(x))\cdot h'(x) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}$$

$$f'(x) = 3e^{3x}\cdot\ln(x^2+3) + e^{3x}\cdot\frac{2x}{x^2+3} \quad \checkmark$$

(5) $y = \ln\sqrt{\dfrac{5x+1}{x^2-4}} = \ln\left(\dfrac{5x+1}{x^2-4}\right)^{1/2}$

$$= \frac{1}{2}\ln\left(\frac{5x+1}{x^2-4}\right)$$

out: $g(x) = \frac{1}{2}\ln x$

$g'(x) = \dfrac{1}{2x}$

in: $h(x) = \dfrac{5x+1}{x^2-4}$

needs quotient rule.

$y' = g'(h(x)) \cdot h'(x)$

$$= \frac{1}{2 \cdot \frac{5x+1}{x^2-4}} \cdot h'(x).$$

Lesson 11: The chain rule; Derivative of the natural log.

The Chain rule:

Suppose $y = g(h(x))$, then

$$\frac{dy}{dx} = \frac{dy}{dh} \cdot \frac{dh}{dx} \qquad y' = g'(h(x)) \cdot h'(x)$$

Examples

① $f(x) = (3x+1)^3(x^4 - 3x)^{4/3}$

$$f'(x) = \frac{d}{dx}\left((3x+1)^3\right)(x^4-3x)^{4/3} + (3x+1)^3 \frac{d}{dx}\left((x^4-3x)^{4/3}\right)$$

$$\frac{d}{dx}\left((3x+1)^3\right)$$

out: $g(x) = x^3$, $g'(x) = 3x^2$ in: $h(x) = 3x+1$, $h'(x) = 3$

$$\| $$

$$g'(h(x)) \cdot h'(x) = 3(3x+1)^2 \cdot 3$$

$$= 9(3x+1)^2$$

$$\frac{d}{dx}\left((x^4-3x)^{4/3}\right)$$

out: $g(x) = x^{4/3}$, $g'(x) = \frac{4}{3}x^{1/3}$ in: $h(x) = x^4 - 3x$, $h'(x) = 4x^3 - 3$

$$\|$$

$$g'(h(x)) \cdot h'(x) = \frac{4}{3}(x^4-3x)^{1/3} \cdot (4x^3 - 3)$$

$$f'(x) = 9(3x+1)^2 \cdot (x^4-3x)^{4/3} + (3x+1)^3 \cdot \frac{4}{3}(x^4-3x)^{1/3}(4x^3-3)$$

(2) $y = 3\csc(5x^3 - 2x^2 + 7)$

out: $g(x) = 3\csc x$

$g'(x) = -3\csc x \cot x$

in: $h(x) = 5x^3 - 2x^2 + 7$

$h'(x) = 15x^2 - 4x + 0$

$$y' = g'(h(x)) \cdot h'(x)$$

$$= -3\csc(5x^3 - 2x^2 + 7)\cot(5x^3 - 2x^2 + 7)(15x^2 - 4x) \checkmark$$

(3) $y = 6\tan^2(4x)$ $\quad y'\left(\frac{\pi}{12}\right)$

$= 6(\tan(4x))^2$

out: $g(x) = 6x^2$

$g'(x) = 12x$

in: $h(x) = \tan(4x)$

$h'(x) = \sec^2(4x) \cdot 4$

$$y' = g'(h(x)) \cdot h'(x) = 12(\tan(4x)) \cdot \sec^2(4x) \cdot 4$$

$$= 48\sec^2(4x)\tan(4x) \checkmark$$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$$y'\left(\frac{\pi}{12}\right) = 48\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)$$

$$= 48\left(\frac{1}{\cos(\pi/3)}\right)^2 \frac{\sin(\pi/3)}{\cos(\pi/3)}$$

$$= 48\left(\frac{1}{1/2}\right)^2 \cdot \frac{\sqrt{3}/2}{1/2}$$

$$= 48 \cdot 2^2 \cdot \sqrt{3}$$

$$= 192\sqrt{3}$$

Derivative of $\ln(x)$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

Examples:

(4) $f(x) = e^{3x}\ln(x^2+3)$

$$f'(x) = \frac{d}{dx}[e^{3x}]\cdot\ln(x^2+3) + e^{3x}\cdot\frac{d}{dx}[\ln(x^2+3)]$$

$\frac{d}{dx}[e^{3x}]$ out: $g(x) = e^x$ in: $h(x) = 3x$

$g'(x) = e^x$ $h'(x) = 3$

$\|$

$$g'(h(x))\cdot h'(x) = e^{3x}\cdot 3 = 3e^{3x}$$

$\frac{d}{dx}[\ln(x^2+3)]$ out: $g(x) = \ln x$ in: $h(x) = x^2+3$

$g'(x) = \frac{1}{x}$ $h'(x) = 2x$

$\|$

$$g'(h(x))\cdot h'(x) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}$$

$$f'(x) = 3e^{3x}\cdot\ln(x^2+3) + e^{3x}\cdot\frac{2x}{x^2+3}$$

✓

(5) $y = \ln\sqrt{\dfrac{5x+1}{x^2-4}} = \ln\left(\dfrac{5x+1}{x^2-4}\right)^{1/2} = \dfrac{1}{2}\ln\left(\dfrac{5x+1}{x^2-4}\right)$

out: $g(x) = \frac{1}{2}\ln x$ in: $h(x) = \dfrac{5x+1}{x^2-4}$

$g'(x) = \frac{1}{2}\cdot\frac{1}{x}$ $h'(x) = \dfrac{5(x^2-4)-(5x+1)(2x)}{(x^2-4)^2}$

$y' = g'(h(x))\cdot h'(x)$

$$= \frac{1}{2}\cdot\frac{1}{\frac{5x+1}{x^2-4}}\cdot\frac{5(x^2-4)-(5x+1)(2x)}{(x^2-4)^2}$$

$$= \frac{1}{2}\cdot\frac{x^2-4}{5x+1}\cdot\frac{5(x^2-4)-2x(5x+1)}{(x^2-4)^2}$$

$$= \frac{1}{2}\cdot\frac{5(x^2-4)-2x(5x+1)}{(5x+1)(x^2-4)}$$ ✓